β-Supergravity - an Effective Field Theory with non-geometric Fluxes

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit β-Supergravitation, einer zehn-dimensionalen Theorie mit nicht-geometrischen Flüssen. Zunächst wird eine Einführung in Verallgemeinerte Geometrie und Doppelfeldtheorie gegeben, welche Reformulierungen von üblichen Supergravitationstheorien erlauben. Letztere ist versehen mit einer starken Zwangsbedingung, welche mittels einer unter T-Dualität invarianten zwei-dimensionalen konformen Feldtheorie untersucht wird. Aufbauend auf früheren Ergebnissen wird β-Supergravitation in Bezug auf beide Theorien eingeordnet und das Studium des NSNS Sektors ermöglicht dann die nicht-geometrischen Q- und R-Flüsse exakt zu identifizieren. Unter Verwendung des Verallgemeinerten Geometrie Formalismus wird der Lagrangian reproduziert und die Bewegungsgleichungen abgeleitet. Interessanterweise tauchen dabei neue Strukturen auf und der Q-Fluss geht in eine Nachbildung des Levi-Civita-Zusammenhangs, welcher die Definition eines zweiten Krümmungsskalar ermöglicht, ein. Dies macht β-Supergravitation zu einem vielversprechenden Anhebungskandidaten für einige vier-dimensionale geeichte Supergravitationstheorien.

Geometrische Hintergründe mit nicht-geometrischen Flüssen sind offensichtlich ein Anwendungsgebiet für β-Supergravitation. Hier zeigt die Besprechung von T-dualen, torusförmigen Hintergründen ein konsistentes Supergravitationslimit für die nicht-geometrische Konfiguration auf. Und die Symmetrien, insbesondere β Eichtransformationen, berechtigen die Einführung eines verallgemeinerten kotangential Bündels. Allerdings kann eine konsistente Klasse von Vakua, welche nicht-geometrisch für gewöhnliche Supergravitation ist, nur mittels β-Transformationen, die eine Symmetrie des Lagrangian unter Isometrien darstellen, definiert werden. Weiter liegen diese in einem geometrischen Orbit unter T-Dualität. Anschließende Untersuchungen überprüfen die Existenz von zehn-dimensionalen Lösungen der Bewegungsgleichungen, welche auf den NSNS Sektor beschränkt sind.

Eine interessante Anwendung findet β-Supergravitation in der Beschreibung von NS-Branen und Bianchi Identitäten für NSNS (nicht)-geometrische Flüsse. Dazu gehören die $NS5$-Brane, der Kaluza-Klein Monopol und die exotische 5_2^2- bzw. Q-Brane. Insbesondere erhalten auf zehn Dimensionen verallgemeinerte Bianchi Identitäten Korrekturen durch Quellterme von einzelnen NS-Branen. In Abwesenheit von Quellen, können diese Bianchi Identitäten mittels eines nilpotenten $Spin(D, D) \times \mathbb{R}^+$ Dirac Operators erzeugt werden.

β-Supergravitation erlaubt es weiter zehn-dimensionale supersymmetrische Vakua mit NSNS nicht-geometrischen Flüssen zu studieren. Hier können die internen Killing Spinor Gleichungen, welche die Supersymmetriebedingungen festlegen, mittels sogenannter einfacher Spinoren, welche eine $SU(3) \times SU(3)$ Struktur in Verallgemeinerter Komplexer Geometrie definieren, umformuliert werden. Der verallgemeinerte Dirac Operator $\mathcal{D}$, welcher von nicht-geometrischen Flüssen abhängt und den üblichen Operator $\mathrm{d} - H\wedge$ ersetzt, spielt dabei eine entscheidende Rolle. Ein allgemeiner Ausdruck für ein Superpotential schließt ebenfalls diesen Operator mit ein und wird mit der Literatur verglichen. Abschließend erfolgt eine geometrische Charakterisierung von Hintergründen, welche die Supersymmetrie erhalten.

Abstract

In this thesis ten-dimensional theory, named β-supergravity, is presented that contains non-geometric fluxes. Being inspired by Generalized Geometry and Double Field Theory, a review of both is given with regard to a reformulation of standard supergravities. The latter has to be equipped with the so-called strong constraint that we trace in a two-dimensional T-duality invariant conformal field theory. Building on earlier work, β-supergravity is classified with respect to the two former theories and study its NSNS sector, where the non-geometric Q- and R-fluxes are precisely identified. Using the Generalized Geometry formalism, the Lagrangian is reproduced and its equations of motion are derived. Interestingly, new structures appear and the Q-flux is captured in an analogue of the Levi-Civita spin connection that gives rise to a second curvature scalar. This makes β-supergravity a promising candidate for uplifting some four-dimensional gauged supergravities.

Evidently, geometric backgrounds with non-geometric fluxes are an interesting field of applying β-supergravity. Reviewing the toroidal example a consistent supergravity limit for non-geometric configuration is recovered. The study of the symmetries of β-supergravity, in particular β gauge transformations, introduces the notion of a generalized cotangent bundle. However, only β-transforms being a manifest symmetry of the Lagrangian with isometries allow to determine a well-defined class of vacua that are non-geometric in standard supergravity, but lie on a geometric T-duality orbit. Further investigations are related to ten-dimensional purely NSNS solution solving the equations of motion.

An interesting area of application of β-supergravity are NS-branes, including the $NS5$-brane, the Kaluza-Klein monopole and the exotic 5^2_2- or Q-brane, together with Bianchi identities for NSNS (non)-geometric fluxes. Four-dimensional Bianchi identities are generalized to ten dimensions with non-constant fluxes and introduce corrections by source terms in presence of an NS-brane. In the absence of sources, our Bianchi identities are recovered by squaring a nilpotent $Spin(D,D) \times \mathbb{R}^+$ Dirac operator.

β-supergravity further allows to study ten-dimensional supersymmetric vacua with NSNS non-geometric fluxes. Specifying a compactification ansatz, internal Killing spinor equations providing supersymmetry conditions are reformulated in terms of pure spinors defining an $SU(3) \times SU(3)$ structure in Generalized Complex Geometry. This involves the generalized Dirac operator $\mathcal{D}$ depending on non-geometric fluxes and replacing the standard $\mathrm{d} - H\wedge$ acting on pure spinors. A proposed general expression for the superpotential also involves $\mathcal{D}$ and is verified to agree with formulas of the literature. Finally preserving supersymmetry, a geometrical characterization of backgrounds is presented.

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Contents

Chapter 1

Introduction

The perception of space and time moves people in astonishing ways. Throughout history the modeling determined the faith of man. Whereas for the earliest cultures it was mainly a question of observing and predicting the movement of celestial objects, already in ancient Greece astronomy evolved as an independent field of mathematics by adopting geometrical principles. However, it was not before Kepler that physics attributed mathematical predictions to a cause. The concepts for space and time in which Newton formulated the law of universal gravitation preceded by the measurements of gravitational acceleration by Galilei are to the moment under severe discussion. Abandoning absolute space and a uniformly passing time the concept of an inertial reference frame in special relativity, in which the laws of physics are invariant and the speed of light is the same for all observers, introduces the notion of an intertwined space-time. Finally, Einstein's theory of general relativity transforms space-time into a dynamical object itself. Matter and energy content affect the shape and curvature of it and vice versa the motion of a free-falling object follows a geodesic line in a curved space-time. Interestingly, black holes experience in the form of the Schwarzschild solution to the classical Einstein field equations a singularity at the origin that infringes the well-definedness of space-time. This is a first hint at a missing understanding of space and time.

The elementary particles building up the matter content of the universe are governed by the electromagnetic, the weak and the strong force described in the Standard Model of particle physics, a quantum field theory. Its particular gauge group gives rise to the electroweak theory unifying the photon with the $W^{\pm}$- and Z^0-bosons and to Quantum Chromodynamics governing quarks and gluons. It allows to predict high energy scattering processes with a high accuracy up to around 100 GeV.

Only recently, the observation of the Higgs particle[1] at the Large Hadron Collider (LHC) at Cern completed the Standard Model. The necessity of a scalar boson is provided by the Higgs mechanism spontaneously inducing electroweak symmetry breaking at a scale $m_{ew} = 246\ GeV$. This gives rise to masses for gauge fields of the weak force and the different types of matter particles observed in nature due to a nontrivial vacuum expectation value

[1]The detection of a new particle with mass $m_H = 126\ GeV$ was published by the ATLAS collaboration in[1].

of the Higgs field. An open question concerning the quantum correction to the mass of the Higgs boson is known by the name of Higgs hierarchy problem. In principle, these corrections which result from fermions running in the loop are of the same order as the experimentally measured Higgs mass unless an important fine-tuning takes place.

In the same way as the electromagnetic and the weak force are unified in the Standard Model it is believed that at an energy scale around $m_{GUT} = 10^{16}\ GeV$ the coupling strength of the strong force joins the one of the electroweak . Theories favoring this kind of behavior of the coupling constants are called Grand Unified Theories (GUTs) and embed the Standard Model gauge group into a larger group like $SU(5)$ or $SO(10)$. Supersymmetry (SUSY) extends the symmetries of these kinds of theories even further and relates fermions and bosons which then can be arranged in so-called multiplets containing particles of different spin. Besides enforcing the coupling constants of the Standard Model to coincide SUSY is favored for naturally extending the Poincaré group and resolving the Higgs hierarchy problem. There is hope that the idea of SUSY can be verified at the LHC in the near future, because the breaking scale of SUSY should not be much higher than the scale of the Standard Model in order to solve the Higgs hierarchy problem. Following the scheme of unification it is certainly desirable to find a Theory of Everything (ToE) in an even higher energy regime that allows the gravitational force to join in.

Yet, gravity is in certain ways different from the three fundamental forces of nature which might have severe consequence for our understanding of the geometry of space-time. First, Einstein's theory of gravity is a classical field theory valid at large distances. Its coupling strength, the Newton constant G_N, is 10^{33} times weaker than the Fermi constant G_F being responsible for the weak interaction. This certainly explains why gravity plays no role in the subatomic processes described by the Standard Model. However, for phenomena like the Planck epoch of the early universe or the vicinity of black hole singularities with energies up to the Planck scale $m_P = 10^{19}\ GeV$ gravity becomes relevant. Here, high curvatures appear in very small regions of space demanding for a proper quantum description of gravity. On the experimental side energies around the Planck scale are out of range of any present or future collider experiment and recent excitement about measured imprints of quantum gravity[2] in the observed microwave background predicted by inflation models turned out to be premature. Unfortunately, also the theoretical access to a quantum theory of gravity is obscure since a consistent quantization method for general relativity has not been found so far due to its non-renormalisability. Speaking loosely, this has to do with the modeling of space-time at very tiny length scales. It is commonly believed that a shift in the perception of space and time might resolve the issue of the incompatibility of Quantum Mechanics and Einstein gravity at the Planck scale.

Interestingly, standard cosmology as defined by recent experiments holds another puzzle related to the faith of space-time in our universe. Cosmological measurements of the expansion of the observed universe indicate that it is actually of de Sitter type with a

[2] The detection of a B-mode polarization in the BICEP2 experiment arising in inflation models due to primordial gravitational waves was published in [2]. Yet the signal was traced back to cosmic dust in our galaxy with new data from the Planck experiment [3, 4].

positive cosmological constant Λ of mass scale $m_\Lambda = \sqrt{\Lambda} \approx 10^{-12}\ GeV$. This so-called cosmological constant problem has to be clearly addressed in any quantum theory modeling the gravitational force. Moreover, the Standard Model is certainly not able to describe the nature of the corresponding dark energy and the discrepancies in energy scales with regard to the Planck scale or even that of the other three distinct forces.

There are several ways to try to accomplish a consistent theory of quantum gravity. Besides string theory, which we are going to pursue in this thesis, loop quantum gravity, non-commutative geometry, group field theory or asymptotic safety are favored alternative approaches. They all teach us of new physics beyond the Standard Model. Distinct from mentioned features, like SUSY or extended gauge symmetries, additional space-time dimensions and a possible non-commutative structure of the space-time directions themselves, only to name a few, tell us to stay open minded when thinking about the fundamental geometry of nature.

1.1 String theory

A promising candidate for a quantum theory of gravity is string theory. Its advent was in the late 1960s when theoretical physicists were looking for a theory describing the interaction of hadrons. With the rapid success of Quantum Chromodynamics describing the strong nuclear force string theory was more and more examined with regard to being a candidate for a quantum theory of gravity. The reason for this is a spin-2 state in the spectrum of the string which can be identified with the graviton. Hence, string theory contains gravity. Further, string theory can incorporate some important properties of the Standard Model, like gauge interactions, chirality and symmetry breaking. It also naturally includes the idea of SUSY as an extension of the Standard Model. The underlying idea of string theory is to develop a quantum description of one-dimensional objects, so-called strings. This ansatz stands in sharp contrast to usual field theories which consider point-like particles. Observing space-time through one-dimensional probes will turn out to contribute to a completely different perception of the fundamental structure of nature.

Bosonic string theory

Moving strings sweep out a two-dimensional surface, known as the world-sheet, embedded in some bigger space-time, called target space. Their coordinates $X^m(\sigma,\tau)$ in target space can be interpreted as fields living on the world-sheet and provide a map between these two concepts. An appropriate action, called the Nambu-Goto action, is found by generalizing the world-line of a relativistic point-particle to a one-dimensional string

$$S = -T \int d\sigma^2 \sqrt{-\det h} \quad \text{with } h_{\alpha\beta} = \partial_\alpha X^m \partial_\beta X^n \eta_{mn}\ , \tag{1.1.1}$$

where $\sigma^\alpha = (\sigma,\tau)$ denotes the two-dimensional world-sheet coordinates and h is the induced metric by the pullback of the Minkowski metric η in target space to the world-sheet. We further use the abbreviation $\partial_\alpha = \frac{\partial}{\partial \sigma^\alpha}$. T denotes the string tension which is historically related to the Regge slope $\alpha^{'}$ by

$$T = \frac{1}{2\pi\alpha^{'}} \ . \tag{1.1.2}$$

A dimensional analysis shows that we can associate with $\alpha^{'}$ the string scale l_s by $l_s^2 = \alpha^{'}$. Since we later are going to identify some excitation of the string with the graviton, the natural energy scale of string theory should be around the Planck scale m_P.

The Nambu-Goto action (1.1.1) can be rewritten in the more convenient form of the Polyakov action

$$S = -\frac{T}{2}\int d\sigma^2 \sqrt{-\det h}\ h^{\alpha\beta}\eta_{mn}\partial_\alpha X^m \partial_\beta X^n \ , \tag{1.1.3}$$

experiencing conformal symmetry. Therefore, the use of two-dimensional conformal field theory (CFT) techniques is important for observing the string from the world-sheet perspective. Now, h is an independent variable determined by its own equation of motion. Variation of the action with respect to the string coordinates leads to the equations of motion for X^m with additional boundary conditions for the bosonic string

$$\partial_\alpha \partial^\alpha X^m = 0 \ , \qquad \partial_\sigma X^m \delta X_m|_{\sigma=0}^{\sigma=\pi,2\pi} = 0 \ . \tag{1.1.4}$$

In addition, we have to consider the variation of the metric $h_{\alpha\beta}$ on the world-sheet. It follows that the stress-energy tensor, which we obtain in this way, imposes further constraints on the string coordinates

$$T_{\alpha\beta} = \eta_{mn}\partial_\alpha X^m \partial_\beta X^n - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\eta_{mn}\partial_\rho X^m \partial_\sigma X^n = 0 \ . \tag{1.1.5}$$

First, we focus on the closed string coordinate with the two ends of the string being identified

$$X^m(\sigma,\tau) = X^m(\sigma + 2\pi, \tau). \tag{1.1.6}$$

The boundary condition (1.1.4) is satisfied trivially. The solution to the free wave equation (1.1.4) can be expressed by a factorization of the string coordinate into a left- and right-moving part

$$\begin{aligned} X_L^m(\sigma+\tau) &= \frac{1}{2}x^m + \frac{1}{2}\alpha^{'} p^m(\sigma+\tau) + i\sqrt{\frac{\alpha^{'}}{2}}\sum_{l\neq 0}\frac{1}{l}\tilde{\alpha}_l^m e^{-il(\sigma+\tau)} \ , \\ X_R^m(\sigma-\tau) &= \frac{1}{2}x^m - \frac{1}{2}\alpha^{'} p^m(\sigma-\tau) + i\sqrt{\frac{\alpha^{'}}{2}}\sum_{l\neq 0}\frac{1}{l}\alpha_l^m e^{+il(\sigma-\tau)} \ , \end{aligned} \tag{1.1.7}$$

where α_l^m and $\tilde{\alpha}_l^m$ denote the oscillator modes at level l respectively. The center of mass position and momentum are given by x^m and p^m. We have yet to impose the constraint (1.1.5) to obtain a classical solution of the string. This constraint is more conveniently expressed in the form of vanishing Fourier modes L_n and $\tilde{L}_n$

$$\begin{aligned} L_n =& \frac{1}{2}\sum_{l\in\mathbb{Z}} \alpha_{n-l}\cdot\alpha_l, \\ \tilde{L}_n =& \frac{1}{2}\sum_{l\in\mathbb{Z}} \tilde{\alpha}_{n-l}\cdot\tilde{\alpha}_l, \end{aligned} \tag{1.1.8}$$

where the zero modes α_0^m and $\tilde{\alpha}_0^m$ are identified with the momentum p^m. Of particular interest is the following constraint at level 0

$$L_0 = \tilde{L}_0 = 0 \ . \tag{1.1.9}$$

Sometimes called level-matching condition for relating left- and right-moving oscillators, it allows to write down an expression for the effective mass of the string in terms of excited oscillation modes

$$M^2 = \frac{4}{\alpha'}\sum_{l>0}\alpha_{-l}\cdot\alpha_l = \frac{4}{\alpha'}\sum_{l>0}\tilde{\alpha}_{-l}\cdot\tilde{\alpha}_l \ . \tag{1.1.10}$$

Next, we take a look at the open bosonic string. Locally, the open string is also governed by the Polyakov action (1.1.3), but now we have to consider the two endpoints of the string separately

$$X^m(\sigma,\tau), \quad \text{at } \sigma = 0, \pi. \tag{1.1.11}$$

Varying the action (1.1.3) with respect to the string coordinate we observe the boundary term (1.1.4), which vanishes for the closed string upon identification of the endpoints. For the open string this leads to two possible boundary conditions:

- Neumann boundary conditions

  $$\partial_\sigma X^m = 0, \quad \text{at } \sigma = 0, \pi. \tag{1.1.12}$$

  Here, the endpoints of the string are allowed to move freely at the speed of light.

- Dirichlet boundary conditions

  $$\delta X_m = 0, \quad \text{at } \sigma = 0, \pi. \tag{1.1.13}$$

  Here, the endpoints of the string lie at a constant position $X^m = c^m$ in space.

Having Neumann boundary and Dirichlet boundary conditions together the endpoints of the open string are fixed to some hypersurface of a certain dimension, also known as D-brane. There are hints that D-branes should be considered as independent dynamic objects in string theory. The first excited states of the open string describe massless oscillations within and transversal to the brane. In particular, it is a $U(1)$ gauge theory that one obtains on a single brane whereas in certain scenarios the Standard Model arises by placing several stacks of branes in some directions of the target space.

Superstring theory

Up to now the formulation of the string only includes bosonic coordinates and does not account for the existence of fermions in target space. Furthermore, the existence of a tachyonic ground state in bosonic string theory is a major drawback as it points towards instabilities of the theory. Superstring theory addresses both questions simultaneously introducing additional fermionic fields on the world-sheet. The action for the superstring in superconformal gauge then takes the form

$$S = -\frac{T}{2}\int d\sigma^2 \eta_{mn}(\partial_\alpha X^m \partial^\alpha X^n + \alpha' i \bar{\Psi}^m \rho^\alpha \partial_\alpha \Psi^n) \tag{1.1.14}$$

in Minkowski space-time, where Ψ^a are two-component Majorana spinors on the world-sheet and ρ^α are 2×2-Dirac matrices. However, introducing SUSY on the world-sheet by adding fermionic degrees of freedom does not right away answer the question of whether or not we observe fermionic states or SUSY in target space. The equations of motion for the fermionic string take the form of the two-dimensional Dirac equation and has to be supplemented by a corresponding boundary condition

$$\rho^\alpha \partial_\alpha \Psi^m = 0 \ , \qquad \Psi^m_+ \delta\Psi_{+m} - \Psi^m_- \delta\Psi_{-m}\Big|_{\sigma=0}^{\sigma=\pi,2\pi} = 0 \ . \tag{1.1.15}$$

The two components of the Majorana spinor Ψ^m correspond to left- and right-moving fermionic coordinates Ψ^m_L and Ψ^m_R.

We leave out the discussion of the open string and focus on the case of the closed string. Now, the two possible boundary conditions for the left- and right-moving part of the fermionic string are given by

$$\Psi^m_L(\sigma,\tau) = \pm\Psi^m_L(\sigma+2\pi,\tau) \quad \text{and} \quad \Psi^m_R(\sigma,\tau) \ = \pm\Psi^m_R(\sigma+2\pi,\tau) \ . \tag{1.1.16}$$

Since the boundary conditions can be chosen independently one distinguishes the four sectors RR, RNS, NSR and NSNS by a periodic Ramond (R) boundary condition, or anti-periodic Neveu-Schwarz (NS) boundary condition.

Considering the mode expansion of the fermionic string in detail allows to access the spectrum in the different sectors. It turns out that the ground state in the R sector is degenerated and lives in the spinor representation of $SO(1,9)$. Whereas in the NS sector we observe a unique tachyonic ground state. Moreover, it appears that the spectrum is not supersymmetric at first. The proper way to address these problems is a consistent truncation of the spectrum by the GSO-projection. Fixing a definite chirality in the R sector then amounts to establishing SUSY in target space which is not obvious, but can be tracked at every mass level. The spectrum is constructed by tensoring states of the respective sectors chosen for the left- and right-moving coordinate of the string. The NSNS sector contains a scalar field called the dilaton ϕ, an antisymmetric two-form b_{mn} and a symmetric traceless rank-two tensor g_{mn}. The RNS and NSR sectors in each case contain a spin 3/2 gravitino and the spin 1/2 dilatino. The RR sector yields a set of antisymmetric p-forms C_p.

Depending on choosing the left- and right-moving ground state to be of the opposite or same chirality in the R sector one distinguishes string theory of type IIA and respectively type IIB. There exist three more consistent types of string theory. Type I is a string theory of unoriented strings sweeping out world-sheets such as the Möbius band for open strings or the Klein bottle in the case of closed strings. Heterotic String theories including the gauge groups $E_8 \times E_8$ or $SO(32)$ are constructed by allowing only the right-moving string coordinate to involve fermions.

There are important constraints on these string theories imposed by consistency of the quantum description. Lifting classical symmetries to the quantum level so-called anomalies occur. The absence of the Weyl anomaly for example fixes the space-time dimensions to 10 for superstring theories and to 26 for bosonic string theory. This so-called critical dimension of the string raises questions about the interpretation of the extra dimensions and the identification of our four dimensional space-time. Also, rather remarkably, dualities between the distinct string theories and new symmetries can be found.

Stringy symmetries and dualities

String theory provides a variety of new symmetries. T- and S-duality are only two of several dualities encountered in string theory that form an intricate web between the 5 different types of string theory. T-duality identifies certain backgrounds which are indistinguishable for the string. Whereas S-duality relates theories at weak and strong string coupling constant. These are all signs for an eleven-dimensional non-perturbative theory, called M-Theory, where branes play the role of the fundamental objects.

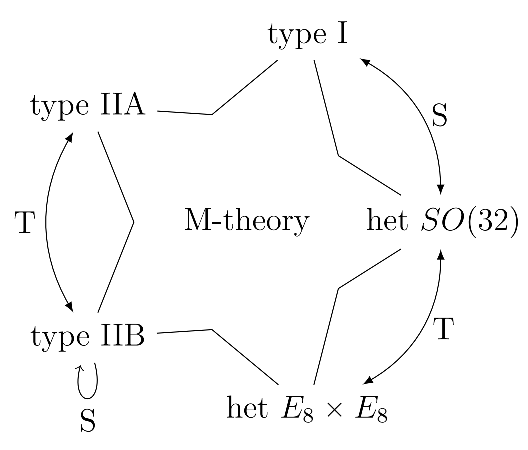

Figure 1.1: Web of dualities between string theories.

In this thesis we are particularly interested in T-duality relating string theories defined on certain distinguished backgrounds. The fact that T-duality is a stringy symmetry, only present when probing space-time with a string, is due to the string being an extended object. Nontrivial winding effects of the string around compactified directions give rise to this new duality.

The study of the spectrum of one bosonic string coordinate on a circle of radius R elucidates the role of the duality transformations[3]. The compactification of one bosonic string coordinate to a circle leads to the periodic identification $X = X + 2\pi R m$, with $m \in \mathbb{Z}$. A consistent mode expansion, respecting (1.1.5), then yields

$$\begin{aligned} X_L(\sigma+\tau) &= x_L + \sqrt{\frac{\alpha'}{2}} p_L(\sigma+\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{l\neq 0} \frac{1}{l} \tilde{\alpha}_l e^{-il(\sigma+\tau)} \ , \\ X_R(\sigma-\tau) &= x_R - \sqrt{\frac{\alpha'}{2}} p_R(\sigma-\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{l\neq 0} \frac{1}{l} \alpha_l e^{+il(\sigma-\tau)} \ . \end{aligned} \tag{1.1.17}$$

The center of mass is $x = x_R + x_L$ and the momenta p_L, p_R read

$$p_L = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R} n + \frac{R}{\sqrt{\alpha'}} m) \ , \quad p_R = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R} n - \frac{R}{\sqrt{\alpha'}} m) \ . \tag{1.1.18}$$

In addition to the momentum in the compact direction taking integer values $n \in \mathbb{Z}$, the winding number $m \in \mathbb{Z}$ accounts for the possibility of the string winding around the circle.

Having a formula for the left- and right-moving momenta at hand we can add 25 non-compact string coordinates to match with the critical dimension of bosonic string theory. The mass shell condition (1.1.10) in the compactified theory is then written as

$$M_{25}^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \ , \tag{1.1.19}$$

where N and $\tilde{N}$ are the number operators counting oscillation modes. The duality symmetry can be seen thanks to the following exchange

$$\frac{R}{\sqrt{\alpha'}} \leftrightarrow \frac{\sqrt{\alpha'}}{R}, \ n \leftrightarrow m \ . \tag{1.1.20}$$

Therefore, T-duality is most easily observed as an exchange of momentum modes with winding modes of the string. However, the statement that string theory defined on a circle of radius R is equivalent to defining it on a circle of inverse radius α'/R is more familiar. Hence, physics at small scales cannot be distinguished from physics at large scales. One interpretation of this is that string theory implements a natural minimal length scale $R_{min} \equiv \sqrt{\alpha'}$.

Based on the work by Narain, the $\mathbb{Z}_2$-symmetry which inverts the radius R was recognized to sit inside a larger group for a d-dimensional toroidal background. The $2d$ vector (p_L, p_R) then spans an even self-dual lattice, called Narain lattice. All even self-dual lattices are related by transformations h forming the group $O(d, d, \mathbb{R})$

$$h^t \eta h = \eta \quad \text{with} \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{1.1.21}$$

[3] An exhaustive review on T-duality can be found in [5].

Furthermore, there exist a subgroup $O(d,\mathbb{R}) \times O(d,\mathbb{R})$ providing a symmetry of the truncated Hamiltonian

$$H_0 = \tfrac{1}{2}p_L^2 + \tfrac{1}{2}p_R^2 \ . \tag{1.1.22}$$

The moduli space is then given by the coset manifold $O(d,d,\mathbb{R})/O(d,\mathbb{R}) \times O(d,\mathbb{R})$. Finally, the discrete duality subgroup $O(d,d,\mathbb{Z})$ leads to physically identical theories. This group is known as the T-duality group in string theory and corresponds to the enlargement of the $\mathbb{Z}_2$-symmetry. Whereas $O(d,d,\mathbb{R})$ is the respective symmetry group appearing in low-energy effective theories of string theory. Analyzing $O(d,d,\mathbb{R})$ the following elements are symmetry generators

$$g_\Theta = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix} \ , \ g_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \ , \ g_{D_i} = \begin{pmatrix} 1-e_i & e_i \\ e_i & 1-e_i \end{pmatrix} \ , \tag{1.1.23}$$

where Θ_{ij} is a antisymmetric $d \times d$-matrix taking constant values, $A \in GL(d,\mathbb{R})$ and e_i is a $d \times d$-matrix where the ii-th entry is one and all other entries are zero. The last element can be identified with a radial inversion in the i-th toroidal direction.

T-duality can also be derived from a world-sheet approach using a non-linear σ-model. Buscher showed, see [6] and [7], that T-duality is a symmetry of the path integral if there exists an abelian isometry in a compactified dimension. The rules for calculating the T-dual background are known by the name of Buscher rules. This provides a shortcut for generating new string backgrounds.

The starting point is a σ-model for the string propagating in curved space with an abelian isometry in one direction, denoted by θ,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|h|}\left(\left(h^{\alpha\beta}g_{mn}(X) + i\epsilon^{\alpha\beta}b_{mn}(X)\right)\partial_\alpha X^m\partial_\beta X^n + \alpha'\phi(X)R^{(2)}\right) \ , \tag{1.1.24}$$

where $h_{\alpha\beta}$ is the world-sheet metric, $g_{mn}(X)$ is the target space metric and $b_{mn}(X)$ is the B-field. The dilaton field $\phi(X)$ is coupled to the scalar curvature $R^{(2)}$ of the world-sheet. Gauging the abelian isometry direction θ by introducing a gauge field together with a Lagrange multiplier, arranging for pure gauge and integrating out the gauge field in a second step yields a dual σ-model. Remarkably, the form of the original σ-model can be restored if the following substitutions are applied

$$\begin{aligned} &\tilde{g}_{\theta\theta} \to \frac{1}{g_{\theta\theta}} \ , \quad \tilde{g}_{\theta m} \to \frac{b_{\theta m}}{g_{\theta\theta}}, \quad \tilde{b}_{\theta m} \to \frac{g_{\theta m}}{g_{\theta\theta}} \ , \quad \tilde{\phi} \to \phi - \frac{1}{2}\log g_{\theta\theta} \ , \\ &\tilde{g}_{mn} \to g_{mn} - \frac{g_{\theta m}g_{\theta n} - b_{\theta m}b_{\theta n}}{g_{\theta\theta}} \ , \quad \tilde{b}_{mn} \to b_{mn} - \frac{g_{\theta m}b_{\theta n} - g_{\theta m}b_{\theta n}}{g_{\theta\theta}} \ . \end{aligned} \tag{1.1.25}$$

These transformations[4] and their generalizations to several isometries are called Buscher rules. Given a string background described by (g,b,ϕ) the dual background $(\tilde{g},\tilde{b},\tilde{\phi})$ is

[4]The transformation of the dilaton ϕ has to be worked out separately by demanding invariance of the partition function under T-duality.

provided by the above fractional linear transformation. Later, the T-duality transformation take a linear form in the $O(d, d)$ framework.

A further symmetry directly related to the geometry of the internal space of the hidden dimensions is mirror symmetry. According to this symmetry String theories of type II should lead to equivalent effective field theories if two mirror Calabi-Yau (CY) manifolds make up the additional space. In certain situations the transformation between mirror pairs is a T-duality as stated by the SYZ-conjecture [8].

1.2 Low-energy effective theories

Despite huge progress in understanding the structure behind string theory a lot of open questions and problems remain a major sign that string theory is one of the most complex theories encountered in physics. One important direction to follow concerns the arising of physics at low energies in the limit $\alpha' \to 0$ from string theory. In one way or another string theory should incorporate the Standard Model of particle physics and gravity described by general relativity. Hence, the infinite tower of states in the spectrum of the string should be truncated to be identified with the finite particle spectrum observed in target space. A natural choice is to keep the massless modes of the string since higher excitations come with masses near the Planck scale m_P and cannot be produced with current energies. Among these, the graviton should give rise to Einstein's theory of gravity. More generally, one could ask: What are the imprints of string theory that could be observed in experiment? Is it possible to see any sign for the existence of extra dimensions? Since SUSY is naturally embedded in string theory we should further find a new scale m_{SUSY}, probably around $1\ TeV$, where it is spontaneously broken and above which new supersymmetric particles should be observed. Moreover, the faith of symmetries only present in string theory is not clear either. Are these in some form present in the low-energy effective theories? By answering the last question one hope is to reveal a new framework underlying string theory paving the way to explore novel regions of the vacuum structure of the string.

1.2.1 Supergravity

SUSY plays an important role in going beyond the Standard Model of elementary particles. Theoretically, this led to the Minimal Supersymmetric Standard Model (MSSM) which ascribes to each particle a supersymmetric partner.[5] A favored characteristic of this model is the unification of the three fundamental forces at high energies. But SUSY is also interesting from the point of view of extending gravity theories in various dimensions. Implementing SUSY as a local symmetry the corresponding theory, called supergravity (SUGRA), inherits space-time diffeomorphism invariance by the supersymmetric extension of the Poincaré algebra. Hence, people reflected about SUGRA theories totally independent of string theory. The field content of such theories is arranged in supermultiplets,

[5] So far, SUSY has not been observed at LHC. Simplest natural versions have been ruled out, but more evolved models have yet a chance.

representations of the SUSY algebra, and depends on the number of supersymmetries $\mathcal{N}$ and the dimension d.

Another view on SUGRA is based on taking superstring theory as the correct quantum theory of gravity and considering SUGRA as a low-energy description of the string. In other words, string theories provide ultraviolet completions of SUGRA theories. However, a consistent method for integrating out the massive excitations as in conventional quantum field theories is not an option in string theory. Nevertheless, any construction of the corresponding SUGRA theory tries to reproduce the massless field content of a particular superstring theory. Besides this correspondence of both spectra, another hint on the low-energy effective action is to study string scattering amplitudes at low energies or calculating the β-functions of string theory. Demanding conformal invariance on the world-sheet for the non-linear σ-model (1.1.24) the β-functions have to vanish and coincide with the equations of motion of SUGRA theories.

Surprisingly, eleven dimensions constitute an upper bound for consistent SUGRA theories and additionally give rise to a single unique SUGRA believed to be the low-energy effective description of M-theory. Of particular interest to us are the ten-dimensional SUGRA theories of type II. Here, we present the bosonic sector of type IIA which should be complemented with a fermionic action for the gravitini $\psi_m^{1,2}$ and the dilatini $\rho^{1,2}$

$$
\begin{aligned}
S_{\mathrm{IIA}} &= S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}} \ , \\
S_{\mathrm{NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi}(\mathcal{R}(g) + 4|d\phi|^2 - \frac{1}{2}|H|^2) \ , \\
S_{\mathrm{R}} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}(|F_2|^2 + |\bar{F}_4|^2) \ , \\
S_{\mathrm{CS}} &= -\frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4 \ .
\end{aligned}
\tag{1.2.1}
$$

In these expressions $\mathcal{R}(g)$ is the standard Ricci scalar, the RR fields are denoted by C_p with the corresponding field strengths F_{p+1}[6], the NS-NS field is B_2 with corresponding field strength H and a useful redefinition is $\bar{F}_4 = dC_3 - C_1 \wedge H_3$. The equations of motion derived from this action are

$$
\begin{aligned}
0 =& -\frac{g_{mn}}{2}(\mathcal{R}(g) + 4|d\phi|^2 - \frac{1}{2}|H_3|^2 - \frac{1}{2}e^{2\phi}|F_2|^2 - \frac{1}{2}e^{2\phi}|\bar{F}_4|^2) \\
&+ \mathcal{R}_{mn} - \frac{1}{2}\iota_m H \cdot \iota_n H + 2g_{mn}(2|d\phi|^2 - \nabla^2\phi) + 2\nabla_m\nabla_n\phi \\
&- \frac{1}{2}e^{2\phi}\iota_m F_2 \cdot \iota_n F_2 - \frac{1}{2}e^{2\phi}\iota_m\bar{F}_4 \cdot \iota_n\bar{F}_4 \ , \\
0 =& \mathcal{R}(g) - \frac{1}{2}|H|^2 + 4(\nabla^2\phi - |d\phi|^2) \ , \\
0 =& d(e^{-2\phi} * H) \ , \\
0 =& d(*F_p) \ .
\end{aligned}
\tag{1.2.2}
$$

[6]By Poincaré duality there are the field strengths $\tilde{F}_6 = *\bar{F}_4$ and $\tilde{F}_8 = *F_2$. In principle, the field strength $F_{10} = dC_9$ can also be defined, but it does not carry propagating degrees of freedom.

Conventions for the notation can be found in appendix A.2. The equations of motion should be supplemented with Bianchi identities (BIs) for the fluxes in the NSNS sector as well as in the RR sector

$$dH = 0 \ , \quad (d - H) \wedge F_p = 0 \ . \tag{1.2.3}$$

A further important detail of any SUGRA theory are the SUSY variations connecting the bosonic and fermionic sector. For type IIA and IIB standard SUGRA with two pairs of chiral fermions the NSNS-flux contributions to their SUSY variations are

$$\begin{aligned} \delta\psi_m^{1,2} &= \left(\nabla_m \mp \frac{1}{8} H_{mnp}\Gamma^{np}\right)\epsilon^{1,2} \ , \\ \delta\rho^{1,2} &= \Gamma^m \left(\nabla_m \mp \frac{1}{24} H_{mnp}\Gamma^{np} - \partial_m\phi\right)\epsilon^{1,2} \ , \end{aligned} \tag{1.2.4}$$

where ∇_m is the standard covariant derivative, the Γ^m satisfy a Dirac algebra and $\epsilon^{1,2}$ denotes the SUSY fermionic parameters, while the sign differs respectively for $1, 2$. The SUSY variations of the bosonic field content are not considered in this thesis and therefore left out.

Whereas type IIA can be derive in a dimensional reduction of eleven-dimensional SUGRA the guidelines for constructing a SUGRA theory for string theory type IIB are solely given by SUSY and gauge invariance. In the same way, it is possible to write down SUGRA theories for the remaining heterotic and type I string theories. Of particular interest to us is S_{NS}, which coincides for all five string theories, and the corresponding equations of motion.

The open string sector also allows for a low-energy effective description in target space which is exact in α' in contrast to the effective action (1.2.1) for the closed string. The so-called Dirac-Born-Infeld (DBI) action[7]

$$S_{\text{DBI}} = -T_p \int_\Sigma d^{p+1}\xi e^{-\phi(X)} \sqrt{-\det\left(g_{\alpha\beta}(X) + 2\pi\alpha' F_{\alpha\beta}(X) + b_{\alpha\beta}(X)\right)} \tag{1.2.5}$$

governs the abelian gauge field A_m with field strength F_{mn} on a single D-brane and couples it to the massless fields $g_{mn}(X)$, $b_{mn}(X)$ and $\phi(X)$ of the closed string. The coordinates $X^m(\xi)$ embed the surface Σ into ten-dimensional target space and define the pullbacks $g_{\alpha\beta}$, $b_{\alpha\beta}$ and $F_{\alpha\beta}$ onto the D-brane, similar to(1.1.1). The DBI action is a non-linear extension of Maxwell's theory.

1.2.2 Compactification

Requiring extra dimensions for string theory to be anomaly free leaves us with the question of identifying our four-dimensional space-time. There are two distinct ideas on the realization of hidden dimensions adding up to the critical dimension of the string. The first

[7] In order to respect SUSY the DBI action is complemented by a Chern-Simons action $S_{CS}[C_p]$ containing the corresponding RR p-form C_p.

scenario embeds space-time as a four-dimensional hypersurface in ten dimensions. The second possibility is to consider the extra dimensions to be small enough that we have not yet been able to observe them in experiment. The latter is known as compactification where extra dimension coil up to form an additional space. Remarkably, the distinction into extended and compact directions is to this moment obscure. Applying a simple compactification ansatz by hand ten-dimensional space splits into four extended space-time directions together with a compact internal manifold $\mathcal{M}_6$ for the extra dimensions

$$\mathcal{M}_{10} = \mathbb{K}_4 \times \mathcal{M}_6 \ . \tag{1.2.6}$$

This idea is more commonly known under the name of Kaluza-Klein compactifications. Typically for convenience, one chooses Minkowski space-time $\mathbb{R}^{1,3}$ for the external part. However, recent cosmological data demands a de Sitter space. More generally, the two spaces could be connected by a warping meaning that the size of the inner manifold changes with the position in the four-dimensional space-time. This choice causes ten-dimensional fields, e.g. the metric g, to split into a four-dimensional part plus additional vector and scalar degrees of freedom. In particular, the scalar fields in the lower dimensional theory, known as moduli, are of certain interest because these specify the inner space. For example, Kähler moduli describe the size and complex structure moduli the shape of the internal manifold. Unfortunately, a potential fixing a specific vacuum expectation value for these moduli is missing. Therefore, important properties of the internal space are not fixed. Hence, a vast number of distinguishable compact internal manifolds are allowed solutions. This is known as the string landscape problem which reduces the predictivity of string theory tremendously. A prefered choice for the internal space is a CY threefold defined by preserving a certain number of SUSY when the external space is maximally symmetric. Especially in heterotic string theory, this scenario [9] is able to provide prefered four-dimensional $\mathcal{N} = 1$ vacuum configurations even if it demands the NS-flux H to vanish.

Despite the huge string landscape the only scalar field observed so far in four dimensions is the Higgs fields. Therefore, a mechanism is needed that allows to stabilize the moduli. Switching on non-vanishing vacuum expectation values for the NS-flux and for the fluxes in the RR sector a scalar potential for the moduli in $\mathcal{N} = 1$ four-dimensional theories is generated

$$V = e^K (K^{mn} D_m W \bar{D}_n W - 3|W|^2) \ , \tag{1.2.7}$$

where W is the superpotential provided by the Gukov-Vafa-Witten formula [10] and contains the standard fluxes. Moreover, K is the Kähler potential[8] and $K^{mn} = \partial_m \bar{\partial}_n K$ denotes the Kähler metric. For the standard example of a type IIB compactification on a $T^6/\mathbb{Z}_2$ orientifold with three identical 2-tori the superpotential does include H-flux and F_3-flux [11]

$$W = P_1(\tau) + S P_2(\tau) \ , \tag{1.2.8}$$

[8] Additional terms in the superpotential arise due to non-perturbative effects by D-brane instantons and in the Kähler potential due to world-sheet instantons.

where τ denotes the complex structure modulus, S the axion-dilaton and the polynomials P_1, P_2 are cubic in τ. The Kähler modulus U does not enter the superpotential and is hence not stabilized. This opens up the wide field of flux compactifications; see [12] for a review. In certain scenarios some or all moduli gain a nontrivial vacuum expectation value and hence moduli stabilization is possible [13, 14, 15]. Restrictions to turning on fluxes in compact spaces result from energy considerations of the configurations. Negative tension sources, so-called orientifolds, are needed for compensating the positive contributions of fluxes. The BIs (1.2.3) in the case of a geometric compactification with integrally quantized fluxes impose further conditions. Of special interest to us are also conditions from demanding a certain number of SUSY in the compactified theory. When fluxes are switched on SUSY can be partially broken. Eventually, the lower-dimensional SUSY variations (1.2.4) including the NSNS H-flux and the geometric f-flux have to be satisfied to obtain a consistent compactification.

1.3 Non-geometry

Considering string theory as a two-dimensional CFT defined from some world-sheet into some target space, a conventional geometric description of space-time is not obvious. The mathematical tools of differential geometry in form of an underlying manifold structure are not applicable in most cases. These kind of "non-geometric" vacua, however, should be included in the string landscape if one hopes to achieve a deeper understanding of string theory itself. Naturally, this brings along the task to identify situations where new effects beyond standard geometry appear. First hints can of course be found when differential geometry breaks down, but in the long run one hopes to develop a new appropriate framework in which these new backgrounds fit in consistently. Symmetries only present in string theory do play a major role in detecting non-geometric string vacua. Following the idea of observing effects of the string with effective theories one immediately comes up with the question: What happens to stringy symmetries at low energy? In the following we present how T-duality applied to well-understood situations in low-energy effective theories gives us a first handle on a tiny segment of non-geometric vacua in the vast string landscape.

1.3.1 Non-geometric fluxes and backgrounds

In the previous section we briefly mentioned the need for moduli stabilization in any compactification and the fail of convenient CY manifolds without fluxes in regard to solving the problem. Including fluxes in the compactification improves the situation, but still makes it hard to derive models for string phenomenology within the standard set of ingredients. The recent discovery of so-called non-geometric fluxes led to the observation of backgrounds with full moduli stabilization [16, 17, 18]. In addition, (metastable) de Sitter solutions have been found [19, 20, 21, 22, 23, 24, 25]. Finding a few particular backgrounds where also the issue of stability is resolved remains of great interest. In general, having more freedom should ease the way to constructing more realistic models imposing further constraints, e.g. a correct particle spectrum. A long these lines non-geometric fluxes provide additional tools for finding phenomenologically interesting solutions. Before we draw the connection of these non-standard fluxes to the breakdown of geometric concepts in certain backgrounds we sketch their appearance in four-dimensional compactified theories.

Appearance in the superpotential and gauged supergravity

Although complex structure moduli experience moduli stabilization in a type IIB compactification on a $T^6/\mathbb{Z}_2$ orientifold the volume of the inner manifold parametrized by the Kähler moduli is arbitrary. The situation in type IIA is slightly better. Here, applying an additional "twisting" to the T^6 which is measured by the so-called geometric f-flux[9] allows to stabilize all moduli [27]. Then, the superpotential includes besides the NSNS H-flux the even RR fluxes

$$W = P_1(\tau) + SP_2(\tau) + UP_3(\tau) \ , \tag{1.3.1}$$

where now also the Kähler modulus U enters, P_1 is still cubic, but $P_{2,3}$ are linear. It is now interesting to observe the effect of T-duality on this superpotential since there is an obvious mismatch with the type IIB side (1.2.8). Eventually, declaring T-duality to be a symmetry of the four-dimensional theory demands further terms in the superpotential which come with coefficients representing new non-standard fluxes. The NSNS sector contains besides the standard H- and f-flux two further non-geometric fluxes labeled Q and R.

Closely related to flux compactification are so-called gauged SUGRA theories[10] in lower dimensions. These theories are based on gauging a subgroup of a global symmetry and should arise in more complicated compactifications. A first example is $SO(8)$ gauged SUGRA which descends from eleven-dimensional SUGRA reduced on S^7. Here, $SO(8)$ is considered to be embeded in E_7 which is the global symmetry of the ungauged theory after toroidal compactification. Non-standard fluxes then appear as structure constants in the T-duality invariant extension of the gauging algebra of four-dimensional gauged SUGRAs [27, 29, 30]

$$\begin{aligned}
[Z_a, Z_b] &= H_{abc}X^c + f^c{}_{ab}Z_c \\
[Z_a, X^b] &= -f^b{}_{ac}X^c + Q_a{}^{bc}Z_c \\
[X^a, X^b] &= Q_c{}^{ab}X^c - R^{abc}Z_c \ ,
\end{aligned} \tag{1.3.2}$$

where the symmetry generators X and Z are derived from ten-dimensional diffeomorphisms and b-field gauge transformations. This further fixes the specific index positions on the fluxes. More systematically, the embedding tensor encodes all possible gaugings consistent with SUSY. Besides the standard geometric embeddings describing diffeomorphisms and p-form gauge transformations other embeddings are possible. From the point of view of flux compactifications, the embedding tensor allows to group standard fluxes and non-geometric

[9] Criteria for compactness in the presence of geometric fluxes can be found in the review [26].
[10] A review can be found in [28].

fluxes corresponding to the later subset of embeddings consistently. These non-standard embeddings do not seem to have an uplift to higher-dimensional SUGRA theories and might indicate compactification on non-geometric backgrounds [31].

3-torus with H-flux and T-dual configurations

T-duality provides us with a way of generating new consistent string backgrounds from yet discovered ones. On the level of SUGRA theories this is done by properly applying Buscher's rules (1.1.25). In the following we track the appearance of non-geometric fluxes to inconsistencies arising for certain backgrounds. The standard toy model guiding our intuition is the three-torus with H-flux[11] and its T-dual versions [32, 33].

Starting with a rectangular flat three-torus T^3 where the three directions x^m for $m = 1, 2, 3$ are periodically identified $x^m \sim x^m + 1$ and switching on a Kalb-Ramond b-field linearly depending on a single direction leads to a flat metric g and constant H-flux. In particular, following the monodromies the metric g stays invariant and the shift in the b-field can be absorbed by a gauge transformation.

The premise of an existing isometry for Buscher's rules is certainly satisfied and the transformation can even be applied twice for having two isometry directions. Performing the transformation once one finds the so-called twisted torus. It is characterized by a vanishing b-field and hence a zero H-flux. More importantly, the metric g is no longer flat, but inherits a non-trivial dependency on the direction the b-field previously depended on. This twist can be described using f-flux. The setup of the twisted torus is still geometric in the sense that the monodromy of the metric g is a simple diffeomorphism.

A second T-duality transformation in the remaining isometry direction does lead to an inconsistent global description. Now, the metric g and the b-field both depend non-trivially on one direction and the monodromies are no longer diffeomorphisms and gauge transformations but local stringy T-duality transformations. Locally, the fields still describe a geometry but globally the geometric picture breaks down. Later, we show that this background can be associated with Q-flux.

Even if there is no isometry left one can speculated about the existence of a third T-dual version of the three-torus with H-flux. It is usually associated with having R-flux which can be seen as an indication for the loss of a local geometric description [27].

This sequence of T-dual backgrounds with the corresponding geometric and non-geometric fluxes can be conveniently summerized by the following T-duality chain of fluxes

$$H_{abc} \xleftrightarrow{\mathcal{T}_1} f^a{}_{bc} \xleftrightarrow{\mathcal{T}_2} Q_c{}^{ab} \xleftrightarrow{\mathcal{T}_3} R^{abc} \ . \tag{1.3.3}$$

This chain also applies to a set of branes in the NSNS sector. Here, the origin is the $NS5$-brane and T-duality transformations lead to dual branes with turned-on non-geometric fluxes.

The lesson of the three-torus with H-flux and its T-dual versions is to put symmetries only present in string theory on an equal footing with diffeomorphisms and gauge transformations, when one wants to construct non-geometric backgrounds. More generally, this

[11] In order to provide a consistent string background further ingredients should be added.

means that we should naturally glue string configurations described on local patches with the help of stringy symmetries. On each of these patches a generic background consists of a metric g, a two-form b-field and the dilaton ϕ. In order to gain the global picture transition functions are needed for patching local descriptions. Now, for a manifold $\mathcal{M}$ the allowed transition functions are limited to diffeomorphisms and gauge transformations, but when having a torus bundle with fiber T^d stringy extensions become interesting. In particular, it is interesting to do the gluing between fibers with transition functions lying in the continuous T-duality group $O(d, d, \mathbb{R})$. From this point of view, it is easy to understand that some string configurations look ill-defined. Transition functions gluing the torus fibers when going around a loop in the base simply do no longer lie in the geometric subgroup $G_{geom} = GL(d) \ltimes \Lambda_2 \subset O(d, d, \mathbb{R})$. These kind of new backgrounds are known as T-folds [34]. Certainly, the notion of a standard manifold is violated in this constructions and for vanishing b-field one finds configurations that allow the patching of big circles to small circles. At the level of the string such configurations generically cause a mixing of momenta and winding numbers.

The observation of non-geometric fluxes and backgrounds is very much linked to starting with a consistent geometric vacuum solution. However, a T-duality invariant superpotential, an embedding tensor in the full T-duality group $O(d, d)$ and transition functions beyond diffeomorphisms and gauge transformations allow to directly construct configurations where it has to be examined whether these arise from geometric ones by applying T-duality transformations. In most cases this will not be the case and the question of consistent uplifts to full string vacua is non-trivial to answer. Additional requirements for a consistent string background are for example modular invariance at higher order in loops. T-dual backgrounds of geometric configurations meet all these conditions. The string theory perspective on this is that even if locally there exists a different σ-model on each patch with a distinct global target space geometry, these all represent one CFT describing the perturbed string. T-duality may change the geometry of the target space from patch to patch but it leaves the CFT invariant. Known examples for consistent non-geometric string configurations are provided by asymmetric orbifolds [35, 36]. Monodromies in this kind of spaces act asymmetrically on the left and right string coordinate. Besides T-duality, other stringy symmetries like S-, U-duality and mirror symmetry could be used analogously to construct even more classes of string backgrounds with exciting new features.

1.3.2 T-duality covariant formalisms

So far T-duality played an important role in the lower-dimensional theories. Non-geometric fluxes arise solely after compactifying a certain SUGRA theory and demanding T-duality invariance of the superpotential. Our understanding of them in terms of non-geometric backgrounds is very limited. A most promising facilitation at this stage would be to have a theory in ten dimensions that contains and provides expressions for the non-geometric fluxes right from the beginning. A justified hope is to draw the connection from the appearance of four-dimensional fluxes to a yet to be determined compactification of this new theory on certain backgrounds experiencing non-geometric features. In other words, trying to find a manifestly T-duality invariant theory is of great interest. Two such formalisms trying to make use of stringy symmetries led to a fast developing field during the last decade. Double Field Theory [37, 38] and Generalized Geometry [39] both provide frameworks that make T-duality manifest and allow to tackle problems of non-geometry directly in ten dimensions.

Generalized geometry

The idea of Generalized Geometry (GG) heavily relies on previous work in the field of mathematics. Hitchin and Gualtieri explored generalizations of symplectic and complex structures and unified these in the framework of Generalized Complex Geometry (GCG) [40],[41]. Based on a new bundle structure over a standard manifold, called the generalized tangent bundle, it is possible to deal with standard vectors and one-forms on an equal footing. In such a framework the group $O(d, d, \mathbb{R})$ arises naturally as the structure group by the a bilinear product acting on generalized vectors consisting of a vector and a one-form part.

Physical motivation for further studying GCG holds the field of supersymmetric flux compactifications.[12] As pointed out before, switched on fluxes enter the SUSY variations and lead to curved manifolds beyond the CY spaces. It turned out that GCG is valuable in reformulating SUSY transformations and sometimes allows to solve the SUSY conditions and BIs instead of the equations of motion when searching for background solutions. In particular, the notion of a generalized Calabi-Yau condition (GCY) [40, 41] which is necessary but not sufficient for realizing unbroken SUSY on manifolds with fluxes provides a interesting new class of backgrounds.

However, it was only lately that an associated theory on the generalized tangent bundle was proposed in [39]. It is a generalization of Einstein gravity based on Riemannian geometry in the sense of providing a generalized connection in analogy to the Levi-Civita connection and realizing the bigger structure group $O(d, d, \mathbb{R})$ as its symmetry group. Therefore, the framework of GG nicely unifies diffeomorphisms and gauge transformations governing the field content of any SUGRA theory. The objects that transform under a generalized Lie derivative associated with the Courant bracket are the so-called generalized metric, encoding the standard metric and the Kalb-Ramond field, and a generalized dilaton in the NSNS sector. Including the other sectors GG is able to provide a rewriting for SUGRA theories of type IIA and type IIB. Eventually, it provides generalized curvature quantities and allows to rewrite apart from SUSY transformations also the equations of motion in a remarkable simple form.

Extensions of GG to different situation have recently appeared. In particular, its application to the low-energy effective theories of M-theory in several dimensions was discussed in [43]. Replacing the $O(d, d, \mathbb{R})$ structure group on the generalized tangent bundle with exceptional groups $E_{d(d)}$[44, 45] allows to implement U-duality as symmetry of the new theory [46]. Further interesting work using GG was established with respect to backgrounds

[12] A review on GCG and its application in the context of supersymmetric flux compactifications can be found in [42].

of AdS type [47], preserving a certain amount of SUSY and allowing for generalized special holonomy [48]. Latest developments try to find an appropriate bundle for treating higher α'-corrections to SUGRA theories [49].

Double field theory

Even more inspired by the setup of string theory is Double Field Theory (DFT)[13] [37, 54]. The existence of left- and right-moving string coordinates and the possibility of additional winding in toroidal backgrounds makes it tempting to consider so-called doubled spaces on which the T-duality group $O(d,d)$ acts by exchanging to sets of coordinates. Loosing the familiar notion of manifold, necessary for GG, DFT establishes a framework that allows to learn something about so-called doubled geometries encoding a whole class of T-dual backgrounds.

The gauge structure of DFT is analogously to GG governed by a bracket, generalizing the Courant-bracket. The associated generalized Lie derivative now contains furthermore a dual partial derivative related to a second set of coordinates corresponding to winding. Therefore, the formalism of DFT is crucially linked to its field content, e.g. a generalized metric, depending on two coordinate sets. Apart from the questions of how to define a doubled space beyond the notion of a standard manifold the framework based on a generalized connection and derived generalized curvature objects [55] matches the one of GG.

DFT is equiped with a constraint that comes in different strengths and that has to be imposed for consistency of the theory. The level-matching condition in string theory here leads to the weak constraint. It is not sufficient for the consistency of DFT whereas the strong constraint, equivalently called section condition for fixing a section in the doubled space on which then the theory lives, does the job. In this sense DFT is not a truely doubled theory and coincides with GG for a specific choice of the section. The gauge structure of DFT, however, is consistent using a weaker form, called the closure constraint. This triggered the development of the so-called flux formulation of DFT [50, 51] where a generalized flux encodes all standard and non-geometric fluxes present in the lower dimensional theories. Practically, generalized Scherk-Schwarz compactifications[14] [56, 57, 58] provide a method that allows to reach all gaugings in the embedding tensor of gauged SUGRAs among which some configurations violate the strong constraint. It is believed that the DFT flux formulation is able to capture some of the genuinely non-geometric backgrounds that do not have a T-dual geometric description.

In recent years more and more attention was also paid to other stringy dualities with regard to manifestly duality invariant formulations. Following earlier work on exceptional symmetry groups [59], being related to M-theory and U-duality [34, 44], arising in compactifications of eleven dimensional SUGRA on tori, manifestly U-duality covariant formulations in various dimensions have been formulated and summarized under the name Exceptional

[13]Review papers can be found in [50, 51, 52, 53]

[14]Generalized Scherk-Schwarz compactifications are problematic on their own, since the construction of the twist tensor on a doubled space is not rigorously defined.

Field Theories (EFT) [60, 61, 62, 63, 64, 65]. Moreover, there has been progress in revealing new tensor hierarchy structures for these kind of theories [66].

Finally, let us list some further interesting developments. Heterotic versions [67, 68] of DFT and supersymmetric extensions [69, 70, 71] have been formulated. There has also been renewed interest in Kaluza-Klein compactification [72, 73] and Scherk-Schwarz compactification [58] with respect to identifying lifts for some lower-dimensional gauged SUGRA theories with non-geometric fluxes. And very recently DFT was extended to group manifolds [74, 75] for non-trivial backgrounds.

Both of these T-duality covariant formalisms are closely related to the authors work and provide the background for the theory presented in this thesis.

1.4 β-supergravity

Apart from GG being a useful framework for generalizing various SUGRA theories it does not tell much about the appearance of non-geometric fluxes and backgrounds. DFT was in principle believed to provide the capacity to include non-geometric fluxes, however, it was not clear at all in the beginning how non-geometric fluxes should appear in its framework. Moreover, the loss of mathematical rigorousness accompanied with the drop of a well-defined underlying manifold clearly limits its power in yielding consistent compactifications for the new class of non-geometric backgrounds. Hence, at that time the relation between the four- and ten-dimensional perspectives on fluxes was not well established and a ten-dimensional theory with a consistent mathematical basis relying explicitly on non-geometric fluxes was a much prefered situation for compactification. This theory is now known by the name of β-supergravity.

It is based on a ten-dimensional local reformulation of standard SUGRA which gives non-geometric fluxes a manifest ten-dimensional origin. Even more interestingly, β-supergravity allows to reformulate a non-geometric background of standard SUGRA into a geometric one of β-supergravity for which then a consistent compactification method can be applied. In this way some vacua of four-dimensional gauged SUGRAs with non-geometric fluxes thus get a clear ten-dimensional uplift. There is justified hope that ten-dimensional backgrounds with non-geometric fluxes experience moduli stabilization.

A local field redefinition of standard supergravity fields

The main idea behind the appearance of non-geometric fluxes in β-supergravity is a specific field redefinition inspired by GCG [76, 77, 78]. In the NSNS sector the standard metric g, the b-field and the dilaton ϕ are traded for a new set of fields, a new metric $\tilde{g}$, an antisymmetric bivector β and a new dilaton $\tilde{\phi}$. In GCG terms, this field redefinition is a reparametrization of the generalized metric $\mathcal{H}$

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & -bg^{-1} \\ g^{-1}b & g^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{g} & \tilde{g}\beta \\ -\beta\tilde{g} & \tilde{g}^{-1} - \beta\tilde{g}\beta \end{pmatrix} \ . \tag{1.4.1}$$

Then, the field redefinition[15] can be read off from the components

$$\left.\begin{array}{l}\tilde{g}^{-1} = (g+b)^{-1} g (g-b)^{-1} \\ \beta = -(g+b)^{-1} b (g-b)^{-1}\end{array}\right\} \Leftrightarrow (g+b)^{-1} = (\tilde{g}^{-1} + \beta) \ , \qquad e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|} = e^{-2\phi}\sqrt{|g|} = e^{-2d} \tag{1.4.2}$$

and can be completed with a new dilaton $\tilde{\phi}$ preserving the measure which now includes the absolute value of the determinant $|\tilde{g}|$ of the metric $\tilde{g}$.

This in principle allows a direct rewriting of the standard SUGRA Lagrangian, but is not very enlightening when it comes to making the non-geometric Q-flux and corresponding geometric structures appear.

A generalized geometric framework

Instead of applying the field redefinition directly to standard SUGRA and afterwards observing what kind of new structures appear, GG reverses this procedure and provides the mathematical tools right from the start. This framework proved successful in reformulating standard SUGRA theories before [39, 46]. Here, the concept of a generalized tangent bundle on which the field redefinition can be stated as choosing an alternative generalized vielbein including the bivector β plays an important role

$$\tilde{\mathcal{E}} = \begin{pmatrix} \tilde{e} & \tilde{e}\beta \\ 0 & \tilde{e}^{-T} \end{pmatrix} \ . \tag{1.4.3}$$

The standard vielbein for a Minkowski metric η is denoted by $\tilde{e}$. Trading the two-form b for a bivector β results in changing the fibrational structure and should be related to a generalized cotangent bundle E_{T*}

$$\begin{array}{ccc} T\mathcal{M} & \hookrightarrow & E_{T*} \\ & & \downarrow \\ & & T^*\mathcal{M} \end{array} \ , \tag{1.4.4}$$

which should be supplemented with a cocycle condition on β.

The standard vielbein $\tilde{e}$ gives rise to the following definitions for standard and non-geometric fluxes in flat indices, where the new bivector β enters the Q- and R-fluxes

$$\begin{array}{ll} H_{abc} = 3\nabla_{[a} b_{bc]} \ , & f^a{}_{bc} = 2\tilde{e}^a{}_m \partial_{[b} \tilde{e}^m{}_{c]} \ , \\ Q_c{}^{ab} = \partial_c \beta^{ab} - 2\beta^{d[a} f^{b]}{}_{cd} \ , & R^{abc} = 3\beta^{d[a}\nabla_d \beta^{bc]} \ . \end{array} \tag{1.4.5}$$

Both the geometric f-flux and the Q-flux are not tensorial and the definitions in flat indices, which had to be deduced previously, appear naturally in this framework and match with the literature [78, 56, 82].

[15] An alternative field redefinition was proposed in [79, 80]. Both field redefinitions were then interpreted in terms of local $O(d,d)$ transformations and Lie algebroids in [81].

The study of the $O(d,d) \times \mathbb{R}^+$ structure on E_{T^*} given by coupling generalized vectors allows to identify associated covariant derivatives. In analogy to the standard Levi-Civita spin connection, generalized metric compatibility and a generalized torsion constraint lead to a partly unique generalized covariant derivative preserving an $O(d-1,1) \times O(1,d-1)$ structure

$$D_{\mathcal{A}} W^{\mathcal{B}} = \begin{cases} D_a w^b = \nabla_a w^b - \eta_{ad} \check{\nabla}^d w^b + \frac{1}{6} \eta_{ad} \eta_{cf} R^{dbf} w^c - \frac{1}{9}(\delta_a^b \Lambda_c - \eta_{ac} \eta^{be} \Lambda_e) w^c \\ D_a w^{\overline{b}} = \nabla_a w^{\overline{b}} - \eta_{ad} \check{\nabla}^d w^{\overline{b}} - \frac{1}{2} \eta_{ad} \overline{\eta_{cf}} R^{d\overline{bf}} w^{\overline{c}} \\ D_{\overline{a}} w^b = \nabla_{\overline{a}} w^b + \overline{\eta_{ad}} \check{\nabla}^{\overline{d}} w^b - \frac{1}{2} \overline{\eta_{ad}} \eta_{cf} R^{\overline{d}bf} w^c \\ D_{\overline{a}} w^{\overline{b}} = \nabla_{\overline{a}} w^{\overline{b}} + \overline{\eta_{ad}} \check{\nabla}^{\overline{d}} w^{\overline{b}} + \frac{1}{6} \overline{\eta_{ad} \eta_{cf}} R^{\overline{dbf}} w^{\overline{c}} - \frac{1}{9}(\delta_{\overline{a}}^{\overline{b}} \Lambda_{\overline{c}} - \overline{\eta_{ac}}\, \overline{\eta^{be}} \Lambda_{\overline{e}}) w^{\overline{c}} \end{cases} . \tag{1.4.6}$$

The quantity Λ_c is related to the dilaton and the latter has an interesting interpretation as a conformal weight. More important, the non-geometric fluxes enter differently. The R-flux simply replaces the former H-flux. However, the Q-flux appears inside a new covariant derivative related to the bivector β

$$\check{\nabla}^b V_a \equiv -\beta^{bd} \partial_d V_a - \omega_{Q}{}_a^{bc} V_c \quad \text{with} \quad \omega_{Q}{}_a^{bc} = \frac{1}{2} \left(Q_a{}^{bc} + \eta_{ad} \eta^{ce} Q_e{}^{db} + \eta_{ad} \eta^{be} Q_e{}^{dc} \right) . \tag{1.4.7}$$

The generalized covariant derivatives $D_{\mathcal{A}}$ on spinors $\epsilon^{\pm}$ for a $Spin(d-1,1) \times Spin(1,d-1)$ structure provide the Killing spinor equations governing a supersymmetric completion of β-supergravity. Furthermore, they allow to compute a curvature scalar

$$S \epsilon^+ = -4 \left(\gamma^a D_a \gamma^b D_b - \overline{\eta^{ab}} D_{\overline{a}} D_{\overline{b}} \right) \epsilon^+ . \tag{1.4.8}$$

Analogue to standard SUGRA, S now gives rise to the Lagrangian of β-supergravity up to a total derivative

$$\tilde{\mathcal{L}}_\beta = e^{-2d} \left(\mathcal{R}(\tilde{g}) + 4(\partial\tilde{\phi})^2 + 4(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a)^2 + \mathcal{R}_Q - \frac{1}{2} R^{acd} f^b{}_{cd} \eta_{ab} - \frac{1}{2} R^2 \right) , \tag{1.4.9}$$

where $\mathcal{T}^a$ is related to the new dilaton $\tilde{\phi}$ and a new curvature scalar for the Q-flux mimics the structure of the Ricci scalar

$$\begin{aligned} \mathcal{R}_Q &\equiv 2\eta_{bc} \beta^{ad} \partial_d \omega_{Q}{}_a^{bc} + \eta_{bc} \omega_{Q}{}_a^{ad} \omega_{Q}{}_d^{bc} - \eta_{bc} \omega_{Q}{}_a^{db} \omega_{Q}{}_d^{ac} \\ &= 2\eta_{ab} \beta^{ad} \partial_d Q_c{}^{bc} - \eta_{cd} Q_a{}^{ac} Q_b{}^{bd} - \frac{1}{4} \left(2\eta_{cd} Q_a{}^{bc} Q_b{}^{ad} + \eta^{ad} \eta_{be} \eta_{cg} Q_a{}^{bc} Q_d{}^{eg} \right) . \end{aligned} \tag{1.4.10}$$

The Lagrangian of β-supergravity explicitly contains the Q- and R-fluxes and its similarity with the four-dimensional scalar potential of gauged SUGRAs makes it a good candidate to uplift four-dimensional gauged SUGRAs. We remark that β-supergravity can also be derived from a DFT approach [56, 82, 83, 50]. Imposing the condition $\beta^{mn}\partial_n \cdot = 0$ and $\partial_p \beta^{np} = 0$ on the field content, $\tilde{\mathcal{L}}_\beta$ reduces to the Lagrangian obtained in [84].

The equations of motion for β-supergravity are also provided by the GG framework. The vanishing of the scalar S in GG is related to the equation of motion for the dilaton.

The equations of motion for the metric $\tilde{g}$ and the field β follow from another generalized curvature quantity

$$\frac{1}{2}R_{a\bar{b}}\gamma^a\epsilon^+ = [\gamma^a D_a, D_{\bar{b}}]\epsilon^+ \ . \tag{1.4.11}$$

The full set of equations of motion for β-supergravity follow then as

$$\mathcal{R} + 4(\partial\tilde{\phi})^2 + 4(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a)^2 + \mathcal{R}_Q - \frac{1}{2}R^{acd}f^b{}_{cd}\eta_{ab} - \frac{1}{2}R^2 = 0 \tag{1.4.12}$$

$$\begin{aligned}\frac{1}{2}\mathcal{R}_{ba} - \frac{1}{2}\eta_{e(a}\eta_{b)g}\breve{\mathcal{R}}^{ge} + \frac{1}{8}\eta_{ae}\eta_{bg}\eta_{if}\eta_{cd}R^{igc}R^{dfe}& \\ +\nabla_b\nabla_a\tilde{\phi} - \eta_{e(a}\eta_{b)g}\breve{\nabla}^g(\breve{\nabla}^e\tilde{\phi}) - \eta_{e(a}\eta_{b)g}\breve{\nabla}^g\mathcal{T}^e = 0 \ ,&\end{aligned} \tag{1.4.13}$$

$$\begin{aligned}\frac{1}{4}\eta_{ae}\eta_{bg}\eta^{df}\partial_d Q_f{}^{eg} - \frac{1}{2}\eta_{e[a}\partial_{b]}Q_d{}^{de} - \frac{1}{4}\beta^{gc}\partial_c f^e{}_{ab}\eta_{ge} + \frac{1}{2}\beta^{gc}\partial_c f^d{}_{d[a}\eta_{b]g}& \\ +\frac{1}{4}f^g{}_{cd}Q_{[a}{}^{dc}\eta_{b]g} + \frac{1}{2}\eta_{e[a}f^h{}_{b]d}Q_i{}^{ec}\eta_{ch}\eta^{di} + \frac{1}{2}\eta_{e[a}f^h{}_{b]c}Q_h{}^{ec}& \\ +\frac{1}{4}\eta_{bg}\eta_{ae}\eta^{ch}f^d{}_{dc}Q_h{}^{eg} - \frac{1}{4}\eta_{ch}Q_d{}^{dc}f^h{}_{ab}& \\ -\eta_{e[a}\nabla_{b]}(\breve{\nabla}^e\tilde{\phi}) - \eta_{e[a}\nabla_{b]}\mathcal{T}^e + \eta_{g[b}\breve{\nabla}^g\nabla_{a]}\tilde{\phi}& \\ -\frac{1}{2}\eta_{ae}\eta_{bg}\eta_{fc}R^{gfe}\mathcal{T}^c + \frac{1}{4}\eta_{ae}\eta_{bg}\eta_{df}e^{2\tilde{\phi}}\breve{\nabla}^d(e^{-2\tilde{\phi}}R^{gfe}) = 0 \ .&\end{aligned} \tag{1.4.14}$$

These provide the frame for identifying new vacua of β-supergravity.

Finally, β-supergravity can be related to four dimensional gauged SUGRAs by a dimensional reduction of $\tilde{\mathcal{L}}_\beta$. Extensions of $\tilde{\mathcal{L}}_\beta$ to a complete β-supergravity respecting SUSY are expected. However, the simultaneous inclusion of H-flux seems to be only consistent under some harsh restrictions.

Vacua of β-supergravity

The study of possible backgrounds of β-supergravity in the NSNS sector is by its own interesting. The toroidal example allows to obtain a first intuitive picture of the situation. Using the field content of $\tilde{g}$, β and $\tilde{\phi}$ the non-geometric configuration after two T-dualities can be expressed as a well-defined background of β-supergravity in the sense that it respects the symmetries of the Lagrangian.[16] It is not surprising that this also restores a SUGRA limit which is lost within the standard SUGRA formulation. Eventually, T-duality should not alter physics. Another example of such a situation is the Q-brane [85] which we will address later.

The symmetries of the Lagrangians of standard SUGRA and β-supergravity coincide, since the former only differ by a total derivative. The appearance of the symmetries however changes. Among standard diffeomorphisms β-supergravity contains so-called β gauge transformations. Not surprisingly, these act on both new fields $\tilde{g}$ and β as they are derived

[16] Formulated differently, a globally consistent configuration should be described by only one theory on every patch [81].

by the very same field redefinition. One possibility to observe well-defined backgrounds is that the transition functions between patches are realized by this new class of gauge transformation. However, the transformation behavior on the field $\tilde{g}$ collides with the interpretation of a consistent metric field, since the chances to compensate its transformation by a standard diffeomorphism are not too promissory. The second possibility includes declaring a new symmetry of the Lagrangian of β-supergravity for a restricted subcase. The appearance of the bivector β in the Lagrangian $\tilde{\mathcal{L}}_\beta$ allows for the following symmetry

$$\beta^{pq} \rightarrow \beta^{pq} + \varpi^{pq} \quad \text{with } \forall\ m, p, q,\ \varpi^{pr}\partial_r \cdot = 0\ ,\ \partial_m \varpi^{pq} = 0\ . \tag{1.4.15}$$

The constant shift ϖ can be related to the β-transform of a generic T-duality and moreover leaves the Q- and R-fluxes invariant. It turns out that this symmetry is equivalent to having a certain number of isometries and can therefore be seen as a reminiscent of T-duality. Now, this enables us to define geometric backgrounds of β-supergravity to transform under diffeomorphisms and β-transforms. Finally, we come up with a more general definition of geometric and non-geometric backgrounds:

- A field configuration is geometric if the fields are globally defined on the manifold considered so do not need to be glued, or if the transformations used to glue them from one patch to the other are symmetries of the theory, and the metric, dilaton and fluxes glue at most with diffeomorphisms.

- A field configuration is non-geometric if the transformations used to glue the fields from one patch to the other are symmetries of the theory, and if the metric, dilaton or fluxes glue with something else than diffeomorphisms.

Clearly, these definitions are theory dependent and bring forward the two biggest achievements of β-supergravity. As observed for the toroidal example β-supergravity allows to restore a geometric target space description for non-geometric backgrounds of standard SUGRA and provides a ten-dimensional uplift to some four-dimensional solutions of gauged SUGRAs with non-geometric fluxes.

However, since T-duality is at the bottom of the motivation and construction of β-supergravity by generating non-geometric background of standard SUGRA it is a main question to clarify the relation of (non)-geometric vacua of standard SUGRA and (non)-geometric vacua of β-supergravity. Eventually, geometric vacua of β-supergravity do only give rise to non-geometric vacua of standard SUGRA which are related to geometric T-dual ones. In other words, this class of vacua lies on a geometric T-duality orbit.

Finally, it is in principle a defined task to look for local solutions in β-supergravity. The new set of equations of motion of β-supergravity have to be solved in a respective patch. Naively, the non-geometric fluxes seem to provide more freedom in doing so. Unfortunately, simple compactification ansätze for finding new backgrounds fail and can be traced back to conceptional reasons. There is little hope that β-supergravity really holds new physics, but it nicely restores geometry for some non-geometric examples.

Bianchi identities and *NS*-branes

Using β-supergravity, one can now study backgrounds with non-geometric fluxes directly in ten dimensions. Of particular interest are BIs for the NSNS fluxes bringing constraints that have to be satisfied by the vacua in addition to the equations of motion. For specific backgrounds corresponding to *NS*-branes the BI receive corrections indicating that these branes actually source those fluxes.

For a vanishing H-flux and non-constant fluxes a ten-dimensional generalization of the BIs observed in gauged SUGRAs can be written down in β-supergravity

$$\partial_{[b} f^a{}_{cd]} - f^a{}_{e[b} f^e{}_{cd]} = 0 \ , \tag{1.4.16}$$

$$\partial_{[c} Q_{d]}{}^{ab} - \beta^{e[a} \partial_e f^{b]}{}_{cd} - \frac{1}{2} Q_e{}^{ab} f^e{}_{cd} + 2 Q_{[c}{}^{e[a} f^{b]}{}_{d]e} = 0 \ , \tag{1.4.17}$$

$$\partial_d R^{abc} - 3\beta^{e[a} \partial_e Q_d{}^{bc]} + 3 R^{e[ab} f^{c]}{}_{de} - 3 Q_d{}^{e[a} Q_e{}^{bc]} = 0 \ , \tag{1.4.18}$$

$$\beta^{e[a} \partial_e R^{bcd]} + \frac{3}{2} R^{e[ab} Q_e{}^{cd]} = 0 \ . \tag{1.4.19}$$

Interestingly, the above identities were found computing the Lagrangian for β-supergravity and precisely hold for the flux expressions (1.4.5). More systematically, BIs can be obtained by the Jacobi identities of some algebra or the square of a nilpotent derivative $\mathcal{D}$[17]

$$\mathcal{D}^2 = 0 \ \Leftrightarrow \ \text{BI (1.4.16)} - \text{(1.4.19)} \ + \ \text{scalar condition} \ . \tag{1.4.20}$$

For $\mathcal{D}$ to include also non-standard fluxes it is associated with the $Spin(D,D) \times \mathbb{R}^+$ covariant derivative $D_{\mathcal{A}}$ in GG [78, 87, 39, 88, 89, 71, 50]. Moreover, $\mathcal{D}$ turns out be a generalized Dirac operator. It can be represented in terms of forms and contractions on a form A_p using a Clifford map

$$\text{for standard supergravity:} \quad \mathcal{D} A_p = 2 e^{\phi} \left(\mathrm{d} - H \wedge\right) \left(e^{-\phi} A_p\right) \tag{1.4.21}$$

$$= 2 \left(\partial_a \cdot e^a \wedge - f \diamond - H \wedge - \mathrm{d}\phi \wedge\right) A_p \ , \tag{1.4.22}$$

$$\text{for } \beta\text{-supergravity:} \quad \mathcal{D} A_p = 2 e^{\tilde{\phi}} \left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T} \vee + R \vee\right) \left(e^{-\tilde{\phi}} A_p\right) \tag{1.4.23}$$

$$= 2 \left(\partial_a \cdot \tilde{e}^a \wedge + \beta^{ab} \partial_b \cdot \iota_a - f \diamond - Q \diamond + R \vee - \mathrm{d}\tilde{\phi} \wedge + (\check{\nabla} \tilde{\phi} - \tau) \vee\right) A_p \ , \tag{1.4.24}$$

where a dot in the derivatives indicates an action only on the form coefficient in flat indices. The convention for the action of fluxes is given in the appendix A.3. Besides allowing to derive the BI identities including non-geometric fluxes, the Dirac operator $\mathcal{D}$ later plays an important role in the SUSY considerations of β-supergravity.

A set of backgrounds that provides corrections in terms of a source term to the above BIs are *NS*-branes. Starting from the *NS*5-brane, a known vacuum of standard SUGRA, T-dual backgrounds can be constructed in a two step mechanism of smearing and applying a T-duality transformation in the gained isometry direction. In this process the Kaluza-Klein (*KK*) monopole and the 5_2^2-brane [90, 91, 92, 50, 93, 94, 95], also called Q-brane [85]

[17] The nilpotency of a "derivative" $\mathcal{D}_\sharp$ built from constant fluxes [16, 86] lead to sourceless BIs.

were found. In standard SUGRA the latter appears to be a non-geometric background [90, 92], but a geometric description is restored in β-supergravity [85, 50]. In analogy to the $NS5$-brane, the BIs for the two derived backgrounds (see also [96]) get corrected by source terms

$$\begin{aligned} &KK\text{-monopole}: \qquad \partial_{[b}f^a{}_{cd]} - f^a{}_{e[b}f^e{}_{cd]} = \frac{C_K}{3}\ \epsilon_{3\perp bcd}\ \epsilon_{1||e}\ \eta^{ea}\ \delta^{(3)}(r_3)\ , && (1.4.25)\\ &Q\text{-brane}: && (1.4.26)\\ &\partial_{[c}Q_{d]}{}^{ab} - \beta^{e[a}\partial_e f^{b]}{}_{cd} - \frac{1}{2}Q_e{}^{ab}f^e{}_{cd} + 2Q_{[c}{}^{e[a}f^{b]}{}_{d]e} = \frac{C_Q}{2}\ \epsilon_{2\perp cd}\ \epsilon_{2||ef}\ \eta^{ea}\eta^{fb}\ \delta^{(2)}(r_2)\ , \end{aligned}$$

which boil down to Poisson equations on warp factors f_K and f_Q

$$KK\text{-monopole}: \quad \Delta_3 f_K = c_K\ \delta^{(3)}(r_3)\ , \qquad Q\text{-brane}: \quad \Delta_2 f_Q = c_Q\ \delta^{(2)}(r_2)\ . \qquad (1.4.27)$$

All respectively remaining BIs should be satisfied with a vanishing RHS.

In principle, there also exist an R-brane for which we determine the warp factor f_R. However, in this case a smearing process which has to be applied on the β is not established.

Aspects of supersymmetry

SUSY provides technical simplifications when searching for new vacua of a theory. Information and constraints on the properties of possible vacua can be formulated geometrically. For standard ten-dimensional type II SUGRA corresponding methods were illustrated in [97, 26] to characterize new classes of vacua. Preservation of SUSY in the lower-dimensional theory demands the fermionic SUSY variations to vanish with regard to a compactification ansatz for a certain background. This leads to reformulated SUSY conditions [97], the so-called pure spinors conditions, in terms of GCG [40, 41]. In particular, the internal six-dimensional manifold can be classified for a Minkowski SUSY vacuum. If further the four-dimensional cosmological constant is zero the pure spinor conditions generalize the CY condition for flux-less backgrounds to a twisted generalized Calabi-Yau condition in the presence of fluxes.

The starting point when considering SUSY for β-supergravity are the SUSY variations of the fermionic field content governed by the generalized covariant derivative $D_{\mathcal{A}}$ (1.4.6)

$$\begin{aligned} \delta\psi_m^{1,2} &= \tilde{e}^a{}_m \left(\nabla_a \pm \eta_{ad}\check{\nabla}^d - \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\Gamma^{bc}\right)\epsilon^{1,2}\ , && (1.4.28)\\ \delta\rho^{1,2} &= \left(\Gamma^a\nabla_a \mp \Gamma^a\eta_{ad}\check{\nabla}^d + \frac{1}{24}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\Gamma^{abc} - \Gamma^a\partial_a\tilde{\phi} \mp \Gamma^a\eta_{ab}(\beta^{bc}\partial_c\tilde{\phi} - \mathcal{T}^b)\right)\epsilon^{1,2}\ , \end{aligned}$$

where $\rho^{1,2} \equiv \Gamma^a\psi_a^{1,2} - \lambda^{1,2}$ and $\epsilon^{1,2}$ the SUSY fermionic parameters and upper/lower signs refer to the indices $1, 2$. Then, analogous pure spinor conditions[18] for β-supergravity can

[18] Contrary to the standard pure spinor conditions the conditions (1.4.29) and (1.4.30) are necessary but not sufficient to preserve SUSY which is probably due to the absence of a RR contribution.

be formulated using a specific compactification ansatz

$$\frac{1}{2}\mathcal{D}\Phi_1 + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_1 = 2\varepsilon\ e^{-A}\mu\ \ \mathrm{Re}(\Phi_2) \tag{1.4.29}$$

$$\frac{1}{2}\mathcal{D}\Phi_2 + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_2 = 3\varepsilon\ e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_1) + e^{-A}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a\right)(e^{A})\overline{\Phi_2}\ , \tag{1.4.30}$$

where $\mathcal{D}$ is precisely the generalized Dirac operator given in (1.4.23) and the other derivatives act solely on the warp factor e^{2A} incorporated in the compactification. For type II SUGRAs we list the following convention

$$\mathrm{IIA}:\ \Phi_1 = \Phi_+\ ,\ \Phi_2 = \Phi_-\ ,\ \varepsilon = +1\ , \qquad \mathrm{IIB}:\ \Phi_1 = \Phi_-\ ,\ \Phi_2 = \Phi_+\ ,\ \varepsilon = -1\ . \tag{1.4.31}$$

$\Phi_\pm$ denote pure spinors in GCG, in particular $O(6,6)$ spinors, but can be conveniently interpreted as polyforms.

A geometrical characterization of the class of backgrounds in β-supergravity satisfying the above pure spinor equations can be established to some extent. In the case of Mink SUSY vacuum and $\check{\nabla}^a \cdot \iota_a A_p = 0$ the first pure spinor condition (1.4.29) provides a β-twisted GCY condition. The second condition (1.4.30) yields a β-twisted generalized Kähler condition in the absence of RR fluxes and a constant warp factor.

Finally, pure spinor conditions and the Dirac operator also govern the structure of the superpotential W for $\mathcal{N} = 1$ four-dimensional effective theories obtained from ten-dimensional standard SUGRA in presence of an $SU(3) \times SU(3)$ structure. This leads us to propose the following superpotential for β-supergravity with only NSNS contributions

$$\tilde{W}_{\mathrm{NS}} = \frac{C}{2}\int_{\mathcal{M}} \langle e^{-\tilde{\phi}}\Phi_1^0, \mathcal{D}\,\mathrm{Im}\,\Phi_2^0\rangle\ , \tag{1.4.32}$$

Here, the Mukai product enters and the warp factor is taken to be constant. For an $SU(3)$ structure, the pure spinors are taken in the simple form

$$\Phi_+^0 = e^{\mathrm{i}\theta_+}e^{-iJ}\ , \qquad \Phi_-^0 = \mathrm{i}\Omega\ . \tag{1.4.33}$$

Typically, superpotentials for standard SUGRA theories are reproduced using the above formula. However more interestingly, the formula (1.4.32) allows to switch on non-geometric fluxes in the scalar potential, possibly derived from β-supergravity, and agrees well with expressions of [98, 86, 99] in type IIA and IIB, corresponding to an O6-plane and an O3- or O7-plane. We even obtain new expressions in the O5- or O9-plane (or heterotic) case.

A completion of the above Lagrangian $\tilde{\mathcal{L}}_\beta$ to other sectors should be possible by further reformulating standard SUGRA. However, it is has not been worked out and we comment on it in the conclusion 7.

1.5 Structure of the thesis

This thesis is based on the papers [100, 101, 102, 103] where the focus lies on the last three.

We start with a review of Generalized Geometry (GG) and Double Field Theory (DFT) in chapter 2. Here, we are interested in providing the basic definitions and notations. We layout the details on the reformulation of type II supergravities (SUGRAs) within GG and in DFT using the generalized metric formalism. Finally, we present the results of a T-duality invariant conformal field theory (CFT) with respect to determining the origin of the strong constraint in DFT.

Chapter 3 recapitulates earlier work on field redefinitions, inspired by GG and then starts off presenting in detail the construction of β-supergravity following the GG formalism. In this process new structures for the Q-flux are obtained in the form of a second covariant derivative involving an analogue of the Levi-Civita connection. The worked out formalism then features generalized covariant derivatives that enter the definitions of generalized curvature quantities which allow to compute the Lagrangian and the equations of motion of β-supergravity.

At length, we provide a discussion on the relation of geometric and non-geometric backgrounds within standard SUGRA and β-supergravity in chapter 4. We present the toroidal example, the symmetries of the Lagrangian and investigate a well-defined class of geometric backgrounds in β-supergravity. Eventually, we turn to the study of the equations of motion with respect to pure NSNS solutions.

An interesting set of backgrounds are provided by NS-branes in chapter 5. Alongside the $NS5$-brane and the Kaluza-Klein (KK) monopole, in particular the Q-brane is studied and the latter experiences a nice description in terms of the fields of β-supergravity. We also introduce ten-dimensional Bianchi identities (BIs) involving non-constant (non)-geometric fluxes that receive corrections from source terms generated by the respective NS-brane. These BIs are then obtained from a generalized nilpotent Dirac operator.

In chapter 6 we investigate the supersymmetry (SUSY) conditions of β-supergravity using a generic compactification ansatz. Introducing pure spinors a reformulation is achieved that makes use of the generalized Dirac operator. In a similar fashion, the pure spinors and the Dirac operator lead to a generic expression for a superpotential that can be evaluate for β-supergravity. We end with the proposal of a geometrical characterization, analogue to a Calabi-Yau (CY) condition, for backgrounds preserving SUSY.

The conclusion 7 summarizes the standing of β-supergravity with respect to its capability to investigate genuinely non-geometric backgrounds and finishes with an outlook on a complete supersymmetric version of β-supergravity.

Chapter 2

Generalized Geometry & Double Field Theory

The development of $O(d,d)$ covariant formalisms describing effects of the string within low-energy effective theories on the target space has attracted quite some interest in recent years. Mathematically motivated, Generalized Geometry (GG) has been studied in relation to string theory and supergravity (SUGRA) theories with background fluxes. On the other side, Double Field Theory (DFT) was directly constructed from considerations in string field theory. Both formalisms treat generalized geometric quantities, e.g. a generalized metric, and construct generalized curvatures by specifying a generalized connection. This leads to two Lagrangians formulated in terms of these objects which coincide if DFT is equipped with the strong constraint.[1] Their equivalence is also reflected in their agreement with the standard $\mathcal{L}_{\text{NSNS}}$ (3.1.5) up to a total derivative.

Beyond the level of the formalism several differences between the GG and DFT occur. These mostly concern the geometry of the underlying space on which the two theories live. GG is based on the mathematical concept of the generalized tangent bundle E_T defined in Generalized Complex Geometry (GCG). A doubling in the fiber over a conventional manifold by the sum of the tangent and cotangent bundle enables us to implement the group $O(d,d)$. In particular, the order of the fibration will be of interest to us when we trade the standard b-field for a new field β. In contrast, DFT introduces a truely doubled space along two sets of coordinates $(x^m, \tilde{x}_m)$. The notion of a doubled geometry extends and contains previous examples of non-geometric backgrounds like the T-fold [34]. In some cases an ordinary manifold of doubled dimension [30] is expected; however, recent work [104, 105] indicates that the coordinate transformations are more subtle. Mathematical investigations on these kind of spaces have been considered in [106, 107, 108]. In this thesis we are mostly interested in the local formalisms of both T-duality covariant theories with respect to providing background for constructing β-supergravity.

[1] The strong constraint is a priori a local condition. For a global equivalence of the two formalisms the constraint should be solved on each patch in the same way, e.g. $\tilde{\partial} = 0$. This is at least possible for geometric backgrounds.

2.1 Generalized Geometry

The foundation of GCG was laid by Hitchin [40] and Gualtieri [41]. The notion of a generalized complex structure on the generalized tangent bundle allows to unify complex and symplectic structures and presents an extension. Its relevance to theoretical physics is based on the fact that the T-duality group $O(d,d)$ naturally appears in this framework. Moreover, supersymmetry (SUSY) can be naturally addressed by falling back on pure spinors of GCG, which also provide a set of tools for flux compactifications. First applications of GCG followed in the context of supersymmetric type II backgrounds [42] and string σ-models. It was only later that the formalism was used in the reformulation of SUGRA theories of type II [39], which triggered the investigation of further SUGRA theories and extensions to M-theory.

The presence of the T-duality group raises hope that GG could shed light on the realization of non-geometric string backgrounds [78] and further aspects of non-geometry in general.

2.1.1 Structures on the generalized tangent bundle

GG observes basic structures on the *generalized tangent bundle* E

$$0 \to T^*\mathcal{M} \to E \to T\mathcal{M} \to 0 \ , \tag{2.1.1}$$

based on the idea of treating vectors and one-forms on an equal footing. The extension of the conventional tangent bundle $T\mathcal{M}$ by the cotangent bundle $T^*\mathcal{M}$ leads usually to the above fibration. Then, gluing conditions on the sections of E have to be specified for moving from patch to patch. On an overlap $U_\alpha \cap U_\beta$, this is defined in the following way

$$v_\alpha + \xi_\alpha = v_\beta + (\xi_\beta - i_{v_\beta} d\Lambda_{\alpha\beta}) \ . \tag{2.1.2}$$

Hence, $v_{\alpha(\beta)} \in TU_{\alpha(\beta)}$ globally specifies a vector, while the one-form part $\xi_{\alpha(\beta)} \in T^*U_{\alpha(\beta)}$ allows for two-form shifts $d\Lambda$. Mathematically more precise, one speaks of introducing a gerbe on the manifold $\mathcal{M}$ which is related to the appearance of the b-field in physics terms. The gerbe brings along a cocycle condition that is possibly curved by the two-form b and further makes sure that patching around a non-trivial loop in the manifold $\mathcal{M}$ works.

Moreover, there is a natural action of one-forms on vectors and it is therefore reasonable to define a symmetric bilinear form on the sections of the generalized tangent bundle E

$$\langle V, W \rangle = \langle v + \xi, w + \eta \rangle = \frac{1}{2}(\xi(w) + \eta(v)) \ . \tag{2.1.3}$$

A Lie algebra with $T \in \mathfrak{o}(d,d)$ respecting the symmetric bilinear form, given by a metric η, encodes the symmetries of the bundle

$$T = \begin{pmatrix} \alpha & \beta \\ \omega & -\alpha^T \end{pmatrix} \ , \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{2.1.4}$$

where $\alpha \in \mathfrak{gl}(d,\mathbb{R})$, $\omega^T = -\omega$ a two-form and $\beta^T = -\beta$ a bivector. The symmetry generators of the tangent bundle are then observed by exponentiation:

- $GL^+(d,\mathbb{R})$-transformations

$$e^{\alpha} = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & (e^{\alpha^T})^{-1} \end{pmatrix} \ , \tag{2.1.5}$$

  but more generally an element A of the full $GL(d,\mathbb{R})$ can be embedded by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} . \tag{2.1.6}$$

- B-transformations

$$e^{\omega} = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \ , \tag{2.1.7}$$

  where $v + \xi \mapsto v + (\xi + i_v\omega)$.

- β-transformations

$$e^{\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \ , \tag{2.1.8}$$

  where $v + \xi \mapsto (v + \beta(\xi)) + \xi$.

Another interesting structure in GG is the $O(d,d)$ *Clifford algebra*

$$\{\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}}\} = 2\eta_{\mathcal{A}\mathcal{B}} \ , \tag{2.1.9}$$

where $\eta_{\mathcal{A}\mathcal{B}}$ denotes the $O(d,d)$ metric in flat generalized indices. Realizing the action of a generalized vector on a polyform $\Psi \in \Lambda^* T^*\mathcal{M}|_{U_\alpha}$ locally, a natural representation is found

$$V \cdot \Psi = V^{\mathcal{A}}\Gamma_{\mathcal{A}} \cdot \Psi = (v_\alpha + \xi_\alpha) \cdot \Psi = i_{v_\alpha}\Psi + \xi_\alpha \wedge \Psi \ . \tag{2.1.10}$$

When we discuss SUSY we are going to make heavy use of the Clifford algebra and the Clifford map relating pure spinors in GG to polyforms.

Next, we observe that GG replaces the $GL(d)$ freedom to choose a basis in $T\mathcal{M}$ by the enlarged group $O(d,d) \times \mathbb{R}^+$ which acts on

$$\tilde{E} = L \otimes E \ , \tag{2.1.11}$$

where the line bundle L takes care of the dilaton. We can fix a *conformal frame* $\{e^{-2d}\mathring{\mathcal{E}}_{\mathcal{A}}\}$ of the generalized tangent bundle $\tilde{E}$ that satisfies the following orthogonality condition

$$\langle \mathring{\mathcal{E}}_{\mathcal{A}}, \mathring{\mathcal{E}}_{\mathcal{B}} \rangle = \eta_{\mathcal{A}\mathcal{B}} \ , \tag{2.1.12}$$

where d denotes the generalized dilaton. These frames can be rotated into a distinct frame $\{e^{-2d}\mathring{\mathcal{E}}'_{\mathcal{A}}\}$ by an element $M \in O(d,d) \times \mathbb{R}^+$

$$\mathring{\mathcal{E}}'_{\mathcal{A}} = \mathring{\mathcal{E}}_{\mathcal{B}} M^{\mathcal{B}}{}_{\mathcal{A}}, \quad \text{since} \quad M^{\mathcal{C}}{}_{\mathcal{A}} M^{\mathcal{D}}{}_{\mathcal{B}} \eta_{\mathcal{C}\mathcal{D}} = \sigma^2 \eta_{\mathcal{A}\mathcal{B}} \ , \tag{2.1.13}$$

for some scalar factor σ.

A specific frame respecting the embedding of the tangent bundle into the generalized tangent bundle, i.e. defining a map $T\mathcal{M} \to E$, is provided by the *conformal split frame* $\{\mathcal{E}_{\mathcal{A}}\}$

$$e^{-2d}\ \mathcal{E}_{\mathcal{A}} = \begin{cases} e^{-2d}\ \mathcal{E}_a = e^{-2\phi}\sqrt{|g|}\ (\partial_a + b_{ab}e^b) & \text{for } \mathcal{A} = a \\ e^{-2d}\ \mathcal{E}^a = e^{-2\phi}\sqrt{|g|}\ e^a & \text{for } \mathcal{A} = a + d \end{cases} , \tag{2.1.14}$$

where $\{\partial_a\}$ is a basis for $T\mathcal{M}$ and $\{e^a\}$ is the dual basis on $T^*\mathcal{M}$. In particular, $O(d,d)\times\mathbb{R}^+$ gets broken and the class of conformal split frames is preserved by elements of the form

$$M = (\det A) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} . \tag{2.1.15}$$

This group is known as the geometric subgroup $G_{geom} \subset O(d,d)\times\mathbb{R}^+$ and contains exactly the elements used for gluing generalized vectors along different patches of the generalized tangent bundle E.

The symmetries of the generalized tangent bundle can be nicely packaged within a *generalized Lie derivative*

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \zeta - i_w d\lambda , \tag{2.1.16}$$

for two generalized vectors $V = v + \lambda$ and $W = w + \zeta$. It incorporates in contrast to the standard Lie derivative also the b-field gauge transformations besides standard diffeomorphisms. The associate bracket for the generalized Lie derivative, the *Courant bracket*, is provided by

$$\begin{aligned} [V,W]_{Courant} =& \frac{1}{2}(L_V W - L_W V) \\ =& [v,w]_{Lie} + \mathcal{L}_v\zeta - \mathcal{L}_w\lambda - \frac{1}{2}d(i_v\zeta - i_w\lambda) . \end{aligned} \tag{2.1.17}$$

In contrast to the standard Lie bracket it does not satisfy the Jacoby identity. We will come upon the $O(d,d)$ notation of these structures in DFT.

In analogy to Riemannian geometry one can introduce a generalized connections on the generalized tangent bundle. The corresponding *generalized covariant derivative* acting on generalized objects in frame indices is denoted by

$$D_{\mathcal{M}} W^{\mathcal{A}} = \partial_{\mathcal{M}} W^{\mathcal{A}} + \tilde{\Omega}_{\mathcal{M}}{}^{\mathcal{A}}{}_{\mathcal{B}} W^{\mathcal{B}} . \tag{2.1.18}$$

Compatibility with the $O(d,d)\times\mathbb{R}^+$ structure restricts the connection coefficients in the following way

$$\tilde{\Omega}_{\mathcal{M}}{}^{\mathcal{A}}{}_{\mathcal{B}} = \Omega_{\mathcal{M}}{}^{\mathcal{A}}{}_{\mathcal{B}} - \Lambda_{\mathcal{M}}\delta^{\mathcal{A}}{}_{\mathcal{B}} , \tag{2.1.19}$$

where Λ takes care of the line bundle factor and Ω satisfies

$$\Omega_{\mathcal{M}}{}^{\mathcal{A}\mathcal{B}} = -\Omega_{\mathcal{M}}{}^{\mathcal{B}\mathcal{A}} . \tag{2.1.20}$$

GG also provides the notion of a *generalized torsion* defined by using the generalized Lie derivative and the generalized covariant derivative

$$T(V)\cdot\alpha = L^D_V\alpha - L_V\alpha\ , \tag{2.1.21}$$

where the index D denotes that the partial derivative has been replaced by the generalized covariant one. Then in the frame $\{\Phi^{-1}\mathring{\mathcal{E}}_{\mathcal{A}}\}$, the generalized torsion becomes

$$T_{\mathcal{ABC}} = -3\tilde{\Omega}_{[\mathcal{ABC}]} + \tilde{\Omega}_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{B}}\eta_{\mathcal{AC}} - e^{4d}\langle e^{-2d}\ \mathring{\mathcal{E}}_{\mathcal{A}}, L_{\mathring{\mathcal{E}}_{\mathcal{B}}}(e^{-2d}\ \mathring{\mathcal{E}}_{\mathcal{C}}\rangle\ . \tag{2.1.22}$$

The rescaling is due to the generalized Lie derivative encoding non-weighted gauge transformations. Furthermore, we can observe a splitting of the generalized torsion

$$\begin{aligned}(T_1)_{\mathcal{MNP}} &= -\,3\tilde{\Omega}_{[\mathcal{MNP}]}\ ,\\ (T_2)_{\mathcal{M}} &= -\,3\tilde{\Omega}_{\mathcal{Q}}{}^{\mathcal{Q}}{}_{\mathcal{M}}\ ,\end{aligned} \tag{2.1.23}$$

which are related to H-flux and the dilaton ϕ for the split frame (2.1.14)

$$T_1 = -4H,\quad T_2 = -4d\phi\ . \tag{2.1.24}$$

Remarkably, generalizing the definition of the Riemannian curvature tensor to

$$R(U,V,W) = [D_U, D_V]W - D_{[U,V]}W\ , \tag{2.1.25}$$

where now the Courant-bracket enters, does not provide a well-defined tensor. However, we are going to see in the next section that generalized Ricci curvatures do exist.

2.1.2 Preserving an $O(p,q)\times O(q,p)$ structure

Differential geometry provides the concept of a Riemannian metric g on a manifold $\mathcal{M}$. In the presence of a metric g the choice of frame is reduced by local Lorentz symmetry $O(d)\subset GL(d,\mathbb{R})$. Further conditions, like metric compatibility and vanishing torsion, allow to uniquely fix a connection on the tangent bundle $T\mathcal{M}$, namely the Levi-Civita connection. Together with the definition of the Ricci scalar, these are the ingredients for Einstein gravity.

In GG the maximal compact subgroup we are interested in is $O(p,q)\times O(q,p)\subset O(d,d)\times\mathbb{R}^+$. This local symmetry arises due to specifying a generalized metric $\mathcal{H}$ in addition to the $O(d,d)$ metric η. Its construction relies on splitting the generalized tangent bundle into subbundles $C_\pm$ on which η gains positive and respectively negative definiteness

$$\mathcal{H} = \eta|_{C_+} - \eta|_{C_-}\ . \tag{2.1.26}$$

The relation to the field content of standard gravity theories in terms of the metric g and the b-field is then found by specifying a map from $T\mathcal{M}$ and $T^*\mathcal{M}$ into $C_\pm$

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & -bg^{-1}\\ g^{-1}b & g^{-1}\end{pmatrix}\ . \tag{2.1.27}$$

The generalized metric $\mathcal{H}$ is symmetric in its indices, squares to the identity, i.e. $(\eta\mathcal{H})^2 = 1$. Further, it has determinant one and is positive definite if g is. As in Riemannian geometry the introduction of two sets of the standard vielbein $e_a^{\pm}$ and their embedding into a generalized frame $\mathring{\mathcal{E}}_{\mathcal{A}}$ determine the local $O(p,q) \times O(q,p)$ structure, which can be seen as a double Lorentz symmetry. The action of an element $K \in O(p,q) \times O(q,p)$ provides the transformations between these frames

$$K = \frac{1}{2} \begin{pmatrix} O_+ + O_- & O_+ - O_- \\ O_+ - O_- & O_+ + O_- \end{pmatrix} \ . \tag{2.1.28}$$

This specific embedding is later decisive in picking an $O(p,q) \subset O(p,q) \times O(q,p)$, done by aligning the two vielbeine $e_a^+ = e_a^-$. The degrees of freedom then reduce to those of a standard metric g and a b-field.

Having the generalized metric $\mathcal{H}$ at hand we are in the position to construct the analogue of the Levi-Civita connection in GG. Generalized metric compatibility together with a condition on the conformal part, related to the dilaton, have to be imposed

$$D\mathcal{H} = 0 \quad \text{and} \quad De^{-2d} = 0 \ . \tag{2.1.29}$$

These conditions are naturally satisfied by embedding the Levi-Civita connection ∇ in the following way

$$D^{\nabla}_{\mathcal{M}} W^a = \begin{cases} \nabla_m w_+^a & \text{for} \quad \mathcal{M} = m \\ 0 & \text{for} \quad \mathcal{M} = m + d \end{cases} , \quad D^{\nabla}_{\mathcal{M}} W^{\bar{a}} = \begin{cases} \nabla_m w_-^{\bar{a}} & \text{for} \quad \mathcal{M} = m \\ 0 & \text{for} \quad \mathcal{M} = m + d \end{cases} , \tag{2.1.30}$$

where w_+^a and $w_-^{\bar{a}}$ are related to a vector $w \in \Gamma(T\mathcal{M})$ in the basis e^+ or e^-. This generalized connection is obviously compatible with an $O(p,q) \times O(q,p)$ structure. Unfortunately, it turns out that the generalized covariant derivative $D^{\nabla}_{\mathcal{M}}$ is not generalized torsion free. However, this can be corrected by adding a part which respects compatibility and compensates for the generalized torsion

$$\tilde{D}^{\nabla} = D^{\nabla} + \Sigma, \quad \text{where} \quad \Sigma_{\mathcal{M}ab} = -\Sigma_{\mathcal{M}ab},\ \Sigma_{\mathcal{M}\bar{a}\bar{b}} = -\Sigma_{\mathcal{M}\bar{a}\bar{b}} \ . \tag{2.1.31}$$

In this way, we certainly loose the uniqueness of the generalized analogue of the Levi-Civita connection. Nevertheless, we observe that some parts are fully determined and allow to define generalized curvature objects. For this we list the following covariant derivatives respecting a $Spin(p,q) \times Spin(q,p)$ structure

$$D_{\bar{a}}\epsilon^+ = (\nabla_{\bar{a}} - \tfrac{1}{8} H_{\bar{a}bc}\gamma^{bc})\epsilon^+ \ , \tag{2.1.32}$$

$$D_a\epsilon^- = (\nabla_{\bar{a}} + \tfrac{1}{8} H_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}})\epsilon^- \ , \tag{2.1.33}$$

$$\gamma^a D_a\epsilon^+ = (\gamma^a\nabla_a - \tfrac{1}{24} H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi)\epsilon^+, \tag{2.1.34}$$

$$\gamma^{\bar{a}} D_{\bar{a}}\epsilon^- = (\gamma^{\bar{a}}\nabla^{\nabla}_{\bar{a}} + \tfrac{1}{24} H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{a}}\partial_{\bar{a}}\phi)\epsilon^- \ , \tag{2.1.35}$$

involving $Spin(p,q)$ spinors $\epsilon^{\pm}$ and respective gamma matrices γ^a and $\gamma^{\bar{a}}$. A generalized Ricci tensor can be defined by using the above determined parts of the generalized connection

$$\tfrac{1}{2}R_{a\bar{b}}\gamma^a\epsilon^+ = [\gamma^a D_a, D_{\bar{b}}]\epsilon^+ \quad \text{or} \quad \tfrac{1}{2}R_{\bar{a}b}\gamma^{\bar{a}}\epsilon^- = [\gamma^{\bar{a}} D_{\bar{a}}, D_b]\epsilon^- \ . \tag{2.1.36}$$

Even more pleasant is the existence of a generalized curvature scalar

$$-\tfrac{1}{4}S\epsilon^+ = (\gamma^a D_a \gamma^b D_b - D^{\bar{a}} D_{\bar{a}})\epsilon^+ \quad \text{or} \quad -\tfrac{1}{4}S\epsilon^- = (\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a)\epsilon^- \ , \tag{2.1.37}$$

allowing for the formulation of an action principle in an Einstein-Hilbert like form

$$S_{\text{NSNS}} = \frac{1}{2\kappa^2}\int e^{-2d} S \ . \tag{2.1.38}$$

So far, we kept two sets of vielbeine $e^{\pm}$ respecting the $O(p,q)\times O(q,p)$, which we now align to find the following curvature expressions in GG

$$\begin{aligned} R_{ab} &= \mathcal{R}_{ab} - \tfrac{1}{4}H_{acd}H_b{}^{cd} + 2\nabla_a\nabla_b\phi + \tfrac{1}{2}e^{2\phi}\nabla^c(e^{-2\phi}H_{cab}) \ , \\ S &= \mathcal{R}(g) + 4\nabla^2\phi - 4(\partial\phi)^2 - \tfrac{1}{2}H^2 \ , \end{aligned} \tag{2.1.39}$$

where $\mathcal{R}_{ab}$ is the standard Ricci tensor and $\mathcal{R}$ is the standard scalar curvature of Einstein gravity. In this way, GG presents a reformulation of the NSNS sector of SUGRA of type II up a total derivative. The equations of motion for the metric g, the b-field and the dilaton ϕ in this formalism are packaged into the vanishing of the generalized Ricci tensor and generalized scalar curvature

$$R_{a\bar{b}} = 0 \ , \quad S = 0 \ . \tag{2.1.40}$$

Additional findings regarding the SUSY variations involving the $Spin(p,q) \times Spin(q,p)$ covariant derivative (2.1.35) or the RR sector are detailed in the section 6.1.

2.2 Double Field Theory

DFT is a promising low-energy effective candidate theory capturing stringy symmetries, in particular T-duality, which was developed by Hull and Zwiebach in [37, 38] and later refined in the work of Hohm. As stated in the Introduction there is hope that DFT provides substantial extension to standard SUGRA theories and is able to cover all kind of features of doubled and non-geometric backgrounds.

Earliest developments [109, 110] go back to Siegel who introduced a doubled formalism for an enlarged group $GL(d)\times GL(d)$. New attention to T-duality covariant formalisms of gravity theories came up when the framework of string field theory [111, 112] was used. In numerous papers different forms of DFT were formulated until the most familiar version in terms of the generalized metric [54] arose. Further important formulations[2] are based on a background independent field $\mathcal{E} = g + b$ [115] combining the metric g and the b field or the notion of a generalized flux [50, 51] containing in addition to standard fluxes also non-geometric ones. In the following we present the generalized metric formulation of DFT and later focus on an underlying geometric concept. Finally, we discuss the origin of the strong constraint of DFT within a T-duality invariant conformal field theory (CFT).

2.2.1 Generalized metric formulation

This formulation is based on forming objects transforming under the group $O(d,d)$

$$X^{\mathcal{M}} \equiv \begin{pmatrix} \tilde{x}_m \\ x^m \end{pmatrix} , \quad \partial_M \equiv \begin{pmatrix} \tilde{\partial}^m \\ \partial_m \end{pmatrix} , \quad \xi^M \equiv \begin{pmatrix} \tilde{\xi}_m \\ \xi^m \end{pmatrix} \ . \tag{2.2.1}$$

The doubling of the space is respected by introducing $\tilde{x}_i$ usually taking into account the possibility of the string to wind around some compact direction. Naturally, there is a partial derivative $\tilde{\partial}^i$ associated to the winding coordinates. Moreover, we combine the two gauge parameters ξ_m and $\tilde{\xi}^m$ to a generalized objects providing infinitesimal double diffeomorphisms, the generalized gauge transformations in DFT.

The degrees of freedom of DFT are summerized in the *generalized metric* $\mathcal{H}$ and a *generalized dilaton* d

$$\mathcal{H}^{\mathcal{M}\mathcal{N}} = \begin{pmatrix} g_{mn} - b_{mp}g^{pq}b_{qn} & -b_{mp}g^{pn} \\ g^{mp}b_{pn} & g^{mn} \end{pmatrix} \ , \quad d \tag{2.2.2}$$

depending on the doubled coordinates $X^{\mathcal{M}}$ and transforming with the parameters $\xi^{\mathcal{M}}$.

The construction of an action in terms of $\mathcal{H}$ is based on building $O(d,d)$ scalars out of the generalized metric $\mathcal{H}$, the dilaton d, the partial derivatives $\partial_{\mathcal{M}}$ and the $O(d,d)$ metric η by contracting all indices. The implementation of a discrete $\mathbb{Z}_2$-symmetry for the b-field, which is not a T-duality transformation, rules out terms including the metric η or derivatives with upper index $\partial^{\mathcal{M}}$. Then, the action in terms of the generalized metric takes the form

$$\begin{aligned} S = \int dx d\tilde{x} e^{-2d}\Big(&\frac{1}{8}\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{K}\mathcal{L}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{K}\mathcal{L}} - \frac{1}{2}\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{K}\mathcal{L}}\partial_{\mathcal{L}}\mathcal{H}_{\mathcal{M}\mathcal{K}} \\ &2\partial_{\mathcal{M}}d\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{M}\mathcal{N}} + 4\mathcal{H}^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d\Big) \ , \end{aligned} \tag{2.2.3}$$

which is obviously invariant under global T-duality transformations and gauge-invariant under the following transformations

$$\delta_\xi \mathcal{H}^{\mathcal{M}\mathcal{N}} = \xi^{\mathcal{P}}\partial_{\mathcal{P}}\mathcal{H}^{\mathcal{M}\mathcal{N}} + (\partial^{\mathcal{M}}\xi_{\mathcal{P}} - \partial_{\mathcal{P}}\xi^{\mathcal{M}})\mathcal{H}^{\mathcal{P}\mathcal{N}} + (\partial^{\mathcal{N}}\xi_{\mathcal{P}} - \partial_{\mathcal{P}}\xi^{\mathcal{N}})\mathcal{H}^{\mathcal{M}\mathcal{P}} \ , \tag{2.2.4}$$

[2] Stringy differential geometry introduced in [113, 114] is a third formalism based on a projection-compatible semi-covariant derivative. Its geometric objects coincide after projection with the corresponding quantities in DFT.

if additionally the constraint

$$\partial^{\mathcal{M}} A \partial_{\mathcal{M}} B = 0 \; , \tag{2.2.5}$$

on fields A and B satisfying $\partial^{\mathcal{M}}\partial_{\mathcal{M}}A = \partial^{\mathcal{M}}\partial_{\mathcal{M}}B = 0$ is imposed. This is the so-called *strong constraint* derived by demanding that products of fields and gauge parameters of the theory vanish under $\partial^{\mathcal{M}}\partial_{\mathcal{M}}\cdot = 0$. It is related to the level-matching condition in string theory and makes sure that the theory only depends on a section of half the dimension in the doubled geometry. Therefore, the theory is not really doubled when applying the strong constraint. In the next section, we clarify the origin of the strong constraint by means of introducing a T-duality invariant CFT.

The gauge transformation of the generalized metric motivates the introduction of a *generalized Lie derivative* for DFT

$$\hat{\mathcal{L}}_\xi V^{\mathcal{M}} = \xi^{\mathcal{P}}\partial_{\mathcal{P}} V^{\mathcal{M}} + (\partial^{\mathcal{M}}\xi_{\mathcal{N}} - \partial_{\mathcal{N}}\xi^{\mathcal{M}})V^{\mathcal{N}} \; . \tag{2.2.6}$$

Remarkably, it leaves the $O(d,d)$ metric η invariant which is not possible in standard Riemannian geometry. Moreover, one finds that the transformed generalized metric $\mathcal{H}' = \mathcal{H} + \mathcal{L}_\xi \mathcal{H}$ is again an element of $O(d,d)$.

The generalized Lie derivative is connected to a bracket structure on the gauge parameters by calculating the commutator of two generalized Lie derivatives on a general field $[_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}] = \hat{\mathcal{L}}_{\xi_{12}}$ with $\xi_{12} = [\xi_1, \xi_2]_C$,where the *C-bracket* is the natural bracket on generalized vector fields or here the $Od,d)$ gauge parameter ξ

$$[\xi_1, \xi_2]_C^{\mathcal{M}} \equiv \xi^{\mathcal{N}}_{[1}\partial_{\mathcal{N}}\xi^{\mathcal{M}}_{2]} - \frac{1}{2}\eta^{\mathcal{MN}}\eta_{\mathcal{PK}}\xi^{\mathcal{P}}_{[1}\partial_{\mathcal{N}}\xi^{\mathcal{K}}_{2]} \; . \tag{2.2.7}$$

It is interesting to observe that (2.2.1) holds only upon using the constraint (2.2.5). However, the strong form is only sufficient and not necessary to achieve a closed gauge algebra. There exist possible field configurations that violate the strong constraint [56, 57, 116]. These backgrounds satisfy the *closure constraints*, taking a weaker form in contrast to the strong constraint, and are furthermore of truely doubled type.

Furthermore, in [52] it was shown that a weaker constraint can be achieved for the RR fields of DFT in the sense that in addition to the coordinate dependence on a totally null subspace a linear dependence on coordinates of an orthogonal space is possible.

Rather surprising is the fact that one can identify a gauge parameter leaving all fields invariant

$$\xi^{\mathcal{M}} = \eta^{\mathcal{MN}}\partial_{\mathcal{N}}\chi \; . \tag{2.2.8}$$

We further stress that the Jacobi identity for the C-bracket is violated. Hence, this bracket coincides in this feature with the Courant-bracket from GG and reduces to it when the strong constraint is applied in the form $\tilde{\partial}^m \cdot = 0$. Finally, the C-bracket as well as the Courant-bracket do provide a symmetry algebra since the failure of the Jacobi identity is precisely a transformation of the form (2.2.8).

A geometric framework underlying DFT can be traced back to the early work of Siegel based on $GL(d) \times GL(d)$ structure [109, 110] and recent work [55]. In principle, the construction of a connection and associated curvature objects follows the description laid out in the section 2.1 for GG, except some minor technical differences regarding the doubling of the coordinate space and the implementation of the strong constraint.

In the end, the action (2.2.3) of DFT is equivalent to a generalized Einstein-Hilbert form

$$S_{\mathcal{R}} = \int dx d\tilde{x}\ e^{-2d}\ \mathcal{R}(\mathcal{H}, d)\ , \tag{2.2.9}$$

where the curvature scalar depends on both the generalized metric and the dilaton

$$\begin{aligned}\mathcal{R} \equiv\ & 4\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\partial_{\mathcal{N}}d - \partial_{\mathcal{M}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{MN}} - 4\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d + 4\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{N}}d \\ & + \frac{1}{8}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{KL}} - \frac{1}{2}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{L}}\mathcal{H}_{\mathcal{MK}}\ .\end{aligned} \tag{2.2.10}$$

This quantity was earlier found to be the equation of motion for the dilaton d. In a similar way, a generalized Ricci tensor encoding the equation of motion for the generalized metric $\mathcal{H}$ can be constructed. Ongoing considerations concern the existence of a generalized Riemann curvature tensor [117]. So far, no successful construction is known in DFT for the same reasons as in GG. However, efforts to include higher α'-corrections in DFT seem to provide further insight into this topic.

2.2.2 T-duality invariant conformal field theory

A promising approach towards α'-corrections in DFT which in addition sheds light on the origin of the strong constraint is a simple T-duality invariant CFT. Following Tseytlin [118, 119], T-duality can be realized as a world-sheet symmetry. In particular, the left- and right-moving components of the string are treated on an equal footing and T-duality acts as a simple reflection on the right-moving degrees [120]. We present the basic concepts of this CFT and use it to study the string theoretic origin of the strong constraint which arises in DFT from the level-matching condition only in the weak form. The relation of this duality invariant CFT to DFT is reinforced by comparing tree-level scattering amplitudes of three massless states with the expanded action of DFT [38], reviewed in appendix B.

The free boson and T-duality

Recapitulating the world-sheet sigma model (1.1.24) presented in the Introduction, we are in the following interested in the free bosonic string coordinate and hence

$$S = \frac{1}{2\pi\alpha'}\int_{\Sigma} dz\, d\bar{z}\, g_{mn}(X)\, \partial X^m\, \bar{\partial} X^n\ , \tag{2.2.11}$$

where conformal gauge is fixed and the target space metric g is allowed to depend on the string coordinate. Then following the reasoning in [118, 119], the splitting of the string coordinate into left- and right-moving components, given in (1.1.17) on a circle with additional winding modes, should hold more generally for non-compact directions and not only in the case of toroidal compactification.[3] This leads us to introduce the T-duality invariant propagators for the standard and winding coordinates

$$\begin{aligned}
\big\langle X^m(z_1,\bar z_1) X^n(z_2,\bar z_2)\big\rangle &= -\frac{\alpha'}{2}\, g^{mn} \ln |z_{12}|^2 \ , \\
\big\langle \widetilde X^m(z_1,\bar z_1) \widetilde X^n(z_2,\bar z_2)\big\rangle &= -\frac{\alpha'}{2}\, g^{mn} \ln |z_{12}|^2 \ , \\
\big\langle X^m(z_1,\bar z_1) \widetilde X^n(z_2,\bar z_2)\big\rangle &= -\frac{\alpha'}{2}\, g^{mn} \ln \frac{z_{12}}{\bar z_{12}} \ ,
\end{aligned} \tag{2.2.12}$$

where $z_{ij} = z_i - z_j$. Manifest T-duality apparently follows from exchanging X and $\widetilde X$, because T-duality simply flips the sign of the right-moving component [120]

$$X^m(z,\bar z) = X_L^m(z) + X_R^m(\bar z) \quad \xleftrightarrow{\text{T}-\text{duality}} \quad \widetilde X^m(z,\bar z) = X_L^m(z) - X_R^m(\bar z) \ . \tag{2.2.13}$$

Next, we determine elementary properties of this theory in the absence of compactified directions.

Vertex operators and descendants

We write down the manifest duality-invariant primary field, solely depending on X and $\widetilde X$

$$V_{p,w}(z,\bar z) =: e^{ip_m X^m(z,\bar z)}\, e^{iw_m \widetilde X^m(z,\bar z)} : \ . \tag{2.2.14}$$

For later reasons, it can be interpreted as a tachyonic state in this CFT. Its weight is determined by

$$(h,\bar h) = \left(\frac{\alpha'}{4}(p+w)^2, \frac{\alpha'}{4}(p-w)^2\right) \tag{2.2.15}$$

and its mass given by

$$M^2 = -\frac{2}{\alpha'}(h+\bar h) = -(p^2+w^2) \ . \tag{2.2.16}$$

Interestingly, the operator product expansion of two such fields

$$\begin{aligned}
V_{p_1,w_1}(z_1,\bar z_1)\, V_{p_2,w_2}(z_2,\bar z_2) =& |z_{12}|^{\alpha'(p_1\cdot p_2 + w_1\cdot w_2)} \left(\frac{z_{12}}{\bar z_{12}}\right)^{\frac{\alpha'}{2}(p_1\cdot w_2 + w_1\cdot p_2)} \\
&\times V_{p_1+p_2, w_1+w_2}(z_2,\bar z_2) + \dots \ ,
\end{aligned} \tag{2.2.17}$$

experiences a logarithmic branch point which vanishes under the quantization condition

$$\alpha'(p_1\cdot w_2 + w_1\cdot p_2) \in \mathbb{Z} \ , \tag{2.2.18}$$

that is used to restore locality.

From (2.2.14), we can derive the first descendant states in this T-duality invariant CFT:

[3] The resulting theory is no longer governed by the sigma model (2.2.11).

- The first excited level yields a form field $\mathcal{A}_{p,w}$ and its complex conjugate $\bar{\mathcal{A}}_{p,w}$

$$\begin{aligned}\mathcal{A}_{p,w}(z,\bar{z}) &= A_m :\partial X^m(z)\, V_{p,w}(z,\bar{z}): \ , \\ \bar{\mathcal{A}}_{p,w}(z,\bar{z}) &= \bar{A}_m :\bar{\partial} X^m(\bar{z})\, V_{p,w}(z,\bar{z}):\end{aligned} \tag{2.2.19}$$

  with A and $\bar{A}$ one-forms. Here, $\mathcal{A}$ is primary with conformal weight $(h,\bar{h}) = (1 + \frac{\alpha'}{4}(p+w)^2, \frac{\alpha'}{4}(p-w)^2)$ if it is transversely polarized in the sense $A_m(p^m + w^m) = 0$. Similarly, $\bar{\mathcal{A}}$ is primary with $(h,\bar{h}) = (\frac{\alpha'}{4}(p+w)^2, 1+\frac{\alpha'}{4}(p-w)^2)$ for $\bar{A}_m(p^m - w^m) = 0$. These states lead to the well-known enhancement of the gauge group for heterotic torus compactification.

- At the next level one finds a $(0,2)$-tensor field $\mathcal{E}_{p,w}$

$$\mathcal{E}_{p,w}(z,\bar{z}) = E_{mn} :\partial X^m(z)\,\bar{\partial} X^n(\bar{z})\, V_{p,w}(z,\bar{z}): \tag{2.2.20}$$

  with the polarization E_{mn}. It is a primary field with $(h,\bar{h}) = (1+\frac{\alpha'}{4}(p+w)^2, 1+\frac{\alpha'}{4}(p-w)^2)$ for transverse polarization in the sense $E_{mn}(p^m + w^m) = 0 = E_{mn}(p^n - w^n)$. It is precisely the scattering amplitude of three (2.2.20) that allows to relate this duality invariant CFT to DFT, as shown in the appendix B.

The Virasoro constraints, which are quantum analogues of the vanishing of the classical energy momentum tensor, determine the physical states to be primary fields with conformal weight $(h,\bar{h}) = (1,1)$. Consequently, level-matching is established if for the above states additional constraints hold, as summarized in table 2.1. We see that both $V_{p,w}$ and the

state	level-matching	primary	mass
$V_{p,w}$	$p\cdot w = 0$	—	$M^2 = -\frac{4}{\alpha'}$
$\mathcal{A}_{p,w}$	$p\cdot w = -\frac{1}{\alpha'}$	$A_m(p^m + w^m) = 0$	$M^2 = -\frac{2}{\alpha'}$
$\bar{\mathcal{A}}_{p,w}$	$p\cdot w = \frac{1}{\alpha'}$	$\bar{A}_m(p^m - w^m) = 0$	$M^2 = -\frac{2}{\alpha'}$
$\mathcal{E}_{p,w}$	$p\cdot w = 0$	$E_{mn}(p^m + w^m) = 0 = E_{mn}(p^n - w^n)$	$M^2 = 0$

Table 2.1: The physical state condition requires the operators to be level-matched primaries of conformal weight $(1,1)$. This fixes the mass of the states.

two states $\mathcal{A}_{p,w}$ and $\bar{\mathcal{A}}_{p,w}$ are tachyonic and that $\mathcal{E}_{p,w}$ is the first massless state which corresponds to the graviton, the b-field or the dilaton depending on the polarization.

Next, we will consider the one-loop partition function whose modular invariance imposes additional constraints relating the holomorphic with the anti-holomorphic sector.

The one-loop partition function

We start by computing the torus partition function for the above CFT and analyze its modular properties. On a torus parametrized by τ we have

$$Z(\tau,\bar{\tau}) = \mathrm{tr}_{\mathcal{H}}\big(q^{L_0 - \frac{c}{24}}\,\bar{q}^{\bar{L}_0 - \frac{c}{24}}\big)\ , \tag{2.2.21}$$

where the trace is over all states in the Hilbert space $\mathcal{H}$ and we denote $q = e^{2\pi i\tau}$. As usual, the trace splits into a trace over the oscillators and an integral over the continuous momenta and windings

$$Z(\tau,\bar\tau) = \frac{f(\tau,\bar\tau)}{|\eta(\tau)|^{2d}} \ , \tag{2.2.22}$$

where the Dedekind eta function $\eta(\tau)$ keeps track of the oscillator part and

$$f(\tau,\bar\tau) = \langle p,w|p,w\rangle \frac{1}{2}\left(\int \frac{d^d p_L}{(2\pi)^d}\, e^{i\frac{\pi}{2}\alpha' p_L^2 \tau}\right)\left(\int \frac{d^d p_R}{(2\pi)^d}\, e^{-i\frac{\pi}{2}\alpha' p_R^2 \bar\tau}\right) \ . \tag{2.2.23}$$

Analyzing modular invariance for (2.2.22), we note that $|\eta(\tau)|$ is invariant under the modular T-transformation $\tau \to \tau + 1$, but the integral for $\mathrm{Im}(\tau) > 0$ is not. Therefore, T-invariance yields the level matching condition

$$\alpha'\, p\cdot w \in \mathbb{Z} \quad \Longleftrightarrow \quad \frac{\alpha'}{4}(p_L^2 - p_R^2) \in \mathbb{Z} \ , \tag{2.2.24}$$

i.e. the two integrals are not independent. We can incorporate this constraint in the integration by inserting the delta function $\delta(p_L^2 - p_R^2 - \frac{4}{\alpha'}m)$ in (2.2.23). Evaluating the remaining integral for $\mathrm{Im}(\tau) > 0$, we obtain up to constant factors

$$f(\tau,\bar\tau) \sim e^{2\pi i\, m\,\tau}\int \frac{d^d p_L}{(2\pi)^d}\, |p_L|^{d-1}\, e^{-\pi\alpha' p_L^2\, \mathrm{Im}(\tau)} \sim \frac{\Gamma\left(d-\frac{1}{2}\right)}{\mathrm{Im}(\tau)^{\frac{d}{2}}}\, \frac{e^{2\pi i\, m\,\tau}}{\mathrm{Im}(\tau)^{\frac{d-1}{2}}} \ . \tag{2.2.25}$$

This is now T-invariant, however modular invariance of (2.2.22) under the S-transformation $\tau \to -\frac{1}{\tau}$ is spoiled by the factor $e^{2\pi i\, m\,\tau}\,\mathrm{Im}(\tau)^{\frac{1-d}{2}}$. The absence of the former factor demands the second integral to be

$$\int \frac{d^d p_R}{(2\pi)^d}\, e^{-i\frac{\pi}{2}\alpha' p_R^2 \bar\tau}\, \delta(p_L, p_R) = g(\bar\tau)\, e^{-i\frac{\pi}{2}\alpha' p_L^2 \bar\tau} \ . \tag{2.2.26}$$

This can be achieved by setting $p_L^2 - p_R^2 = \frac{4}{\alpha'}m$ to zero and by introducing a relation

$$p_R = \mathcal{M}\, p_L \quad \text{with} \quad \mathcal{M} \in O(d) \tag{2.2.27}$$

between the left- and right-moving momentum. The delta function is then $\delta^d(p_R - \mathcal{M}p_L)$. Altogether, the torus partition function, denoting $\langle p,w|p,w\rangle = V_d$, reads

$$Z(\tau,\bar\tau) = \frac{V_d/2}{(2\pi\sqrt{\alpha'})^d\, \mathrm{Im}(\tau)^{\frac{d}{2}}\, |\eta(\tau)|^{2d}} \ . \tag{2.2.28}$$

Let us make some remarks on the results of this section. First, T-invariance only required $\alpha' p\cdot w \in \mathbb{Z}$, whereas additional S-invariance finally led to the weak constraint $p\cdot w = 0$. We see from this, that the spectrum only contains states with matching number of left- and right-oscillators.[4] Eventually, the strong constraint does not follow from modular invariance and locality just implies $\alpha'(p_i \cdot w_j + w_i \cdot p_j) \in \mathbb{Z}$. In the next section, we will analyze string diagrams containing momenta and winding of several states in contrast to the one-loop partition function. In particular, we determine the strong constraint using the scattering amplitude of four tachyons.

Tachyons scattering

The correlation function of N tachyon vertex operators $V_{p_i,w_i}(z_i, \bar{z}_i) \equiv V_i$ (2.2.14) of the T-duality invariant CFT is given by

$$\Big\langle V_1 \ldots V_N \Big\rangle = \prod_{1 \leqslant i < j \leqslant N} |z_{ij}|^{\alpha'(p_i \cdot p_j + w_i \cdot w_j)} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\frac{\alpha'}{2}(p_i \cdot w_j + w_i \cdot p_j)} \delta\Big(\sum p_i\Big) \delta\Big(\sum w_i\Big) \ . \tag{2.2.29}$$

The difference to the standard tachyon correlator is the $\frac{z_{ij}}{\bar{z}_{ij}}$-factor[5]. In particular, it is the pole structure of the Virasoro-Shapiro amplitude that encodes the on-shell physical states and provides further constraints.

The duality invariant Virasoro-Shapiro amplitude

The full string scattering amplitude of N tachyons is given by

$$\begin{aligned} A_N(p_i, w_i) = g_s^N \, C_{S^2} \int \prod_{i=1}^{N} d^2 z_i \, \textstyle\prod_{j=1}^3 \delta(z_j - z_j^0) \, |z_{12} z_{13} z_{23}|^2 \\ \times \Big\langle V_1 \ldots V_N \Big\rangle (z_1, \ldots z_N) \ . \end{aligned} \tag{2.2.30}$$

Here, the conformal group $PSL(2,\mathbb{C})$ allowed to fix three of the N insertion points on the sphere, i.e. $z_1 = 0$, $z_2 = 1$ and $z_3 \to \infty$. Moreover, the factor $|z_{12}\, z_{23}\, z_{13}|^2$ is related to the correlator of three c-ghost vertices $\left|\Big\langle c(z_1)\, c(z_2)\, c(z_3) \Big\rangle\right|^2$. The prefactors are a factor of the closed string coupling constant g_c for every closed string vertex operator and C_{S^2} accounting for various normalizations.[6]

The three-tachyon amplitude is given by

$$A_3(p_i, w_i) = g_c^3 \, C_{S^2} \Big\langle (c\, \bar{c}\, V_1)(c\, \bar{c}\, V_2)(c\, \bar{c}\, V_3) \Big\rangle = g_c^3 \, C_{S^2} \ , \tag{2.2.31}$$

where the δ-distributions implementing momentum and winding conservation have to be understood as implicit. The three-point amplitude is therefore identical to the standard one for three tachyons without a winding dependence.

[4] $\mathcal{A}_{p,w}$ and its complex conjugate listed in table 2.1 do not contribute to the spectrum.
[5] $SL(2,\mathbb{C})$-invariance can be checked explicitly.
[6] More details on the conventions can be found in [121].

With the help of (2.2.29), the four-point amplitude reads

$$\begin{aligned} A_4(p_i,w_j) &= g_c^4\, C_{S^2} \int d^2 z \left\langle (c\,\bar{c}\,V_1)\,(c\,\bar{c}\,V_2)\,(c\,\bar{c}\,V_3)\,V_4 \right\rangle \\ &= g_c^4\, C_{S^2} \int d^2 z \Big\{ z^{\alpha'(p_1\cdot w_4 + w_1\cdot p_4)} (1-z)^{\alpha'(p_2\cdot w_4+w_2\cdot p_4)} \\ &\qquad\qquad \times |z|^{\alpha'(p_1-w_1)\cdot(p_4-w_4)}\, |1-z|^{\alpha'(p_2-w_2)\cdot(p_4-w_4)} \Big\} \;. \end{aligned} \tag{2.2.32}$$

At this point, we introduce two sets of Mandelstam variables respectively for left- and right-moving momenta

$$\begin{aligned} s &= -(p_{L3}+p_{L4})^2 \;, & \mathfrak{s} &= -(p_{R3}+p_{R4})^2 \;, \\ t &= -(p_{L2}+p_{L4})^2 \;, & \mathfrak{t} &= -(p_{R2}+p_{R4})^2 \;, \\ u &= -(p_{L1}+p_{L4})^2 \;, & \mathfrak{u} &= -(p_{R2}+p_{R4})^2 \;. \end{aligned} \tag{2.2.33}$$

Level-matching and the mass-shell condition further yield $s+t+u = \mathfrak{s}+\mathfrak{t}+\mathfrak{u} = -\frac{16}{\alpha'}$ and the difference between these sets is given by an integer

$$(p_{Li}+p_{Lj})^2 - (p_{Ri}+p_{Rj})^2 = 4(p_i\cdot w_j + w_i\cdot p_j) \in \frac{4}{\alpha'}\,\mathbb{Z} \;. \tag{2.2.34}$$

The amplitude can be conveniently rewritten using the function $\alpha(s) = -1-\frac{\alpha'}{4}s$

$$A_4(p_i,w_j) = 2\pi\, g_c^4\, C_{S^2}\, \frac{\Gamma\big(\alpha(s)\big)\,\Gamma\big(\alpha(t)\big)\,\Gamma\big(\alpha(u)\big)}{\Gamma\big(\alpha(\mathfrak{t})+\alpha(\mathfrak{u})\big)\,\Gamma\big(\alpha(\mathfrak{s})+\alpha(\mathfrak{u})\big)\,\Gamma\big(\alpha(\mathfrak{s})+\alpha(\mathfrak{t})\big)} \;. \tag{2.2.35}$$

In order to be able to make a statement on the symmetries between the different channels s,t and u, we use the relation (2.2.34) and $\alpha(\mathfrak{s}) = \alpha(s) - n_{34}$, where

$$n_{ij} = \alpha'(p_i\cdot w_j + w_i\cdot p_j) \quad \text{with} \quad n_{14}+n_{24}+n_{34} = 0 \;, \tag{2.2.36}$$

in order write the amplitude in terms of left-moving variables only

$$A_4(p_i,w_j) = \frac{2\pi\, g_c^4\, C_{S^2}\; \Gamma\big(\alpha(s)\big)\,\Gamma\big(\alpha(t)\big)\,\Gamma\big(\alpha(u)\big)}{\Gamma\big(\alpha(t)+\alpha(u)+n_{34}\big)\,\Gamma\big(\alpha(s)+\alpha(u)+n_{24}\big)\,\Gamma\big(\alpha(s)+\alpha(t)+n_{14}\big)} \;. \tag{2.2.37}$$

Clearly, a similar expression in terms of right-moving variables exists.

Now, we recognize that channel duality for (2.2.37) requires $n_{14} = n_{24} = n_{34}$ and consequently implies $n_{ij} = 0$ through (2.2.36). We argue in the following more rigorously that this constraint is identical to the strong constraint.

Pole structure and the strong constraint

We are now interested in the intermediate states of the four tachyon amplitude in the different channels and the question whether these are physical. In particular, the poles of this amplitude determine where the physical states become on-shell and encode the mass spectrum of the theory. For example, the n^{th} pole[7] in the s-channel is located at

$$s = \frac{4}{\alpha'}(n-1) \quad \Longleftrightarrow \quad \mathfrak{s} = \frac{4}{\alpha'}(n + n_{34} - 1)\ . \tag{2.2.38}$$

Then, a physical intermediate state has mass and level-matching condition

$$(M^{int})^2 = -\big((p^{\text{int}})^2 + (w^{\text{int}})^2\big) = \frac{4}{\alpha'}\big(n + \frac{n_{34}}{2} - 1\big) \quad \text{and} \quad p^{\text{int}} \cdot w^{\text{int}} = \frac{n_{34}}{\alpha'}\ , \tag{2.2.39}$$

where the momentum and winding of this state are determined by $s = -(p_{L3} + p_{L4})^2 \equiv -(p_L^{\text{int}})^2$ and $k_{L/R}^{\text{int}} = p^{\text{int}} \pm w^{\text{int}}$. Hence, the level-matching condition allows for asymmetrically excited states which violate the condition (2.2.27), derived from modular invariance. Including the t- and u-channel, the physical spectrum is required to satisfy $n_{ij} = 0$ and we derived the strong constraint

$$p_i \cdot w_j + p_j \cdot w_i = 0 \quad \forall i, j\ . \tag{2.2.40}$$

The better known form (2.2.5) [37, 115]

$$\partial_m f_i\, \tilde{\partial}^m f_j + \tilde{\partial}^m f_i\, \partial_m f_j = 0 \tag{2.2.41}$$

arises by defining the functions $f_i(x, \tilde{x}) = \exp(ip_i \cdot x + iw_i \cdot \tilde{x})$.

To summarize, while modular invariance of the partition function determined the physical spectrum, consistency with the pole structure of the Virasoro-Shapiro amplitude allowed to derive the strong constraint. Let us now combine the condition (2.2.27) with the constraint (2.2.40). In terms of left- and right-moving momenta $K_i = (k_{Li}, k_{Ri})^t$ the strong constraint reads $\langle K_i, K_j \rangle_d = 0\ \forall i, j$. Combining it with $k_{R_i} = \mathcal{M}_i k_{Li}$, we obtain the joint condition

$$k_{Li}{}^t\big(\mathbb{1} - \mathcal{M}_i^t\, \mathcal{M}_j\big) k_{Lj} = 0 \tag{2.2.42}$$

which for fixed i, j must hold for all left-moving momenta. This implies $\mathcal{M}_i = \mathcal{M}_j$ for all i, j so that both constraints can be summarized by the consistency condition

$$k_{Ri} = \mathcal{M}\, k_{Li} \quad \text{with } \mathcal{M} \in O(d)\ \forall i\ . \tag{2.2.43}$$

Constraints from torus compactifications

So far, we considered continuous momentum and winding in non-compact spaces which yield the strong constraint. However, Scherk-Schwarz reductions of DFT [116] allow to work with the weaker closure constraint. Here, momentum and winding are quantized

[7] The function $\Gamma(x)$ develops single poles at $x = -n$ for $n \in \mathbb{N}$ with residue $\frac{(-1)^n}{n!}$ and has no zeros.

due to compact directions. We summarize the results of the previous analysis for torus compactification, detailed in [100], in the following.

Modular invariance of the partition function under T- and S-transformations demands the lattices representing the toroidal compactification to be even and self-dual, which is a well-known result [122]. Hence, the spectrum in the internal sector is less constrained. For the external non-compact part left- and right-moving momenta have to be related by and $O(D-d)$ rotation. The analogue analysis of the pole structure is reducible to internal and external components of momenta and winding since contractions do not mix. In the external direction we derive the previous result of the strong constraint (2.2.40). However, we find that asymmetric excitations in the internal directions are indeed valid.

One-loop modular invariance and the pole structure of the four tachyon amplitude do clarify the need for the strong constraint in non-compact directions and are in agreement with a weaker constraint in the internal sector, as proposed for Scherk-Schwarz reductions.[8] We recall that this T-duality invariant CFT matches with DFT at the two derivative level which makes this theory interesting for studying possible higher α'-corrections to DFT.

[8] In the presence of fluxes additional constraints are required.

Chapter 3

β-supergravity

In the following we present a ten-dimensional theory that contains non-geometric Q- and R-fluxes. Motivations for constructing such a theory are laid out in the Introduction. This theory can be thought of as a reformulation of the standard ten-dimensional supergravity (SUGRA) theories and we name it *β-supergravity* for containing the new field variable β. We are going to study here its NSNS sector, which is common to all standard ten-dimensional SUGRA theories. Questions towards β-supergravity are posed with regard to an underlying geometric framework and its capability to providing new insight into matters of non-geometry. We start a general discussion on background solutions of β-supergravity and their relations to non-geometric configurations in standard SUGRA in the next chapter 4. In particular, the set of $NS5$-branes, including the Q-brane, is chosen as an application of the formalism of β-supergravity in the context of Bianchi identities (BIs) and possible corrections to these when observing branes with non-geometric fluxes in chapter 5. Finally, we use in chapter 6 ideas present in Generalized Complex Geometry (GCG) to classify vacua of β-supergravity by means of writing down the supersymmetry (SUSY) variations in form of pure spinor conditions. This chapter focuses on earlier constructions using a field redefinition that involves a bivector β, the relation of the theory to Double Field Theory (DFT) and presents a derivation of β-supergravity using the Generalized Geometry (GG) formalism.

3.1 First steps towards β-supergravity

Inspired by [76, 77, 78], where GCG tools were used to study non-geometry, in [84] a specific field redefinition was considered which was performed on the standard NSNS fields. The metric g_{mn}, the Kalb-Ramond field b_{mn}, and the dilaton ϕ, get replaced by a new set of fields, given by a new metric $\tilde{g}_{mn}$, an antisymmetric bivector β^{mn}, and a new dilaton $\tilde{\phi}$. This field redefinition is an $O(2d-2,2)$ transformation (more precisely here an $O(d-1,1) \times O(1,d-1)$, as detailed in appendix C.2) taking us from one generalized vielbein $\mathcal{E}$ to another one $\tilde{\mathcal{E}}$, while preserving the generalized metric $\mathcal{H}$, i.e. a change of generalized frame

$$\mathcal{E} = \begin{pmatrix} e & 0 \\ e^{-T}b & e^{-T} \end{pmatrix} \ , \ \tilde{\mathcal{E}} = \begin{pmatrix} \tilde{e} & \tilde{e}\beta \\ 0 & \tilde{e}^{-T} \end{pmatrix} \ , \ \mathbb{I} = \begin{pmatrix} \eta_d & 0 \\ 0 & \eta_d^{-1} \end{pmatrix} \ , \tag{3.1.1}$$

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & -bg^{-1} \\ g^{-1}b & g^{-1} \end{pmatrix} = \mathcal{E}^T \ \mathbb{I} \ \mathcal{E} = \tilde{\mathcal{E}}^T \ \mathbb{I} \ \tilde{\mathcal{E}} = \begin{pmatrix} \tilde{g} & \tilde{g}\beta \\ -\beta\tilde{g} & \tilde{g}^{-1} - \beta\tilde{g}\beta \end{pmatrix} \ . \tag{3.1.2}$$

Here, the vielbeins e and $\tilde{e}$ correspond to metrics $g = e^T\eta_d e$ and $\tilde{g} = \tilde{e}^T\eta_d\tilde{e}$, where η_d denotes the flat metric. The field redefinition[1] can be read from (3.1.2) and rewritten in various manners, in particular

$$\left.\begin{aligned} \tilde{g}^{-1} &= (g+b)^{-1}g(g-b)^{-1} \\ \beta &= -(g+b)^{-1}b(g-b)^{-1} \end{aligned}\right\} \Leftrightarrow (g+b)^{-1} = (\tilde{g}^{-1} + \beta) \ . \tag{3.1.3}$$

Additionally, the new dilaton $\tilde{\phi}$ is chosen such that the following measure is preserved

$$e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|} = e^{-2\phi}\sqrt{|g|} = e^{-2d} \ , \tag{3.1.4}$$

where $|\tilde{g}|$ denotes the absolute value of the determinant of the metric $\tilde{g}$.

The main idea was then to directly apply the field redefinition to the standard ten-dimensional NSNS Lagrangian

$$\mathcal{L}_{\text{NSNS}} \equiv e^{-2\phi}\sqrt{|g|}\left(\mathcal{R}(g) + 4(\partial\phi)^2 - \frac{1}{2}H^2\right) \ , \tag{3.1.5}$$

where $\mathcal{R}$ denotes the Ricci scalar corresponding to the Levi-Civita connection (A.3.1), the H-flux is $H_{mnp} \equiv 3\partial_{[m}b_{np]}$, and the squares are defined in (A.3.28). The actual computation is rather involved and has been performed in two steps. To obtain the new Lagrangian $\tilde{\mathcal{L}}$ two simplifying assumptions $\beta^{pq}\partial_q \cdot = 0$ and $\partial_q\beta^{pq} = 0$ were implemented in [84]. Then, the final Lagrangian $\tilde{\mathcal{L}}$ contains a Ricci scalar of the new metric $\mathcal{R}(\tilde{g})$, a standard kinetic term for the new dilaton $\tilde{\phi}$, and a square of the quantity $\partial_m\beta^{pq}$, which was identified with a ten-dimensional flux ${Q_m}^{pq}$. This specific identification was first motivated by the correct index structure and secondly was able to generate the four-dimensional Q-flux term in the potential in a dimensional reduction. Later in section 4.1.1, the use of this formula on the toroidal example will provide the expected value for the Q-flux. The full computation of $\tilde{\mathcal{L}}$ without simplifying assumption was finally performed in [82, 83]. The direct computation, starting from $\mathcal{L}_{\text{NSNS}}$ and applying the field redefinition results in a new Lagrangian denoted

[1] An alternative field redefinition was proposed in [79, 80]. Both field redefinitions were then interpreted in terms of local $O(d,d)$ transformations and Lie algebroids in [81].

here $\tilde{\mathcal{L}}_0$ which is equal to $\mathcal{L}_{\rm NSNS}$ up to a total derivative

$$\begin{aligned}\tilde{\mathcal{L}}_0 = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\ \bigg(&\mathcal{R}(\tilde{g}) + 4(\partial\tilde{\phi})^2 - \frac{1}{2}R^2 \\ &+ 4\tilde{g}_{mn}\beta^{mp}\beta^{nq}\partial_p d\ \partial_q d - 2\partial_p d\ \partial_q\left(\tilde{g}_{mn}\beta^{mp}\beta^{nq}\right) \\ &- \frac{1}{4}\tilde{g}_{mp}\tilde{g}_{nq}\tilde{g}^{rs}\ \partial_r\beta^{pq}\ \partial_s\beta^{mn} + \frac{1}{2}\tilde{g}_{mn}\partial_p\beta^{qm}\ \partial_q\beta^{pn} \\ &+ \tilde{g}_{nq}\tilde{g}_{rs}\beta^{nm}\big(\partial_p\beta^{qr}\ \partial_m\tilde{g}^{ps} + \partial_p\tilde{g}^{qr}\ \partial_m\beta^{ps}\big) \\ &- \frac{1}{4}\tilde{g}_{mp}\tilde{g}_{nq}\tilde{g}_{rs}\big(\beta^{ru}\beta^{sv}\partial_u\tilde{g}^{pq}\ \partial_v\tilde{g}^{mn} - 2\beta^{mu}\beta^{nv}\partial_u\tilde{g}^{qr}\ \partial_v\tilde{g}^{ps}\big)\bigg) \\ = \mathcal{L}_{\rm NSNS} - \partial_m&\left(e^{-2d}\big(\tilde{g}^{mn}\tilde{g}^{pq}\partial_n\tilde{g}_{pq} - g^{mn}g^{pq}\partial_n g_{pq} + \partial_n(\tilde{g}^{mn} - g^{mn})\big)\right)\ .\end{aligned} \tag{3.1.6}$$

Indeed, the simplified Lagrangian of [84] is equal to the first two terms, and the first term of the third line. Besides, several new terms appear, in particular the square of a ten-dimensional R-flux

$$R^{mnp} \equiv 3\beta^{q[m}\partial_q\beta^{np]} = 3\beta^{q[m}\nabla_q\beta^{np]}\ , \tag{3.1.7}$$

as in [56]. Since ∇_m denotes the standard covariant derivative with Levi-Civita connection defined in (A.3.1) the R-flux above is a tensor. We come back to discuss the structure of the terms involving β two second order.

The Lagrangian (3.1.6) can also be obtained from an alternative method [83, 82] which relies on DFT [37, 38, 115, 54]. It is known that by applying the strong constraint in the form $\tilde{\partial} = 0$ to the Lagrangian $\mathcal{L}_{\rm DFT}$ allows to recover the standard NSNS Lagrangian $\mathcal{L}_{\rm NSNS}$, up to a total derivative. Hence, performing similar steps after applying the field redefinition in terms of the reparametrization (3.1.2) leads also to $\tilde{\mathcal{L}}_0$, up to a total derivative. These two methods are depicted by the two left columns and lines of the diagram (3.1.8). The plain equalities of this diagram were established in [84, 83, 82] and the dashed ones have been presented in [101].

$$\begin{array}{ccccc} \mathcal{L}_{\rm DFT}(g,b,\phi) & = & \mathcal{L}_{\rm DFT}(\tilde{g},\beta,\tilde{\phi}) & = & \mathcal{L}_{\rm DFT}(\mathcal{R},\check{\mathcal{R}}) + \partial(\dots) + \tilde{\partial}(\dots) \\ \tilde{\partial}=0 \Big\| & & \tilde{\partial}=0 \Big\| & & \tilde{\partial}=0 \ \Vert \\ \mathcal{L}_{\rm NSNS} + \partial(\dots) & = & \tilde{\mathcal{L}}_0 + \partial(\dots) & = = = = = = = = & \tilde{\mathcal{L}}_\beta + \partial(\dots) \end{array} \tag{3.1.8}$$

The ten-dimensional theory given by the Lagrangian $\tilde{\mathcal{L}}_0$ was also proposed to yield an uplift to some four-dimensional gauged SUGRA theories. In particular in [82], a partial dimensional reduction showed that the Q- and R-flux non-geometric terms of the four-dimensional scalar potential could be reproduced. This cannot be achieved from the standard $\mathcal{L}_{\rm NSNS}$. The precise identification of the fluxes is nevertheless unclear in this reduction. Observing the scalar potential being quadratic in the fluxes is not sufficient in order to clarify the identification of fluxes. More information is usually provided by the superpotential. Fortunately, the R-flux term in the four-dimensional scalar potential only resulted from the R^2 term in $\tilde{\mathcal{L}}_0$, which makes the above identification rather likely. On the contrary, the structure of the last three lines of $\tilde{\mathcal{L}}_0$ in (3.1.6) does not allow for a simple identification of ten-dimensional Q-flux and naive suggestions like $\nabla\beta$ failed to produce a squared expression for the Q-flux.

Some progress in the structure of the Lagrangian was nevertheless obtained in [83, 82], at the level of DFT. Indeed, as depicted in the first line of the diagram (3.1.8), the DFT Lagrangian, expressed in terms of the new fields $\tilde{g}, \beta, \tilde{\phi}$, was reformulated in a covariant manner with respect to the standard diffeomorphisms. The key ingredient of this reformulation was a new covariant derivative $\check{\nabla}^m$ involving the derivative $\tilde{\partial}^m - \beta^{mn}\partial_n$ and a connection $\check{\Gamma}^{mn}_p$. It enters various DFT quantities, i.e. a Ricci-like tensor $\check{\mathcal{R}}^{mn}$ and an associated scalar $\check{\mathcal{R}}$. The latter appears in the reformulated DFT Lagrangian, together with the standard Ricci scalar $\mathcal{R}(\tilde{g})$. Applying the constraint $\tilde{\partial} = 0$ on this last DFT Lagrangian $\mathcal{L}_{\mathrm{DFT}}(\mathcal{R}, \check{\mathcal{R}})$ a first expression of $\tilde{\mathcal{L}}_\beta$ at the SUGRA level, which formally inherits the structure of $\mathcal{L}_{\mathrm{DFT}}(\mathcal{R}, \check{\mathcal{R}})$, was obtained

$$\tilde{\mathcal{L}}_\beta = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\left(\mathcal{R}(\tilde{g}) + \check{\mathcal{R}}(\tilde{g}) + 4(\partial\tilde{\phi})^2 - \frac{1}{2}R^2 + 4(\beta^{mp}\partial_p\tilde{\phi} - \mathcal{T}^m)^2\right) . \tag{3.1.9}$$

The squares are defined in (A.3.28), and further objects involving the new covariant derivative $\check{\nabla}$ on a (co)-vector V are defined as

$$\check{\mathcal{R}} = \tilde{g}_{mn}\check{\mathcal{R}}^{mn} \ , \ \check{\mathcal{R}}^{mn} = -\beta^{pq}\partial_q\check{\Gamma}^{mn}_p + \beta^{mq}\partial_q\check{\Gamma}^{pn}_p + \check{\Gamma}^{mn}_p\check{\Gamma}^{qp}_q - \check{\Gamma}^{qm}_p\check{\Gamma}^{pn}_q \ , \tag{3.1.10}$$

$$\check{\Gamma}^{mn}_p = \frac{1}{2}\tilde{g}_{pq}\left(-\beta^{mr}\partial_r\tilde{g}^{nq} - \beta^{nr}\partial_r\tilde{g}^{mq} + \beta^{qr}\partial_r\tilde{g}^{mn}\right) + \tilde{g}_{pq}\tilde{g}^{r(m}\partial_r\beta^{n)q} - \frac{1}{2}\partial_p\beta^{mn} \ , \tag{3.1.11}$$

$$\mathcal{T}^n \equiv \check{\Gamma}^{pn}_p = \partial_p\beta^{np} - \frac{1}{2}\beta^{nm}\tilde{g}_{pq}\partial_m\tilde{g}^{pq} = \nabla_p\beta^{np} \ , \tag{3.1.12}$$

$$\check{\nabla}^m V^p = -\beta^{mn}\partial_n V^p - \check{\Gamma}^{mp}_n V^n \ , \ \check{\nabla}^m V_p = -\beta^{mn}\partial_n V_p + \check{\Gamma}^{mn}_p V_n \ . \tag{3.1.13}$$

As noticed in [83, 82], the trace $\mathcal{T}^n$ of the connection is hence obviously a tensor. Moreover, the definition of $\check{\nabla}$ can be naturally extended for tensors with more indices. The above definitions enter in the derivation of the equations of motion from (3.1.9) in curved indices, which was done in[101].

By construction, $\tilde{\mathcal{L}}_\beta$ should be equal to $\tilde{\mathcal{L}}_0$, and to $\mathcal{L}_{\mathrm{NSNS}}$, up to total derivatives. This is depicted in the second line of the diagram (3.1.8). The explicit verification can be found in the appendix of [101]. A first advantage of the reformulated $\tilde{\mathcal{L}}_\beta$ compared to $\tilde{\mathcal{L}}_0$ is its manifestly diffeomorphism covariance. In addition, as noticed already at the level of DFT in [82], $\check{\mathcal{R}}$ captures most of the terms of the last three lines of $\tilde{\mathcal{L}}_0$ providing some structure for the four-dimensional Q-flux terms. Still, this interesting repackaging does not allow to identify directly the Q-flux. In [83, 82] it was noticed that the $\partial\beta$ essentially appear within the new connection $\check{\Gamma}$ and thus the Q-flux was believed to be part of a connection coefficient. We investigate the question of the Q-flux not being a tensor in more detail in the next section.

3.2 Generalized Geometry derivation

In order to compute the Lagrangian $\tilde{\mathcal{L}}_\beta$ we develop an underlying geometric concept with corresponding quantities. We essentially follow the GG paper [39] that treats the generalized vielbein $\mathcal{E}(e,b)$ in standard parameterization. An analogous DFT formalism with similar objects was developed before in [55], and its relation to [39] has been established in [69]. These three papers are, to some extent, based on the early work [109, 110]. Previous constructions of geometric objects in terms of generalized vielbeins can also be found in [78, 114]. More recent related work for the $O(d,d)$ covariant formalisms appeared in [117, 123, 124, 50, 125], where a specific form of the generalized connection is sometimes chosen. Most of this recent work remains however at the generalized or doubled level, without specifying a generalized vielbein for reasons of preserving $O(d,d)$.[2]

As explained, the field redefinition considered here corresponds to a change of generalized vielbein, from $\mathcal{E}(e,b)$ to $\tilde{\mathcal{E}}(\tilde{e},\beta)$. Such a change in the two above formalisms should thus lead to $\tilde{\mathcal{L}}_\beta$ (3.1.9). Although this result is already known for DFT as depicted in the diagram (3.1.8), it has not been established in a formulation where one relies solely on generalized geometric objects. Hence, in the following we show how choosing the generalized vielbein $\tilde{\mathcal{E}}(\tilde{e},\beta)$ in the GG formalism leads to a scalar $S = e^{2d}\ (\tilde{\mathcal{L}}_\beta + \partial(\dots))$ that allows us to identify the Lagrangian $\tilde{\mathcal{L}}_\beta$. We explicitly construct the geometric objects corresponding to the choice $\tilde{\mathcal{E}}$ and enlighten the structures appearing. For instance, the derivative $\check{\nabla}$ and the connection ω_Q (1.4.7) appear naturally. The role of the trace of the connection $\mathcal{T}^m$ (3.1.12) gets clarified and finally we obtain specific derivatives on spinors which will be useful in the study of SUSY in chapter 6.

3.2.1 The $O(d,d) \times \mathbb{R}^+$ structure

We start by considering a manifold $\mathcal{M}$ of dimension d. Then, associated with the tangent space over each patch of the manifold comes a frame that we denote here ∂_a for convenience. When going from one patch to the next, the frames transform into each other with elements of $GL(d,\mathbb{R})$ acting on the flat index a. Globally, the union of tangent spaces forms the tangent bundle $T\mathcal{M}$, whose structure group is then $GL(d,\mathbb{R})$. It is common to introduce a globally preserved metric η_d on these tangent spaces which then reduces the structure group to $O(d-1,1)$.[3] In GG a $2d$-dimensional generalization of the tangent bundle, that we call here *generalized bundle* E, mimics the standard construction with generalized frames that transform according to the structure group $O(d,d)$. The latter arises due to a $2d \times 2d$ metric

$$\eta_{(u/d)} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{3.2.1}$$

with components denoted by $\eta_{\mathcal{AB}}$, where we introduce a generalized flat index $\mathcal{A}$. The metric above also reflects the natural coupling of vectors and one-forms (∂_a, e^b) leading to

[2] One exception is [50] and we recover some of their generalized connection components.

[3] We use in this paper standard Minkowski signature for clarity, but there is actually no restriction on it, as indicated in [39].

$O(d,d)$. A simple local realization of E is given by the direct sum $T\mathcal{M} \oplus T^*\mathcal{M}$.

In order to implement the dilaton present in any SUGRA, the bundle E has to be extended by a conformal weight, following [39]. The corresponding structure group of the extended bundle is $O(d,d) \times \mathbb{R}^+$, and the different objects involved get weighted by a conformal factor related to the dilaton. In particular, we now talk of a *generalized conformal frame*, that we denote $e^{-2d}\, \mathring{\mathcal{E}}_{\mathcal{A}}$.[4]

In the following, we are interested in a particular type of generalized frames that allow a splitting of the generalized bundle. Here, splitting means that an isomorphism $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$ can be found and refers to a local relation, which might be difficult to define globally. In particular, the isomorphism locally realizes a map from a generalized frame to (∂_a, e^a). Choosing locally a set of coordinates the vielbein $\mathring{e}$ relates $\partial_a = \mathring{e}^m{}_a \partial_m$ and similarly one-forms with $\mathrm{d}x^m$. For completeness, we also introduce matrix notation $\mathring{e}^{-T}\partial$ for ∂_a, detailed in appendix A.3. Then, a *generalized conformal split frame* $e^{-2d}\, \mathring{\mathcal{E}}_{\mathcal{A}}$ can be denoted

$$e^{-2d}\, \mathring{\mathcal{E}}^{-T} \begin{pmatrix} \partial \\ \mathrm{d}x \end{pmatrix} , \tag{3.2.2}$$

where the matrix $\mathring{\mathcal{E}}$ of components $\mathring{\mathcal{E}}^A{}_M$ is a generalized vielbein. This notation leads to d-dimensional blocks in matrices and we thus clarify the index placement for vectors and one-forms by fixing the following up/down (u/d) notation

$$U_{\mathcal{A}} = \begin{pmatrix} u_a \\ u^a \end{pmatrix} , \ V^{\mathcal{A}} = \begin{pmatrix} v^a \\ v_a \end{pmatrix} , \qquad \eta_{\mathcal{AB}} = \frac{1}{2} \begin{pmatrix} 0 & \delta_a^b \\ \delta_b^a & 0 \end{pmatrix} , \tag{3.2.3}$$

and indicate the indices for the $O(d,d)$ metric $\eta_{(u/d)}$.[5]

Local expressions for a generalized conformal frame are provided by the two generalized vielbeins $\mathcal{E}$ and $\tilde{\mathcal{E}}$ of (3.1.1)

$$e^{-2d}\, \mathcal{E}^{-T} = e^{-2\phi}\sqrt{|g|} \begin{pmatrix} e^{-T} & e^{-T}b \\ 0 & e \end{pmatrix} , \ (\mathcal{E}^{-T})_A{}^M = \mathcal{E}^M{}_A = \begin{pmatrix} e^m{}_a & e^n{}_a b_{nm} \\ 0 & e^a{}_m \end{pmatrix} , \tag{3.2.4}$$

$$e^{-2d}\, \tilde{\mathcal{E}}^{-T} = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|} \begin{pmatrix} \tilde{e}^{-T} & 0 \\ \tilde{e}\beta & \tilde{e} \end{pmatrix} , \ (\tilde{\mathcal{E}}^{-T})_A{}^M = \tilde{\mathcal{E}}^M{}_A = \begin{pmatrix} \tilde{e}^m{}_a & 0 \\ \tilde{e}^a{}_n \beta^{nm} & \tilde{e}^a{}_m \end{pmatrix} . \tag{3.2.5}$$

[4]To distinguish the standard generalized frame from generic quantities we denote the later by using a ring on top.

[5]The $O(d,d)$ structure group considered here is a priori different from the T-duality group. Indeed, our $O(d,d)$ acts on the flat index $\mathcal{A}$, i.e. "on the left" of a generalized vielbein $\mathring{\mathcal{E}}$, while a standard T-duality acts on the "generalized curved space" index $\mathcal{M}$, i.e. "on the right" of $\mathring{\mathcal{E}}$, also observed from the generalized $\mathcal{H}$ in (3.1.2). The two groups may however be related. From our $O(d,d)$ metric $\eta_{\mathcal{AB}}$, one can define a "curved space" metric $\eta_{\mathcal{MN}} = \mathring{\mathcal{E}}^{\mathcal{A}}{}_{\mathcal{M}}\, \eta_{\mathcal{AB}}\, \mathring{\mathcal{E}}^{\mathcal{B}}{}_{\mathcal{N}}$. The vielbeins considered in (3.1.1) are elements of $O(d,d)$ and for those, $\eta_{\mathcal{MN}}$ is then equal to the $O(d,d)$ metric. One can thus consider $O(d,d)$ transformations on the curved space index. We come back to this idea in section 4.1.

A third notation in terms of up and down indices is given by

$$e^{-2d}\ \mathcal{E}_{\mathcal{A}} = \begin{cases} e^{-2d}\ \mathcal{E}_a = e^{-2\phi}\sqrt{|g|}\ (\partial_a + b_{ab}e^b) \\ e^{-2d}\ \mathcal{E}^a = e^{-2\phi}\sqrt{|g|}\ e^a \end{cases} , \tag{3.2.6}$$

$$e^{-2d}\ \tilde{\mathcal{E}}_{\mathcal{A}} = \begin{cases} e^{-2d}\ \tilde{\mathcal{E}}_a = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\ \partial_a \\ e^{-2d}\ \tilde{\mathcal{E}}^a = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\ (\tilde{e}^a + \beta^{ab}\partial_b) \end{cases} . \tag{3.2.7}$$

The standard example (3.2.6) was studied in [39], while the second (3.2.7) was left for investigation. We now focus on the latter and work out corresponding generalized geometric structures.

Locally on each patch, these frames clearly provide an isomorphism to $T\mathcal{M} \oplus T^*\mathcal{M}$. However, the existence of a consistent global splitting requires more attention and depends on the transformation behavior of a frame from one patch to another. In the case of the frame (3.2.6) a global meaning is provided by gauge transformations of the b-field, that can be defined properly in this context [39]. Whether a similar global completion can be found for (3.2.7) is less straightforward and possibly involves well-defined β gauge transformation. We discuss this point in section 4.1.2. In what follows, we consider all objects to be local quantities and push questions of an underlying global geometry aside. Stated differently, we work out the consequence of the conformal frame (3.2.7) formally.

Furthermore, we remark that a splitting reduces the structure group to a subgroup G_{split} preserving the form of the splitting. For (3.2.6), the subgroup contains b-field gauge transformations and diffeomorphisms. The reduction of the structure group of the generalized bundle E also manifests itself in a refinement of the bundle itself. For (3.2.6), E becomes the generalized tangent bundle E_T of GCG and (3.2.7) in principle should be associated with a group G_{split} and a proposed *generalized cotangent bundle* E_{T^*}, discussed in section 4.1.2.

Let us now define various generalized geometric objects, that are compatible with the $O(d,d)\times\mathbb{R}^+$ structure of the extended generalized bundle. We mostly follow [39]. To start with, we introduce the bilinear product of two generalized vectors V and W

$$\langle \mathring{\mathcal{E}}_{\mathcal{A}}, \mathring{\mathcal{E}}_{\mathcal{B}}\rangle \equiv \eta_{\mathcal{AB}}\ , \quad \langle V, W\rangle = V^{\mathcal{A}}\eta_{\mathcal{AB}}W^{\mathcal{B}} \text{ for } V = V^{\mathcal{A}}\mathring{\mathcal{E}}_{\mathcal{A}}\ , W = W^{\mathcal{B}}\mathring{\mathcal{E}}_{\mathcal{B}}\ , \tag{3.2.8}$$

where also a conformal factor can be inserted. Furthermore, we define a *generalized covariant derivative* acting on a generalized vector component V^B in flat indices

$$D_{\mathcal{A}}V^{\mathcal{B}} = \partial_{\mathcal{A}}V^{\mathcal{B}} + \hat{\Omega}_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}}V^{\mathcal{C}}\ , \tag{3.2.9}$$

where $\partial_{\mathcal{A}}$ denotes a generalized partial derivative and $\hat{\Omega}_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}}$ is a generalized spin connection. Demanding compatibility of the latter with $O(d,d)\times\mathbb{R}^+$ requires to separate it into two pieces

$$\hat{\Omega}_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} = \Omega_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} - \Lambda_{\mathcal{A}}\delta^{\mathcal{B}}_{\mathcal{C}}\ , \tag{3.2.10}$$

where Ω is the spin connection for E, and Λ is related to the conformal weight. Furthermore, the $O(d,d)$ structure, or equivalently compatibility with the metric (3.2.1), yields the antisymmetry property analogous to the standard spin connection in (A.3.13)

$$\eta^{\mathcal{DC}}\Omega_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} = -\eta^{\mathcal{BC}}\Omega_{\mathcal{A}}{}^{\mathcal{D}}{}_{\mathcal{C}} \ . \tag{3.2.11}$$

On the tangent bundle, there exists a uniquely fixed torsion-free connection compatible with a $O(d-1,1)$ structure called the Levi-Civita connection. Our goal is to impose analogue requirements on the generalized objects. We start with the definition of the generalized torsion T. The standard torsion is obtained by the difference of two Lie derivatives, where for one of them, the partial derivative is replaced by the covariant derivative. The *generalized torsion* is then defined similarly [39] in terms of the generalized Lie derivative L [78] with the covariant derivative D (3.2.9)

$$T(V,W) \equiv L_V^D W - L_V W \ . \tag{3.2.12}$$

This definition is bilinear in V and W and its components on a generic frame are then defined using (3.2.8) as

$$T^{\mathcal{A}}{}_{\mathcal{BC}} \equiv \eta^{\mathcal{AD}}\langle \mathring{\mathcal{E}}_{\mathcal{D}}, T(\mathring{\mathcal{E}}_{\mathcal{B}}, \mathring{\mathcal{E}}_{\mathcal{C}})\rangle \ . \tag{3.2.13}$$

We find the following result in terms of the generalized connection using (3.2.12)

$$T_{\mathcal{ABC}} = -3\hat{\Omega}_{[\mathcal{ABC}]} + \hat{\Omega}_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{B}}\eta_{\mathcal{AC}} - e^{4d}\langle e^{-2d}\ \mathring{\mathcal{E}}_{\mathcal{A}}, L_{\mathring{\mathcal{E}}_{\mathcal{B}}}(e^{-2d}\ \mathring{\mathcal{E}}_{\mathcal{C}})\rangle \ , \tag{3.2.14}$$

where certain indices are lowered with $\eta_{\mathcal{AB}}$. Setting to zero the generalized torsion then fixes some components of the generalized connection in terms of a given frame. In the following we work this out for the frames (3.2.6) and (3.2.7).

In case of a splitting, we can specify the generalized partial derivative[6]

$$\partial_{\mathcal{M}} = \begin{cases} \partial_m \\ \partial^m \equiv 0 \end{cases} \quad , \qquad \partial_{\mathcal{A}} = \mathring{\mathcal{E}}^{\mathcal{M}}{}_{\mathcal{A}}\ \partial_{\mathcal{M}} \ . \tag{3.2.15}$$

While $\partial_{\mathcal{A}}$ is simply ∂_a for the frame with b-field (3.2.6), we find $\partial^a = \beta^{ab}\partial_b$ for the frame (3.2.7) with β, as can be seen from the generalized vielbeins in the form (3.2.4) and (3.2.5). This gives a natural origin to $\beta\partial$ and it will lead to the new covariant derivative $\check{\nabla}$ defined in (3.1.13).

Using the u/d notation (3.2.3) the antisymmetry (3.2.11) on the components takes the form

$$\Omega_{\mathcal{A}c}{}^{b} = -\Omega_{\mathcal{A}}{}^{b}{}_{c} \ , \qquad \Omega_{\mathcal{A}}{}^{bc} = -\Omega_{\mathcal{A}}{}^{cb} \ , \ \Omega_{\mathcal{A}bc} = -\Omega_{\mathcal{A}cb} \ . \tag{3.2.16}$$

We can then work out more concretely the generalized derivative (3.2.9) of a generalized vector expanded on a conformal frame

$$\begin{aligned}(D_{\mathcal{A}}V^{\mathcal{B}})e^{-2d}\ \mathring{\mathcal{E}}_{\mathcal{B}} = e^{-2d}\,&((\partial_{\mathcal{A}}v^b + \Omega_{\mathcal{A}}{}^{b}{}_{c}v^c + \Omega_{\mathcal{A}}{}^{bc}v_c)\mathring{\mathcal{E}}_b \\ &+(\partial_{\mathcal{A}}v_b - \Omega_{\mathcal{A}}{}^{c}{}_{b}v_c + \Omega_{\mathcal{A}bc}v^c)\mathring{\mathcal{E}}^b - \Lambda_{\mathcal{A}}V^{\mathcal{B}}\mathring{\mathcal{E}}_{\mathcal{B}} \ .\end{aligned} \tag{3.2.17}$$

[6] In principle one could introduce the abstract derivative ∂^m which is set to zero in [39]. It could serve as the $\tilde{\partial}^m$ of DFT.

For the frame (3.2.6), another natural requirement [39] is that $D_{\mathcal{A}}$ reproduces the standard covariant derivative ∇_a (A.3.8). More precisely, given the generalized derivative (3.2.15) D_a should reproduces ∇_a while D^a should not get any contribution. Within D_a, there are three types of connection coefficients involved. $\Omega_a{}^b{}_c$ presents a standard spin connection and its antisymmetric part matches that of the Levi-Civita connection after imposing the torsion-free condition. The symmetric part should then be fixed such that the full ω^b_{ac} is reproduced. Hence, one finds the standard ∇_a on both contravariant and covariant objects, namely $\nabla_a v^b$ and $\nabla_a v_b$. The second component $\Omega_a{}^{bc}$ is a non-standard term and we set it to zero. The third type of connection coefficient Ω_{abc} will be later related to the H-flux for the frame (3.2.6). For D^a we find the component $\Omega^a{}_{bc}$ analogue to $\Omega_a{}^{bc}$ in D_a. Consequently, we set it to zero as well

$$\Omega^a{}_{bc} = 0 \ , \ \Omega_a{}^{bc} = 0 \ . \tag{3.2.18}$$

Most of the other contributions to D^a for the frame (3.2.6) vanish thanks to the torsion-free condition. The fixing indicated then realizes the requirement of reproducing the standard ∇_a with $D_{\mathcal{A}}$. For the frame (3.2.7) the torsion condition will be different for D^a, however we stick here to the same fixing

$$\begin{cases} (D_a V^{\mathcal{B}})e^{-2d}\ \mathring{\mathcal{E}}_{\mathcal{B}} = e^{-2d}\left((\partial_a v^b + \Omega_a{}^b{}_c v^c)\mathring{\mathcal{E}}_b + (\partial_a v_b - \Omega_a{}^c{}_b v_c)\mathring{\mathcal{E}}^b + \Omega_{abc}v^c\mathring{\mathcal{E}}^b - \lambda_a V^{\mathcal{B}}\mathring{\mathcal{E}}_{\mathcal{B}}\right) \\ (D^a V^{\mathcal{B}})e^{-2d}\ \mathring{\mathcal{E}}_{B} = e^{-2d}\left((\partial^a v^b - \Omega^a{}_c{}^b v^c)\mathring{\mathcal{E}}_b + (\partial^a v_b + \Omega^a{}_b{}^c v_c)\mathring{\mathcal{E}}^b + \Omega^{abc}v_c\mathring{\mathcal{E}}_b - \xi^a V^{\mathcal{B}}\mathring{\mathcal{E}}_{\mathcal{B}}\right) \end{cases} \tag{3.2.19}$$

where we denote

$$\Lambda_a \equiv \lambda_a \ , \ \Lambda^a \equiv \xi^a \ . \tag{3.2.20}$$

On the contrary to the frame (3.2.6), the ∂^a is non-trivial for the frame (3.2.7). The form of the derivatives in (3.2.19) suggests that the antisymmetric part of $\Omega^a{}_b{}^c$ should be given by that of ω_Q (A.3.21). In order to reproduce the full ω_Q we then fix accordingly its symmetric part. This will lead to the new covariant derivative $\check{\nabla}^a$ (A.3.20) being reproduced by D^a.[7]

Let us now work out the torsion-free condition for the frame (3.2.7). We first compute

$$\begin{aligned} e^{4d}\langle e^{-2d}\ \tilde{\mathcal{E}}_{\mathcal{A}}, L_{\tilde{\mathcal{E}}_{\mathcal{B}}} e^{-2d}\ \tilde{\mathcal{E}}_{\mathcal{C}}\rangle =& \tfrac{1}{2}(f^a{}_{bc} + f^c{}_{ab} + f^b{}_{ca} + Q_a{}^{bc} + Q_b{}^{ca} + Q_c{}^{ab} - R^{abc}) \\ &+ (f^d{}_{db} - 2\partial_b\tilde{\phi} + Q_d{}^{bd} + \beta^{dg}f^b{}_{dg} - 2\beta^{bd}\partial_d\tilde{\phi})\eta_{\mathcal{AC}} \ , \end{aligned} \tag{3.2.21}$$

where the right-hand side lists the components according to up or down indices on the left-hand side. The fluxes f, Q and R appearing here are precisely those defined in (1.4.5). Then, using the connection coefficients of (3.2.19), and setting the torsion to zero in (3.2.14), one first obtains[8]

$$\begin{aligned} &f^a{}_{bc} = 2\Omega_{[b}{}^a{}_{c]} \ , \quad f^c{}_{ab} = 2\Omega_{[a}{}^c{}_{b]} \ , \quad f^b{}_{ca} = 2\Omega_{[c}{}^b{}_{a]} \ , \\ &Q_a{}^{bc} = 2\Omega^{[b}{}_a{}^{c]} \ , \quad Q_b{}^{ca} = 2\Omega^{[c}{}_b{}^{a]} \ , \quad Q_c{}^{ab} = 2\Omega^{[a}{}_c{}^{b]} \ , \\ &R^{abc} = 3\Omega^{[abc]} \ , \quad \Omega_{[abc]} = 0 \ . \end{aligned} \tag{3.2.22}$$

[7]For the frame (3.2.6), the symmetric part of $\Omega^a{}_b{}^c$ should rather be set to zero to get $D^a = 0$.

[8]These results as well as (3.2.24) are in agreement with those of [50].

As discussed below (3.2.17) and (3.2.19), and given the properties (A.3.13) and (A.3.23) compared to the relations just derived, we identify for the frame (3.2.7)

$$\Omega_b{}^a{}_c = \omega^a_{bc} \ , \quad \Omega^b{}_a{}^c = \omega_{Q}{}^{bc}_a \ . \tag{3.2.23}$$

From those, we deduce $f^d{}_{db} = \Omega_d{}^d{}_b$ and $Q_d{}^{db} = \Omega^d{}_d{}^b$. The torsion-free condition then finally gives

$$\lambda_b = 2\partial_b\tilde{\phi} \ , \quad \xi^b = -2Q_d{}^{bd} - \beta^{cd}f^b{}_{cd} + 2\beta^{bd}\partial_d\tilde{\phi} \ . \tag{3.2.24}$$

The sign of $Q_d{}^{bd}$ in (3.2.21) looks rather surprising, as it differs from that of $f^d{}_{db}$ when viewed as the trace of a connection, and leads to the $-2Q_d{}^{bd}$ in ξ^b. It is however the correct result, and one accomplishes a better understanding by noticing that the tensor $\mathcal{T}^m$ (3.1.12) can be expressed in flat indices as

$$\mathcal{T}^a = -Q_b{}^{ba} + \frac{1}{2}\beta^{cd}f^a{}_{cd} \ . \tag{3.2.25}$$

One can then rewrite

$$\lambda_a = 2\partial_a\tilde{\phi} \ , \quad \xi^a = 2(\beta^{ad}\partial_d\tilde{\phi} - \mathcal{T}^a) \ , \tag{3.2.26}$$

which give precisely the two dilaton terms in the Lagrangian $\tilde{\mathcal{L}}_\beta$ in (3.1.9). $\mathcal{T}^a$ plays the role of the conformal weight together with the dilaton, and appears in the corresponding combination given by ξ^a. This is the non-standard conformal weight, obtained with the frame (3.2.7), that matches with the non-standard dilaton term in the Lagrangian. The standard term for the frame (3.2.6) is λ_a, corresponding to the standard dilaton kinetic term.

There are further components of the connection which are unfixed by the torsion-free condition. Those do not appear when computing $-3\Omega_{[\mathcal{ABC}]} + \Omega_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{B}}\eta_{\mathcal{AC}}$, i.e. the parts of Ω_{abc} or Ω^{abc} that are not fully antisymmetric. For (3.2.6) in [39], these undetermined components eventually do not contribute to the scalar S respectively the Lagrangian. Inspired by this situation, we choose here for simplicity to set them to zero for the frame (3.2.7). In the end, we recover $\tilde{\mathcal{L}}_\beta$ from S, despite this restriction.

Finally, we obtain for the frame (3.2.6)[9]

$$\begin{cases} (D_aV^{\mathcal{B}})e^{-2d}\ \mathcal{E}_{\mathcal{B}} = e^{-2d}\left((\nabla_a v^b)\mathcal{E}_b + (\nabla_a v_b)\mathcal{E}^b - \frac{1}{3}H_{abc}v^c\mathcal{E}^b - \lambda_a V^{\mathcal{B}}\mathcal{E}_{\mathcal{B}}\right) \\ (D^aV^{\mathcal{B}})e^{-2d}\ \mathcal{E}_{\mathcal{B}} = 0 \end{cases} , \tag{3.2.27}$$

while the frame (3.2.7) leads to

$$\begin{cases} (D_aV^{\mathcal{B}})e^{-2d}\ \tilde{\mathcal{E}}_{\mathcal{B}} = e^{-2d}\left(\ (\nabla_a v^b)\tilde{\mathcal{E}}_b + (\nabla_a v_b)\tilde{\mathcal{E}}^b - \lambda_a V^{\mathcal{B}}\tilde{\mathcal{E}}_{\mathcal{B}}\ \right) \\ (D^aV^{\mathcal{B}})e^{-2d}\ \tilde{\mathcal{E}}_{\mathcal{B}} = e^{-2d}\left(-(\check{\nabla}^a v^b)\tilde{\mathcal{E}}_b - (\check{\nabla}^a v_b)\tilde{\mathcal{E}}^b + \frac{1}{3}R^{abc}v_c\tilde{\mathcal{E}}_b - \xi^a V^{\mathcal{B}}\tilde{\mathcal{E}}_{\mathcal{B}}\right) \end{cases} , \tag{3.2.28}$$

where we introduced a new covariant derivative with connection coefficients (3.2.23)

$$\check{\nabla}^a v^b = -\beta^{ac}\partial_c v^b + \omega_{Q}{}^{ab}_c v^c \ , \quad \check{\nabla}^a v_b = -\beta^{ac}\partial_c v_b - \omega_{Q}{}^{ac}_b v_c \ . \tag{3.2.29}$$

[9]The torsion-free condition has been worked out in [39] with $\lambda_a = 2\partial_a\phi$.

3.2.2 Preserving an $O(d-1,1) \times O(1,d-1)$ structure

After having presented the $O(d,d) \times \mathbb{R}^+$ structure group of the extension of E, the associated generalized geometric objects, and given some concrete examples for those, we are now interested in a specific $O(d-1,1) \times O(1,d-1)$ subgroup. This structure was considered for $\mathcal{N}=1$ SUSY [69, 70], but also allowed to reproduce type II SUGRAs [55, 39, 89, 71]. Preserving such a structure brings in more constraints. For instance, the metric and dilaton are fixed by this structure, meaning that the conformal weight is globally defined, and one then only focuses on the bundle E. Another example is that the two orthogonal groups lead to two $Spin(d-1,1)$ groups, and associated spinors turned out to be related to the two supersymmetries of type II theories. Finally, the generalized curvature scalar S defined in terms of these spinors was shown to be related to the standard NSNS Lagrangian (3.1.5). Here, we are interested in the frame (3.2.7) with β and the new derivative obtained in (3.2.28). Analogously, the $O(d-1,1) \times O(1,d-1)$ structure leads eventually to a scalar S related to the Lagrangian $\tilde{\mathcal{L}}_\beta$ (1.4.9). In chapter 6 we study SUSY of β-supergravity using former results.

We define the subgroup $O(d-1,1) \times O(1,d-1)$ as follows. The $O(d,d)$ metric $\eta_{(\mathrm{u/d})}$ (3.2.1), preserved on the generalized bundle E, has positive and negative eigenvalues. It is possible to form two sets of signature $(d-1,1)$ and $(1,d-1)$, as given by the diagonalized $O(d,d)$ metric η

$$\eta = \begin{pmatrix} \eta_d & 0 \\ 0 & -\overline{\eta}_d \end{pmatrix} \ , \ \eta_{\mathcal{AB}} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\overline{\eta_{ab}} \end{pmatrix} \ , \tag{3.2.30}$$

where we consider η_{ab} and $\overline{\eta_{ab}}$ to be the same in value, with $(d-1,1)$ signature. The two sets are distinguished by an unbarred/barred notation. A conformal generalized frame can then locally be separated into these two sets, and denoted accordingly $e^{-2d}\ \mathring{\tilde{\mathcal{E}}}_a\ ,\ e^{-2d}\ \mathring{\tilde{\mathcal{E}}}_{\overline{a}}$. Whether these two sets remain separated spaces globally is however not guaranteed by the $O(d,d)$ structure group. Preserving such a frame is equivalent to reducing $O(d,d)$ to $O(d-1,1) \times O(1,d-1)$, since the metric (3.2.30) is left invariant. In that case, the generalized bundle E is isomorphic to the direct sum of two sub-bundles, denoted as

$$E \simeq C_+ \oplus C_- \ . \tag{3.2.31}$$

$O(d-1,1)$ and $O(1,d-1)$ act on these spaces respectively and the corresponding indices are unbarred respectively barred.

Of the various quantities defined in section 3.2.1, we would now like to consider those that preserve this $O(d-1,1) \times O(1,d-1)$ structure. In order to do so, we first rotate the previous $O(d,d)$ representation to a new one where the embedding of $O(d-1,1)\times O(1,d-1)$ is diagonal. In particular, this means switching from the up/down basis with metric $\eta_{(\mathrm{u/d})}$ to the unbarred/barred basis with metric η. Secondly, we project out quantities that do not respect the $O(d-1,1) \times O(1,d-1)$ structure.

We will perform this procedure for the frame (3.2.7) with β introducing two different sets of vielbeins, namely $\tilde{e}^a{}_m$ respectively $\tilde{e}^{\overline{a}}{}_m$ on the sub-bundles $C_\pm$. Nevertheless, these give rise to the same metric $\tilde{g}_{mn}$ [39]. Eventually, we consider an alignment of vielbeins, i.e.

that $\tilde{e}^a{}_m = \tilde{e}^{\overline{a}}{}_m$ for $a = \overline{a}$ in value, in order to reduce to the standard degrees of freedom of a SUGRA theory.

The described procedure is based on introducing a matrix P transforming the previous up/down $\eta_{\text{u/d}}$ (3.2.1) into the unbarred/barred diagonal η (3.2.30). More details on this can be found in the appendix C.1.

$$\eta = P\eta_{\text{u/d}}P^T \ , \qquad P = \begin{pmatrix} \mathbb{1} & \eta_d \\ \mathbb{1} & -\overline{\eta}_d \end{pmatrix} \ , \ P^{-1} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \eta_d^{-1} & -\overline{\eta}_d^{-1} \end{pmatrix} \ . \tag{3.2.32}$$

Any object in the fundamental representation of $O(d,d)$ carrying an index $\mathcal{A}$ is then rotated as follows

$$V_{\mathcal{B}} = P_{\mathcal{B}}{}^{\mathcal{A}} V_{(u/d)\mathcal{A}} \ , \quad V^{\mathcal{B}} = V^{\mathcal{A}}_{(u/d)}(P^{-1})_{\mathcal{A}}{}^{\mathcal{B}} = (P^{-T})^{\mathcal{B}}{}_{\mathcal{A}} V^{\mathcal{A}}_{(u/d)} \tag{3.2.33}$$

$$\text{with } P_{\mathcal{B}}{}^{\mathcal{A}} = \begin{pmatrix} \delta^a_b & \eta_{bc}\delta^c_a \\ \delta^a_{\overline{b}} & -\overline{\eta_{bc}}\delta^{\overline{c}}_a \end{pmatrix} \ , \ (P^{-T})^{\mathcal{B}}{}_{\mathcal{A}} = \frac{1}{2}\begin{pmatrix} \delta^b_{\underline{a}} & \eta^{bc}\delta^a_c \\ \delta^{\overline{b}}_{a} & -\overline{\eta^{bc}}\delta^a_{\overline{c}} \end{pmatrix} \tag{3.2.34}$$

so that bilinears are preserved. In particular, the δ's in P and P^{-T} allow to pass from the up/down to the unbarred/barred indices, however we leave them out in order simplify formulas in the following. A first important example of this rotation is obtained by acting on the frames (3.2.6) and (3.2.7) which then take the form

$$e^{-2d} \ \mathcal{E}_{\mathcal{A}} = \begin{cases} e^{-2d} \ \mathcal{E}_a = e^{-2\phi}\sqrt{|g|}(\partial_a + b_{ab}e^b + \eta_{ab}e^b) \\ e^{-2d} \ \mathcal{E}_{\overline{a}} = e^{-2\phi}\sqrt{|g|}(\partial_{\overline{a}} + b_{\overline{ab}}e^{\overline{b}} - \overline{\eta_{ab}}e^{\overline{b}}) \end{cases} \ , \tag{3.2.35}$$

$$e^{-2d} \ \tilde{\mathcal{E}}_{\mathcal{A}} = \begin{cases} e^{-2d} \ \tilde{\mathcal{E}}_a = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}(\partial_a + \eta_{ab}\beta^{bc}\partial_c + \eta_{ab}\tilde{e}^b) \\ e^{-2d} \ \tilde{\mathcal{E}}_{\overline{a}} = e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}(\partial_{\overline{a}} - \overline{\eta_{ab}}\beta^{\overline{bc}}\partial_{\overline{c}} - \overline{\eta_{ab}}\tilde{e}^{\overline{b}}) \end{cases} \ , \tag{3.2.36}$$

where we did not write out the δ's.

Next, we redefine a covariant derivative $D_{\mathcal{A}}(W^{\mathcal{B}})e^{-2d} \ \mathring{\mathcal{E}}_{\mathcal{B}}$, where $\mathcal{A},\mathcal{B}$ are now unbarred/barred indices. A priori, one would have for the unbarred $\mathcal{A} = a$

$$D_a(W^{\mathcal{B}})e^{-2d} \ \mathring{\mathcal{E}}_{\mathcal{B}} = e^{-2d}\left(\partial_a(w^b)\mathring{\mathcal{E}}_b + \partial_a(w^{\overline{b}})\mathring{\mathcal{E}}_{\overline{b}} + \hat{\Omega}_a{}^{\mathcal{B}}{}_{\mathcal{C}}W^{\mathcal{C}}\mathring{\mathcal{E}}_{\mathcal{B}}\right) \ , \tag{3.2.37}$$

where ∂_a, respectively $\hat{\Omega}_a{}^{\mathcal{B}}{}_{\mathcal{C}}$ are defined as the unbarred component obtained from the rotation of the up/down ∂, respectively $\hat{\Omega}$. As mentioned earlier, one could also start from quantities D, ∂ and $\hat{\Omega}$ with a generalized curved index $\mathcal{M}$ and contract these with the proper generalized vielbein. In any case, the last term in (3.2.37) splits into four terms, according to the unbarred or barred choices for $\mathcal{B},\mathcal{C}$. Preserving the $O(d-1,1)\times O(1,d-1)$ structure $\hat{\Omega}_{\mathcal{M}}{}^{\overline{b}}{}_c$ and $\hat{\Omega}_{\mathcal{M}}{}^b{}_{\overline{c}}$ transforming in the bi-fundamental representation of $O(d,d)$ are easily seen to be off-diagonal with respect to the $O(d-1,1)\times O(1,d-1)$ diagonal structure. Such a component in the covariant derivative (3.2.37) would introduce mixed contributions from $C_\pm$ and the components on $\mathring{\mathcal{E}}_b$ would additionally contain $w^{\overline{c}}$ thanks to $\hat{\Omega}_{\mathcal{M}}{}^b{}_{\overline{c}}$. These have to be projected out in order to preserve the $O(d-1,1)\times O(1,d-1)$ structure and are thus set to zero. Separating the components on $\mathring{\mathcal{E}}_b$ and $\mathring{\mathcal{E}}_{\overline{b}}$, one obtains the following $O(d-1,1)\times O(1,d-1)$ derivative

$$D_{\mathcal{A}}W^{\mathcal{B}} = \begin{cases} D_a w^b = \partial_a w^b + \hat{\Omega}_a{}^b{}_c w^c \\ D_a w^{\overline{b}} = \partial_a w^{\overline{b}} + \hat{\Omega}_a{}^{\overline{b}}{}_{\overline{c}} w^{\overline{c}} \\ D_{\overline{a}} w^b = \partial_{\overline{a}} w^b + \hat{\Omega}_{\overline{a}}{}^b{}_c w^c \\ D_{\overline{a}} w^{\overline{b}} = \partial_{\overline{a}} w^{\overline{b}} + \hat{\Omega}_{\overline{a}}{}^{\overline{b}}{}_{\overline{c}} w^{\overline{c}} \end{cases} , \tag{3.2.38}$$

where again all indices are unbarred or barred.

We leave the details of the determination of the derivative ∂, the connection Ω, and the piece due to the conformal weight for the frame (3.2.7) to the appendix C.1.[10] The definition of the latter is slightly changed [39] with respect to (3.2.10) and we discuss this in the appendix. In particular, this involves rotating the contributions to the $O(d,d)\times\mathbb{R}^+$ derivative obtained in (3.2.28) and leaves us with

$$D_{\mathcal{A}}W^{\mathcal{B}} = \begin{cases} D_a w^b = \nabla_a w^b - \eta_{ad}\check{\nabla}^d w^b + \frac{1}{6}\eta_{ad}\eta_{cf}R^{dbf}w^c - \frac{1}{9}(\delta_a^b\Lambda_c - \eta_{ac}\eta^{be}\Lambda_e)w^c \\ D_a w^{\overline{b}} = \nabla_a w^{\overline{b}} - \eta_{ad}\check{\nabla}^d w^{\overline{b}} - \frac{1}{2}\eta_{ad}\overline{\eta_{cf}}R^{d\overline{b}\overline{f}}w^{\overline{c}} \\ D_{\overline{a}} w^b = \nabla_{\overline{a}} w^b + \overline{\eta_{ad}}\check{\nabla}^{\overline{d}} w^b - \frac{1}{2}\overline{\eta_{ad}}\eta_{cf}R^{\overline{d}bf}w^c \\ D_{\overline{a}} w^{\overline{b}} = \nabla_{\overline{a}} w^{\overline{b}} + \overline{\eta_{ad}}\check{\nabla}^{\overline{d}} w^{\overline{b}} + \frac{1}{6}\overline{\eta_{ad}\eta_{cf}}R^{\overline{dbf}}w^{\overline{c}} - \frac{1}{9}(\delta_{\overline{a}}^{\overline{b}}\Lambda_{\overline{c}} - \overline{\eta_{ac}}\overline{\eta^{be}}\Lambda_{\overline{e}})w^{\overline{c}} \end{cases} \tag{3.2.39}$$

as given in (1.4.6), where

$$\Lambda_{\mathcal{C}} = \begin{cases} \Lambda_c = \lambda_c + \eta_{cd}\xi^d \\ \Lambda_{\overline{c}} = \lambda_{\overline{c}} - \overline{\eta_{cd}}\xi^{\overline{d}} \end{cases} , \tag{3.2.40}$$

with λ and ξ given in (3.2.24).

Following [39, 69], we introduce for the $O(d-1,1)\times O(1,d-1)$ structure an associated $Spin(d-1,1)\times Spin(1,d-1)$ structure with respective spinors ϵ^+ and ϵ^-. The definition of spinorial derivatives for (3.2.38) is the standard one

$$D_{\mathcal{M}}\epsilon^+ = \partial_{\mathcal{M}}\epsilon^+ + \frac{1}{4}\hat{\Omega}_{\mathcal{M}}{}^b{}_c\eta_{bd}\gamma^{dc}\epsilon^+ , \tag{3.2.41}$$

$$D_{\mathcal{M}}\epsilon^- = \partial_{\mathcal{M}}\epsilon^- + \frac{1}{4}\hat{\Omega}_{\mathcal{M}}{}^{\overline{b}}{}_{\overline{c}}\overline{\eta_{bd}}\gamma^{\overline{dc}}\epsilon^- , \tag{3.2.42}$$

where the γ matrices and their properties are discussed in appendix A.2. Interestingly, one can build from these derivatives the generalized curvature scalar S mentioned in the Introduction which is related to the Lagrangian $\mathcal{L}_{\rm NSNS}$ up to a total derivative in the standard NSNS case. As in (1.4.8), the scalar S is defined by

$$-\frac{1}{4}S\epsilon^+ = \left(\gamma^a D_a\gamma^b D_b - \overline{\eta^{ab}}D_{\overline{a}}D_{\overline{b}}\right)\epsilon^+ , \tag{3.2.43}$$

[10] The same procedure applied to the frame (3.2.6) reproduces the result of [39].

or equivalently for the spinor ϵ^- with unbarred and barred indices exchanged. Our goal is to compute the scalar S for the frame (3.2.7) with β, i.e. using the spinorial forms (3.2.41) and (3.2.42) of the derivatives (3.2.39). In particular, the derivatives needed to compute S are given by

$$\gamma^a D_a \epsilon^+ = \left(\gamma^a \nabla_a - \gamma^a \eta_{ad} \check{\nabla}^d + \frac{1}{24} \eta_{ad}\eta_{be}\eta_{cf} R^{def} \gamma^{abc} - \frac{1}{2}\gamma^c \Lambda_c\right) \epsilon^+ \ , \tag{3.2.44}$$

$$D_{\overline{a}} \epsilon^+ = \left(\nabla_{\overline{a}} + \overline{\eta_{ad}} \check{\nabla}^{\overline{d}} - \frac{1}{8}\overline{\eta_{ad}} \eta_{be}\eta_{cf} R^{\overline{d}ef} \gamma^{bc}\right) \epsilon^+ \ , \tag{3.2.45}$$

$$D_{\overline{a}} w^{\overline{a}} = \nabla_{\overline{a}} w^{\overline{a}} + \overline{\eta_{ad}} \check{\nabla}^{\overline{d}} w^{\overline{a}} - \Lambda_{\overline{a}} w^{\overline{a}} \ , \tag{3.2.46}$$

where we used (A.2.7). Here, ∇ and $\check{\nabla}$ on the spinors are now the naturally defined spinorial covariant derivatives obtained from (3.2.39) and (3.2.41). The last expression (3.2.46) is needed in order to determine the non-spinorial part of the connection of the covariant derivative acting on $\overline{\eta^{ab}} D_{\overline{b}} \epsilon^+$. To present details of the computation more conveniently we rewrite the above derivatives

$$\gamma^a D_a \epsilon^+ = \left(\gamma^a \partial_a + \gamma^a \eta_{ad} \beta^{dc} \partial_c + X_{abc}\gamma^{abc} + X_a \gamma^a\right) \epsilon^+ \ , \tag{3.2.47}$$

$$D_{\overline{a}} \epsilon^+ = \left(\partial_{\overline{a}} - \overline{\eta_{ad}} \beta^{\overline{dc}} \partial_{\overline{c}} + Y_{\overline{a}bc}\gamma^{bc}\right) \epsilon^+ \ , \tag{3.2.48}$$

$$D_{\overline{a}} \left(\overline{\eta^{ab}} D_{\overline{b}} \epsilon^+\right) = \left(\partial_{\overline{a}} - \overline{\eta_{ad}} \beta^{\overline{dc}} \partial_{\overline{c}} + Y_{\overline{a}bc}\gamma^{bc} + Z_{\overline{a}}\right) \overline{\eta^{ab}} \left(\partial_{\overline{b}} - \overline{\eta_{be}} \beta^{\overline{ef}} \partial_{\overline{f}} + Y_{\overline{b}ef}\gamma^{ef}\right) \epsilon^+ \tag{3.2.49}$$

with abbreviations

$$X_{abc} = \frac{1}{4} \eta_{be} \left(\omega_{ac}^{e} - \eta_{ad} \omega_{Q}{}_{c}^{de} + \frac{1}{6} \eta_{ad}\eta_{cf} R^{def}\right) \ , \tag{3.2.50}$$

$$X_a = \frac{1}{2} \left(\omega_{da}^{d} + \eta_{ac} \omega_{Q}{}_{d}^{dc} - \Lambda_a\right) \ , \tag{3.2.51}$$

$$Y_{\overline{a}bc} = \frac{1}{4} \eta_{be} \left(\omega_{\overline{a}c}^{e} + \overline{\eta_{ad}} \omega_{Q}{}_{c}^{\overline{d}e} - \frac{1}{2} \overline{\eta_{ad}} \eta_{cf} R^{\overline{d}ef}\right) \ , \tag{3.2.52}$$

$$Z_{\overline{a}} = \omega_{\overline{da}}^{\overline{d}} - \overline{\eta_{ac}} \omega_{Q}{}_{\overline{d}}^{\overline{dc}} - \Lambda_{\overline{a}} \ , \tag{3.2.53}$$

where (A.2.7) and the antisymmetry properties of ω and ω_Q were used. With these convenient notations we compute S from (3.2.43) and obtain at first

$$
\begin{aligned}
-\frac{1}{4}S\epsilon^{+} =\Bigg[&\gamma^a\gamma^b(\partial_a+\eta_{ad}\beta^{de}\partial_e)(\partial_b+\eta_{bc}\beta^{cf}\partial_f)-\overline{\eta^{ab}}(\partial_{\overline{a}}-\overline{\eta_{ad}}\beta^{\overline{de}}\partial_{\overline{e}})(\partial_{\overline{b}}-\overline{\eta_{bc}}\beta^{\overline{cf}}\partial_{\overline{f}})\\
&+6\eta^{ab}\gamma^{cf}X_{[bcf]}(\partial_a+\eta_{ad}\beta^{de}\partial_e)+2\eta^{ac}X_c(\partial_a+\eta_{ad}\beta^{de}\partial_e)\\
&-2\overline{\eta^{ab}}Y_{\overline{a}de}\gamma^{de}(\partial_{\overline{b}}-\overline{\eta_{bc}}\beta^{\overline{cf}}\partial_{\overline{f}})-Z_{\overline{a}}\overline{\eta^{ab}}(\partial_{\overline{b}}-\overline{\eta_{bc}}\beta^{\overline{cf}}\partial_{\overline{f}})\\
&+\gamma^a\gamma^{bcf}(\partial_a+\eta_{ad}\beta^{de}\partial_e)(X_{bcf})+\gamma^a\gamma^c(\partial_a+\eta_{ad}\beta^{de}\partial_e)(X_c)\\
&+\frac{1}{2}X_{ade}X_{bcf}\{\gamma^{ade},\gamma^{bcf}\}+X_{ade}X_c\{\gamma^{ade},\gamma^c\}+X_aX_c\gamma^a\gamma^c\\
&-\overline{\eta^{ab}}(\partial_{\overline{a}}-\overline{\eta_{ad}}\beta^{\overline{de}}\partial_{\overline{e}})(Y_{\overline{b}cf})\gamma^{cf}-\frac{1}{2}\overline{\eta^{ab}}Y_{\overline{a}de}Y_{\overline{b}cf}\{\gamma^{de},\gamma^{cf}\}-Z_{\overline{a}}\overline{\eta^{ab}}Y_{\overline{b}cf}\gamma^{cf}\Bigg]\epsilon^{+}\ ,
\end{aligned}
\tag{3.2.54}
$$

where we used (A.2.7).

The first three lines of (3.2.54) are acting on the spinor and have to vanish since we expect S to be a scalar. Moreover, the last three lines of (3.2.54) containing various orders of fully antisymmetrized products of γ matrices should vanish either. The scalar S of the above expression simply represents the zeroth order in γ. We detail the verification in appendix C.3. This leaves us with only a scalar multiplying ϵ^+ as in (C.3.20), namely

$$
\begin{aligned}
-\frac{1}{4}S\epsilon^{+}=-\frac{1}{4}\Bigg(&\mathcal{R}(\tilde{g})+\mathcal{R}_Q-\frac{1}{2}R^{acd}f^b{}_{cd}\eta_{ab}-\frac{1}{2}R^2\\
&-4(\partial\tilde{\phi})^2+4\nabla^2\tilde{\phi}-4(\beta^{ab}\partial_b\tilde{\phi}-\mathcal{T}^a)^2-4\eta_{ab}\check{\nabla}^a(\beta^{bc}\partial_c\tilde{\phi}-\mathcal{T}^b)\Bigg)\epsilon^{+}\ .
\end{aligned}
\tag{3.2.55}
$$

Remarkably, this scalar contains only an even number of β in each term, i.e. all odd orders get canceled. Using (A.3.17) and the Leibniz rule, we rewrite the second line of (3.2.55) and get eventually

$$
\begin{aligned}
S=&\mathcal{R}(\tilde{g})+4(\partial\tilde{\phi})^2+4(\beta^{ab}\partial_b\tilde{\phi}-\mathcal{T}^a)^2+\mathcal{R}_Q-\frac{1}{2}R^{acd}f^b{}_{cd}\eta_{ab}-\frac{1}{2}R^2\\
&+e^{2d}\ \partial_p\left(4e^{-2d}\tilde{g}^{pq}\partial_q\tilde{\phi}-4e^{-2d}\beta^{pm}\tilde{g}_{mq}(\beta^{qr}\partial_r\tilde{\phi}-\mathcal{T}^q)\right)\ .
\end{aligned}
\tag{3.2.56}
$$

Using the analogue relation of the scalar S to the Lagrangian $\mathcal{L}_{NSNS}$ given in [39], we can write

$$
S=e^{2d}\ (\tilde{\mathcal{L}}_\beta+\partial(\dots))\ ,
\tag{3.2.57}
$$

and recover the Lagrangian $\tilde{\mathcal{L}}_\beta$ for β-supergravity, as presented in (1.4.9). This is indeed, the correct Lagrangian as shown in [101] by a direct rewriting of the Lagrangian $\mathcal{L}_0$ in flat indices.

In addition, it was shown in [39] that such a curvature scalar encodes the dilaton equation of motion for standard supergravity, by considering $S=0$. Here, we obtain the analogous result reproducing the dilaton equation of motion (1.4.12) of β-supergravity. Besides the scalar curvature (3.2.43), generalized geometry allows for the definition of an analogue of the Ricci tensor. To derive the two other equations of motion in analogy to the standard Ricci scalar encoding the Einstein equation, we calculate the *generalized Ricci tensor*

$$\frac{1}{2}R_{a\bar{b}}\gamma^a\epsilon^+ = [\gamma^a D_a, D_{\bar{b}}]\epsilon^+ \ , \tag{3.2.58}$$

that depends on the spinorial form of the derivatives (3.2.39).[11] For standard SUGRA, it was shown in [39] that setting the symmetric part to zero, $R_{(ab)} = 0$, corresponds to the Einstein equation, while the antisymmetric part $R_{[ab]} = 0$ yields the equation of motion for the b-field. In analogy, we obtain the equations of motion for $\tilde{g}$ and β taking respectively the symmetric or antisymmetric part of R_{ab}. Using additionally to the generalized covariant derivative expressions (3.2.44),(3.2.45) and (3.2.46) defined above,

$$D_a w^{\bar{b}} = \nabla_a w^{\bar{b}} - \eta_{ad}\check{\nabla}^d w^{\bar{b}} - \frac{1}{2}\eta_{ad}\overline{\eta_{cf}}R^{d\overline{bf}}w^{\bar{c}} \ , \tag{3.2.59}$$

the generalized Ricci tensor (3.2.58) becomes

$$\begin{aligned}\frac{1}{2}R_{a\bar{b}}\gamma^a\epsilon^+ =& \left(\gamma^a\partial_a + \gamma^a\eta_{ad}\beta^{dc}\partial_c + X_{acd}\gamma^{acd} + X_a\gamma^a\right)\left(\partial_{\bar{b}} - \overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\bar{e}} + Y_{\bar{b}gh}\gamma^{gh}\right)\epsilon^+ \\ &- \gamma^a\omega^{\bar{c}}_{a\bar{b}}\left(\partial_{\bar{c}} - \overline{\eta_{cg}}\beta^{\overline{ge}}\partial_{\bar{e}} + Y_{\bar{c}gh}\gamma^{gh}\right)\epsilon^+ + \gamma^a\eta_{ad}\omega_Q{}^{d\bar{c}}{}_{\bar{b}}\left(\partial_{\bar{c}} - \overline{\eta_{cg}}\beta^{\overline{ge}}\partial_{\bar{e}} + Y_{\bar{c}gh}\gamma^{gh}\right)\epsilon^+ \\ &- \frac{1}{2}\gamma^a\eta_{ad}\overline{\eta_{bf}}R^{d\overline{fc}}\left(\partial_{\bar{c}} - \overline{\eta_{cg}}\beta^{\overline{ge}}\partial_{\bar{e}} + Y_{\bar{c}gh}\gamma^{gh}\right)\epsilon^+ \\ &- \left(\partial_{\bar{b}} - \overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\bar{e}} + Y_{\bar{b}gh}\gamma^{gh}\right)\left(\gamma^a\partial_a + \gamma^a\eta_{ad}\beta^{dc}\partial_c + X_{acd}\gamma^{acd} + X_a\gamma^a\right)\epsilon^+ \ .\end{aligned} \tag{3.2.60}$$

We leave the computational details of the above expression to appendix C.4, and give here the result. After aligning the vielbeins, and considering only the first order in γ-matrices, $\frac{1}{2}R_{ab}\gamma^a$ gives

$$\begin{aligned}\Big(&\frac{1}{2}\mathcal{R}_{ba} - \frac{1}{2}\eta_{e(a}\eta_{b)g}\check{\mathcal{R}}^{ge} + \frac{1}{8}\eta_{ae}\eta_{bg}\eta_{if}\eta_{cd}R^{igc}R^{dfe} \\ &+ \nabla_b\nabla_a\tilde{\phi} - \eta_{e(a}\eta_{b)g}\check{\nabla}^g(\check{\nabla}^e\tilde{\phi}) - \eta_{e(a}\eta_{b)g}\check{\nabla}^g\mathcal{T}^e \\ &+ \frac{1}{4}\eta_{ae}\eta_{bg}\eta^{df}\partial_d Q_f{}^{eg} - \frac{1}{2}\eta_{e[a}\partial_{b]}Q_d{}^{de} - \frac{1}{4}\beta^{gc}\partial_c f^e{}_{ab}\eta_{ge} + \frac{1}{2}\beta^{gc}\partial_c f^d{}_{d[a}\eta_{b]g} \\ &+ \frac{1}{4}\eta_{bg}\eta_{ae}\eta^{ch}f^d{}_{dc}Q_h{}^{eg} - \frac{1}{4}\eta_{ch}Q_d{}^{dc}f^h{}_{ab} \\ &+ \frac{1}{4}f^g{}_{cd}Q_{[a}{}^{dc}\eta_{b]g} + \frac{1}{2}\eta_{e[a}f^h{}_{b]d}Q_i{}^{ec}\eta_{ch}\eta^{di} + \frac{1}{2}\eta_{e[a}f^h{}_{b]c}Q_h{}^{ec} \\ &- \eta_{e[a}\nabla_{b]}(\check{\nabla}^e\tilde{\phi}) - \eta_{e[a}\nabla_{b]}\mathcal{T}^e + \eta_{g[b}\check{\nabla}^g\nabla_{a]}\tilde{\phi} \\ &- \frac{1}{2}\eta_{ae}\eta_{bg}\eta_{fc}R^{gfe}\mathcal{T}^c + \frac{1}{4}\eta_{ae}\eta_{bg}\eta_{df}e^{2\tilde{\phi}}\check{\nabla}^d(e^{-2\tilde{\phi}}R^{gfe})\Big)\gamma^a \ .\end{aligned} \tag{3.2.61}$$

[11] Analogous quantities to S and $R_{a\bar{b}}$ were considered before in [109, 110, 54, 126, 55, 113, 114]. Their relations to the Lagrangian and the equations of motion were as well studied. The DFT quantities were shown in [69] to match those of (3.2.43) and (3.2.58) for standard SUGRA.

The first order in γ^a will be enough to recover the equations of motion, i.e. the higher orders in γ^a vanish, as they did for S.

As explained above, setting $R_{ab} = 0$ and therefore demanding the expression (3.2.61) to vanish, we should obtain the equations of motion for $\tilde{g}$ and β. More precisely, setting the symmetric part of (3.2.61) to vanish gives the Einstein equation (1.4.13)

$$\begin{aligned} &\frac{1}{2}\mathcal{R}_{ba} - \frac{1}{2}\eta_{e(a}\eta_{b)g}\check{\mathcal{R}}^{ge} + \frac{1}{8}\eta_{ae}\eta_{bg}\eta_{if}\eta_{cd}R^{igc}R^{dfe} \\ &+\nabla_b\nabla_a\tilde{\phi} - \eta_{e(a}\eta_{b)g}\check{\nabla}^g(\check{\nabla}^e\tilde{\phi}) - \eta_{e(a}\eta_{b)g}\check{\nabla}^g\mathcal{T}^e = 0 \ . \end{aligned} \tag{3.2.62}$$

Similarly, the antisymmetric part of (3.2.61) gives

$$\begin{aligned} &\frac{1}{4}\eta_{ae}\eta_{bg}\eta^{df}\partial_d Q_f{}^{eg} - \frac{1}{2}\eta_{e[a}\partial_{b]}Q_d{}^{de} - \frac{1}{4}\beta^{gc}\partial_c f^e{}_{ab}\eta_{ge} + \frac{1}{2}\beta^{gc}\partial_c f^d{}_{d[a}\eta_{b]g} \\ &+\frac{1}{4}\eta_{bg}\eta_{ae}\eta^{ch}f^d{}_{dc}Q_h{}^{eg} - \frac{1}{4}\eta_{ch}Q_d{}^{dc}f^h{}_{ab} \\ &+\frac{1}{4}f^g{}_{cd}Q_{[a}{}^{dc}\eta_{b]g} + \frac{1}{2}\eta_{e[a}f^h{}_{b]d}Q_i{}^{ec}\eta_{ch}\eta^{di} + \frac{1}{2}\eta_{e[a}f^h{}_{b]c}Q_h{}^{ec} \\ &-\eta_{e[a}\nabla_{b]}(\check{\nabla}^e\tilde{\phi}) - \eta_{e[a}\nabla_{b]}\mathcal{T}^e + \eta_{g[b}\check{\nabla}^g\nabla_{a]}\tilde{\phi} \\ &-\frac{1}{2}\eta_{ae}\eta_{bg}\eta_{fc}R^{gfe}\mathcal{T}^c + \frac{1}{4}\eta_{ae}\eta_{bg}\eta_{df}e^{2\tilde{\phi}}\check{\nabla}^d(e^{-2\tilde{\phi}}R^{gfe}) = 0 \ . \end{aligned} \tag{3.2.63}$$

This last result denotes the equation of motion for β (1.4.14). In [101], further verifications for the presented equations of motion of β-supergravity are given by following the standard procedure of varying the Lagrangian $\tilde{\mathcal{L}}_\beta$ with respect to the fields $\tilde{g}$, β and $\tilde{\phi}$. The resulting equations of motion have then been transformed and matched to the set (1.4.12), (1.4.13) and (1.4.14) in flat indices. In particular, the equation of motion for β takes here a more convenient form since the Q-flux appears explicity. A further interesting observation, discussed in the appendix C.5, is a slight mismatch between the equations of motion derived from the simplified Lagrangian in [84] and the reduced equations of motion of β-supergravity. Hence, deriving the equations of motion and implementing the simplifying assumption is not a commutative procedure.

The spinorial derivatives given in (3.2.41) and (3.2.42), and their explicit expressions, such as (3.2.44) and (3.2.45), serve further purposes. As mentioned previously, it has been noticed that these quantities, for the frame (3.2.6) with b-field, are those entering the SUSY variations of the fermions in type II SUGRAs. In other words, these quantities lead to the Killing spinor equations. The expressions worked out here therefore play the analogous, important, role in β-supergravity and we come back to these questions in chapter 6

This concludes our derivation of the NSNS part of the Lagrangian $\tilde{\mathcal{L}}_\beta$ and its equations of motion for $\tilde{g}$, β and $\tilde{\phi}$ for β-supergravity from generalized curvature quantities within the GG formalism.

Chapter 4

Geometric vacua of β-supergravity

In this chapter, we address questions of non-geometry and how β-supergravity provides a geometric description for a certain set of backgrounds despite encountering the problem of detecting vacua covering truely new physics. In particular, we come back to the toroidal example and how a geometry can be restored for the non-geometric configuration that leads again to a consistent supergravity (SUGRA) limit within β-supergravity. More generally, we observe the symmetries of β-supergravity and relate them to transition function patching possible vacuum solutions. Hence, we first study the gauge symmetries of the Lagrangian and possible distinct symmetries that might arise in certain constrained situations. While the former are believed to add nothing new to the situation with regard to the global consistency of backgrounds, the latter allow to establish β-supergravity as a useful framework for recovering an underlying geometry for specific previously non-geometric backgrounds. We then analyze the relation of this class with respect to being T-dual or not to geometric vacua in standard SUGRA and the associated hope of finding genuinely non-geometric classes with new physical effects. Finally, we discuss the task of directly identifying new solutions within β-supergravity from its equation of motion.

4.1 From non-geometry to geometry

The previous derivation of the Lagrangian $\tilde{\mathcal{L}}_\beta$ was based on either working locally or assuming that the underlying differential geometry is governed by the metric $\tilde{g}_{mn}$ and possible non-geometric fluxes. The relation of the latter situation to a non-geometry in the sense of [127, 29, 128] has been discussed further in [84, 82, 129] and is certainly the most interesting for applications. Here, we focus on the toroidal example [32, 33] that nicely illustrates various aspects of such a discussion and how a SUGRA limit can be restored in the case of non-geometric configurations. Then, more generally, we address the underlying gauge symmetries of $\tilde{\mathcal{L}}_\beta$ and answer the question of whether these realize the transition functions used for patching non-geometric backgrounds. Eventually, we try to relate this discussion to a new generalized bundle structure that we would like to call generalized cotangent bundle.

4.1.1 The toroidal example and the supergravity limit

In the Introduction we started discussing the toroidal example with its three T-dual NSNS field configurations related by Buscher transformations. These configurations alone do not provide consistent SUGRA solutions [130], but have to be completed with further ingredients of other sectors or should be considered as backgrounds within a dilute flux approximation [131].[1] We list the toroidal example in terms of the standard field content in table 4.1. First, we have the *standard three-torus with constant H-flux* and the dilaton ϕ_0 along the directions $x^1 = x,\ x^2 = y,\ x^3 = z$ with radii $R_{m=1,2,3}$, where the b-field is linear. The second configuration is known as the *twisted torus* and is described by a fibration of a circle along x over the base torus with directions $y,\ z$. Moreover, due to its fibration, further discussed in section 4.1.2, it falls into the class of nilmanifolds and is also known as the Heisenberg manifold. Interestingly, the b-field is vanishing and instead a non-zero structure constant f can be computed using (4.1.8) and (A.3.10). The duality with the first configuration arises by performing a Buscher transformation along the isometry direction x. A further transformation along the isometry in y, then leads to the interesting *non-geometric configuration*. In order to observe the non-geometry of the latter, one should study the patching of fields around the base circle along z. This is most conveniently done computing the generalized metric $\mathcal{H}$ given in (3.1.2) on two different patches. These configurations are then related as follows

$$T_{\mathcal{C}}^T\ \mathcal{H}\big|_{z=0}\ T_{\mathcal{C}} = \mathcal{H}\big|_{z=2\pi}\ , \quad \text{where}\ \ T_{\mathcal{C}} = \begin{pmatrix} \mathbb{1}_3 & \varpi \\ 0 & \mathbb{1}_3 \end{pmatrix}\ ,\ \varpi = \begin{pmatrix} 0 & 2\pi H & 0 \\ -2\pi H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{4.1.1}$$

We see that the transition matrix $T_{\mathcal{C}} \in O(3,3)$ needed for this kind of patching does not take the form of a diffeomorphism or a b-field gauge transformation. Therefore, such a situation where the gluing, needed for global consistency of the background, happens due to a purely stringy symmetry are said to be non-geometric. Although we are able to identify the transition matrices as being stringy, for now the situation for compactifying such a background is still unsatisfying. Since the notion of a standard manifold does no longer apply to these kind of spaces it is not known how to use them in the compact internal directions. This is also reflected in the dilaton and the volume form related to $\sqrt{|g|}$ being globally ill-defined as these depend on a non single-valued function f_0.

These properties of the non-geometric field configuration are also known to be problematic for the SUGRA limit. The function f_0 makes the radii go from below the string scale to above, spoiling a possible large volume limit. Indeed, for $f_0(z=0) = 1$ and choosing for convenience the two fiber radii to be of the same order such that $R_1 \sim R_2 \ll 1$, a large volume limit $g_{11} \sim g_{22} \gg 1$ can be achieved. However, on a second patch we have $f_0(z = 2\pi) = 1\Big/\left(1 + \left(\frac{2\pi H}{R_1 R_2}\right)^2\right)$, where $2\pi H$ is quantized, when going around the base circle, leading to

[1] Interestingly, the study of the quantum properties of the closed string on these approximated backgrounds leads to non-commutativity found for the non-geometric configuration in [131]. We also stick to the conventions $\alpha' = \frac{1}{2}$ and $2\pi H$ being quantized.

Configuration	Fields	Flux
H-flux torus	$g = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}\ ,\ b = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,$ $e^{-2\phi} = e^{-2\phi_0}\ ,$	$H_{123} = \frac{H}{R_1R_2R_3}$
Twisted torus	$g = \begin{pmatrix} \frac{1}{R_1^2} & -\frac{Hz}{R_1^2} & 0 \\ -\frac{Hz}{R_1^2} & R_2^2 + \left(\frac{Hz}{R_1}\right)^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}\ ,\ b = 0\ ,$ $e^{-2\phi} = e^{-2\phi_0} R_1^2\ ,$	${f^1}_{23} = -\frac{H}{R_1R_2R_3}$
Non-geometry	$g = f_0 \begin{pmatrix} \frac{1}{R_1^2} & 0 & 0 \\ 0 & \frac{1}{R_2^2} & 0 \\ 0 & 0 & \frac{R_3^2}{f_0} \end{pmatrix}\ ,\ b = f_0 \begin{pmatrix} 0 & -\frac{Hz}{R_1^2R_2^2} & 0 \\ \frac{Hz}{R_1^2R_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,$ $e^{-2\phi} = e^{-2\phi_0} R_1^2 R_2^2\ f_0^{-1}\ ,\ \text{with}\ f_0 = \left(1 + \left(\frac{Hz}{R_1R_2}\right)^2\right)^{-1}$	

Table 4.1: The toroidal example, with the standard NSNS fields and fluxes

the violation $g_{11} \sim g_{22} \sim \left(\frac{R_1}{2\pi H}\right)^2 \ll 1$ of the previous large volume limit. Furthermore, this variation of f_0 within the dilaton prevents us from defining a small string coupling constant. Note that despite these two issues with the SUGRA limit, this non-geometric configuration is thought to lead to an admissible string background, because it is T-dual to standard geometric situations [34].

The field redefinition (3.1.3) and (3.1.4) have been proposed [84] to cure the above problems of non-geometry, by restoring a standard geometry and introducing new fluxes. For the toroidal example, the new fields, computed from the standard NSNS ones of the non-geometric configuration, are given by

$$\tilde{g} = \begin{pmatrix} \frac{1}{R_1^2} & 0 & 0 \\ 0 & \frac{1}{R_2^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}\ ,\ \beta = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ e^{-2\tilde{\phi}} = e^{-2\phi_0} R_1^2 R_2^2\ , \tag{4.1.2}$$

where the Q-flux takes the value ${Q_3}^{12} = \frac{H}{R_1R_2R_3}$. A standard geometry of a three-torus is restored, together with a well-defined dilaton. When going around the circle along z, the field β patches with a constant shift which provides a globally well-defined Q-flux. There are further non-trivial checks indicating that this field configuration is a good one to consider. The dependence in radii of the metric and the dilaton are the expected ones from T-duality. Also, the ten-dimensional Q-flux, computed from (1.4.5), has the same value as the fluxes of the other field configurations, as expected from the four-dimensional T-duality chain [27]. Additionally, it is worth emphasizing that we restore, together with a standard geometry and the well-definedness of fields, a SUGRA limit. A large volume limit is certainly possible in the regime $R_1 \sim R_2 \ll 1$, which is the expected T-dual regime to the other two geometric field configurations. This regime is also compatible with a small string coupling constant given by the new dilaton. There is reason to believe that such properties are more general. An indication for this can be observed from the field redefinition (3.1.3) from g to $\tilde{g}$ which involves two inverse powers of the metric and hence corrects the dependence on radii such that a large volume limit is possible. This toroidal example therefore illustrates the role of $\tilde{\mathcal{L}}_\beta$, and β-supergravity, in providing a ten-dimensional geometric description of some non-geometries.[2] We address the question whether our formalism may describe more throughout this chapter.

We have discussed how a large volume limit can be restored using $\tilde{\mathcal{L}}_\beta$ and its fields. For a complete SUGRA limit, one should though consider all higher order corrections in α' to such an effective theory, and verify that they are subdominant. This could be worked out from a world-sheet perspective. Performing the field redefinition (3.1.3) on the standard bosonic string σ-model (1.1.24) gives the following action

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{|h|}\ h^{\alpha\beta}\left(\left(\tilde{g}^{-1}+\beta\right)^{-1}(X)\right)_{mn}\ \partial_\alpha X^m\ \partial_\beta X^n\ , \qquad (4.1.3)$$

where we use the conventions of [131]. This action may play a role in such a verification. Also, the vanishing β-functionals are usually given by the standard SUGRA equations of motion. As $\mathcal{L}_{\rm NSNS}$ and $\tilde{\mathcal{L}}_\beta$ only differ by a total derivative, their equations of motion should be the same, up to the field redefinition. Therefore, the vanishing β-functionals of (4.1.3), completed with the dilaton $\tilde{\phi}$, should be given by the equations of motion derived in this paper, namely (1.4.12) - (1.4.14).

4.1.2 β gauge transformation and generalized cotangent bundle

Let us now discuss the symmetries of $\tilde{\mathcal{L}}_\beta$ and some related aspects. Among its gauge symmetries we find diffeomorphisms that leave $\tilde{\mathcal{L}}_\beta$ invariant. The field redefinition (3.1.3) and (3.1.4) simply relates tensors and therefore the notion of a diffeomorphism remains the same through this procedure. The other gauge symmetry of the standard $\mathcal{L}_{\rm NSNS}$, given by the b-field gauge transformation, written for convenience with a shift matrix s

$$\begin{cases} g \to g \\ b \to b+s \end{cases}\ , \quad \text{where} \quad s_{mn} = \partial_{[m}\xi_{n]}\ , \qquad (4.1.4)$$

[2] In [84, 82, 129] the role of the total derivative difference between $\mathcal{L}_{\rm NSNS}$ and the new Lagrangian was underlined. This $\partial(\dots)$ being ill-defined would allow to have only one of the two Lagrangians well-defined, and thus a preferred description.

and ξ_m is subject to further constraints, as seen in [39]. By performing the field redefinition (3.1.3) on (4.1.4), we can rewrite this transformation, using matrix notation, in terms of the new fields

$$\begin{cases} \tilde{g} \to (\mathbb{1} + (\tilde{g}^{-1} + \beta)s)^T \ \tilde{g} \ (\mathbb{1} + (\tilde{g}^{-1} + \beta)s) \\ \beta \to (\mathbb{1} + (\tilde{g}^{-1} + \beta)s)^{-1} \left(\beta - (\tilde{g}^{-1} + \beta)s(\tilde{g}^{-1} + \beta)^T\right) (\mathbb{1} + (\tilde{g}^{-1} + \beta)s)^{-T} \end{cases} . \quad (4.1.5)$$

As $\tilde{\mathcal{L}}_\beta$ differs only by a total derivative from $\mathcal{L}_{\mathrm{NSNS}}$, the transformation (4.1.5) should be a gauge symmetry of our Lagrangian $\tilde{\mathcal{L}}_\beta$ and we call it β *gauge transformation*. The novelty, with respect to (4.1.4), is that $\tilde{g}$ changes as well as β under this gauge transformation. This is somehow expected, as the redefinition (3.1.3) mixes the two fields. Although not problematic for the theory, this behavior of $\tilde{g}$ is troublesome for the interpretation of this field, that we called so far a metric. A metric of a standard manifold should only transform under diffeomorphisms and consequently the underlying geometry may differ from a conventional one if it has an additional transformation. We come back to this point further down.

Let us now turn to transition functions and the comparison with the symmetries of the theory. This is crucial to distinguish geometry from non-geometry, as recently discussed in [81]. We introduced with the Generalized Geometry (GG) formalism in section 3.2.1 the notion of generalized bundle E defined over a set of patches. Its structure group $O(d,d)$ is by definition made of transition functions that relate the generalized frames when going from one patch to the other. An element of the structure group, viewed as a transition matrix, therefore acts on the generalized flat index $\mathcal{A}$. For a generalized frame allowing for a splitting, one can consider locally a generalized vielbein, as given in (3.2.2). A transition matrix T then relates two generalized vielbeins on patches ζ and ϑ as

$$T^{\mathcal{B}}{}_{\mathcal{A}} \ \mathring{\mathcal{E}}^{\mathcal{A}}{}_{\mathcal{M}}\big|_{\zeta} = \mathring{\mathcal{E}}^{\mathcal{B}}{}_{\mathcal{M}}\big|_{\vartheta} \ . \quad (4.1.6)$$

As discussed in section 3.2.1, preserving a specific form of the generalized frame, in particular the splitting, reduces generically the structure group to G_{split}. The bundle then should reduce accordingly.[3] Preserving the form of the frame with b-field (3.2.6) amounts to maintaining the block structure of the generalized vielbein $\mathcal{E}$ given in (3.1.1). The reduced structure group is then $G_{\mathrm{split}} = GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2}$, where schematically $GL(d,\mathbb{R})$ gives the diagonal transformation on the vielbein e, and $\mathbb{R}^{d(d-1)/2}$ is the antisymmetric lower off-diagonal block shifting the b-field in $\mathcal{E}$ [39]. The bundle gets reduced to the generalized tangent bundle E_T, that can be viewed as a fibration of the cotangent bundle over the tangent bundle, denoted as

$$\begin{array}{ccc} T^*\mathcal{M} & \hookrightarrow & E_T \\ & & \downarrow \\ & & T\mathcal{M} \end{array} \ . \quad (4.1.7)$$

[3] More precisely, as discussed in section 3.2.1, G_{split} is a subgroup of $O(d,d) \times \mathbb{R}^+$ and it is the conformal extension of E that gets reduced.

This particular ordering in the fibration can be understood when comparing with a standard fibration, such as the one of the twisted torus considered in section 4.1.1. The Cartan one-forms of the latter can be read from the metric in table 4.1, and correspond to the co-frame. These and the frame are given for that example by

$$e^a = e^a{}_m \mathrm{d}x^m = \begin{cases} e^1 = \frac{1}{R_1}(\mathrm{d}x - Hz\mathrm{d}y) \\ e^2 = R_2 \ \mathrm{d}y \\ e^3 = R_3 \ \mathrm{d}z \end{cases} , \quad \partial_a = (e^{-T})_a{}^m \partial_m = \begin{cases} \partial_1 = R_1 \ \partial_x \\ \partial_2 = \frac{1}{R_2}(\partial_y + Hz\partial_x) \\ \partial_3 = \frac{1}{R_3} \ \partial_z \end{cases} \tag{4.1.8}$$

The non-trivial one-form e^1 is given by the sum of a local one-form $\mathrm{d}x$ along the fiber and the connection one-form living typically on the base. This changes for the frame, where the non-trivial one is now ∂_2, given by the sum of a local base vector ∂_y, and the fiber vector ∂_x multiplied by the connection one-form component. If we compare this frame structure to that of the generalized frame with b-field (3.2.6), we deduce for the latter that the one-forms e^a are along the fiber, and the base directions are given by ∂_a. The connective structure is given by the b-field b_{ab}. Hence, we recover the structure (4.1.7) of the generalized tangent bundle.

The same comparison holds for the generalized frame with β (3.2.7) and we obtain the opposite situation where vectors ∂_a are fibered over one-forms $\tilde{e}^a$ in the base by the connective structure β^{ab}. This formula (3.2.7) is only a local expression, but if there is a global completion that preserves this local form of the frame, then the corresponding bundle should be a *generalized cotangent bundle* E_{T^*}, i.e.

$$\begin{array}{ccc} T\mathcal{M} & \hookrightarrow & E_{T^*} \\ & & \downarrow \\ & & T^*\mathcal{M} \end{array} . \tag{4.1.9}$$

The associated structure group G_{split} should by definition preserve the form of the frame (3.2.7). Therefore, it is again given by $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2}$, with the difference that $\mathbb{R}^{d(d-1)/2}$ now matches the antisymmetric upper off-diagonal block shifting β in the generalized vielbein $\tilde{\mathcal{E}}$ of (3.1.1). First hints on such a bundle and its structure group were given in [78].

Having presented bundles and structure groups, we now study the corresponding transition matrices. To ease their comparison to the symmetries discussed above, it is useful to go to generalized curved indices. Having at least locally generalized vielbeins, one can define from (4.1.6) a transition matrix $T_{\mathcal{C}}$ with curved indices

$$(T_{\mathcal{C}})^{\mathcal{M}}{}_{\mathcal{N}} \equiv (\mathring{\mathcal{E}}^{-1})^{\mathcal{M}}{}_{\mathcal{B}}\big|_\zeta \ T^{\mathcal{B}}{}_{\mathcal{A}} \ \mathring{\mathcal{E}}^{\mathcal{A}}{}_{\mathcal{N}}\big|_\zeta = \mathring{\mathcal{E}}^{\mathcal{M}}{}_{\mathcal{B}}\big|_\zeta \ \mathring{\mathcal{E}}^{\mathcal{B}}{}_{\mathcal{N}}\big|_\vartheta \ . \tag{4.1.10}$$

The generalized vielbein, and metric $\mathcal{H} = \mathring{\mathcal{E}}^T \ \mathbb{I} \ \mathring{\mathcal{E}}$ as in (3.1.2), then transform as

$$\mathring{\mathcal{E}}^{\mathcal{A}}{}_{\mathcal{M}}\big|_\zeta \ (T_{\mathcal{C}})^{\mathcal{M}}{}_{\mathcal{N}} = \mathring{\mathcal{E}}^{\mathcal{A}}{}_{\mathcal{N}}\big|_\vartheta \ , \quad T_{\mathcal{C}}^T \ \mathcal{H}\big|_\zeta \ T_{\mathcal{C}} = \mathcal{H}\big|_\vartheta \ , \tag{4.1.11}$$

an example of which was thus given in (4.1.1). If the form of the vielbein $\tilde{\mathcal{E}}$ in (3.1.1) is preserved by T as it should be for E_{T*}, then we can write $T_{\mathcal{C}}$ as follows

$$\tilde{\mathcal{E}} = \begin{pmatrix} \tilde{e} & \tilde{e}\beta \\ 0 & \tilde{e}^{-T} \end{pmatrix} = \begin{pmatrix} \tilde{e} & 0 \\ 0 & \tilde{e}^{-T} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \ , \ \tilde{\mathcal{E}}^{-1} = \begin{pmatrix} \tilde{e}^{-1} & -\beta\tilde{e}^{T} \\ 0 & \tilde{e}^{T} \end{pmatrix} \ , \tag{4.1.12}$$
$$T_{\mathcal{C}} = \tilde{\mathcal{E}}^{-1}\big|_{\zeta}\ \tilde{\mathcal{E}}\big|_{\vartheta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^{-T} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \varpi_{\beta} \\ 0 & \mathbb{1} \end{pmatrix} \text{ with } \Delta = \tilde{e}^{-1}\big|_{\zeta}\ \tilde{e}\big|_{\vartheta}\ ,\ \varpi_{\beta} = \beta\big|_{\vartheta} - \Delta^{-1}\ \beta\big|_{\zeta}\ \Delta^{-T}\ .$$

It is worth noting that $T_{\mathcal{C}}$ has the same form as $\tilde{\mathcal{E}}$, in particular with ϖ_{β} being antisymmetric. We can now compare these transition functions with the symmetries of the theory.

We consider a field configuration given by $\tilde{g}$ and β on a set of patches, together with the transition functions relating them on the overlaps. For instance, going from ζ to ϑ, Δ and ϖ_{β} can be read from (4.1.12). For this configuration to be a geometric one in the sense of $\tilde{\mathcal{L}}_{\beta}$, these transition functions have at least to be realized as symmetries of β-supergravity. This would hold in either of the following two cases:

- The first possibility is that the transition functions are realized by the gauge symmetries discussed above. For example, if an s exists such that

$$\begin{aligned} \Delta\ &= \mathbb{1} + (\tilde{g}^{-1}+\beta)\big|_{\zeta}\ s\ , \\ \varpi_{\beta} &= -\Delta^{-1}(\tilde{g}^{-1}+\beta)\big|_{\zeta}\ s\ (\tilde{g}^{-1}+\beta)^{T}\big|_{\zeta}\ \Delta^{-T} \\ &= -\left((\tilde{g}^{-1}+\beta)^{-1}\big|_{\zeta}\ +s\right)^{-1} s\left((\tilde{g}^{-1}+\beta)^{-1}\big|_{\zeta}\ +s\right)^{-T}\ , \end{aligned} \tag{4.1.13}$$

  then the transition function is completely realized by the β gauge transformation (4.1.5). Diffeomorphisms could as well be considered to realize parts of the transition functions. There is actually an interesting combination of a β gauge transformation and a diffeomorphism. Suppose that on the overlap of two patches, given a matrix s and the local expressions of $\tilde{g}$ and β, one finds a diffeomorphism such that

$$\frac{\partial x'}{\partial x} = \left(\mathbb{1} + (\tilde{g}^{-1}+\beta)\big|_{\zeta}\ s\right)(x)\ . \tag{4.1.14}$$

  Then, transforming the fields under the β gauge transformation (4.1.5) and further under the inverse of the diffeomorphism (4.1.14) would give on that overlap of two patches the effective transformation

$$\begin{cases} \tilde{g} \to \tilde{g} \\ \beta \to \beta - (\tilde{g}^{-1}+\beta)s(\tilde{g}^{-1}+\beta)^{T} \end{cases}\ . \tag{4.1.15}$$

  For the concrete field configuration, the transformation of $\tilde{g}$ under the β gauge transformation is compensated by a diffeomorphism, while β is only shifted.[4] This effective

[4]This effective transformation could be related to the β-diffeomorphism of [80], that acts in a similar fashion. The study of the analogue to the condition (4.1.14) would be interesting.

transformation has the advantage of avoiding an undesired transformation of the metric discussed below (4.1.5). If the transition functions are realized in that manner, not only the field configuration is geometric in the sense of $\tilde{\mathcal{L}}_\beta$, but we have a standard differential geometry described by the metric $\tilde{g}$. The differential conditions (4.1.14) could be an important constraint for any field configuration that should be used in a compactification for instance.

The constraint (4.1.14) may also be a condition to construct a generalized cotangent bundle E_{T^*} over a standard manifold with metric $\tilde{g}$. Restricting transition functions to be part of the set of gauge transformations already allows a priori to construct the bundles corresponding to the theory. For the generalized vielbein $\mathcal{E}$ with b-field, the elements of the transition matrix can be restricted to give only diffeomorphisms and gauge transformations (4.1.4), i.e. the symmetries of the theory, and the generalized tangent bundle E_T then provides a geometric picture of it. Similarly here for $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{L}}_\beta$, we have just discussed how the transition functions of E_{T^*} could be restricted to the symmetries of the theory, for instance through (4.1.13). The difference however with the b-field case is the transformation of the metric. To define a generalized cotangent bundle E_{T^*} over a manifold with metric $\tilde{g}$, the restriction discussed around (4.1.14) might be necessary.

Even if a field configuration $\tilde{g}$ and β is patched as discussed above through the gauge symmetries, so well described by the Lagrangian $\tilde{\mathcal{L}}_\beta$, there is a drawback to such a situation, pointed out in [81]. It is then easy to translate this whole set-up back into the standard g and b. The transition functions, initially realized by the symmetry (4.1.5) and diffeomorphisms, then translate into the symmetry (4.1.4) and diffeomorphisms, i.e. the gauge symmetries of the standard $\mathcal{L}_{\rm NSNS}$. This implies that the field configuration is also geometric in standard NSNS terms. Such a situation is not what was aimed at while introducing the field redefinition. Rather, an interesting case would be a field configuration that is non-geometric in one set of fields becoming geometric in the other set, as in the toroidal example. Therefore, a situation where transition functions are realized by the two gauge symmetries of $\tilde{\mathcal{L}}_\beta$ could occur, but would not be of physical interest.

The toroidal example is not realized this way, as it is non-geometric for standard NSNS fields. More precisely, cutting the base circle into two patches, one can study the transition functions of g and b given in table 4.1 on the overlaps. One sees that no diffeomorphism on the metric can reproduce the change in the function f_0. A reason for this is that f_0 is not periodic in z. It follows that even if the new fields $\tilde{g}$ and β of (4.1.2) are simpler, their transition functions are not realized by diffeomorphisms and β gauge transformation. Interestingly, the generalized metric $\mathcal{H}$ is generically unchanged by our field redefinition (see (3.1.2)), so the transition matrix $T_{\mathcal{C}}$ patching $\mathcal{H}$ as in (4.1.11) is the same for both choices of generalized vielbein and fields. It is given here by (4.1.1). This $T_{\mathcal{C}}$ has the same form as the ones (4.1.12) admissible with $\tilde{\mathcal{E}}$, since it simply shifts β by a constant. As argued above, this constant shift cannot be realized by the gauge symmetries of the theory. This brings us to the second option.

- A second possibility is the presence of an additional symmetry, through which the transition functions are realized. The symmetries of $\mathcal{L}_{\rm NSNS}$, so of $\tilde{\mathcal{L}}_\beta$, are known to be only the gauge symmetries. Another symmetry would then appear only if we specify to a subcase, as a symmetry enhancement. A good example is the case studied in [84], where an additional constraint on any field $\beta^{mn}\partial_n\cdot = 0$ was imposed. This restricts the set of field configurations that can be described and further reduces the Lagrangian $\tilde{\mathcal{L}}_\beta$ to a simpler expression where β enters only through $\partial\beta$. A new symmetry of the theory is then present in the form of constant shifts of β. The toroidal example fits well with that subcase, as it satisfies automatically the constraint. This allows to use the restricted $\tilde{\mathcal{L}}_\beta$ to describe it. Moreover, we just explained that its transition functions are given by a constant shift of β, which are then a symmetry of the theory. This example can be considered as geometric, in the sense of $\tilde{\mathcal{L}}_\beta$. It would be interesting to generalize the situation of [84] via a more general constraint on the fields, e.g. in the form of $\mathcal{T}^a = 0$ or Bianchi identities on the fluxes [27, 132, 133]. The consequences on the construction of a generalized cotangent bundle are also interesting.

4.1.3 A new symmetry of β-supergravity

We are now going to analyze in detail how a new symmetry of β-supergravity might arise and in which sense it helps to realize the transition functions. We start by considering a field configuration in target space as given by a set of fields defined locally on several patches and glue these from one to the other by some transformations. In order for this field configuration to be described by a single theory, as it should be to provide a good description of the physics, or in other words, in order to use only one Lagrangian over the whole space, the gluing transformations should be symmetries of that theory [81]. It is therefore important to first identify these symmetries, as we now turn to. In section 4.2, we will then look at what type of background the symmetries lead to when used as gluing transformations.

We present here a new symmetry of β-supergravity, present under some conditions, that we will later relate to the β-transforms of T-duality. The Lagrangian $\tilde{\mathcal{L}}_\beta$ (3.1.9) only contains β through either $\partial_m\beta^{pq}$ or $\beta^{pr}\partial_r\cdot$, where the dot stands for any field or derivatives thereof. Therefore, the following holds

$$\boxed{\left.\begin{array}{r} \beta^{pq} \longrightarrow \beta^{pq} + \varpi^{pq} \\ \text{with } \forall\ m,p,q,\ \ \varpi^{pr}\partial_r\cdot = 0\ ,\ \ \partial_m\varpi^{pq} = 0 \end{array}\right| \ \text{is a symmetry of } \tilde{\mathcal{L}}_\beta\ . \qquad (4.1.16)}$$

In others words, a constant shift of β by an antisymmetric part ϖ^{pq} satisfying $\varpi^{pr}\partial_r\cdot = 0$ leaves $\tilde{\mathcal{L}}_\beta$ invariant. Unfortunately, it does not seem possible to relax the two requirements on ϖ in (4.1.16) in such a way that ϖ is not required to be constant.[5] The relation we will establish to T-duality suggests that there is no such generalization. Hence, it is now important to understand the two conditions on ϖ in (4.1.16), i.e. how this symmetry can be concretely realized. To that end, let us consider the following equivalence, given a field configuration and an integer $N > 1$

$\exists$ N isometries generated by N independent constant Killing vectors $V_\iota,\ \iota \in \{1 \dots N\}$.	$\Leftrightarrow$	Any constant ϖ^{pq} being only non-zero along a specific $N \times N$ (diagonal) block satisfies $\varpi^{pr}\partial_r \cdot = 0$.	(4.1.19)

A rigorous proof of this equivalence is provided in the appendix of [103]. As shown there, the left-hand side of (4.1.19) is found to be equivalent to the independence of the fields on N coordinates, as is expected for standard Killing vectors. In addition, the right-hand side of (4.1.19) gives conditions on the ϖ that are precisely those needed to realize the symmetry (4.1.16), up to the restriction of having a non-zero block. So this equivalence can be translated in particular into the implication[6]

The fields are independent of N coordinates .	$\Rightarrow$	The shift $\beta^{pq} \to \beta^{pq} + \varpi^{pq}$, with non-zero constant ϖ^{pq} only along the $N \times N$ block, is a symmetry of $\tilde{\mathcal{L}}_\beta$.	(4.1.20)

The symmetry is thus realized by constant shifts along the isometry directions provided the fields are independent of N coordinates. The new symmetry (4.1.16) is therefore tied to having isometries. It is not a symmetry of general β-supergravity, but requires to focus on the subcases in which backgrounds provide additional isometries. In this sense, it is

[5] It is tempting to consider the conditions

$$\forall\ m,p,q,\quad \varpi^{pr}\partial_r\tilde{g}^{mq} + \tilde{g}^{pr}\partial_r\varpi^{mq} = 0\ , \tag{4.1.17}$$

$$\varpi^{pr}\partial_r\beta^{mq} + \beta^{pr}\partial_r\varpi^{mq} = 0\ . \tag{4.1.18}$$

(4.1.17) implies the invariance of $\check{\Gamma}_p^{mn}$ under the shift, and so of $\mathcal{T}^n = \check{\Gamma}_p^{pn}$. In addition, (4.1.18) makes the linear terms in ϖ in the variation of the R-flux vanish. One could then hope for a more general symmetry. However, using the (anti)symmetry of m, q in (4.1.17), one obtains that this condition and (4.1.18) are actually equivalent to the two of (4.1.16), at least for $\tilde{g}$ and β instead of the dot.

[6] The reverse can only be formulated with the $\varpi^{pr}\partial_r\cdot = 0$ condition, because it is not clearly the same as the constant shift being a symmetry.

reminiscent of T-duality for string theory. The actual relation will be the topic in the following.

4.1.4 Elements of the T-duality group

We now turn to T-duality. When the target-space fields are independent of N coordinates in a d-dimensional space-time, the bosonic string σ-model experiences an additional symmetry, namely T-duality[7]. This symmetry translates in the NSNS sector into the action of a constant $O(N,N)$ group on the fields. Therefore, if the latter are independent of N coordinates, the target-space theory, namely $\mathcal{L}_{\text{NSNS}}$ inherits this symmetry in form of an invariance under the $O(N,N)$ transformation up to a total derivative.[8] Since one usually considers a full SUGRA, for instance type IIA/B, that also contains a RR sector which is not preserved by T-duality, such an invariance of the NSNS sector is not often mentioned. Further details are given in the appendix of [103].

Let us now present the action of the T-duality group $O(N,N)$. Its action on the fields is better characterized by considering the generalized metric $\mathcal{H}$ as a $2d\times 2d$ matrix depending on the metric g and b-field, and the quantity d related to the dilaton. In addition, one should consider elements $O \in O(d,d)$ in their fundamental representation preserving the $2d \times 2d$ matrix

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \ O^T \eta O = \eta \ . \tag{4.1.21}$$

The T-duality transformations then consist in taking a trivial embedding of $O(N,N)$ into $O(d,d)$, and acting with the corresponding elements on $\mathcal{H}$. For completeness, the transformed dilaton is defined such that d remains invariant

$$\begin{pmatrix} a & c \\ f & h \end{pmatrix} \in O(N,N) \ , \ O = \left(\begin{array}{cc|cc} a & & c & \\ & \mathbb{1}_{d-N} & & 0_{d-N} \\ \hline f & & h & \\ & 0_{d-N} & & \mathbb{1}_{d-N} \end{array}\right) \in O(d,d) \ , \tag{4.1.22}$$

$$\mathcal{H}' = O^T \mathcal{H} O \ , \ e^{-2d} = e^{-2\phi}\sqrt{|g|} = e^{-2\phi'}\sqrt{|g'|} \ . \tag{4.1.23}$$

Only the components along the N directions are then transformed. A particular example is the Buscher transformation [6, 7] along all N directions given by $a = h = 0_N$, $c = f = \mathbb{1}_N$.

Next, we list the generating elements [5, 134] of the group $O(N,N)$ which reduces in the context of string theory to $O(N,N,\mathbb{Z})$:

- the $GL(N,\mathbb{Z})$ subgroup: for $a \in GL(N,\mathbb{Z})$, one considers[9]

$$O_a = \begin{pmatrix} a & 0_N \\ 0_N & a^{-T} \end{pmatrix} \in O(N,N,\mathbb{Z}) \ . \tag{4.1.24}$$

[7] Reviews on T-duality can be found in [5, 134, 135] and references therein.

[8] Its regime of validity as an effective theory might however be changed accordingly to the transformation.

[9] This subgroup can be further decomposed into generators, see e.g. [5] and references therein.

- the b-transforms: for ϖ an $N \times N$ antisymmetric integer matrix, one considers

$$O_{\varpi} = \begin{pmatrix} \mathbb{1}_N & 0_N \\ \varpi & \mathbb{1}_N \end{pmatrix} \in O(N,N,\mathbb{Z}) \ . \tag{4.1.25}$$

- the Buscher transformations [6, 7]: for c_i the $N \times N$ matrix with only one non-zero entry, equal to 1 and placed in the (i,i) position, one considers

$$O_{t_i} = \begin{pmatrix} \mathbb{1}_N - c_i & c_i \\ c_i & \mathbb{1}_N - c_i \end{pmatrix} \in O(N,N,\mathbb{Z}) \ . \tag{4.1.26}$$

There is yet another set of elements which can be generated from the above elements

- the β-transforms: for an integer $N \times N$ antisymmetric matrix ϖ, one considers

$$\begin{pmatrix} \mathbb{1}_N & \varpi \\ 0_N & \mathbb{1}_N \end{pmatrix} = \begin{pmatrix} 0_N & \mathbb{1}_N \\ \mathbb{1}_N & 0_N \end{pmatrix} \begin{pmatrix} \mathbb{1}_N & 0_N \\ \varpi & \mathbb{1}_N \end{pmatrix} \begin{pmatrix} 0_N & \mathbb{1}_N \\ \mathbb{1}_N & 0_N \end{pmatrix} = O_t^T O_{\varpi} O_t \ , \tag{4.1.27}$$

  where we denote by O_t the Buscher transformation along all N directions

$$O_t = O_{t_1} \dots O_{t_N} = \begin{pmatrix} 0_N & \mathbb{1}_N \\ \mathbb{1}_N & 0_N \end{pmatrix} \ . \tag{4.1.28}$$

At the level of SUGRA, the stringy T-duality group just discussed is extended to $O(N,N,\mathbb{R})$. We then consider the natural extensions of the above elements towards the $GL(N,\mathbb{R})$ subgroup, the real b- and β-transforms, where a and ϖ are now real. Those three sets form three independent subgroups of $SO(N,N,\mathbb{R})$ with determinant equal to 1. Therefore, they do not generate the whole $O(N,N,\mathbb{R})$, in particular no combination can reproduce an O_t with $\det O_t = -1$. In the following we will mainly focus on these three subgroups of $SO(N,N,\mathbb{R})$, but we can keep in mind the possibility of further T-duality transformations.

We now look at the action of these three subgroups on the NSNS fields. We explained above that when fields are independent of N coordinates, the $O(N,N)$ T-duality group is a symmetry of the Lagrangians. So each of these three transformations should then correspond to a symmetry. The action of the three subgroups of interest can be read from (4.1.22) and (4.1.23), but also from the corresponding action on a generalized vielbein $\mathring{\mathcal{E}}$ up to Lorentz transformations

$$\mathring{\mathcal{E}}' = \mathring{\mathcal{E}} O \ . \tag{4.1.29}$$

By considering $\mathcal{E}$ and $\tilde{\mathcal{E}}$ of (3.1.1), b-transforms, respectively β-transforms, simply consist in shifting the b-field, respectively β

$$b\text{-transform:} \quad e' = e, \ b' = b + \begin{pmatrix} \varpi & \\ & 0_{d-N} \end{pmatrix} \ , \tag{4.1.30}$$

$$\beta\text{-transform:} \quad \tilde{e}' = \tilde{e}, \ \beta' = \beta + \begin{pmatrix} \varpi & \\ & 0_{d-N} \end{pmatrix} \ , \tag{4.1.31}$$

along the N directions. In addition, the $GL(N,\mathbb{R})$ action on either set of fields yields

$$\begin{aligned} O_a : e' = e \begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix} , \ b' = \begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix}^T b \begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix} , \\ \tilde{e}' = \tilde{e} \begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix} , \ \beta' = \begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix}^{-1} \beta \begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix}^{-T} . \end{aligned} \tag{4.1.32}$$

Having observed the transformations, we now identify the corresponding symmetries. The b-transforms (4.1.30) are an obvious symmetry of $\mathcal{L}_{\text{NSNS}}$. Constant shifts of b certainly leave the Lagrangian invariant, as the latter only depends on ∂b and moreover such a shift symmetry is a subcase of the known b-field gauge symmetry, since a constant shift can be brought to the form of a $\mathrm{d}\Lambda$. The $GL(N)$ subgroup is also clearly a symmetry. The action (4.1.32) on the fields is a particular example of diffeomorphisms in matrix notation, that are known to be a gauge symmetry of both $\mathcal{L}_{\text{NSNS}}$ and $\tilde{\mathcal{L}}_\beta$. Being more precise, a diffeomorphism generically transforms the b-field as $b_{mn}(x') = b_{pq}(x) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}$. Then, for an O_a transformation to be a diffeomorphism we have to satisfy the following set of differential equations

$$\begin{pmatrix} a & \\ & \mathbb{1}_{d-N} \end{pmatrix}^p{}_m = \frac{\partial x^p}{\partial x'^m} \ . \tag{4.1.33}$$

This can easily be achieved since a is constant. For that reason, $b_{mn}(x') = b_{pq}(x) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}$ can be realized by the action of O_a.

Eventually, the β-transforms (4.1.31) should also be a symmetry when fields are independent of N coordinates. This may look surprising from the $\mathcal{L}_{\text{NSNS}}$ point of view, as it does not seem to match a known symmetry. In particular, translated on the standard SUGRA fields, this transformation acts both on b and g.[10] However, in view of (4.1.20), β-transforms clearly correspond to the new symmetry of $\tilde{\mathcal{L}}_\beta$ discussed in section 4.1.3. It follows then that it is a symmetry of $\mathcal{L}_{\text{NSNS}}$ up to a total derivative. In conclusion, the new symmetry of section 4.1.3 can be viewed as the β-transforms being a specific subgroup of the T-duality group.

Let us finally recall the main idea of this section on the symmetries of $\mathcal{L}_{\text{NSNS}}$ and $\tilde{\mathcal{L}}_\beta$. Considering a restriction, the symmetries of the theory get enhanced and the new symmetries can be used to build interesting geometric vacua of the constrained β-supergravity. We considered here the subcase when fields are independent of N coordinates and obtained the subgroup of β-transforms which is a manifest symmetry of $\tilde{\mathcal{L}}_\beta$. We will see in the following that the β-transforms will play a crucial role in the construction of geometric vacua of β-supergravity.

[10] The two other subgroups of the T-duality group have been shown to correspond to subcases of gauge transformations, so one may wonder whether the same could happen for the β-transforms. We remark that such transformation would act on both g and b. This is related to the footnote 5, and it looks unlikely. It may still be doable in the broader set-up of DFT, when considering $\tilde{\partial} \neq 0$.

4.2 To be or not to be geometric

We discussed above the different symmetries of $\mathcal{L}_{\text{NSNS}}$ and $\tilde{\mathcal{L}}_\beta$ in general, but also when restricting to the presence of some isometries. We now study the effect of using these various symmetries to glue fields of these theories between patches. After proposing a precise definition of geometry and non-geometry, we discuss whether using a given symmetry leads to a geometric or non-geometric field configuration.

4.2.1 Symmetries and (non)-geometry

The original idea of non-geometry [127, 29, 128] can be summerized as looking for field configurations that need transition functions, used to glue them from one patch to the other, that are not among diffeomorphisms and gauge transformations making up the standard symmetries of a geometric configuration. Yet, the transformations have to be symmetries of string theory. We recall that for consistency of a certain theory the transformations must correspond to its symmetries [81], since otherwise one ends up with a distinct theory on every patch.

Keeping this idea in mind, we extend here the notion of a geometric or non-geometric field configuration to our target space theories. In particular, the gluing of fields should then be done by symmetries of the latter. Since working with a certain theory means specifying the available symmetries, the distinction between a geometry and a non-geometry is theory dependent and we thus reformulate and generalize the original idea stated above into the following proposed definitions

Definitions of geometric and non-geometric field configurations

- A field configuration is geometric if the fields are globally defined on the manifold considered so do not need to be glued, or if the transformations used to glue them from one patch to the other are symmetries of the theory, and the metric, dilaton and fluxes glue at most with diffeomorphisms.

- A field configuration is non-geometric if the transformations used to glue the fields from one patch to the other are symmetries of the theory, and if the metric, dilaton or fluxes glue with something else than diffeomorphisms.

In this way the notion is certainly theory dependent. In particular, since the metric describing the manifold may change from one theory to the other in the case of $\mathcal{L}_{\text{NSNS}}$ and $\tilde{\mathcal{L}}_\beta$, the notion of (non-)geometry changes accordingly. This is precisely the interest in changing the theory to describe a background. For one theory a configuration might be non-geometric, but in another theory the geometry gets restored. We exactly observed this behavior for the toroidal example and also the Q-brane, as discussed below (5.2.39), falls into this class.

These definitions also involve the notion of fluxes. In $\mathcal{L}_{\text{NSNS}}$, respectively $\tilde{\mathcal{L}}_\beta$, the H-flux and respectively the R-flux are tensors and hence their transformation under diffeomorphisms is clear. But one also faces the structure constant or geometric flux, and the Q-flux, which are not tensors. Their transformation under diffeomorphisms can still be considered, as they correspond to building blocks of the spin connections ω and ω_Q. For a geometric configuration, it is important that the flux remains invariant under the other symmetries. The H-flux is invariant under the b-field gauge transformations, and the Q- and R-flux are invariant under the β-transform discussed above. The latter is obvious for the R-flux given its definition and for the Q-flux when rewritten as

$$Q_c{}^{ab} = \tilde{e}^q{}_c \tilde{e}^a{}_m \tilde{e}^b{}_n \left(\partial_q \beta^{mn} + 2\tilde{e}^d{}_q \beta^{p[m} \partial_p \tilde{e}^{n]}{}_d \right) \ . \tag{4.2.1}$$

We summarize the previous discussion of symmetries and how they give rise to geometric (G) or non-geometric (NG) field configurations in different theories in the table 4.2, where we implicitly assume that the fields are independent of N coordinates.

Symmetry used as gluing transformation	$\mathcal{L}_{\text{NSNS}}$	$\tilde{\mathcal{L}}_\beta$	Example
diffeomorphism	G	G	twisted torus
b-field gauge transfo.	G	NG (or $\times$)	T^3 + constant H
β-transform	NG (or $\times$)	G	toroidal example
b-field gauge transfo. and β-transform	NG (or $\times$)	NG (or $\times$)	
Buscher transformation	NG	NG	radial inversion
more combinations	?	?	

Table 4.2: Geometric (G) or non-geometric (NG) field configuration, according to the symmetry used to glue its fields, and to the theory

We note that diffeomorphisms can certainly be added to all configurations listed in the table as these do not alter the classification as geometric or non-geometric for the two considered theories and therefore are implicit here. Furthermore, we denote by a $\times$ in table 4.2 a tiny possibility for a field configuration to be geometric, as discussed in section 4.1.2. The b-field gauge transformation, translated after field redefinition into a β gauge transformation, also acts on the new metric $\tilde{g}$ as expected from the non-linearity of the field redefinition. Depending on the transformation and the background, the transformation of $\tilde{g}$ could be compensated by a diffeomorphism. In that case, the field configuration would be geometric, provided the fluxes also transform properly. Such a situation is rather unlikely, but we cannot fully exclude it at the moment. A similar reasoning holds for the β-transform, that would act not only on the b-field but also on the metric g, as can be seen from the field redefinition. Moreover, we mentioned in section 4.1.4 the possibility of other elements of the T-duality group $O(N,N)$ that we have not considered. These could be built for instance by further combinations of the elements already studied here. The effect of such a generic element is not easy to guess, so we cannot conclude in full generality.

This explains the meaning of the last line of table 4.2.

To conclude this study, we refer to the reasoning detailed in the Introduction, and one can see that the results of table 4.2 are in good agreement with it. In particular, it is worth considering a subcase that gives rise to more symmetries, and allows to go beyond the situations of the first two lines of table 4.2. Considering the independence on N coordinates gives the new symmetry of β-transforms. The latter allows, as indicated in the third line, to get field configurations that are geometric for $\tilde{\mathcal{L}}_\beta$ while being non-geometric for $\mathcal{L}_{\rm NSNS}$. In that case, it is worth changing theory and we established a well-defined class of backgrounds for which β-supergravity provides a better description than standard supergravity.

4.3 Geometric backgrounds of β-supergravity and T-duality orbits

As explained in the Introduction, backgrounds that are geometric for $\tilde{\mathcal{L}}_\beta$ and non-geometric for $\mathcal{L}_{\rm NSNS}$ are the most interesting ones for β-supergravity. In the previous section we just realized such backgrounds by restricting fields to be independent of N coordinates and using β-transforms as gluing transformations. Such a situation is also the topic of this section. The additional isometries allow to perform T-duality transformations and we analyze the T-duality orbit of these backgrounds in general and then in a compact case.

4.3.1 Always on a geometric orbit?

We consider a geometric background for $\tilde{\mathcal{L}}_\beta$ in terms of the fields $\tilde{g},\ \beta,\ \tilde{\phi}$. Through the field redefinition, it is expressed with $g,\ b,\ \phi$ and is then non-geometric for $\mathcal{L}_{\rm NSNS}$.[11] As it is independent of N coordinates, one can further T-dualize along these directions. Using the Buscher rules along all N directions the fields $g,\ b,\ \phi$ are transformed to the T-dual fields $g',\ b',\ \phi'$, as depicted in table 4.3.

Let us now show that $g',\ b',\ \phi'$ provide a geometric background of $\mathcal{L}_{\rm NSNS}$. The fields $\tilde{g},\ \beta,\ \tilde{\phi}$ glue with a β-transform and possibly a diffeomorphism A. These transformations can be decomposed into blocks along the N directions. Here, we introduce a for the $N\times N$ block. We denote by z^p the $d-N$ coordinates on which the fields depend and by y^r the

[11] Despite its similarity with a Buscher T-duality along all d directions, let us stress that the field redefinition (3.1.3) is not such a transformation. The indices of $\tilde{g}^{-1}+\beta$ are up, while those of a T-dual metric and b-field are down. In particular, T-duality relates a b-field to a b-field, there is no notion of bivector appearing. Another way to see this is by considering the subcase $b=\beta=0$, giving $g=\tilde{g}$, while a T-duality along all directions would invert the metric. This difference is crucial for the large volume limit, as discussion in 4.1.1. Additionally, in SUGRA, a T-duality along all directions would require the fields to be constant, while the field redefinition can be performed without restriction. In DFT, such a T-duality would replace the coordinates x^m by $\tilde{x}_m$, but the field redefinition does not change the coordinate dependence.

Theories	$\tilde{\mathcal{L}}_\beta$		$\mathcal{L}_{\text{NSNS}}$
T-duality frames	$\tilde{g},\ \beta,\ \tilde{\phi}$ (G)	$\xleftrightarrow{\text{field redef.}}$	$g,\ b,\ \phi$ (NG)
			$\updownarrow$ T-d. // N dir.
			$g',\ b',\ \phi'$ (G)

Table 4.3: Different descriptions of a geometric background of $\tilde{\mathcal{L}}_\beta$

N coordinates on which they don't. Then, a generic diffeomorphism $A^m{}_n = \frac{\partial x^m}{\partial x'^n}$ becomes here

$$A = \begin{pmatrix} a & j \\ i & k \end{pmatrix} \ , \qquad \begin{pmatrix} \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} a & j \\ i & k \end{pmatrix} \begin{pmatrix} \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix} \ . \tag{4.3.1}$$

The independence of the fields on N coordinates y^r leads to a constraint on the possible diffeomorphisms. The z and z' should mix at most among themselves, i.e. should not involve any y or y' dependence and therefore we have to demand $\frac{\partial z^p}{\partial y'^r} = 0$, i.e. $i^p{}_r = 0$. As a cross-check, one should have $\frac{\partial}{\partial y'^r} k^p{}_q = \frac{\partial}{\partial y'^r}\frac{\partial z^p}{\partial z'^q} = 0$. As A is a diffeomorphism, this equals $\frac{\partial^2 z^p}{\partial z'^q \partial y'^r} = \frac{\partial}{\partial z'^q} i^p{}_r$, that indeed vanishes for $i=0$. So A is restricted as follows[12]

$$A = \begin{pmatrix} a & j \\ 0 & k \end{pmatrix} \ , \ A^{-T} = \begin{pmatrix} a^{-T} & 0 \\ -k^{-T} j^T a^{-T} & k^{-T} \end{pmatrix} \ . \tag{4.3.2}$$

We now consider the gluing of the fields $\tilde{g},\ \beta,\ \tilde{\phi}$. This is best observed using the generalized metric

$$\mathcal{H}(z_2) = O^T \mathcal{H}(z_1) O \ , \tag{4.3.3}$$

$$O = \left(\begin{array}{cc|cc} \mathbb{1}_N & & \varpi & \\ & \mathbb{1}_{d-N} & & 0_{d-N} \\ \hline 0_N & & \mathbb{1}_N & \\ & 0_{d-N} & & \mathbb{1}_{d-N} \end{array}\right) \left(\begin{array}{cc|cc} a & j & 0_N & \\ & k & & 0_{d-N} \\ \hline 0_N & & a^{-T} & \\ & 0_{d-N} & -k^{-T} j^T a^{-T} & k^{-T} \end{array}\right) \tag{4.3.4}$$

with $\varpi^T = -\varpi$ giving the β-transform. As already mentioned, the field redefinition does not change $\mathcal{H}$, so the gluing of the fields $g,\ b,\ \phi$ is expressed in the same manner. Let us

[12] The restriction on the dependence on coordinates enforces $i=0$, and this will allow us to obtain a geometric T-dual. This is a crucial point, as $i \neq 0$ would have lead to a non-trivial β-transform block after the T-duality, which would have implied a non-geometric T-dual. Another take on this is to consider the Maurer-Cartan one-forms that are globally defined, $\tilde{e}^a(x') = \tilde{e}^a(x)$. This provides the diffeomorphism matrix, as $\mathrm{d}x^n = \tilde{e}^n{}_a(x)\tilde{e}^a{}_m(x')\mathrm{d}x'^m$. Considering a multiple step fibration, such as the nilmanifold n 3.14, one may think that it is possible to find a vielbein leading to $i \neq 0$. But this involves a dependence on coordinates that are not well-defined, namely those corresponding to fibered directions. In addition,these make the fields depend on the wrong coordinates after gluing. Considering a correct coordinate dependence restores $i=0$.

now perform the Buscher T-duality along the N directions. Following (4.1.22) and (4.1.23), we use again $\mathcal{H}$ to get the T-dual $\mathcal{H}'$ as

$$\mathcal{H}' = T^T \mathcal{H} T \ , \tag{4.3.5}$$

where T is given below (4.1.23). By T-dualizing $\mathcal{H}$ on each patch, i.e. on both sides of (4.3.3), we deduce the gluing of $\mathcal{H}'$

$$\mathcal{H}'(z_2) = (TOT)^T \mathcal{H}'(z_1) TOT \ , \tag{4.3.6}$$

where we used that $T^T = T^{-1} = T$. This gluing is therefore given by

$$TOT = \left(\begin{array}{cc|cc} \mathbb{1}_N & & 0_N & \\ & \mathbb{1}_{d-N} & & 0_{d-N} \\ \hline \varpi & jk^{-1} & \mathbb{1}_N & \\ -(jk^{-1})^T & 0_N & & \mathbb{1}_{d-N} \end{array}\right) \left(\begin{array}{cc|cc} a^{-T} & & 0_N & \\ & k & & 0_{d-N} \\ \hline 0_N & & a & \\ & 0_{d-N} & & k^{-T} \end{array}\right) . \tag{4.3.7}$$

We recognize the combination of a b-shift and a diffeomorphism, where the former is due to the initial β-transform and the off-diagonal piece j of the diffeomorphism. We conclude that the fields g', b', ϕ' form a geometric background for $\mathcal{L}_{\rm NSNS}$.

We have shown that the backgrounds that glue with β-transform and diffeomorphism, i.e. geometric for $\tilde{\mathcal{L}}_\beta$ and non-geometric for $\mathcal{L}_{\rm NSNS}$, are T-dual to geometric ones for $\mathcal{L}_{\rm NSNS}$. Hence, these geometric backgrounds of $\tilde{\mathcal{L}}_\beta$ are in a sense not new, or do not reveal new physics. These backgrounds lie always on a geometric orbit from the point of view of a four dimensional gauged SUGRA theory. The converse claim may still be of interest. Consider a geometric background of a four-dimensional gauged SUGRA. On its T-duality orbit, there are geometric and possibly non-geometric backgrounds. If one geometric point on this orbit can be lifted to a ten-dimensional background that glues as in (4.3.7), then we know that there exists on that orbit a non-geometric one that can be lifted and described by β-supergravity.

It is disappointing that the backgrounds of β-supergravity considered above do not lead to new physics. However, we can at least list some possible circumventions:

- As indicated in table 4.2, there might be other T-duality elements that could be used to glue fields. They may, as for the β-transform, allow geometric backgrounds for $\tilde{\mathcal{L}}_\beta$ and non-geometric for $\mathcal{L}_{\rm NSNS}$. Then, if a study as the above on the T-duals does not give rise to any geometric point, the corresponding backgrounds would be fully new.
- We only studied the NSNS sector and considering backgrounds involving other sectors, such as RR, may alter the above conclusion.
- One may find another restriction than the independence of coordinates, that would as well enhance the symmetries. The new symmetries could then be used again for gluing fields, possibly in the desired way. In particular, if there is no assumption on the coordinate dependence anymore, then the T-duality can a priori not be performed, preventing from the above conclusion.

- $\mathcal{L}_{\mathrm{NSNS}}$ contains a discrete symmetry, namely $\mathbb{Z}_2$ transforming $b \to -b$. This alters the sign on the H-flux and could therefore lead to a non-geometric field configuration, following the definitions of section 4.2.1. This $\mathbb{Z}_2$ translates for $\tilde{\mathcal{L}}_\beta$ into a sign on β only. The effect on the fluxes is a sign on the Q-flux, but not on the R-flux. Then, with a vanishing Q-flux, such a field configuration would be geometric for $\tilde{\mathcal{L}}_\beta$.

- The notion of geometry used above is close to that of standard differential geometry and smooth manifolds. If singularities are present, the conclusions may be altered. Nevertheless, in the case of the Q-brane and $NS5$-brane, the previous reasonings can be applied everywhere away from the singularity, and the latter is treated in the same way for both $\tilde{g},\ \beta,\ \tilde{\phi}$ and $g',\ b',\ \phi'$.

4.3.2 Pure NSNS solutions

Having discussed the general situation for geometric and non-geometric field configurations in β-supergravity we now turn to the question of finding new ten-dimensional solutions, satisfying a standard compactification ansatz. Interestingly, having a theory in ten dimensions expressed in terms of Q- and R-fluxes allows, for the first time, to look directly there for solutions with non-geometric fluxes. Such vacua would be geometric for $\tilde{\mathcal{L}}_\beta$ and take the form of a given compactification ansatz. Interestingly, this ansatz is not too restrictive and the equations of motion indicate the possibility of getting non-trivial solutions. This stands in harsh contrast to the case for $\mathcal{L}_{\mathrm{NSNS}}$ and hence justifies the interest in getting such vacua of $\tilde{\mathcal{L}}_\beta$. Indeed, considering a standard compactification ansatz together with the standard ten-dimensional NSNS Lagrangian $\mathcal{L}_{\mathrm{NSNS}}$ (3.1.5) only leads to trivial solutions. In other words, leaving out Ramond-Ramond or gauge flux contributions, branes or orientifold planes and setting the dilaton to be constant, leads to a solution with a vanishing H-flux, a flat internal manifold and a flat four-dimensional space-time. A way to reach this conclusion is to follow the analogous reasoning [136] to the one made below, where we essentially combine conditions obtained from the Einstein and dilaton equations of motion. The more general framework of [137] gives the same result. The NSNS sector of β-supergravity alone may though turn out to be too restricted to get new non-trivial solutions either. Finally, we layout a distinct argumentation along the T-dual relation of geometric vacua of $\tilde{\mathcal{L}}_\beta$ to geometric ones of $\mathcal{L}_{\mathrm{NSNS}}$ that additionally lends credence to the observation.

Let us now indicate a possibility to get pure NSNS solutions of β-supergravity. Such solutions should satisfy the equations of motion given in (1.4.12), (1.4.13) and (1.4.14). We consider the following compactification ansatz. Ten-dimensional space-time is splitted in an four-dimensional external and a six-dimensional compact internal part. Accordingly, the metric factorizes into two parts $\tilde{g}_{(4)}$ and $\tilde{g}_{(6)}$, where the four-dimensional metric $\tilde{g}_{(4)}$ depends only on the four-dimensional coordinates and we do not take a warp factor into consideration. This structure is certainly reflected in the vielbeins. Furthermore, we pick β to depend only on internal coordinates and to have only components along the internal space, such that the non-geometric fluxes are purely internal as well. We also impose unimodularity conditions, i.e. the vanishing of the traces $f^a{}_{ab} = 0$, $Q_a{}^{ab} = 0$, for both the geometric flux and the Q-flux for reasons of preserving the compactness of the internal manifold. Finally, we choose to have a constant dilaton $\tilde{\phi}$ and additionally set $\mathcal{T}^a = 0$.[13] These two conditions simplify the ten-dimensional equations of motion to

$$\mathcal{R} + \check{\mathcal{R}} - \frac{1}{2}R^2 = 0 \ , \tag{4.3.8}$$

$$\mathcal{R}_{ba} - \eta_{e(a}\eta_{b)g}\check{\mathcal{R}}^{ge} + \frac{1}{4}\eta_{ae}\eta_{bg}\eta_{if}\eta_{cd}R^{igc}R^{dfe} = 0 \ , \tag{4.3.9}$$

$$\begin{aligned} &\eta_{ae}\eta_{bg}\eta^{df}\partial_d Q_f{}^{eg} - \beta^{gc}\partial_c f^e{}_{ab}\eta_{ge} + \eta_{ae}\eta_{bg}\eta_{df}\check{\nabla}^d R^{gfe} \\ &+ f^g{}_{cd}Q_{[a}{}^{dc}\eta_{b]g} + 2\eta_{e[a}f^h{}_{b]d}Q_i{}^{ec}\eta_{ch}\eta^{di} + 2\eta_{e[a}f^h{}_{b]c}Q_h{}^{ec} = 0 \ . \end{aligned} \tag{4.3.10}$$

The compactification ansatz leads to a vanishing four-dimensional Ricci tensor from the Einstein equation and thus we have

$$\mathcal{R}_{(4)} = 0 \ . \tag{4.3.11}$$

For a maximally symmetric four-dimensional space-time, this condition makes it Minkowski. Taking the ten-dimensional trace of the Einstein equation (4.3.9), one obtains

$$\mathcal{R}_{(4)} + \mathcal{R}_{(6)} - \check{\mathcal{R}} + \frac{3}{2}R^2 = 0 \ . \tag{4.3.12}$$

These two conditions, together with the dilaton equation of motion (4.3.8), are solved by the following constraints on the internal quantities

$$-2\mathcal{R}_{(6)} = R^2 = \check{\mathcal{R}} \ . \tag{4.3.13}$$

On the contrary to the situation with standard NSNS fields as in (3.1.5), the internal quantities here can a priori be found non-vanishing! This is essentially due to the presence of three types of fluxes, instead of two for the standard NSNS case. This asymmetry may look surprising when simply counting the degrees of freedom from the fundamental fields, since b and β have the same number. An asymmetry nevertheless appears when looking at the placement of indices of these two fields, with respect to that of the derivative ∂_m. This difference allows at the SUGRA level to define two fluxes from β and only one from b, as clearly seen from their definitions. A related question is that of the independence of Q and R. At least, we see from (A.3.7) that an R-flux can be present without a Q-flux. The other way is obvious. Thanks to the three fluxes and associated quantities in (4.3.13), the system is not over constrained. On the contrary to the standard NSNS case, we conclude that interesting pure NSNS solution could in principle be found in β-supergravity.

[13]The condition $\mathcal{T}^a = \nabla_b\beta^{ab} = 0$ is not too constraining for the fluxes. Indeed, this trace of $\nabla\beta$ does not appear directly in the fluxes. There is also no combination of components of the fluxes that gives $\mathcal{T}^a$. Hence, it appears like an independent quantity, that we then fix to a desired value. This is consistent with its interpretation as a conformal weight. Note also that $\mathcal{T}^a = 0$ is an interesting intermediate condition between no assumption and the simplifying assumption of [84]. The latter implied not only $\mathcal{T}^a = 0$, but also a vanishing R-flux, while we can still have here a non-zero R-flux.

Solving the condition (4.3.13) is nevertheless not simple. As $R^2 \geqslant 0$, a non-trivial solution would have a negatively curved internal manifold, $\mathcal{R}_{(6)} < 0$. All nilmanifolds except the torus, as well as some solvmanifolds, verify this requirement. They additionally satisfy by definition the unimodularity condition on f. So this is an interesting set to look for solutions.[14] A larger set of interesting Lie group based manifolds is described in [140]. The condition (4.3.13) however implies as well that $\check{\mathcal{R}} \geqslant 0$. Imposing in addition the tracelessness condition for the Q-flux gives, from (A.3.25) and (A.3.27),

$$\check{\mathcal{R}} = -\frac{1}{4}\left(\eta^{ad}\eta_{be}\eta_{cg}Q_a{}^{bc}Q_d{}^{eg} + 2\eta_{cd}Q_a{}^{bc}Q_b{}^{ad} + 2R^{acd}f^b{}_{cd}\eta_{ab}\right) . \tag{4.3.14}$$

Getting the above positive is not easy, as the first term is negative. The last term in $R^{acd}f^b{}_{cd}$ could certainly help, so it should better be non-vanishing. This simple analysis already leads to non-trivial constraints on the field configuration. We tried to solve the $\check{\mathcal{R}} > 0$ condition on a few manifolds of [136], namely one or two Heisenberg manifolds, i.e. the twisted torus of section 4.1.1, denoted by the algebra $\mathfrak{g}_{3.1}$, and those associated to $\mathfrak{g}_{3.4}^{-1}$, $\mathfrak{g}_{3.5}^0$, as well as $\mathfrak{g}_{5.17}^{0,0,r}$ also called s 2.5.[15] With reasonable ansätze for the fluxes, the sum of the last two terms in (4.3.14) was either zero or negative, or when it was not the case, $\check{\mathcal{R}}$ still failed to be positive. Finding pure NSNS solutions with this compactification ansatz therefore looks difficult, even if a priori possible.

For completeness we mention the equation of motion of β (4.3.10), which would bring additional constraints. It is in principle, convenient to have (4.3.10) with explicit non-geometric fluxes present where β only appears implicit. We did not take it into consideration here, since all attempts to non-trivially satisfy the equations (4.3.8) and (4.3.9) failed so far.

Now we turn to an argumentation that points in the same directions as the results that we found for observing the equations of motion with regard to possible non-trivial NSNS vacua of β-supergravity. In the above, we worked out a well-defined class of backgrounds that are geometric for $\tilde{\mathcal{L}}_\beta$, and could thus serve as candidates for the vacua we are now interested in. However, we have also shown that these backgrounds are T-dual to geometric ones of $\mathcal{L}_{\mathrm{NSNS}}$, as described by the chain of relations in table 4.3. Let us now study how the compactification ansatz evolves through that chain. In this way we will constrain further the possibility of getting geometric vacua of $\tilde{\mathcal{L}}_\beta$ that are suited for compactification.

We recall that due to $\tilde{\mathcal{L}}_\beta$ and $\mathcal{L}_{\mathrm{NSNS}}$ differing only by a total derivative, and to T-duality being a symmetry of the equations of motion, a vacua of $\tilde{\mathcal{L}}_\beta$ given by $\tilde{g}$, β, $\tilde{\phi}$ leads to g, b, ϕ and g', b', ϕ' of table 4.3 being as well vacua of $\mathcal{L}_{\mathrm{NSNS}}$. Let us now look at the consequence of the compactification ansatz described above. The field redefinition and the T-duality certainly respect the splitting and the coordinate dependence and thus the fields g' and b' show the same structure. Finally, we had $\tilde{\phi}$ = constant in our ansatz and are interested in whether this is the case for ϕ'. For this, we recall that the dilaton goes

[14] A review on solvmanifolds can be found in [136], and more examples are present in [138, 139].

[15] The last two manifolds can be negatively curved for appropriately chosen radii.

through the following chain of equalities

$$e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|} = e^{-2\phi}\sqrt{|g|} = e^{-2\phi'}\sqrt{|g'|} \ . \tag{4.3.15}$$

Having ϕ' constant would put a severe constraint on the possibility of getting $\tilde{g},\ \beta,\ \tilde{\phi}$ as the type of vacua we are interested in. Indeed, one can show that a constant ϕ' only leads to a trivial solution of $\mathcal{L}_{\rm NSNS}$, namely a flat space-time and manifold with vanishing H-flux. The corresponding background in terms of $\tilde{g},\ \beta,\ \tilde{\phi}$ is then most likely trivial as well. For instance, constant $g',\ b',\ \phi'$ or even a pure gauge b', do not allow for much freedom to get interesting $\tilde{g},\ \beta,\ \tilde{\phi}$. Therefore, ϕ' should better be non-constant. However, we are going to see that this is not compatible with $\tilde{\phi}$ being constant. We would require the ratio

$$\frac{\sqrt{|\tilde{g}|}}{\sqrt{|g'|}} \tag{4.3.16}$$

to be non-constant. Note that $\tilde{g}$ and g' being part of geometric backgrounds, they are globally well-defined. For $\tilde{\phi}$ being constant, we deduce that ϕ' is also globally well-defined.[16] It looks rather a difficult to get (4.3.16) non-constant.[17]

The ratio (4.3.16) can in principle be computed in terms of one or the other set of fields, since we know how the fields are related in table 4.3. A difficulty however comes from the fact that the field redefinition involves the whole fields while the T-duality only acts on certain blocks. That makes a generic computation not possible, as the inverse and the determinant of a matrix divided in blocks cannot generically be expressed in terms of those blocks. Hence, we consider the following subcase

$$\tilde{g} = \begin{pmatrix} \tilde{g}_N & \\ & \tilde{g}_{d-N} \end{pmatrix} \ , \ \beta = \begin{pmatrix} \beta_N & \\ & \beta_{d-N} \end{pmatrix} \ , \tag{4.3.17}$$

where these fields do not have off-diagonal components. One then computes $g,\ b$ and $g',\ b'$. Using some freedom of sign in the field redefinition [84], g' can be simplified to

$$g' = \begin{pmatrix} \tilde{g}_N^{-1} & \\ & (\tilde{g}_{d-N}^{-1} + \beta_{d-N})^{-1}\tilde{g}_{d-N}^{-1}(\tilde{g}_{d-N}^{-1} - \beta_{d-N})^{-1} \end{pmatrix} \ . \tag{4.3.18}$$

This result can easily be understood. The field redefinition is similar to a T-duality in all directions, although the indices are placed differently. This is an important distinction between the former two which is crucial for the existence of the large volume limit. Yet,

[16] We also note that g is part of a non-geometric background. Because of the equalities (4.3.15), if $|g|$ is ill-defined, then so is ϕ. A good SUGRA limit is then lost in the non-geometric background, but β-supergravity can restore it, as argued before. In addition, an ill-defined ϕ is likely to be non-constant, so the compactification ansatz cannot be used for this set of fields. Then, $g,\ b,\ \phi$ does not allow to conclude on the (non-)existence of solutions of $\tilde{\mathcal{L}}_{\beta}$, on the contrary here to $g',\ b',\ \phi'$.

[17] One could also deviate from the compactification ansatz by considering warp factors and a non-constant dilaton. Compact NSNS solutions with these features exist, such as wrapped *NS*-branes, or non-Kähler backgrounds of heterotic string. The SUGRA limit of those is nevertheless more delicate.

the similarity explains why the block along the N directions is barely changed by the combination of the field redefinition and the T-duality, while the other block only goes through the field redefinition. Interestingly, β_N does not contribute. From this result, we deduce

$$\frac{\sqrt{|\tilde{g}|}}{\sqrt{|g'|}} = |\tilde{g}_N| \times |\mathbb{1}_{d-N} + \tilde{g}_{d-N}\beta_{d-N}| \ . \tag{4.3.19}$$

Although not impossible, having this quantity non-constant is rather unlikely, at least in usual set-ups where we look for solutions. First, β_{d-N} is likely to be constant, as it does not transform under gluing. Secondly, the metric $\tilde{g}_{d-N}$ is usually constant, as for instance that of a base circle. This makes the second factor constant. The metric $\tilde{g}_N$ can certainly be non-constant: for twisted tori, it goes through a non-trivial gluing. Its determinant is however usually constant, giving for instance a constant internal volume.[18] This implies that the above ratio is constant.

We conclude that, even though we made some assumptions such as (4.3.17), it looks unlikely to get a non-constant ϕ'. As explained above, purely NSNS solutions of β-supergravity that are geometric, non-trivial, and satisfy the compactification ansatz, are thus out of reach, at least in the usual set-ups. This holds despite the apparent possibility offered by the equations of motion of $\tilde{\mathcal{L}}_\beta$. It would be interesting to reach the same conclusion using only those equations.

In summary, we found that the framework of β-supergravity establishes geometric descriptions of backgrounds which have been non-geometric in standard SUGRA. However, we experienced two major drawbacks when working out the details. First, we noticed that such geometric solutions in β-supergravity are linked to geometric ones in standard SUGRA via the field redefinition and the Buscher rules. Possibilities to avoid lying on the geometric T-duality orbit seem to be scare, but nevertheless should be worked out. And secondly, we lack interesting backgrounds beyond the toroidal example. This will then actually be the topic of the next chapter, where we present another class of T-dual backgrounds which can be conveniently described with β-supergravity.

[18] One may wonder whether a constant internal volume can be thought of as unimodularity, $f^a{}_{ab} = 0$, related to the compactness of the internal manifold. One has $\partial_m \ln|e| = -\tilde{e}^a{}_n \partial_m \tilde{e}^n{}_a$, which is $f^a{}_{ab}$ up to a term in $\partial_p \tilde{e}^p{}_b$. In our context, the only non-trivial ∂_p are those along the $d-N$ directions. However, the inverse vielbein $\tilde{e}^p{}_b$ along those is most likely constant, as is $\tilde{g}_{d-N}$. So $\partial_m|e| = 0$ indicating constant volume and $f^a{}_{ab} = 0$ would be equivalent.

Chapter 5

Bianchi identities and *NS*-branes

In this section we consider a particular type of backgrounds, known as *NS*-branes. As motivated in the Introduction, starting from the *NS*5-brane T-duality generates certain brane configurations that are associated with non-geometric fluxes, like the Q-flux or even R-flux. We discuss here in the following the *NS*5-brane, the Kaluza-Klein (KK) monopole and finally the 5^2_2-brane [90, 92], also known as Q-brane [85] for carrying non-geometric Q-flux. Moreover, we are interested in the Bianchi Identities (BIs) for these backgrounds and how these get corrected by possible source terms. Another important aspect is the introduction of a Dirac-like derivative for β-supergravity that leads to the BIs with non-geometric fluxes turned on based on a nilpotency condition.

5.1 NSNS Bianchi identities without sources

We start by reviewing the appearance of NSNS BIs through the literature. BIs in the NSNS sector arise in different ways, as mentioned in the Introduction. We recall here various approaches. In general, the Jacobi identities of algebras with associated brackets lead to BIs. Less known is the use of a nilpotency condition on a generalization of the standard exterior derivative including both geometric and non-geometric fluxes. We follow in particular the second approach by squaring a $Spin(d,d) \times \mathbb{R}^+$ derivative and derive the BIs (1.4.16) - (1.4.19).

5.1.1 Sourceless NSNS Bianchi identities in the literature

We start by making a few remarks on the BIs for the NSSN fluxes in the absence of sources (1.4.16) - (1.4.19)

$$\partial_{[b}f^{a}{}_{cd]} - f^{a}{}_{e[b}f^{e}{}_{cd]} = 0\ , \tag{5.1.1}$$

$$v\partial_{[c}Q_{d]}{}^{ab} - \beta^{e[a}\partial_{e}f^{b]}{}_{cd} - \frac{1}{2}Q_{e}{}^{ab}f^{e}{}_{cd} + 2Q_{[c}{}^{e[a}f^{b]}{}_{d]e} = 0\ , \tag{5.1.2}$$

$$\partial_{d}R^{abc} - 3\beta^{e[a}\partial_{e}Q_{d}{}^{bc]} + 3R^{e[ab}f^{c]}{}_{de} - 3Q_{d}{}^{e[a}Q_{e}{}^{bc]} = 0\ , \tag{5.1.3}$$

$$\beta^{e[a}\partial_{e}R^{bcd]} + \frac{3}{2}R^{e[ab}Q_{e}{}^{cd]} = 0\ . \tag{5.1.4}$$

These conditions are actual identities, as they are fulfilled by inserting the definitions of fluxes. Remarkably, we obtained them through direct computation in the first place in appendix C.3. Equation (5.1.1) corresponds to the first BI of the Riemann tensor, respectively known as the torsionless Cartan equation

$$\frac{1}{2}\mathcal{R}^{a}{}_{[bcd]} = \partial_{[c}\omega^{a}_{db]} - \frac{1}{2}\omega^{a}_{e[b}f^{e}{}_{cd]} + \omega^{e}_{[cd}\omega^{a}_{b]e} = \frac{1}{2}\left(\partial_{[c}f^{a}{}_{db]} + f^{e}{}_{[cd}f^{a}{}_{b]e}\right)\ , \tag{5.1.5}$$

which also follows from $\mathrm{d}(\mathrm{d}\tilde{e}^{a})$. Analogously, (5.1.3) should be interpreted as the BI for the Riemann tensor associated to $\check{\mathcal{R}}$, given in (3.44) or (3.47) of [82]. (5.1.4) also arises as the BI for the R-flux $\check{\nabla}^{[m}R^{npq]} = 0$ obtained in [83, 82]. We come back to the second equation (5.1.2) around (5.1.39). In the following we review the appearance of NSNS BIs in the literature and comment on the relations to the above set of BIs.

Algebraic interpretation

In this approach the geometric and non-geometric fluxes appear as structure constants of a certain algebra. In particular the NSNS BIs follow then from the Jacobi identities of the algebra. For four-dimensional gauged supergravity (SUGRA) theories such algebras control the possible gaugings in standard geometric backgrounds. Here, the generators Z and X entering the algebra descend from ten-dimensional diffeomorphisms and b-field gauge transformations [141, 142, 143, 144]. Expecting T-duality covariance, extensions with non-geometric fluxes [27, 30][1] became interesting and lead to the famous algebra (1.3.2).[2] Then, the Jacobi identities of (1.3.2) generate the following set of NSNS BIs [27]

$$f^{e}{}_{[ab}H_{cd]e} = 0 \tag{5.1.6}$$

$$H_{e[bc}Q_{d]}{}^{ae} + f^{a}{}_{e[b}f^{e}{}_{cd]} = 0 \tag{5.1.7}$$

$$\frac{1}{2}H_{ecd}R^{abe} - \frac{1}{2}Q_{e}{}^{ab}f^{e}{}_{cd} + 2Q_{[c}{}^{e[a}f^{b]}{}_{d]e} = 0 \tag{5.1.8}$$

$$R^{e[ab}f^{c]}{}_{de} - Q_{d}{}^{e[a}Q_{e}{}^{bc]} = 0 \tag{5.1.9}$$

$$R^{e[ab}Q_{e}{}^{cd]} = 0\ . \tag{5.1.10}$$

[1] Our conventions differ by a minus sign for the R-flux with those of [27].

[2] Further interesting generalizations to other sectors of SUGRA are found in [145].

For vanishing H-flux and constant fluxes, these BIs exactly match our relations (5.1.1) - (5.1.4). Hence, the latter present a generalization for non-constant fluxes.[3]

A similar generalization has already been given in [133].[4] Based on a quasi-Poisson structure β, a Lie bracket on the generators $Z_a = \partial_a$, $X^a = \beta^{ab}\partial_b$ is able to reproduce the algebra (1.3.2) for $H = 0$.[5] Including additionally H-flux then reduces to a simple deformation of the algebra. In the end, the derived NSNS BIs for $H = 0$ with non-constant fluxes are provided by the Jacobi identities and match with (5.1.1) - (5.1.4) as verified in the appendix of [102]. Thus our BIs can be interpreted as ten-dimensional identities derived from Lie brackets.

Finally, let us list further brackets leading to similar results. In Generalized Complex Geometry (GCG) [78] the algebra (1.3.2) for $H = 0$ arises from the Courant bracket on generalized $O(d, d)$ frames. The Courant bracket on standard frames and co-frames in [133] and in [146] for Dirac structures yields also the respective algebra. In particular, the Jacobiators in [133] encode the respective BIs. In Double Field Theory (DFT) the C-bracket [110, 38], which reduces to the Courant bracket after using the strong constraint, presents a further $O(d, d)$ covariant extension [50] of the algebra. In [147], the Roytenberg bracket, generalizing the Courant bracket, also allows to write the algebra (1.3.2). Furthermore, Exceptional Field Theory provides a generalized Lie derivative, introduced in [148], leading to BIs including (5.1.1). Finally, in [36] a conformal field theory (CFT) approach allows to obtain the algebra (1.3.2) directly from the actions of (asymmetric) orbifolds.

Nilpotent derivative

An alternative to evaluating the Jacobi identities for a certain bracket is related to a generalization of the standard exterior derivative. In this case the nilpotency condition on this derivative leads to constraints that equal the BIs. This idea is most easily illustrated by squaring the derivative $\mathrm{d} - H\wedge$ on a p-form A. A generalization of such an exterior derivative including all NSNS geometric and non-geometric fluxes was proposed in [16]

$$\begin{aligned}
&\mathcal{D}_{\rm stw} A = (-H\wedge - f\cdot - Q\cdot + R\vee)A \ , \qquad (5.1.11)\\
&f\cdot = \frac{1}{2!} f^a{}_{bc}\ \tilde{e}^b \wedge \tilde{e}^c \wedge \iota_a \ , \ Q\cdot = \frac{1}{2!} Q_c{}^{ab}\ \tilde{e}^c \wedge \iota_a\ \iota_b \ , \ R\vee = \frac{1}{3!} R^{abc}\ \iota_a\ \iota_b\ \iota_c \ ,
\end{aligned}$$

where ι_a and $\vee$ denote contractions on forms, and we refer to appendix A.2 for more conventions.[6] The four-dimensional perspective explains the absence of a standard derivative, since fluxes are considered to be integrated over an internal space. Then, the nilpotency

[3]It was argued in [27] that the BIs (5.1.6)-(5.1.10) could be obtained from one another by applying T-duality in four dimensions as described there. It would be interesting to study the behavior of (5.1.1) - (5.1.4) under such a transformation.

[4]Relations similar to (5.1.1) - (5.1.4) were also obtained in [132], although they do not match exactly, as the Q-flux defined there is different, and there is no geometric flux turned on.

[5]The definitions of fluxes match ours except a sign for the R-flux.

[6]The derivative here is presented in our conventions with numerical coefficients.

condition $\mathcal{D}^2_{\rm stw} = 0$ was claimed to reproduce the NSNS BIs for constant fluxes (5.1.6) - (5.1.10). In [86] the previous derivative was completed[7] to

$$\mathcal{D}_\sharp A = \Big(-\frac{1}{3!} H_{abc} \tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^c \wedge -\frac{1}{2!} f^a{}_{bc}\ \tilde{e}^b \wedge \tilde{e}^c \wedge \iota_a - \frac{1}{2!} Q_c{}^{ab}\ \tilde{e}^c \wedge \iota_a\ \iota_b + \frac{1}{3!} R^{abc}\ \iota_a\ \iota_b\ \iota_c \\ - \frac{1}{2} f^a{}_{ab}\ \tilde{e}^b \wedge + \frac{1}{2} Q_a{}^{ab}\ \iota_b \Big) A \ . \tag{5.1.12}$$

Later, the terms involving the traces of f and Q will reappear inside a $Spin(d,d) \times \mathbb{R}^+$ derivative. Here, they lead in an explicit calculation of the nilpotency condition for the derivative (5.1.12) to

$$\mathcal{D}^2_\sharp = 0 \ \Leftrightarrow \ \text{BI (5.1.6)} - \text{(5.1.10)} \ \text{ and } \ \frac{1}{3} H_{abc} R^{abc} + \frac{1}{2} f^a{}_{ab} Q_a{}^{ab} = 0 \ . \tag{5.1.13}$$

Interestingly, the BIs are supported by an extra scalar constraint.

For the discussed generalizations of the derivative $\mathrm{d} - H\wedge$, there exists a nice interpretation in terms of $O(d,d)$ quantities. The former derivative constrains the RR fluxes for type II SUGRA through the BI $(\mathrm{d} - H\wedge)F = 0$ in the sourceless case, as mentioned in [16, 149]. The sum of RR fluxes F together with their gauge potential C actually are polyforms and further have an interpretation as $O(d,d)$ spinors, pointed out in [150, 151, 152, 153, 154] and seen from the supersymmetry (SUSY) conditions of [97]. Consequently at the level of the DFT in [87, 50], the RR fluxes $\mathcal{F} = \mathcal{D}\mathcal{C}$ were written down using a Dirac operator $\mathcal{D} = \Gamma^{\mathcal{A}} D_{\mathcal{A}}$ associated to a $Spin(d,d) \times \mathbb{R}^+$ covariant derivative $D_{\mathcal{A}}$ with $Spin(d,d)$ Clifford matrices $\Gamma^{\mathcal{A}}$.[8]

In section 5.1.2, we consider a generic GG definition of such a spinorial derivative for standard SUGRA and β-supergravity. Then, the vanishing of the square of this spinorial derivative should lead to the NSNS BIs. In particular, we generalize in this way the above derivative $\mathcal{D}_\sharp$.

5.1.2 The $Spin(d,d) \times \mathbb{R}^+$ covariant derivative

Following the previous argument, we introduce here a $Spin(d,d) \times \mathbb{R}^+$ covariant derivative at the level of the GG formalism. Our aim is then to reduce it on the two generalized frames (3.2.6) and (3.2.7) for either standard SUGRA or β-supergravity. In a first step, we determine the connection coefficients of this covariant derivative along our previous discussion in section 3.2.1 and verify in a second step that the BIs together with scalar conditions are reproduced using the nilpotency condition on it. We further clarify the relation to the above $\mathcal{D}_\sharp$ of [86].

[7] More precisely, we again rewrite a formula that was given on form components, namely (B.3) of [86], using forms and contractions. Further, our conventions differ by a minus sign for the H-flux.

[8] A related derivative has already appeared in [78, 89, 71].

Considering the $O(d,d)\times\mathbb{R}^+$ generalized covariant derivative of (3.2.9) we write down the corresponding spinorial derivative $D_{\mathcal{A}}$[9], as well as the Dirac operator $\mathcal{D}$, on a spinor $\Psi\in\Gamma(S^{\pm}_{(1/2)})$ [39]

$$\mathcal{D}\Psi=\Gamma^{\mathcal{A}}D_{\mathcal{A}}\Psi=\Gamma^{\mathcal{A}}\Big(\partial_{\mathcal{A}}+\frac{1}{4}\Omega_{\mathcal{A}\mathcal{B}\mathcal{C}}\Gamma^{\mathcal{B}\mathcal{C}}-\frac{1}{2}\Lambda_{\mathcal{A}}\Big)\Psi\ . \tag{5.1.14}$$

Here, the Γ-matrices satisfy the Clifford algebra

$$\{\Gamma^{\mathcal{A}},\Gamma^{\mathcal{B}}\}=2\eta^{\mathcal{A}\mathcal{B}},\quad \eta=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \eta^{-1}=2\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{5.1.15}$$

and $\eta_{\mathcal{A}\mathcal{B}}$ denotes the $O(d,d)$ metric. A particular representation of this algebra is given by the Clifford map

$$\Gamma^{\mathcal{A}}=\begin{cases}\Gamma^a=2\tilde{e}^a\ ,\\ \Gamma_a=2\iota_a\ ,\end{cases}\quad \text{with}\quad \{\tilde{e}^a,\tilde{e}^b\}=0\ ,\quad \{\tilde{e}^a,\iota_b\}=\delta^a_b\ ,\quad \{\iota_a,\iota_b\}=0\ . \tag{5.1.16}$$

This allows us to rewrite the fluxes, forms and contractions, as a generalization of the standard exterior derivative acting on a p-form A. Correspondingly, the spinor Ψ is then understood as a polyform [97]. A first simplification for (5.1.14) is provided by the identity $\Gamma^{\mathcal{A}}\Gamma^{\mathcal{B}\mathcal{C}}=\Gamma^{\mathcal{A}\mathcal{B}\mathcal{C}}+\eta^{\mathcal{A}\mathcal{B}}\Gamma^{\mathcal{C}}-\eta^{\mathcal{A}\mathcal{C}}\Gamma^{\mathcal{B}}$ that relates antisymmetrized products of Γ-matrices

$$\begin{aligned}\mathcal{D}\Psi=\Gamma^{\mathcal{A}}D_{\mathcal{A}}\Psi=&\Big(\Gamma^{\mathcal{A}}\partial_{\mathcal{A}}+\frac{1}{4}\Omega_{\mathcal{A}\mathcal{B}\mathcal{C}}\Gamma^{\mathcal{A}\mathcal{B}\mathcal{C}}+\frac{1}{2}(\Omega_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{C}}-\Lambda_{\mathcal{C}})\Gamma^{\mathcal{C}}\Big)\Psi\\ =&\Big(\Gamma^{\mathcal{A}}\partial_{yA}+\frac{1}{4}\hat{\Omega}_{\mathcal{A}\mathcal{B}\mathcal{C}}\Gamma^{\mathcal{A}\mathcal{B}\mathcal{C}}+\frac{1}{2}\hat{\Omega}_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{C}}\Gamma^{\mathcal{C}}\Big)\Psi\\ \equiv&\Big(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3\Big)\Psi\ .\end{aligned} \tag{5.1.17}$$

We remark that $\mathcal{D}_3$ denotes the trace part due to the extension of the $O(d,d)$ by the conformal factor $\mathbb{R}^+$ that usually combines the determinant of the metric and the dilaton.

We now determine these three terms for the two choices of generalized frames. As outlined in section 3.2.1, this fixes $\partial_{\mathcal{A}}$, $\Omega_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}}$ and $\Lambda_{\mathcal{A}}$. In particular, the Clifford map (5.1.16) allows us to express Γ-matrices as forms and contractions on a one-form in A while we note that a derivative $\partial_a\cdot$ solely acts on the component of A in flat indices. Details on the computation of $\mathcal{D}_2$ are given in appendix D.1.

- Standard supergravity:

  Using the generalized frames with b-field, we obtain

$$\mathcal{D}_1=2\partial_a\cdot e^a\wedge \tag{5.1.18}$$

$$\mathcal{D}_2=-f^c{}_{ab}e^a\wedge e^b\wedge\iota_c-f^d{}_{dc}e^c\wedge-\frac{1}{3}H_{abc}e^a\wedge e^b\wedge e^c\wedge \tag{5.1.19}$$

$$\mathcal{D}_3=f^a{}_{ab}\,e^b\wedge-2\partial_a\phi\,e^a\wedge\ , \tag{5.1.20}$$

[9] In (5.1.14), the index $_{\mathcal{B}}$ of the generalized connection coefficient has been lowered with the $O(d,d)$ metric.

which sums up to $\mathcal{D}$ given by

$$\frac{1}{2}\mathcal{D}A = \left(\partial_a \cdot e^a \wedge -\frac{1}{2}f^c{}_{ab}e^a \wedge e^b \wedge \iota_c - \frac{1}{6}H_{abc}e^a \wedge e^b \wedge e^c \wedge -\partial_a\phi\ e^a \wedge\right) A \tag{5.1.21}$$
$$= e^{\phi}\left(\mathrm{d} - H\wedge\right)\left(e^{-\phi}A\right) . \tag{5.1.22}$$

- β-supergravity:

  Using the generalized frames with β, we obtain

$$\mathcal{D}_1 = 2\partial_a \cdot \tilde{e}^a \wedge +2\beta^{ab}\partial_b \cdot \iota_a \tag{5.1.23}$$
$$\mathcal{D}_2 = -f^c{}_{ab}\tilde{e}^a \wedge \tilde{e}^b \wedge \iota_c - f^d{}_{dc}\tilde{e}^c \wedge -Q_a{}^{bc}\tilde{e}^a \wedge \iota_b\,\iota_c + Q_d{}^{dc}\iota_c + \frac{1}{3}R^{abc}\iota_a\,\iota_b\,\iota_c \tag{5.1.24}$$
$$\mathcal{D}_3 = f^a{}_{ab}\ \tilde{e}^b \wedge -2\partial_a\tilde{\phi}\ \tilde{e}^a \wedge +Q_a{}^{ab}\ \iota_b - 2(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a)\ \iota_a\ . \tag{5.1.25}$$

  Adding up these various pieces, we find

$$\begin{aligned}\mathcal{D} =&2\partial_a \cdot \tilde{e}^a \wedge +2\beta^{ab}\partial_b \cdot \iota_a - f^c{}_{ab}\,\tilde{e}^a \wedge \tilde{e}^b \wedge \iota_c - 2\partial_a\tilde{\phi}\,\tilde{e}^a \wedge \\ &- Q_a{}^{bc}\,\tilde{e}^a \wedge \iota_b\,\iota_c + 2Q_d{}^{dc}\,\iota_c - 2(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a)\,\iota_a + \frac{1}{3}R^{abc}\,\iota_a\,\iota_b\,\iota_c\ ,\end{aligned} \tag{5.1.26}$$

  where the second row could be further simplified using the definition of $\mathcal{T}^a$. We can rewrite this result differently, using the following relations for a 2-form A

$$\frac{1}{2}\iota_a\check{\nabla}^a(A_{bd})\tilde{e}^b \wedge \tilde{e}^d = (-\beta^{ac}\partial_c A_{ad} + Q_a{}^{ac}A_{dc} - \frac{1}{2}Q_d{}^{ac}A_{ac})\tilde{e}^d\ , \tag{5.1.27}$$
$$Q_a{}^{bc}\,\tilde{e}^a \wedge \iota_b\,\iota_c(\frac{1}{2}A_{ef}\tilde{e}^e \wedge \tilde{e}^f) = -Q_a{}^{ef}A_{ef}\tilde{e}^a\ ,\ Q_c{}^{ca}\iota_a(\frac{1}{2}A_{bd}\tilde{e}^b \wedge \tilde{e}^d) = Q_c{}^{ca}A_{ad}\tilde{e}^d\ . \tag{5.1.28}$$

  These relations are derived using the definitions and properties of $\check{\nabla}$, Q, and conventions of appendices A.2 and A.3. From them, we deduce an analogue expression as (5.1.22), as given in (1.4.23),

$$\frac{1}{2}\mathcal{D}A = e^{\tilde{\phi}}(\nabla_a \cdot \tilde{e}^a \wedge -\check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee +R\vee)(e^{-\tilde{\phi}}A)\ , \tag{5.1.29}$$

  where $\nabla_a \cdot \tilde{e}^a \wedge\ = \mathrm{d}$ is the standard exterior derivative. The second term gives an interesting counterpart to the exterior derivative.

The resulting $\mathcal{D}$ for standard SUGRA is a known spinorial derivative [97], and its square gives the standard NSNS BIs. We are now going to show the analogous result for the β-supergravity derivative and our BIs (5.1.1) - (5.1.4). A first hint is given by the comparison to the above derivative $\mathcal{D}_\sharp$ of [86] given in (5.1.12). For constant forms and fluxes, we recognize that in both cases (β or b vanishes), one has

$$\mathcal{D}_\sharp = \frac{1}{2}\mathcal{D}_2\ . \tag{5.1.30}$$

The natural completion of $\mathcal{D}_\sharp$ in the case of non-constant fluxes would have been by derivatives, as given by $\mathcal{D}_1$. Interestingly, the additional traces and dilaton terms of $\mathcal{D}_3$ are also needed in order to recover the full set of BIs. Therefore, we now turn to the study of the nilpotency condition for the above derivative $\mathcal{D}$ of (5.1.26)

$$\mathcal{D}^2 A = 0 \ . \tag{5.1.31}$$

We leave the details of the computation to appendix D.1 and end up with the following set of seven equations

$$-\frac{1}{2}\partial_{[a} f^d{}_{bc]} + \frac{1}{2} f^d{}_{g[a} f^g{}_{bc]} = 0 \tag{5.1.32}$$
$$-\frac{1}{2} Q_d{}^{da} f^g{}_{ga} = 0 \tag{5.1.33}$$
$$-\frac{3}{2}\beta^{de}\partial_{[e} f^b{}_{da]} + \frac{3}{2}\beta^{de} f^b{}_{h[a} f^h{}_{ed]} = 0 \tag{5.1.34}$$
$$-\frac{1}{2}(\partial_{[a} Q_{c]}{}^{de} - \beta^{g[d}\partial_g f^{e]}{}_{ac}) + \frac{1}{4}(-4 f^{[d}{}_{g[a} Q_{c]}{}^{e]g} + f^g{}_{ac} Q_g{}^{de}) = 0 \tag{5.1.35}$$
$$-\frac{1}{2}\beta^{dc}\partial_c Q_d{}^{ab} - \frac{1}{2}\beta^{cd}\beta^{g[a}\partial_g f^{b]}{}_{cd} - \beta^{dc} Q_c{}^{g[a} f^{b]}{}_{dg} + \frac{1}{4}\beta^{dc} Q_g{}^{ab} f^g{}_{cd} = 0 \tag{5.1.36}$$
$$\frac{1}{6}(\partial_a R^{bcd} - 3\beta^{e[b}\partial_e Q_a{}^{cd]}) - \frac{1}{2}(-R^{g[bc} f^{d}{}_{a]g} + Q_a{}^{g[d} Q_g{}^{bc]}) = 0 \tag{5.1.37}$$
$$-\frac{1}{6}\beta^{g[a}\partial_g R^{bcd]} - \frac{1}{4} Q_g{}^{[ab} R^{cd]g} = 0 \ . \tag{5.1.38}$$

Rather remarkably, the dilaton terms completely cancel out. Furthermore we notice the following relations among the above equations. (5.1.34) is a contraction of (5.1.32) by β, and similarly (5.1.36) is a contraction of (5.1.35). We are then left with a set of five independent identities. These are exactly the four BIs listed before: (5.1.32) matches (5.1.1), (5.1.35) matches (5.1.2), (5.1.37) matches (5.1.3), (5.1.38) matches (5.1.4). So the square of the spinorial derivative (5.1.26) precisely produces the BIs. In addition, we find the scalar condition derived in [86], and given in (5.1.13), from the fully contracted terms (5.1.33).

At this point we come back to the interpretation of (5.1.2). Given the results above, and the expression of $\mathcal{D}$ given in (5.1.29), we deduce on a two-form A

$$\Big\{ \nabla_a \cdot \tilde{e}^a \wedge \ , \check{\nabla}^b \cdot \iota_b - \mathcal{T} \vee \Big\} A = -\frac{1}{2}\left(3\ \beta^{eb} S^a_{ebc}\ A_{ad} + S^{ab}_{cd}\ A_{ab}\right) \tilde{e}^c \wedge \tilde{e}^d \ , \tag{5.1.39}$$

where the quantities S are defined in section 5.2.3 and correspond to the LHS of the BI (5.1.1) and (5.1.2). This gives a tensorial form to (5.1.2), since such a form for (5.1.1) was already mentioned around (5.1.5). The cases of (5.1.3) and (5.1.4) were discussed below the latter.

5.2 T-dual *NS*-branes sourcing the Bianchi identities

As presented in the Introduction, the BIs for certain brane configurations receive corrections in the form of source terms. We show in this section that the BIs (5.1.1)-(5.1.4) just studied get corrected for *NS*-branes, namely for the *NS*5-brane, the *KK*-monopole and the *Q*-brane. These are vacua of standard SUGRA and β-supergravity. Up to smearing, they are T-dual to the *NS*5-brane. We first present these solutions following the literature. We then focus on the smearing procedure that allows to perform T-dualities along isometry directions. This clarifies how the different warp factors can be the appropriate Green functions in the Poisson equations of each brane. We finally verify how the branes are related by T-duality. We further show that the above BIs on the brane vacua boil down to the Poisson equations, allowing the emergence of the source term. This study establishes β-supergravity as a convenient framework for describing *Q*-branes.

5.2.1 *NS*-branes solutions

We present here the various *NS*-branes, starting with the *NS*5-brane sourcing H-flux. The *NS*5-brane solution was first given in the limit of zero size instanton in [155], and presented in a broader context in [156] corresponding to the case where the gauge field vanishes. More generalizations and references can be found in [157, 158]. Smearing and T-dualizing this brane along one direction leads to the *KK*-monopole, which was first discovered as a solution to pure five-dimensional general relativity[10]. It can be associated with geometric flux. A further smearing and T-duality along another direction leads to a new brane known as the 5_2^2-brane [90, 92] or *Q*-brane [85]. It belongs to the class of exotic branes [90, 91, 92, 85, 161, 162, 163] which have recently received much attention for being related to standard branes by different U-dualities. Remarkably, the *Q*-brane, being non-geometric in standard SUGRA, becomes a geometric vacuum in β-supergravity [85, 50], where it sources Q-flux.

In the following we list these three brane configurations:

- The *NS*5-brane, being the magnetic counterpart of the fundamental string, is physically a codimension four object, i.e. it is located in four dimensions. The original solution takes the following form[11]

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f_H\, \mathrm{d}\hat{s}_4^2 \ ,\ H_{mnp} = -\sqrt{|g_4|}\epsilon_{4mnpq} g^{qr}\partial_r \ln f_H \ ,\ e^{2\phi} = f_H \qquad (5.2.1)\\ &\text{where } \mathrm{d}\hat{s}_4^2 = \sum_{m=1\dots4} (\mathrm{d}x^m)^2 \ ,\ r_4^2 = \sum_{m=1\dots4} (x^m)^2 \ ,\ f_H = e^{2\phi_H} + \frac{q}{r_4^2} \ , \end{aligned}$$

  where $\mathrm{d}s_6^2$ is the Minkowski metric and $\mathrm{d}\hat{s}_4^2$ denoting the flat Euclidian metric gives the transverse directions. The warp factor f_H depends on the radius r_4 and on two

[10] [159, 160], and [96] provide more references.

[11] We have a factor of 2 difference for the H-flux with respect to the conventions of [156]. Note that the warp factor given here is not considered in [164, 50], as only the *KK*-monopole and T-duals are used there. In particular, only the smeared warp factor of the *NS*5-brane is present there.

constants, the value of the dilaton ϕ_H at ∞, and q, which is related to the tension of the brane. The H-flux is proportional to the volume form coefficient of the transverse four-dimensional space $\sqrt{|g_4|}\epsilon_{4mnpq}$[12]. Given the transverse metric, we can simplify the expression for the H-flux to

$$H_{mnp} = -\epsilon_{4mnpq}\delta^{qr}\partial_r f_H \ . \tag{5.2.2}$$

- The KK-monopole can be considered as a codimension three brane

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f_K \ \mathrm{d}\hat{s}_3^2 + f_K^{-1}(\mathrm{d}x + a\mathrm{d}y)^2 \ , \ H_{mnp} = 0 \ , \ e^{2\phi} = 1 \\ &\text{where } \mathrm{d}\hat{s}_3^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \rho^2\sin^2\varphi \ \mathrm{d}y^2 \ , \ f_K = e^{2\phi_K} - \frac{q_K}{\rho} \ . \end{aligned} \tag{5.2.3}$$

The metric $\mathrm{d}s_6^2$ is still of Minkowski type, and the metric $\mathrm{d}\hat{s}_3^2$ denotes the flat transverse space which we prefer here to express using spherical coordinates $\{\rho, \varphi, y\}$. The radius ρ will sometimes be denoted r_3 below. The warp factor f_K depends on two constants, ϕ_K and q_K that we will relate to the above q in section 5.2.2.[13] Finally, the quantity a mimics a connection one-form coefficient and is a priori not gauge-invariant. Away from the singularity, one has

$$a(\varphi) = q_K \cos\varphi \text{ for } \rho > 0 \ . \tag{5.2.5}$$

A reasonable completion reads

$$a(\rho, \varphi) = \cos\varphi \ \rho^2\partial_\rho f_K \ , \tag{5.2.6}$$

as detailed in section 5.2.3. We later deduce a corresponding geometric flux given by

$$f^x{}_{\varphi y} = f_K^{-\frac{3}{2}}\partial_\rho f_K \ . \tag{5.2.7}$$

- The Q-brane is a codimension two brane and has a proper description only in terms of β-supergravity

$$\begin{aligned} &\mathrm{d}\tilde{s}^2 = \mathrm{d}s_6^2 + f_Q \ \mathrm{d}\hat{s}_2^2 + f_Q^{-1}(\mathrm{d}x^2 + \mathrm{d}y^2) \ , \ \text{only } \beta^{xy} = -\beta^{yx} \neq 0 \ , \ e^{2\tilde{\phi}} = f_Q^{-1} \\ &\text{where } \ \mathrm{d}\hat{s}_2^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 \ , \ f_Q = e^{-2\tilde{\phi}_Q} - q_Q \ln\rho \ . \end{aligned} \tag{5.2.8}$$

Its expression in terms of standard SUGRA is given below in (5.2.39). The metric $\mathrm{d}s_6^2$ is again Minkowski and $\mathrm{d}\hat{s}_2^2$ is the flat metric which we express in polar coordinates

[12] Conventions can be found in appendix A.2.

[13] A warp factor for the KK-monopole depending on x was considered in [165, 166], and related to world-sheet instantons corrections [167], see also [162]. One can verify that it matches ours far away from the brane

$$f_K(\rho, x) = \frac{1}{g^2} + \frac{1}{2\rho}\frac{\sinh\rho}{\cosh\rho - \cos x} \ . \tag{5.2.4}$$

$\{\rho, \varphi\}$ for the transverse directions this time. The radius ρ will sometimes be denoted r_2 below. The warp factor f_Q depends on two constants, $\tilde{\phi}_Q$ and q_Q that we will relate to q in section 5.2.2. $\tilde{\phi}_Q$ may contain a cutoff when $\rho \to \infty$, as mentioned in [90, 92] and we rediscuss this point in section 5.2.2. Finally, analogously to a for the *KK*-monopole, the field β is not a well-defined

$$\beta^{xy} = -\varphi \ \rho \partial_\rho f_Q \ \Rightarrow \ \beta^{xy} = q_Q \ \varphi \ \text{for} \rho > 0 \ . \tag{5.2.9}$$

Instead, the Q-flux turns out to be the better defined quantity and is given in flat indices as[14]

$$Q_\varphi{}^{xy} = -f_Q^{-\frac{3}{2}} \partial_\rho f_Q \ . \tag{5.2.11}$$

We verify explicitly in appendix D.2 that the Q-brane is a solution to the equations of motion of β-supergravity. In [85], using a different method, this result was obtained away from the singularity.

5.2.2 Smearing warp factors and Poisson equations

The brane solutions that we have just presented are related by smearing and T-dualizing along transverse directions. We focus here on the different warp factors, and show how smearing relates one warp factor to the other. This explains how each of them can satisfy the appropriate Poisson equation. We first review the well-known case of p-branes solutions, before turning to *NS*-branes.

A p-brane is a type II SUGRA background that provides an effective description of a D_p-brane in some regime. This solution contains in particular a dilaton that depends on the warp factor $Z_p(r)$, and the metric is given by

$$\mathrm{d}s^2 = Z_p^{-\frac{1}{2}} \mathrm{d}s_{||}^2 + Z_p^{\frac{1}{2}} \mathrm{d}s_\perp^2 \ , \tag{5.2.12}$$

where $\mathrm{d}s_{||}^2$ denotes the Minkowski space-time along the brane and $\mathrm{d}s_\perp^2$ is the flat Euclidean space transverse to the brane. Then, the warp factor depends on r, the Euclidean radius for the latter,

$$Z_p(r) = 1 + \frac{q_p}{r^{7-p}} \ , \ \text{for } p \leqslant 6 \ , \tag{5.2.13}$$

[14] As usual, the three fluxes are the same in flat indices, up to a sign for the structure constant. For the H-flux, one can choose coordinates that isolate the coordinate r_4. The corresponding metric element would still only be given by a warp factor, so one would get

$$H_{mnp} = -\sqrt{|g_3|} \epsilon_{4mnp(r_4)} f_H^{-\frac{3}{2}} \partial_{r_4} f_H \ . \tag{5.2.10}$$

The remaining volume factor is then removed when going to flat indices. Useful conventions for ϵ can be found in appendix A.1. So the three fluxes are the same in flat indices, although one needs to take the same warp factor. This only happens when there is smearing, i.e. in the case of T-duality, as we will show below. It is definitely in that case that we expect the equality of the fluxes, as given in the T-duality chain of [27].

where the constant q_p is related to the tension of the brane. The RR flux F of this background verifies typically a BI of the form

$$\mathrm{d}F = Q\ \delta(x_\perp)\ . \tag{5.2.14}$$

This flux is sourced by the brane, localized in its $9-p$ transverse directions by δ,[15] and carries charge Q. Using for instance the transverse Hodge star $*_\perp$, one can extract the forms to leave only coefficients, in particular the density $\delta^{(9-p)}(x_\perp)$. The BI then typically boils down to the scalar equation

$$\Delta_{9-p} Z_p = \delta^{(9-p)}(r)\ , \tag{5.2.16}$$

where Δ_{9-p} is the Laplacian of the unwarped metric $\mathrm{d}s_\perp^2$. This scalar equation is known as a Poisson equation and a solution with possible boundary conditions is provided by the Green function for the Laplacian which is known in two dimensions as $\ln r$, and for $d_\perp \geqslant 3$ as $\frac{1}{r^{d_\perp - 2}}$. For $d_\perp = 3$, this is the well-known electrostatic potential. As expected, the radial dependence in the transverse space directions $d_\perp = 9-p$ coincides precisely with that of Z_p (5.2.13).

We now consider T-dualities on these branes. T-dualizing along a transverse direction is known to extend a D_p-brane to a D_{p+1}-brane. The correct powers of the warp factor are obtained by applying Buscher rules. However, the warp factor itself should also be changed from Z_p to Z_{p+1}. This is done by a smearing procedure explained in [170] that also takes care of picking the proper radial dependence $r_{9-(p+1)}$. In this step we gain the necessary isometry direction x for performing a T-duality, since a priori Z_p depends on x

$$Z_{p+1}(r_{9-(p+1)}) \sim \int \mathrm{d}x\ Z_p(r_{9-p})\ , \quad r_{9-p}^2 = x^2 + r_{9-(p+1)}^2\ . \tag{5.2.17}$$

The smeared p-brane is then T-dual to the $(p+1)$-brane. Interestingly, the Poisson equations are also consistent under this procedure

$$\begin{aligned} \Delta_{9-p} Z_p &= \left((\partial_x)^2 + \Delta_{9-(p+1)}\right) Z_p = \delta^{(9-p)}(r_{9-p}) \\ \Rightarrow & \int \mathrm{d}x\ \left((\partial_x)^2 + \Delta_{9-(p+1)}\right) Z_p = \int \mathrm{d}x\ \delta^{(9-p)}(r_{9-p}) \\ \Leftrightarrow & \left(0 + \Delta_{9-(p+1)} \int \mathrm{d}x\ \right) Z_p = \delta^{(9-(p+1))}(r_{9-(p+1)}) \\ \Leftrightarrow &\ \ \Delta_{9-(p+1)} Z_{p+1} = \delta^{(9-(p+1))}(r_{9-(p+1)})\ . \end{aligned} \tag{5.2.18}$$

In the last but one line, we use conditions on the warp factor and its derivatives that will be verified in the examples below. In this derivation, we actually only need the warp factor

[15] The $(9-p)$-form $\delta(x_\perp)$ of (5.2.14) can also be viewed as a current, and is defined through

$$\int_{||} A_{p+1} = \int_{10} A_{p+1} \wedge \delta(x_\perp) \tag{5.2.15}$$

for any $(p+1)$-form A_{p+1}, as in [168, 169].

without its pure constant part, since only its derivatives are involved. So that is what we meant in (5.2.17), and what will be used in the following.

The *NS*-branes share many features with the p-brane solutions. They both have warp factors that determine the transverse directions. The constants in the warp factors are related to the tension of the brane, although they scale differently in $e^{\phi_0} = g_s$. Finally, these warp factors take analogous forms, corresponding to the various Green functions in different codimensions and consequently the *NS*-branes will satisfy as well Poisson equations. Moreover, they are also T-dual up to smearing. We will verify explicitly the T-duality relations and derive the Poisson equations from the BIs in section 5.2.3. Next, we relate their warp factors by smearing as just explained for the p-branes.

- The BI for the H-flux of the $NS5$-brane is given by $\mathrm{d}H$

$$\partial_{[m}H_{npq]} = -\partial_{[m}\epsilon_{4npq]r}\delta^{rs}\partial_s f_H \propto \epsilon_{4mnpq}\delta^{rs}\partial_r\partial_s f_H \ , \tag{5.2.19}$$

  where we used the expression of the H-flux (5.2.2). One therefore finds that

$$\mathrm{d}H \propto \hat{\mathrm{vol}}_4\ \Delta_4 f_H\ , \quad \Delta_4 = \sum_{m=1\dots4} (\partial_m)^2\ , \tag{5.2.20}$$

  with the four-dimensional volume form $\hat{\mathrm{vol}}_4$. The BI in presence of a source is given by $\mathrm{d}H \propto \hat{\mathrm{vol}}_4\ \delta^{(4)}(r_4)$, so the warp factor has to solve the Poisson equation

$$\Delta_4 f_H = c_H\ \delta^{(4)}(r_4)\ , \tag{5.2.21}$$

  with a constant c_H. In other words, f_H/c_H should be a Green function for the four-dimensional Laplacian Δ_4. A known Green function for this problem is $\frac{1}{r_4^2}$, so f_H given in (5.2.1) certainly solves the Poisson equation. A cross check of this result is that away from the singularity $r_4 = 0$, the Poisson equation boils down to the Laplace equation, meaning

$$\Delta_4 f_H = 0 \text{ for } r_4 > 0\ . \tag{5.2.22}$$

  One can verify that this holds for f_H of (5.2.1).

- We turn to the KK-monopole. According to the procedure explained above, we smear the $NS5$-brane along the direction x. For this purpose we introduce the new three-dimensional radius $r_3^2 = r_4^2 - x^2$ and smear the warp factor without its constant $f_H - e^{2\phi_H}$ to get the new f_K up to its constant $e^{2\phi_K}$

$$f_K(r_3) - e^{2\phi_K} = \int_{-\infty}^{+\infty} \mathrm{d}x\ \left(f_H(r_4) - e^{2\phi_H}\right) = \left[\frac{q}{r_3}\arctan\left(\frac{x}{r_3}\right)\right]_{-\infty}^{+\infty} = \frac{q\pi}{r_3}\ . \tag{5.2.23}$$

  This new warp factor matches the one given in (5.2.3) with $q_K = -\pi q$. In addition, it is a known solution to the three-dimensional Poisson equation

$$\Delta_3 f_K = c_K\ \delta^{(3)}(r_3)\ , \tag{5.2.24}$$

the well-known electrostatic potential. One can straightforwardly verify that

$$\Delta_3 f_K = 0 \text{ for } r_3 > 0 \ . \tag{5.2.25}$$

This result was expected from the discussion around (5.2.18). One condition for this procedure to work is that the derivative of the warp factor vanishes on the boundary. Here this holds, as $\partial_m f_H = -\frac{2q\ x^m}{r_4^4} \sim_\infty -\frac{2q}{(x^m)^3}$. The same will be true for the other warp factors as the power of x^m in the denominator decreases by one at each step.

- Next, we obtain the warp factor f_Q of the Q-brane by smearing the previous one along a further direction y. We introduce the two-dimensional radius $r_2^2 = r_3^2 - y^2$, and the boundary constant $e^{-2\tilde{\phi}_Q}$. We introduce further ϵ, which will be sent to ∞, and the function $\text{arsinh}x = \ln(x + \sqrt{(x^2+1)})$. Then

$$\begin{aligned} f_Q(r_2) - e^{-2\tilde{\phi}_Q} = \int_{-\epsilon}^{+\epsilon} \mathrm{d}y\ \left(f_K(r_3) - e^{2\phi_K}\right) &= q\pi \left[\ln\left(\frac{y + \sqrt{(y^2+r_2^2)}}{r_2}\right)\right]_{-\epsilon}^{+\epsilon} \\ &= q\pi \left[\ln\left(y + \sqrt{(y^2+r_2^2)}\right)\right]_{-\epsilon}^{+\epsilon} \ . \end{aligned} \tag{5.2.26}$$

The function $\text{arsinh}x$ is odd, from which we get the property

$$\ln\left(-y + \sqrt{(y^2+r_2^2)}\right) = -\ln\left(y + \sqrt{(y^2+r_2^2)}\right) + 2\ln r_2 \ .$$

We deduce

$$f_Q(r_2) - e^{-2\tilde{\phi}_Q} = 2q\pi \ln\left(\epsilon + \sqrt{(\epsilon^2 + r_2^2)}\right) - 2q\pi \ln r_2 \ .$$

This quantity diverges when taking the limit $\epsilon \to \infty$. We therefore need a cutoff, as argued in [92], to remove the divergence.[16] Up to a redefinition of the constant $\tilde{\phi}_Q$ to absorb it, one obtains

$$f_Q(r_2) = e^{-2\tilde{\phi}_Q} - 2q\pi \ln r_2 \ . \tag{5.2.27}$$

This warp factor matches the solution (5.2.8) with $q_Q = 2\pi q$. In addition, it is a known solution to the two-dimensional Poisson equation

$$\Delta_2 f_Q = c_Q\ \delta^{(2)}(r_2) \tag{5.2.28}$$

and one straightforwardly verifies that

$$\Delta_2 f_Q = 0 \text{ for } r_2 > 0 \ . \tag{5.2.29}$$

[16]It would be interesting to study whether the divergence is related to non-geometry, and thus whether the field redefinition could avoid it, by for instance including volume factors in the integral relation (5.2.17).

- It is tempting to consider a hypothetical R-brane with warp factor f_R by smearing also along the direction z. We introduce the one-dimensional radius $r_1^2 = r_2^2 - z^2 = w^2$ and again an ϵ that will be sent to ∞. Then

$$\begin{aligned} f_R(r_1) - e^{2\tilde{\phi}_R} &= \int_{-\epsilon}^{+\epsilon} \mathrm{d}z \ \left(f_Q(r_2) - e^{-2\tilde{\phi}_Q}\right) \\ &= -q\pi \int_{-\epsilon}^{+\epsilon} \mathrm{d}z \ \ln(z^2 + r_1^2) \\ &= -q\pi \left[z \ln(z^2 + r_1^2)\right]_{-\epsilon}^{+\epsilon} + q\pi \int_{-\epsilon}^{+\epsilon} \mathrm{d}z \ z \frac{2z}{z^2 + r_1^2} \\ &= -2q\pi\epsilon \ln(\epsilon^2 + r_1^2) + 2q\pi \int_{-\epsilon}^{+\epsilon} \mathrm{d}z \ \left(1 - \frac{r_1^2}{z^2 + r_1^2}\right) \\ &= -2q\pi \left(\epsilon \ln(\epsilon^2 + r_1^2) - 2\epsilon\right) - 2q\pi r_1 \left[\arctan\left(\frac{z}{r_1}\right)\right]_{-\epsilon}^{+\epsilon} \ . \end{aligned} \tag{5.2.30}$$

As for the Q-brane, the first term diverges. We consider again a cutoff and absorb it in a redefinition of the constant. For $\epsilon \to \infty$ we are then left with

$$f_R(r_1) = e^{2\tilde{\phi}_R} - 2q\pi^2 r_1 = e^{2\tilde{\phi}_R} - 2q\pi^2 |w| \ . \tag{5.2.31}$$

The absolute value is known to be a solution of the one-dimensional Poisson equation

$$\Delta_1 f_R = c_R \ \delta^{(1)}(r_1) \ , \tag{5.2.32}$$

and one can again verify that away from the singularity,

$$\Delta_1 f_R = 0 \text{ for } r_1 > 0 \ . \tag{5.2.33}$$

Although smearing the warp factor seems to work and yields a consistent result, performing a T-duality along z is more challenging. It would require to smear as well the b-field or the β, for which there is no clear procedure. Maybe, one could rather consider a direct T-duality transformation of the flux, as proposed in [171], since the flux is a better defined quantity that does not depend on z.[17]

5.2.3 Smeared branes, T-duality and sourced Bianchi identities

Now that we have the correct warp factors related by smearing at hand for the different branes, we T-dualize the smeared *NS*-branes into one another and show how the BIs (5.1.1) - (5.1.4) lead to the corresponding Poisson equations. We start with the Q-brane, as it involves most of the ingredients needed for the others and also present the procedure for the *KK*-monopole. Finally, we come back to the *NS*5-brane.

[17] Note that it should be different from the one proposed in [85], which instead involves a dual coordinate.

We are going to obtain the Q-brane by T-dualizing the $NS5$-brane along two directions. In a first step we need to smear the latter and thus consider the smeared warp factor f_Q of (5.2.8) rather than the standard f_H of (5.2.1). The most appropriate coordinates for T-duality are then the cylindrical ones $\rho = r_2$ and φ and the Cartesian x, y for the two smeared directions. Unless one uses a procedure like the one of [171], T-duality requires a b-field. Given the expression of the H-flux in (5.2.2) and the relation $H_{mnp} = 3\partial_{[m}b_{np]}$, it is much simpler to obtain a b-field that respects the isometries using those coordinates. Then, starting with (5.2.1), the twice smeared $NS5$-brane is given by

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f\ \mathrm{d}\hat{s}_4^2\ ,\ H_{mnp} = -\rho\ \epsilon_{4mnp\rho}\partial_\rho f\ ,\ e^{2\phi} = f \\ &\text{where}\ \ \mathrm{d}\hat{s}_4^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}x^2 + \mathrm{d}y^2\ ,\ f = f_Q\ , \end{aligned} \tag{5.2.34}$$

in curved cylindrical indices. Fixing $\epsilon_{4\rho\varphi xy} = +1$, see appendix A.2, the only non-trivial component of the H-flux away from the singularity yields

$$H_{\varphi xy} = q_Q \text{ for } \rho > 0\ , \tag{5.2.35}$$

in curved indices. We then choose the gauge

$$b_{xy} = -b_{yx} = q_Q\ \varphi \text{ for } \rho > 0\ , \tag{5.2.36}$$

for the b-field, which respects the isometries. To include the singularity, it is tempting to define

$$b_{mn} = \epsilon_{4\rho\varphi mn}\ a(\rho,\varphi)\ ,\ \text{with } a = -\varphi\ \rho\partial_\rho f\ , \tag{5.2.37}$$

which gives the correct expression when acting with ∂_φ. But it leads to undesired H-flux components at the singularity when acting with ∂_ρ. The same ambiguity will appear below for the KK-monopole and the Q-brane. It is important to note that this b-field is ambiguous and for good fluxes we should set $\partial_\rho a = 0$.

We start by T-dualizing along the x direction. Applying the Buscher rules[18] we get vanishing b-field and

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f\ \mathrm{d}\hat{s}_3^2 + f^{-1}(\mathrm{d}x + a\mathrm{d}y)^2\ ,\ H_{mnp} = 0\ ,\ e^{2\phi} = 1 \\ &\text{where}\ \ \mathrm{d}\hat{s}_3^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}y^2\ . \end{aligned} \tag{5.2.38}$$

This solution corresponds to the KK-monopole (5.2.3) smeared along y, as can be seen from the warp factor and the coordinates. In this case the smeared a can only be understood through the T-duality procedure. Finally, we T-dualize along y and get

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f\ \mathrm{d}\hat{s}_2^2 + f^{-1}(1 + \frac{a^2}{f^2})^{-1}(\mathrm{d}x^2 + \mathrm{d}y^2)\ ,\ e^{2\phi} = f^{-1}(1 + \frac{a^2}{f^2})^{-1}\ , \\ &b_{xy} = -b_{yx} = -af^{-2}(1 + \frac{a^2}{f^2})^{-1}\ ,\ \text{where}\ \mathrm{d}\hat{s}_2^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\ , \end{aligned} \tag{5.2.39}$$

[18] In [131] Buscher rules in terms of g and b have been used that are equivalent to the transformation (4.1.23). We use those, with a minus sign difference for the b-field, due to conventions.

which has been argued in [90] to be non-geometric. Using the field redefinition (3.1.3), we get precisely the Q-brane solution (5.2.8)

$$\begin{aligned} &\mathrm{d}\tilde{s}^2 = \mathrm{d}s_6^2 + f\ \mathrm{d}\hat{s}_2^2 + f^{-1}(\mathrm{d}x^2+\mathrm{d}y^2)\ ,\ \beta^{xy} = -\beta^{yx} = a\ ,\ e^{2\tilde{\phi}} = f^{-1} \\ &\text{where}\ \ \mathrm{d}\hat{s}_2^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\ . \end{aligned} \tag{5.2.40}$$

Going around the singularity, i.e. moving along φ at $\rho > 0$, the constant shift of β along the isometry directions can be compensated by a β-transform. Hence the Q-brane is part of the class studied in sections 4.2 and 4.3. The T-dual background given by the smeared *NS*5-brane also has a linear b-field. As described in those sections, such a situation leads typically to a non-geometry, as in (5.2.39).

Let us now determine the fluxes of this solution using the following vielbein

$$\tilde{e} = \begin{pmatrix} f^{\frac12} & & & \\ & f^{\frac12}\rho & & \\ & & f^{-\frac12} & \\ & & & f^{-\frac12} \end{pmatrix}, \tag{5.2.41}$$

from which we deduce the non-zero structure constants or geometric flux (A.3.10)

$$f^{\varphi}{}_{\rho\varphi} = -\frac{1}{2}f^{-\frac32}\partial_\rho f - f^{-\frac12}\rho^{-1}\ ,\ f^{x}{}_{\rho x} = f^{y}{}_{\rho y} = \frac{1}{2}f^{-\frac32}\partial_\rho f\ ,\ f^{a}{}_{bc} = -f^{a}{}_{cb}\ , \tag{5.2.42}$$

where we slightly abuse our notation. On the LHS we denote flat indices with corresponding curved space coordinate and on the RHS the derivative has a curved index. Next, we compute the Q-flux. It is worth noticing that the Q-brane solution verifies the condition $\beta^{mn}\partial_n\cdot = 0$, as pointed out in [85] which leads to

$$Q_c{}^{ab} \equiv \partial_c\beta^{ab} - 2\beta^{d[a}f^{b]}{}_{cd} \ \overset{\beta^{mn}\partial_n\cdot=0}{=\!=\!=\!=\!=}\ \tilde{e}^p{}_c\tilde{e}^a{}_m\tilde{e}^b{}_n\partial_p\beta^{mn}\ \ , \tag{5.2.43}$$

as can be seen from (4.2.1), while $R^{abc} = 0$. Recalling the ambiguity of the b-field and a in the *NS*5-brane discussed around (5.2.37), one obtains the only non-trivial component of the Q-flux

$$Q_\varphi{}^{xy} = -f^{-\frac32}\partial_\rho f\ , \tag{5.2.44}$$

where we again use the above mentioned notation for flat indices. The Q-flux takes exactly the same value as the smeared *NS*5 H-flux in flat indices and thus gives credit to the applied procedure.

Finally, we turn to the BIs. Given the fluxes just determined and using some antisymmetry arguments, one can see that (5.1.1), (5.1.3) and (5.1.4) are satisfied. Let us instead focus on (5.1.2), and the quantity

$$S^{ab}_{cd} = \partial_{[c}Q_{d]}{}^{ab} - \beta^{e[a}\partial_e f^{b]}{}_{cd} - \frac{1}{2}Q_e{}^{ab}f^e{}_{cd} + 2Q_{[c}{}^{e[a}f^{b]}{}_{d]e}\ . \tag{5.2.45}$$

The second term vanishes here and we fix (c,d,a,b) to the only non-trivial choice (ρ,φ,x,y) up to antisymmetries

$$\begin{aligned} S^{xy}_{\rho\varphi} &= \frac{1}{2} f^{-\frac{1}{2}} \partial_\rho Q_\varphi{}^{xy} - \frac{1}{2} Q_\varphi{}^{xy} (f^\varphi{}_{\rho\varphi} - f^y{}_{\rho y} - f^x{}_{\rho x}) \\ &= -\frac{1}{2} f^{-2} \left(\partial^2_\rho f + \rho^{-1} \partial_\rho f\right) = -\frac{1}{2} f^{-2} \Delta_2 f \ , \end{aligned} \tag{5.2.46}$$

where Δ_2 is the two-dimensional Laplacian using polar coordinates since f does not depend on φ. As argued in (5.2.28), f is here the Green function for Δ_2 up to a constant c_Q. So we propose the following correction of the BI (5.1.2) due to a source

$$S^{ab}_{cd} = -\frac{c_Q}{2} f^{-2} \ \epsilon_{2\perp cd} \ \epsilon_{2||ef} \ \eta^{ea} \eta^{fb} \ \delta^{(2)}(\rho) \ , \tag{5.2.47}$$

where we took into account the constraints on the indices. This results in the BI (1.4.26), and we have just shown that the Q-brane solves it.[19]

Next, we apply a similar procedure that yields the *KK*-monopole from T-dualizing the *NS*5-brane along one direction.The *NS*5-brane gets smeared along x and we use the smeared warp factor f_K of (5.2.3) instead of f_H. Here, spherical coordinates $\rho = r_3, \ \varphi, \ y$ are most appropriate and the smeared *NS*5-brane is given by

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f \ \mathrm{d}\hat{s}_4^2 \ , \ H_{mnp} = -\rho^2 \sin\varphi \ \epsilon_{4mnp\rho} \partial_\rho f \ , \ e^{2\phi} = f \\ &\text{where } \ \mathrm{d}\hat{s}_4^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2 + \rho^2 \sin^2\varphi \ \mathrm{d}y^2 + \mathrm{d}x^2 \ , \ f = f_K . \end{aligned} \tag{5.2.48}$$

Similarly to the discussion for the Q-brane, we introduce in curved indices

$$\begin{aligned} b_{mn} &= \epsilon_{4\rho\varphi mn} \ a(\rho,\varphi) \ , \ \text{with } a = \cos\varphi \ \rho^2 \partial_\rho f \ , \\ b_{xy} &= q_K \cos\varphi \ , \ H_{\varphi xy} = -q_K \sin\varphi \ , \ \text{for } \rho > 0 \ . \end{aligned} \tag{5.2.49}$$

Then, performing the T-duality along x we find

$$\begin{aligned} &\mathrm{d}s^2 = \mathrm{d}s_6^2 + f \ \mathrm{d}\hat{s}_3^2 + f^{-1} (\mathrm{d}x + a \mathrm{d}y)^2 \ , \ H_{mnp} = 0 \ , \ e^{2\phi} = 1 \\ &\text{where } \mathrm{d}\hat{s}_3^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2 + \rho^2 \sin^2\varphi \ \mathrm{d}y^2 \ , \end{aligned} \tag{5.2.50}$$

and recover the *KK*-monopole with f and a precisely those of (5.2.3).

In order to determine the corresponding geometric flux, we consider the following vielbein and its inverse in the basis (ρ,φ,y,x)

$$\tilde{e} = \begin{pmatrix} f^{\frac{1}{2}} & & & \\ & f^{\frac{1}{2}} \rho & & \\ & & f^{\frac{1}{2}} \rho \sin\varphi & \\ & & f^{-\frac{1}{2}} a & f^{-\frac{1}{2}} \end{pmatrix} \ , \ \tilde{e}^{-1} = \begin{pmatrix} f^{-\frac{1}{2}} & & & \\ & f^{-\frac{1}{2}} \rho^{-1} & & \\ & & f^{-\frac{1}{2}} \rho^{-1} \sin^{-1}\varphi & \\ & & -f^{-\frac{1}{2}} a \rho^{-1} \sin^{-1}\varphi & f^{\frac{1}{2}} \end{pmatrix} .$$

[19] A BI with a Q-brane source term was proposed in [172] and we conclude on a mismatch with our proposal (1.4.26) in the appendix of [102].

The non-trivial structure constants (A.3.10) are then found to be

$$f^{\varphi}{}_{\rho\varphi} = f^{y}{}_{\rho y} = \rho\ \partial_\rho(f^{-\frac{1}{2}}\rho^{-1})\ ,\ f^{x}{}_{\rho x} = f^{-1}\partial_\rho f^{\frac{1}{2}}\ ,\ f^{y}{}_{\varphi y} = f^{-\frac{1}{2}}\rho^{-1}\sin\varphi\ \partial_\varphi(\sin^{-1}\varphi)\ ,$$
$$f^{x}{}_{\varphi y} = -f^{-\frac{3}{2}}\rho^{-2}\sin^{-1}\varphi\ \partial_\varphi a = f^{-\frac{3}{2}}\partial_\rho f\ ,\quad f^{a}{}_{bc} = -f^{a}{}_{cb}\ . \tag{5.2.51}$$

As above, we mean flat indices on the LHS, and the derivatives carry curved indices on the RHS. Due to the ambiguity of the b-field of the $NS5$-brane and of a discussed around (5.2.37), we do not consider an $f^{x}{}_{\rho y}$ that would have been non-zero at the singularity. This way all T-dual branes carry one coinciding flux component in flat indices (φ, x, y) as they arise from the potential a. In particular, the value matches up to a sign. The other geometric components of f are mostly artefacts of the metric.

Finally the BIs (5.1.2) - (5.1.4) are trivially satisfied, since the KK-monopole does not give rise to other fluxes than geometric ones. The only interesting equation is then (5.1.1) and involves the quantity

$$S^{a}_{bcd} = \partial_{[b}f^{a}{}_{cd]} - f^{a}{}_{e[b}f^{e}{}_{cd]}\ . \tag{5.2.52}$$

By antisymmetry, $S^{\varphi}_{bcd} = 0$. In addition, one can verify

$$S^{y}_{\rho\varphi y} = \frac{1}{3}\left(f^{-\frac{1}{2}}\partial_\rho f^{y}{}_{\varphi y} + f^{y}{}_{y\varphi}f^{\varphi}{}_{\rho\varphi}\right) = 0\ . \tag{5.2.53}$$

Therefore, the only non-zero S^{a}_{bcd} is given by

$$\begin{aligned} S^{x}_{\rho\varphi y} &= \frac{1}{3}\left(f^{-\frac{1}{2}}\partial_\rho f^{x}{}_{\varphi y} - f^{x}{}_{\varphi y}\left(f^{x}{}_{x\rho} + f^{y}{}_{\rho y} + f^{\varphi}{}_{\rho\varphi}\right)\right) \\ &= -\frac{1}{3}\sin^{-1}\varphi f^{-2}\rho^{-2}\partial_\rho\partial_\varphi a \\ &= \frac{1}{3}f^{-2}\left(\partial_\rho^2 f + \frac{2}{\rho}\partial_\rho f\right) = \frac{1}{3}f^{-2}\Delta_3 f\ , \end{aligned} \tag{5.2.54}$$

where Δ_3 is the three-dimensional Laplacian used in spherical coordinates, since f only depends on ρ. We mentioned that f is the Green function for Δ_3 up to a constant c_K (5.2.24). Therefore, we propose the following correction of the BI (5.1.1) by a source term

$$S^{a}_{bcd} = \frac{c_K}{3}f^{-2}\ \epsilon_{3\perp bcd}\ \epsilon_{1||e}\ \eta^{ea}\ \delta^{(3)}(\rho)\ , \tag{5.2.55}$$

where the constraints on indices were taken into account, and $\epsilon_{1||e}$ is only non-zero and equal to one if e is the direction along the brane. This finally shows that the KK-monopole solves the BI (1.4.25).

For completeness, let us come back to the BI of the H-flux for the $NS5$-brane. We showed below (5.2.19) how this BI in curved indices leads to the Poisson equation. Going to flat indices by simple multiplication with vielbeins we obtain the quantity

$$S_{abcd} = \tilde{e}^{m}{}_{a}\tilde{e}^{n}{}_{b}\tilde{e}^{p}{}_{c}\tilde{e}^{q}{}_{d}\partial_{[m}H_{npq]} = \partial_{[a}H_{bcd]} - \frac{3}{2}f^{e}{}_{[ab}H_{cd]e}\ . \tag{5.2.56}$$

In Cartesian coordinates, the vielbeins are just given by $f^{\frac{1}{2}}$.[20] So from (5.2.19), (5.2.21), and the above, we propose the following contribution of the source

$$S_{abcd} = -\frac{c_H}{4} f^{-2}\ \epsilon_{4\perp abcd}\ \delta^{(4)}(r_4)\ , \tag{5.2.57}$$

where only the numerical factor should be verified, and the convention for ϵ_4 is in appendix A.2. This results in the BI for the *NS*5-brane

$$NS5\text{-brane}: \qquad \partial_{[a}H_{bcd]} - \frac{3}{2} f^{e}{}_{[ab}H_{cd]e} = \frac{c_H}{4}\ \epsilon_{4\perp abcd}\ \delta^{(4)}(r_4)\ . \tag{5.2.58}$$

In summary, we described a set of T-dual brane configurations beginning with the *NS*5-brane and work out their BIs that get corrected by source terms. We presented a consistent smearing procedure which is need to perform the T-duality transformation on the respective *NS*-branes. In particular, the Q-brane receives an underlying geometric picture and turns out to be a solution to β-supergravity. Hypothetically, there are hints for the existence of another brane sourcing R-flux, called R-brane. However, the corresponding T-duality on the Q-brane is not a standard one. Hence, we could only determine its respective warp factor.

[20] For the three branes, we obtained a factor f^{-2} next to the δ in the source contributions to the BI. It would be better to have a generic formula that reproduces this factor, for instance with volumes or vielbeins, but we did not find any.

Chapter 6

Aspects of supersymmetry

So far, our main focus laid on the NSNS sector of β-supergravity and its relation to standard supergravity (SUGRA) theories. However, the latter experience manifest supersymmetry (SUSY) by containing in addition a RR sector with various p-form fields depending on the type of SUGRA theory considered and a specific set of fermionic fields. Despite the naming, it is an open task to introduce these sectors for β-supergravity. In this section, we are rather considering two minor aspects of SUSY instead of trying to complete the full picture. First, we observe that the SUSY variations restricted to only including NS-fluxes are provided by the previously defined $Spin(d-1,1) \times Spin(1,d-1)$ derivatives and can be reformulated in terms of pure spinor equations. This allows, to some extent, to specify supersymmetric vacuum solutions of β-supergravity. A second interesting field with respect to uplifting four-dimensional gauged SUGRA theories is the determination of a proper superpotential in β-supergravity that contains in addition to the standard fluxes also the non-geometric ones. It turns out that the Dirac operator $\mathcal{D}$ defined in the previous chapter plays an important role. We further note that we slightly change the notation provided in the appendix A.1 in the following since the focus of this section is now on six-dimensional objects associated with an internal space.

6.1 From SUSY variations to pure spinor conditions

In [26], the SUSY conditions for four-dimensional $\mathcal{N}=1$ SUSY were reformulated in terms of Generalized Complex Geometry (GCG) as pure spinor equations. Here, we follow their approach by first specifying the SUSY variations in β-supergravity, then introducing a consistent compactification ansatz and rewriting the former on pure spinors. In this way, we hope to characterize, in particular geometrically, the allowed backgrounds and find analogue classes of internal spaces that satisfy a respective modification of the generalized Calabi-Yau (GCY) condition.

6.1.1 Fermionic supersymmetry variations

As explained, β-supergravity is a local reformulation of standard SUGRA and thus it is expected to possess a supersymmetric completion which has yet to be constructed. For this reason we can not read off the SUSY variations of the field content leaving a the supersymmetric version of the Lagrangian $\tilde{\mathcal{L}}_\beta$ invariant. Nevertheless, we gain here the fermionic SUSY variations in an indirect way by observing their appearance in the Generalized Geometry (GG) formulation of standard SUGRA. Some structures appearing in the GG formalism and in Double Field Theory (DFT), namely the $Spin(d-1,1) \times Spin(1,d-1)$ derivatives [109, 110, 55, 114, 39, 69] or generalizations, were noticed to give these variations for standard SUGRA [39, 69, 70, 71, 46, 173, 64, 49]. Moreover, these quantities enter the Lagrangian and the equations of motion. Since we showed in section 3.2 that the corresponding derivatives in β-supergravity exactly play the same role, we naturally assume that the derivatives give analogously in β-supergravity the SUSY variations.

Type IIA and IIB standard SUGRA have two pairs of chiral fermions. We repeat here for convenience the NSNS contribution to their SUSY variations, given by

$$\begin{aligned} \delta\psi_M^{1,2} =& e^A{}_M \left(\nabla_A \mp \frac{1}{8} H_{ABC}\Gamma^{BC} \right) \epsilon^{1,2} \ , \\ \delta\rho^{1,2} =& \Gamma^A \left(\nabla_A \mp \frac{1}{24} H_{ABC}\Gamma^{BC} - \partial_A\phi \right) \epsilon^{1,2} \ , \end{aligned} \tag{6.1.1}$$

where we use a different index notations from now on as listed in appendix A.1. The SUSY fermionic parameters $\epsilon^{1,2}$ are the same as before with $1,2$ refering respectively to the upper/lower sign. In [39], the above variations[1] have been rephrased in terms of the following Spin(9,1)×Spin(1,9) derivatives

$$\begin{aligned} D_A\epsilon^2 &= \left(\nabla_A + \frac{1}{8} H_{A\overline{BC}}\Gamma^{\overline{BC}} \right) \epsilon^2 \ , \\ D_{\overline{A}}\epsilon^1 &= \left(\nabla_{\overline{A}} - \frac{1}{8} H_{\overline{A}BC}\Gamma^{BC} \right) \epsilon^1 \ , \\ \Gamma^A D_A\epsilon^1 &= \left(\Gamma^A\nabla_A - \frac{1}{24} H_{ABC}\Gamma^{ABC} - \Gamma^A\partial_A\phi \right) \epsilon^1 \ , \\ \Gamma^{\overline{A}} D_{\overline{A}}\epsilon^2 &= \left(\Gamma^{\overline{A}}\nabla_{\overline{A}} + \frac{1}{24} H_{\overline{ABC}}\Gamma^{\overline{ABC}} - \Gamma^{\overline{A}}\partial_{\overline{A}}\phi \right) \epsilon^2 \ , \end{aligned} \tag{6.1.2}$$

where the indices $_{A,\ \overline{A}}$ and spinors $\epsilon^1,\ \epsilon^2$ correspond to each Spin group respectively. Then, the SUSY variations in [39] for standard SUGRA take the following form

$$\begin{aligned} &\delta\psi_M^1 = e^{\overline{A}}{}_M D_{\overline{A}}\epsilon^1 \ , \ \delta\psi_M^2 = e^A{}_M D_A\epsilon^2 \ , \\ &\delta\rho^1 = \Gamma^A D_A\epsilon^1 \ , \ \delta\rho^2 = \Gamma^{\overline{A}} D_{\overline{A}}\epsilon^2 \ , \end{aligned} \tag{6.1.3}$$

[1] These conventions match those of [39], except for the $\pm$ there denoted $1,2$ here, and the use here of flat indices.

using (A.2.7) and aligned vielbeins. The fermionic SUSY variations of β-supergravity should be as well given by (6.1.3), where we replace the vielbein e by $\tilde{e}$, as argued before, and we use the following derivatives determined in section 3.2

$$
\begin{aligned}
D_A\epsilon^2 &= \left(\nabla_A - \eta_{AD}\check{\nabla}^D - \frac{1}{8}\eta_{AD}\eta_{\overline{BE}}\eta_{\overline{CF}}R^{D\overline{EF}}\Gamma^{\overline{BC}}\right)\epsilon^2 \ , \\
D_{\overline{A}}\epsilon^1 &= \left(\nabla_{\overline{A}} + \eta_{\overline{AD}}\check{\nabla}^{\overline{D}} - \frac{1}{8}\eta_{\overline{AD}}\eta_{BE}\eta_{CF}R^{\overline{D}EF}\Gamma^{BC}\right)\epsilon^1 \ , \\
\Gamma^A D_A\epsilon^1 &= \left(\Gamma^A\nabla_A - \Gamma^A\eta_{AD}\check{\nabla}^D + \frac{1}{24}\eta_{AD}\eta_{BE}\eta_{CF}R^{DEF}\Gamma^{ABC} - \Gamma^A\partial_A\tilde{\phi} - \Gamma^A\eta_{AB}(\beta^{BC}\partial_C\tilde{\phi} - \mathcal{T}^B)\right)\epsilon^1 \ , \\
\Gamma^{\overline{A}} D_{\overline{A}}\epsilon^2 &= \left(\Gamma^{\overline{A}}\nabla_{\overline{A}} + \Gamma^{\overline{A}}\eta_{\overline{AD}}\check{\nabla}^{\overline{D}} + \frac{1}{24}\eta_{\overline{AD}}\eta_{\overline{BE}}\eta_{\overline{CF}}R^{\overline{DEF}}\Gamma^{\overline{ABC}} - \Gamma^{\overline{A}}\partial_{\overline{A}}\tilde{\phi} + \Gamma^{\overline{A}}\eta_{\overline{AB}}(\beta^{\overline{BC}}\partial_{\overline{C}}\tilde{\phi} - \mathcal{T}^{\overline{B}})\right)\epsilon^2 \ ,
\end{aligned} \tag{6.1.4}
$$

where the covariant derivatives ∇_A and $\check{\nabla}^A$, defined in appendix A.3, on spinors are

$$
\begin{aligned}
\nabla_A\epsilon &= \partial_A\epsilon + \frac{1}{4}\omega^B_{\ AC}\eta_{BD}\Gamma^{DC}\epsilon \ , \\
\check{\nabla}^A\epsilon &= -\beta^{AB}\partial_B\epsilon + \frac{1}{4}\omega_{Q\ C}^{\ AB}\eta_{BD}\Gamma^{DC}\epsilon \ ,
\end{aligned} \tag{6.1.5}
$$

and ω_Q is the spin connection of $\check{\nabla}$, related to the Q-flux as in (E.1.6). Then from (6.1.3) and (6.1.4), with aligned vielbeins, we deduce the NSNS contribution to the fermionic SUSY variations of both type IIA and IIB β-supergravity, given by (1.4.28), which we repeat here for convenience

$$
\begin{aligned}
\delta\psi_M^{1,2} &= \tilde{e}^A{}_M\left(\nabla_A \pm \eta_{AD}\check{\nabla}^D - \frac{1}{8}\eta_{AD}\eta_{BE}\eta_{CF}R^{DEF}\Gamma^{BC}\right)\epsilon^{1,2} \\
\delta\rho^{1,2} &= \left(\Gamma^A\nabla_A \mp \Gamma^A\eta_{AD}\check{\nabla}^D + \frac{1}{24}\eta_{AD}\eta_{BE}\eta_{CF}R^{DEF}\Gamma^{ABC} - \Gamma^A\partial_A\tilde{\phi} \mp \Gamma^A\eta_{AB}(\beta^{BC}\partial_C\tilde{\phi} - \mathcal{T}^B)\right)\epsilon^{1,2} \ .
\end{aligned} \tag{6.1.6}
$$

6.1.2 Compactification ansatz and conditions on SUSY vacua

We now specify an ansatz for the fields, suited to the compactification of a ten-dimensional background on a compact internal six-dimensional manifold $\mathcal{M}$. Then, we will study the decomposition of the previous SUSY variations accordingly, and deduce conditions for a SUSY vacuum. To start with, we consider the following ten-dimensional metric

$$
\mathrm{d}\tilde{s}^2 = e^{2A(y)}\tilde{g}_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + \tilde{g}_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n \ , \tag{6.1.7}
$$

where Latin indices denote internal indices, Greek indices denote four-dimensional ones, as listed in appendix A.1, and e^{2A} is the warp factor. The ten-dimensional vielbeins are then decomposed into

$$
\tilde{e}^{A=\alpha}{}_{M=\mu} = \tilde{e}^\alpha{}_\mu = e^{A(y)}\tilde{e}^{\dot{\alpha}}{}_\mu(x) \ , \quad \tilde{e}^{A=a}{}_{M=m} = \tilde{e}^a{}_m(y) \ . \tag{6.1.8}
$$

The ten-dimensional β^{AB} is chosen a priori non-trivial only along the internal directions, i.e.

$$\beta^{\alpha\beta} = \beta^{\alpha b} = \beta^{a\beta} = 0 \ , \ \beta^{ab}(y) \ . \tag{6.1.9}$$

This ansatz allows to compute the various components of the ten-dimensional fluxes $f,\ Q,\ R$, and those of the spin connections ω and ω_Q. Details can be found in appendix E.1.

We now turn to the spinors, in particular to the ten-dimensional SUSY parameters. In agreement with the above metric, the Lorentz group and its spinorial representation splits into two factors. Accordingly, $\epsilon^{1,2}$ should be decomposed on a basis of internal spinors. However, since we restrict ourselves to backgrounds with $\mathcal{N}=1$ SUSY in four dimensions, we only consider one external spinor ζ_+. The decomposition with two internal spinors $\eta_+^{1,2}$ yields then

$$\text{IIA} \begin{cases} \epsilon^1 = \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1 \\ \epsilon^2 = \zeta_+ \otimes \eta_-^2 + \zeta_- \otimes \eta_+^2 \end{cases} , \qquad \text{IIB} \begin{cases} \epsilon^1 = \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1 \\ \epsilon^2 = \zeta_+ \otimes \eta_+^2 + \zeta_- \otimes \eta_-^2 \end{cases} . \tag{6.1.10}$$

The distinct theories are specified by their chiralities denoted by $\pm$. The six-dimensional and four-dimensional spinors are Weyl, and Euclidian respectively Lorentzian. This implies the complex conjugations $(\eta_+^i)^* = \eta_-^i$ and $(\zeta_+)^* = \zeta_-$. We also consider $\eta_+^i(y)$ and $\zeta_+(x)$. The ten-dimensional spinors are Majorana-Weyl, hence real, as given by the sum of the two terms in ϵ^i. The ten-dimensional Γ-matrices are decomposed similarly, in terms of the six-dimensional γ^a and four-dimensional γ^α. The properties of the latter, together with the components of the spin connections, allow to obtain the various components of the two spinorial covariant derivatives ∇_A and $\check{\nabla}^A$. This is detailed in appendix E.1.

Finally, let us further specify the external part for a four-dimensional maximally symmetric space-time. The three possibilities Mink, AdS and de Sitter forbid to single out a vector, implying $\partial_\alpha\tilde{\phi} = 0$ in the background. In addition, the four-dimensional covariant derivative $\nabla_\alpha\zeta_\pm$, generically decomposed on a spinor basis, is then at most given by

$$\nabla_\alpha \zeta_\pm = \frac{1}{2}\mu_\pm e^{-A} \eta_{\alpha\beta}\gamma^\beta \zeta_\mp \ . \tag{6.1.11}$$

Let us comment on the coefficient. The covariant derivative ∇_α carries a factor e^{-A} which can be seen by multiplication of th standard derivative ∂_μ by the vielbein $\tilde{e}^\mu{}_\alpha$. Similarly the relation for the spin connection $\omega_{\alpha\gamma}^\beta = e^{-A}\omega_{\dot{\alpha}\dot{\gamma}}^{\dot{\beta}}$ holds. The matrices γ^α do not carry such a factor, since they satisfy the Clifford algebra. Therefore, we get $\nabla_\alpha = e^{-A}\nabla_{\dot{\alpha}}$. This factor, manifest on the RHS of (6.1.11), then carries the whole dependence on the internal coordinates. Furthermore, the generic $\mu_\pm$ is restricted to be a complex function of the external coordinates.[2] For Mink and AdS [3], the value $|\mu_\pm|^2$ is actually known to be related to the scalar curvature,[4] i.e. to the cosmological constant. We thus restrict $\mu_\pm$

[2]The factor $\mu_\pm$, defined in (6.1.11), can a priori change according to the theory IIA or IIB, although we do not denote it differently here. In [174], this quantity is related to the vacuum value of the superpotential, with a sign change between the theories. Here, we will rather introduce the sign ε.

[3]De Sitter does not allow to consider such a spinorial equation.

[4]This is probably derived considering the commutator of two ∇.

to be complex constants, differing at most by a phase. SUSY will impose μ_+ to be the complex conjugate of $\mu_- \equiv \mu$.

We are now interested in vacua satisfying the compactification ansatz described above and preserving $\mathcal{N}=1$ SUSY. While not strictly necessary for SUSY, we make the additional helpful assumption that the internal spinors η^i_+ are globally defined and non-vanishing. Given a metric and an orientation on $\mathcal{M}$, this further assumption reduces the structure group of the tangent bundle, for one spinor to $SU(3)$, and for two generic ones to $SU(2)$. Then, the existence of a reduced structure group is a useful topological constraint allowing to equivalently consider specific globally defined forms on $\mathcal{M}$. In GCG, the structure group of $T\mathcal{M}\oplus T^*\mathcal{M}$ is reduced to $SU(3)\times SU(3)$ by the existence of globally defined η^i_+. This equivalently defines the pure spinors $\Phi_\pm$ that can be viewed as polyforms, as described in section 6.1.3. In addition to this topological condition, the spinors will have to satisfy differential conditions. These are derived from the fermionic SUSY variations, that have to vanish in a SUSY background. We obtain these conditions in the following, by setting (6.1.6) to zero for the above compactification ansatz.

We start with the vanishing variation of the gravitini (6.1.6). In type IIB, it gives on the internal directions

$$\begin{aligned}&\zeta_+\otimes\left(\nabla_a \pm \eta_{ad}\check{\nabla}^d - \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\right)\eta^{1,2}_+\\ &+\zeta_-\otimes\left(\nabla_a \pm \eta_{ad}\check{\nabla}^d - \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\right)\eta^{1,2}_- = 0\ ,\end{aligned} \tag{6.1.12}$$

and in type IIA one should change the chirality on η^2. Projecting by chirality imposes both lines to vanish and since the two lines are complex conjugate to each other, the only conditions, in type IIB, left is

$$\left(\nabla_a \pm \eta_{ad}\check{\nabla}^d - \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\right)\eta^{1,2}_+ = 0\ . \tag{6.1.13}$$

On the external directions, we obtain in type IIB

$$\begin{aligned}&\left(\nabla_\alpha\otimes\mathbb{1} + \frac{1}{2}\eta_{\alpha\beta}\gamma^\beta\gamma_{(4)}\otimes\gamma^d\partial_d A \pm \frac{1}{2}\eta_{\alpha\delta}\gamma^\delta\gamma_{(4)}\otimes\gamma^c\eta_{cd}\beta^{de}\partial_e A\right)\epsilon^{1,2} = 0\\ \Leftrightarrow\quad &\frac{1}{2}\eta_{\alpha\beta}\gamma^\beta\zeta_+\otimes\left(\mu_- e^{-A}\eta^{1,2}_- + (\gamma^d\partial_d A \pm \gamma^c\eta_{cd}\beta^{de}\partial_e A)\eta^{1,2}_+\right)\\ &+\frac{1}{2}\eta_{\alpha\beta}\gamma^\beta\zeta_-\otimes\left(\mu_+ e^{-A}\eta^{1,2}_+ - (\gamma^d\partial_d A \pm \gamma^c\eta_{cd}\beta^{de}\partial_e A)\eta^{1,2}_-\right) = 0\ ,\end{aligned} \tag{6.1.14}$$

and for type IIA one should change the chirality of the η^2. Again, both lines should vanish and we deduce $\mu_+^* = \mu_- \equiv \mu$. Then, the only conditions left in type IIB are

$$\mu\ \eta^{1,2}_- + e^A\gamma^d\left(\partial_d A \pm \eta_{dc}\beta^{ce}\partial_e A\right)\eta^{1,2}_+ = 0\ . \tag{6.1.15}$$

We note here that having Mink, i.e. vanishing μ, is equivalent to a constant warp factor which is due to the fact we only have NSNS contributions.[5]

We finally turn to the variation $\delta\rho^{1,2}$ in (6.1.6). A few computations, using in particular $\gamma^\alpha \eta_{\alpha\beta}\gamma^\beta = 4$, lead to

$$\begin{aligned}
\delta\rho^{1,2} =& \Bigg(\gamma^\alpha \nabla_\alpha \otimes \mathbb{1} + \gamma_{(4)} \otimes \frac{1}{24}\eta_{ad}\eta_{be}\eta_{cf} R^{def}\gamma^{abc} \\
&+ \gamma_{(4)} \otimes \gamma^a \Big(\nabla_a + \partial_a(2A-\tilde{\phi}) \mp \eta_{ad}\check{\nabla}^d \mp \eta_{ad}\beta^{de}\partial_e(2A+\tilde{\phi}) \pm \eta_{ad}\mathcal{T}^d\Big)\Bigg)\epsilon^{1,2} \\
=& \zeta_+ \otimes \left(2\mu_- e^{-A}\eta_-^{1,2} + \Big(\frac{1}{24}R^{abc}\gamma_{abc} + \gamma^a\Big(\nabla_a + \partial_a(2A-\tilde{\phi}) \mp \eta_{ad}\check{\nabla}^d \mp \eta_{ad}\beta^{de}\partial_e(2A+\tilde{\phi}) \pm \eta_{ad}\mathcal{T}^d\Big)\Big)\eta_+^{1,2}\right) \\
&+ \zeta_- \otimes \left(2\mu_+ e^{-A}\eta_+^{1,2} - \Big(\frac{1}{24}R^{abc}\gamma_{abc} + \gamma^a\Big(\nabla_a + \partial_a(2A-\tilde{\phi}) \mp \eta_{ad}\check{\nabla}^d \mp \eta_{ad}\beta^{de}\partial_e(2A+\tilde{\phi}) \pm \eta_{ad}\mathcal{T}^d\Big)\Big)\eta_-^{1,2}\right)
\end{aligned}$$

for type IIB, while for type IIA one should change the chirality on the η^2. We further lowered indices on the fully antisymmetric γ using the metric η. Setting this variation to zero imposes both lines to vanish, from which we deduce again $\mu_+^* = \mu_- = \mu$, and the two lines are then complex conjugates. We are left with

$$2\mu\, \eta_-^{1,2} + e^A\Big(\frac{1}{24}R^{abc}\gamma_{abc} + \gamma^a\Big(\nabla_a + \partial_a(2A-\tilde{\phi}) \mp \eta_{ad}\check{\nabla}^d \mp \eta_{ad}\beta^{de}\partial_e(2A+\tilde{\phi}) \pm \eta_{ad}\mathcal{T}^d\Big)\Big)\eta_+^{1,2} = 0\ .$$

In summary, we look for backgrounds that satisfy the above compactification ansatz, admit an $SU(3)\times SU(3)$ structure and verify in type IIB[6] the following three Killing spinor equations or SUSY conditions

$$\mu\, \eta_-^{1,2} + e^A \left(\partial\!\!\!/ A \pm \beta\!\!\!/_\partial A\right)\eta_+^{1,2} = 0\ , \tag{6.1.16}$$

$$\nabla_a \eta_+^{1,2} = \left(\mp\eta_{ad}\check{\nabla}^d + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\right)\eta_+^{1,2}\ , \tag{6.1.17}$$

$$\nabla\!\!\!\!/\, \eta_+^{1,2} = -2\mu e^{-A}\eta_-^{1,2} - \left(\frac{1}{4}R\!\!\!/ + \partial\!\!\!/(2A-\tilde{\phi}) \mp \check{\nabla\!\!\!\!/} \mp \beta\!\!\!/_\partial(2A+\tilde{\phi}) \pm \mathcal{T}\!\!\!\!/\right)\eta_+^{1,2}\ , \tag{6.1.18}$$

where we introduced the notations

$$\begin{gathered}
\partial\!\!\!/ = \gamma^a\partial_a\ ,\ \nabla\!\!\!\!/ = \gamma^a\nabla_a\ ,\ \beta\!\!\!/_\partial = \gamma^a\eta_{ab}\beta^{bc}\partial_c\ ,\ \check{\nabla\!\!\!\!/} = \gamma^a\eta_{ab}\check{\nabla}^b\ , \\
\mathcal{T}\!\!\!\!/ = \gamma^a\eta_{ab}\mathcal{T}^b\ ,\ R\!\!\!/ = \frac{1}{6}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{abc}\ .
\end{gathered} \tag{6.1.19}$$

[5]We reached the same conclusion in [101], using the equations of motion and a few more assumptions, such as a constant dilaton. Here, it comes from SUSY.

[6]For type IIA, one should change the chirality on η^2.

6.1.3 Supersymmetry conditions in terms of pure spinors

Now, we reformulate the previous SUSY conditions using pure spinors $\Phi_\pm$. We follow closely the procedure described in the appendix of [26] and first define $\Phi_\pm$ as the following bispinors

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\dagger}_+ , \quad \Phi_- = \eta^1_+ \otimes \eta^{2\dagger}_- \ . \tag{6.1.20}$$

The product can be expressed thanks to the Fierz identity given in six dimensions by

$$\eta^1_+ \otimes \eta^{2\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\dagger}_\pm \gamma_{a_k \dots a_1} \eta^1_+ \right) \gamma^{a_1 \dots a_k} \ , \tag{6.1.21}$$

where indices are lowered by the flat metric. In addition, the Clifford map relates antisymmetric products of γ-matrices and differential forms

$$C = \sum_k \frac{1}{k!} C^{(k)}_{a_1 \dots a_k} \tilde{e}^{a_1} \wedge \dots \wedge \tilde{e}^{a_k} \ \longleftrightarrow \ \not{C} = \sum_k \frac{1}{k!} C^{(k)}_{a_1 \dots a_k} \gamma^{a_1 \dots a_k} \ . \tag{6.1.22}$$

Thanks to this map, the above pure spinors can be viewed as polyforms, i.e. sums of forms of different degrees. Further, we note here that $\Phi_\pm$ in (6.1.20), expressed with the Fierz identity, should be understood as slashed following the notation of (6.1.22). $\Phi_\pm$ are examples of $Spin(6,6)$ spinors on $T\mathcal{M} \oplus T^*\mathcal{M}$, as considered in GCG. Being bispinors, they are as well $Spin(6) \times Spin(6)$ spinors. They are additionally pure because they are built from two pure spinors.[7] As mentioned in the previous section, we require them to be globally defined which reduces the structure group of $T\mathcal{M} \oplus T^*\mathcal{M}$ to $SU(3) \times SU(3)$. On the tangent bundle, this reduces the structure group to $SU(3)$ or $SU(2)$. For example, $\Phi_\pm$ for an $SU(3)$ structure are given as polyforms in (6.2.3) in a simplified case. More details on such examples are provided in section 6.2.1. The pure spinors are acted on by $Cliff(6,6)$ Γ-matrices, from which one can construct a chirality operator. Through the Clifford map, their chirality is simply related to the degree of the forms, i.e. the summation runs only over forms of even, respectively odd, degree, for positive, respectively negative, chirality. Respectively, Φ_+ or Φ_- are of positive or negative chirality through the number of γ-matrices via the Fierz identity.

Reformulating the SUSY conditions on the η^i (6.1.16) as polyform equations on the pure spinors essentially amounts to compute the exterior derivative

$$\mathrm{d}\Phi_\pm \equiv \tilde{e}^a \wedge \nabla_a \Phi_\pm \ . \tag{6.1.23}$$

In particular, we write (6.1.23) with γ-matrices acting on the η^i, thanks to the Clifford map and the bispinor expressions. We then use the above SUSY conditions, and finally rewrite the resulting expression in terms of forms, using the Clifford map backwards. The whole procedure, with the required properties of the Clifford map, are detailed in appendix E.2. We note here the following subtlety. We obtain at first expressions for $\mathrm{d}\Phi_\pm$ that are simpler

[7] In six dimension any spinor is pure.

than the final ones (1.4.29) and (1.4.30). This is due to the absence of a RR contribution. We nevertheless follow further the procedure of [26] for standard SUGRA, and construct from (6.1.18) another form expression, that should be given by the RR fluxes but is here vanishing. We add this quantity to one of the $\mathrm{d}\Phi_\pm$ obtained, as was done in [26]. This eventually results in (1.4.29) and (1.4.30), that we repeat here for convenience

$$e^{\tilde\phi}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde\phi}\Phi_1\right) + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_1 = 2\varepsilon\ e^{-A}\mu\ \mathrm{Re}(\Phi_2) \tag{6.1.24}$$

$$\begin{aligned} e^{\tilde\phi}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde\phi}\Phi_2\right) &+ e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_2 \\ &= 3\varepsilon\ e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_1) + e^{-A}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a\right)(e^{A})\overline{\Phi_2}\ , \end{aligned} \tag{6.1.25}$$

where $\Phi_{1,2}$ and ε depend on the theory (1.4.31).[8] The sign ε can be viewed as a change of μ in between the two theories, see footnote 2.

Let us comment on these pure spinor conditions, and compare them to those of standard SUGRA

$$e^{\phi}\left(\mathrm{d} - H\wedge\right)\left(e^{-\phi}\Phi_1\right) + e^{-2A}\mathrm{d}(e^{2A}) \wedge \Phi_1 = 2\varepsilon\ e^{-A}\mu\ \mathrm{Re}(\Phi_2) \tag{6.1.26}$$

$$e^{\phi}\left(\mathrm{d} - H\wedge\right)\left(e^{-\phi}\Phi_2\right) + e^{-2A}\mathrm{d}(e^{2A}) \wedge \Phi_2 = 3\varepsilon\ e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_1) + e^{-A}\mathrm{d}(e^{A}) \wedge \overline{\Phi_2} + \mathrm{RR}\ , \tag{6.1.27}$$

where RR denotes a contribution from the RR sector. The NSNS sector of the two theories are known to match for vanishing b and β. Here, one can verify that the pure spinor conditions do agree in that case, which is a non-trivial check of our result. More generally, it is remarkable that the differential operator acting on the pure spinors in both theories is precisely the Dirac operator $\mathcal{D}$ (1.4.21) and (1.4.23).

Next, we focus on the particular case of a Mink space-time ($\mu = 0$). In standard SUGRA this provides an interesting characterization of the background, as already discussed in the Introduction. Here, (6.1.24) and (6.1.25) reduce to

$$e^{\tilde\phi}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde\phi}\Phi_1\right) + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_1 = 0 \tag{6.1.28}$$

$$e^{\tilde\phi}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde\phi}\,\mathrm{Re}\,\Phi_2\right) + e^{-A}\left(\mathrm{d} + 3\check{\nabla}^a \cdot \iota_a\right)(e^{A})\,\mathrm{Re}\,\Phi_2 = 0 \tag{6.1.29}$$

$$e^{\tilde\phi}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde\phi}\,\mathrm{Im}\,\Phi_2\right) + e^{-A}\left(3\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{A})\,\mathrm{Im}\,\Phi_2 = 0\ . \tag{6.1.30}$$

[8]The conditions for a vacuum to preserve SUSY in various theories have been reformulated into analogues of the pure spinor conditions, applying similar techniques to space(-time)s of dimension d where the external part is either Minkowski or Anti de Sitter. For type II and M-theory, such conditions were obtained for d from 11 down to 3 in [175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186]. Rewritings of the previous conditions were worked out in [187, 188, 189]. The heterotic case for $\mathcal{N} = 1$ Mink vacuum was treated in [190]. Finally conditions for type II Mink vacua were also written in terms of Exceptional Generalized Geometry (EGG) in [191, 192, 193].

In contrast to standard SUGRA, the warp factor terms can not be factorized with the dilaton, because of the sign in front of $\check{\nabla}^a(e^{2A})\iota_a$. In other words, the condition (6.1.28) cannot be written as a pure spinor closed under the Dirac operator $\mathcal{D}$ of (1.4.23). Although an analogue of the GCY condition is thus not provided, we still have the analogous to the generalized complex structure condition, as discussed in section 6.3. Since these warp factor terms cannot be absorbed within $\mathcal{D}$, they can be understood as an effect due to the compactification, that goes beyond the manifold $\mathcal{M}$ and the Spin(6,6)$\times\mathbb{R}^+$ structure of $\mathcal{D}$. Nevertheless, in the particular case of backgrounds for which $\beta^{ab}\partial_b A = 0$, e.g. when β is only non-zero along isometry directions, the warp factor terms can be factorized. We are then back to a situation analogous to standard SUGRA, where the pure spinor conditions are expressed purely in terms of $\mathcal{D}$ and additional RR contributions. The corresponding background characterization is discussed in section 6.3.

The pure spinor conditions (6.1.24) and (6.1.25) have been derived using the three SUSY conditions (6.1.16)-(6.1.18), meaning that the former are necessary for SUSY to be preserved in the backgrounds considered. It is important to study whether they are also sufficient. In addition, this would allow us to solve form equations, which is more convenient, instead of the Killing spinor equations. Following the method of [26], we address this question in appendix E.3. We conclude from there that (6.1.24) and (6.1.25) are not sufficient and contain a remaining freedom or ambiguity with respect to the SUSY conditions. However, we argue that this ambiguity should be fixed in presence of RR fluxes. Moreover, the RR contribution is expected to simply consist in an addition to the RHS of (6.1.25), in analogy to standard SUGRA with (6.1.27). Therefore, the results established above remain useful. The discussion on the structures appearing, such as the Dirac operator and the related background characterization, which we address later, are in any case relevant.

6.2 The superpotential

In this section, we come back to pure spinors defining an $SU(3) \times SU(3)$ structure and discuss their properties. These allow to write down an expression for the NSNS part of the superpotential obtained from standard SUGRA, as reviewed. The appearance of the Dirac operator leads us to propose a corresponding expression ensuing from β-supergravity, that includes non-geometric fluxes. This proposal is compared to the literature and further discussed.

6.2.1 $SU(3) \times SU(3)$ structure pure spinors for $\mathcal{N} = 1$ SUSY

The pure spinors $\Phi_\pm$ have been defined as bispinors (6.1.20) in terms of the internal spinors $\eta^{1,2}_\pm$. For globally defined η^i_+, one obtains an $SU(3)\times SU(3)$ structure group on $T\mathcal{M}\oplus T^*\mathcal{M}$, as noted in the sections 6.1.2 and 6.1.3. The corresponding structure group of the tangent bundle then depends on the two internal spinors η^1_+ and η^2_+. If these are parallel, i.e.

proportional, one obtains an $SU(3)$ structure and otherwise one has an $SU(2)$ structure. For the latter, the differential conditions on the pure spinors impose to distinguish further two cases. Orthogonality between η_+^1 and η_+^2, i.e. a relation by a γ-matrix, leads to an orthogonal or static $SU(2)$ structure. More general, if the two spinors are neither parallel nor orthogonal, one finds an intermediate $SU(2)$ structure. If the angle between the spinors is actually varying on $\mathcal{M}$, one talks of a dynamical $SU(2)$ structure. Further references and a discussion can be found in [103].

Thanks to the Fierz identity and the Clifford map, the pure spinors can be viewed as polyforms. The corresponding formulas for $\Phi_\pm$ vary accordingly to the above cases and different expressions can be found e.g. in [194]. For an $SU(3)$ structure one finds

$$\Phi_+ = \frac{|a|^2}{8} e^{\mathrm{i}\theta_+} e^{-iJ} \ , \qquad \Phi_- = -\mathrm{i} e^{\mathrm{i}\theta_-} \frac{|a|^2}{8} \Omega \ , \tag{6.2.1}$$

where J is a real (1,1)-form and Ω is a (3,0)-form, with respect to an almost complex structure. On a Calabi-Yau (CY) manifold these forms satisfying additional constraints are known as the Kähler form and the holomorphic 3-form. The quantity $|a|$ is related to the norm of the internal spinors which we take here to be equal, as is the case in presence of an orientifold plane. This will also allow us to set $|a|^2 = e^A$. In the following we will consider a constant warp factor and constant phases $\theta_\pm$, as done for most of the formulas for the superpotential in the literature. The orientifold when present then fixes the phase θ_+, while θ_-, not being physical, is left free [195]. We thus choose for convenience $\theta_- = \pi$, while θ_+ will be fixed as

$$\text{O3 or O7} : e^{\mathrm{i}\theta_+} = \pm \mathrm{i} \ , \ \text{O5 or O9} : e^{\mathrm{i}\theta_+} = \pm 1 \ , \ \text{O6} : e^{\mathrm{i}\theta_+} \text{ is free.} \tag{6.2.2}$$

Note that O4- or O8-planes do not allow for an $SU(3)$ structure. Also, for an O6-plane, $e^{\mathrm{i}\theta_+}$ is sometimes taken to be 1 in the literature. Given the fixing of these various parameters, we will consider in the following the simpler $SU(3)$ structure pure spinors (1.4.33)

$$\Phi_+^0 = e^{\mathrm{i}\theta_+} e^{-iJ} \ , \qquad \Phi_-^0 = \mathrm{i}\Omega \ . \tag{6.2.3}$$

With the above assumptions, the pure spinor conditions for SUSY in standard SUGRA (6.1.26) and (6.1.27) simplify to

$$e^{\phi} (\mathrm{d} - H\wedge) \left(e^{-\phi}\Phi_1^0\right) = 2\varepsilon \ e^{-A} \mu \ \mathrm{Re}(\Phi_2^0) \tag{6.2.4}$$

$$e^{\phi} (\mathrm{d} - H\wedge) \left(e^{-\phi}\Phi_2^0\right) = 3\varepsilon \ e^{-A} \ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_1^0) + \ \mathrm{RR} \ . \tag{6.2.5}$$

In general, two pure spinors Φ_1 and Φ_2 of GCG defining an $SU(3) \times SU(3)$ structure satisfy some compatibility conditions. Those include conditions on the norms, that need not be specified here, and

$$\langle \Phi_1, (v \vee + \xi\wedge)\Phi_2 \rangle = \langle \overline{\Phi_1}, (v \vee + \xi\wedge)\Phi_2 \rangle = 0 \ , \quad \forall (v+\xi) \in T\mathcal{M} \oplus T^*\mathcal{M} \ , \tag{6.2.6}$$

where the Mukai product is defined as taking the top form, here the six-form,

$$\langle \Psi_1, \Psi_2 \rangle = \Psi_1 \wedge \lambda(\Psi_2)|_{\mathrm{top}} \ , \tag{6.2.7}$$

and λ brings a sign by reversing all the form indices. The compatibility condition (6.2.6) can also be formulated with matrices $\Gamma^{\mathcal{A}}$. In the case of an $SU(3)$ structure, this leads to the condition

$$J \wedge \Omega = 0 \ , \tag{6.2.8}$$

which can also be understood from the almost complex structure.

We now turn to the superpotential for which we will use the various properties just described. The superpotential W of the $\mathcal{N}=1$ four-dimensional effective theory obtained from standard ten-dimensional SUGRA has been formulated in terms of GCG pure spinors for $SU(3)$ structure in [196, 197] and also [198], and then for an $SU(3) \times SU(3)$ in [199, 174, 200, 42]. Up to the RR contribution, this superpotential can be written for a constant warp factor as

$$W_{\rm NS} = C \int_{\mathcal{M}} \langle \Phi_1^0, ({\rm d} - H\wedge) \left(e^{-\phi}\, {\rm Im}\, \Phi_2^0\right)\rangle \ , \tag{6.2.9}$$

with a constant C and the Mukai product defined in (6.2.7). One finds that for a supersymmetric Mink vacuum without RR, $({\rm d} - H\wedge) \left(e^{-\phi}\, {\rm Im}\, \Phi_2^0\right)$ vanishes. And in the case of RR fluxes, their contribution to the superpotential is precisely canceled through (6.2.5). Therefore, the formula (6.2.9) gives $W = 0$ as expected. Another way to see this is to use [26]

$$\int_{\mathcal{M}} \langle \Psi_1, ({\rm d} - H\wedge)\, \Psi_2\rangle = \int_{\mathcal{M}} \langle ({\rm d} - H\wedge)\, \Psi_1, \Psi_2\rangle \ , \tag{6.2.10}$$

to rather get $({\rm d} - H\wedge)\, \Phi_1^0$: the other condition (6.2.4) makes again the superpotential vanish for a Mink vacuum, up to a derivative of the dilaton. This last derivative should however not contribute because of the compatibility condition (6.2.6), as we will see below.[9] Finally, for an AdS vacuum, one obtains from (6.2.4) or (6.2.5) that W is related to μ, thus to the cosmological constant, as expected [174].

6.2.2 Proposed superpotential and comparison to the literature

The formula (6.2.9) was extended in [17, 86, 200] to include non-geometric fluxes. Performing a dimensional reduction, these papers further compared their W to corresponding four-dimensional superpotentials expressed in terms of moduli [27, 98]. To include Q- and R-fluxes, the idea was to replace ${\rm d} - H\wedge$ by a more general derivative operator, such as the $\mathcal{D}_\sharp$ of [86] discussed in details in section 5.1.2. So we naturally propose here for the NSNS part of the superpotential with constant warp factor

$$\tilde{W}_{\rm NS} = \frac{C}{2} \int_{\mathcal{M}} \langle e^{-\tilde{\phi}} \Phi_1^0, \mathcal{D}\, {\rm Im}\, \Phi_2^0\rangle \ , \tag{6.2.11}$$

as already presented in the Introduction (1.4.32), where $\mathcal{D}$ is the Dirac operator. Picking for the latter the standard SUGRA one (1.4.21), the general formula (6.2.11) reproduces

[9] Note also that a constant warp factor typically leads to a constant dilaton in the (SUSY) vacuum.

the standard W (6.2.9). Doing the same in β-supergravity leads to an expression for W with non-geometric fluxes. A difference with previous papers is that Q and R have here a ten-dimensional interpretation. We now compute this superpotential more explicitly for an $SU(3)$ structure, and compare our results to formulas of the literature.

We use the expression (1.4.24) for the Dirac operator in β-supergravity with its various definitions of contractions for the fluxes. We also consider that the coefficients in flat indices of the $SU(3)$ structure forms, J_{ab} and Ω_{abc}, do not depend on internal coordinates. These coefficients are usually replaced by moduli, that only have a four-dimensional dependence. Finally, we recall that J is a real two-form. In type IIB, we obtain at first

$$\begin{aligned}\tilde{W}_{\rm NS} = {\rm i}C \int_{\mathcal{M}} e^{-\tilde{\phi}} \Big(& c_\theta \ f \diamond J + \frac{s_\theta}{2}\ Q \diamond (J \wedge J) + \frac{c_\theta}{3!}\ R \vee (J \wedge J \wedge J) \\ & + c_\theta \ {\rm d}\tilde{\phi} \wedge J + \frac{s_\theta}{2}\ (\tau - \check{\nabla}\tilde{\phi}) \vee (J \wedge J) \Big) \wedge \Omega \ ,\end{aligned} \tag{6.2.12}$$

with $c_\theta = (\theta_+)$, $s_\theta = \sin(\theta_+)$. More explicitly,

$$\begin{aligned} f \diamond J &= \frac{1}{2} f^c{}_{ab} J_{ce}\ \tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^e \\ \frac{1}{2}\ Q \diamond (J \wedge J) &= \frac{1}{2} Q_a{}^{bc} \left(\frac{1}{2} J_{cb} J_{ef} - J_{ce} J_{bf} \right) \tilde{e}^a \wedge \tilde{e}^e \wedge \tilde{e}^f \\ \frac{1}{3!}\ R \vee (J \wedge J \wedge J) &= \frac{1}{2} R^{abc} \left(\frac{1}{2} J_{cb} J_{ae} J_{fg} - \frac{1}{3} J_{ce} J_{bf} J_{ag} \right) \tilde{e}^e \wedge \tilde{e}^f \wedge \tilde{e}^g \\ \frac{1}{2}\ (\tau - \check{\nabla}\tilde{\phi}) \vee (J \wedge J) &= \frac{1}{2} \left(-\frac{1}{2} \beta^{bc} f^a{}_{bc} + \beta^{ab} \partial_b \tilde{\phi} \right) J_{ae} J_{fg}\ \tilde{e}^e \wedge \tilde{e}^f \wedge \tilde{e}^g \ . \end{aligned} \tag{6.2.13}$$

The last equation indicates that $(\tau - \check{\nabla}\tilde{\phi}) \vee (J \wedge J)$ is proportional to J. This implies that the second row of (6.2.12) is proportional to $J \wedge \Omega$. Requiring an $SU(3)$ structure, i.e. enforcing the compatibility condition (6.2.8), then makes this second row vanish.[10] We note that this property needs to remain true despite the moduli fluctuations. Using further (6.2.8), only some terms remain from the Q- and R-fluxes contributions

$$\begin{aligned} \frac{1}{2}\ Q \diamond_r (J \wedge J) &= -\frac{1}{2} Q_a{}^{bc} J_{ce} J_{bf} \tilde{e}^a \wedge \tilde{e}^e \wedge \tilde{e}^f \\ \frac{1}{3!}\ R \vee_r (J \wedge J \wedge J) &= -\frac{1}{3!} R^{abc} J_{ce} J_{bf} J_{ag} \tilde{e}^e \wedge \tilde{e}^f \wedge \tilde{e}^g \ . \end{aligned} \tag{6.2.14}$$

Finally, let us distinguish between the possible phases according to the choice of orientifold. However, we note that the notion of orientifold is not really defined in the context of β-supergravity, since the RR sector has not been studied so far. The distinction between

[10] This reasoning could be generalized to a generic $SU(3) \times SU(3)$ structure with the condition (6.2.6), allowing to discard the dilaton terms, or more precisely the terms due to the $\mathbb{R}^+$ factor. The corresponding terms at the level of DFT are also truncated in [99] thanks to the orientifold projection which is considered to be odd on the winding. Interestingly, we do not need this projection so far here.

O3, O7, and O5, O9, should then be viewed more formally as a choice on the phase θ_+.[11] Absorbing the possible minus sign of (6.2.2) in a redefinition of C, we get

$$\text{O3 or O7:}\quad \tilde{W}_{\rm NS} = {\rm i}C \int_{\mathcal{M}} e^{-\tilde{\phi}} \left(\frac{1}{2}\, Q \diamond_r (J\wedge J)\right) \wedge \Omega\ , \tag{6.2.15}$$

$$\text{O5 or O9:}\quad \tilde{W}_{\rm NS} = {\rm i}C \int_{\mathcal{M}} e^{-\tilde{\phi}} \left(f \diamond J + \frac{1}{3!}\, R \vee_r (J\wedge J\wedge J)\right) \wedge \Omega\ . \tag{6.2.16}$$

Let us compare these formulas to those in the literature. The first superpotential with non-geometric fluxes was proposed in [27] based on duality arguments, and was given in terms of moduli of an STU model. This expression was recovered in [98] for type IIB with an O3-plane from an expression in terms of internal forms. We compare the latter to (6.2.15). Forgetting about the H-flux, we find an exact agreement, fixing $C = -\frac{1}{3}$. The same type of contraction as $Q\diamond$ appears in the superpotential of [98], as well as in [99]. For the case of an O5- or O9-plane, we have not found in the literature an expression in terms of forms to be compared to (6.2.16). To the best of our knowledge this expression is then new. It could be used as well for heterotic (see footnote 11). One should still perform the expansion and integration on a form basis to get the expression in terms of moduli. This is however beyond the scope of this work. From T-duality arguments, the expression (6.2.16) looks in any case very plausible.

We now turn to type IIA, that should be compared in the literature to the case of an O6-plane. The formula (6.2.11) leads to a $\mathcal{D}\,{\rm Re}\,\Omega$. This quantity however does not appear in the literature, except in [196] but without non-geometric fluxes, and in [200] where it is only implicitly proposed. On the contrary, $\mathcal{D}$ is rather acting on Φ_+^0 in [98] and [86]. Such a situation could only be reached after integrating by parts with $\mathcal{D}$, similarly to (6.2.10). While the latter holds for standard SUGRA thanks to the absence of boundary on the compact $\mathcal{M}$ and the H-flux acting with a wedge, the analogous result for $\mathcal{D}$ in β-supergravity is not obvious to derive, because of the contractions on forms. In any case, let us assume here that this property holds, i.e.

$$\int_{\mathcal{M}} \langle \Psi_1, \mathcal{D}\Psi_2\rangle = \int_{\mathcal{M}} \langle \mathcal{D}\Psi_1, \Psi_2\rangle\ , \tag{6.2.17}$$

allowing us to start in type IIA with

$$\tilde{W}_{\rm NS} = \frac{C}{2} \int_{\mathcal{M}} \langle \mathcal{D}(e^{-\tilde{\phi}}\Phi_1^0), {\rm Im}\,\Phi_2^0\rangle\ . \tag{6.2.18}$$

Pursuing the same reasoning as in type IIB, we derive from (6.2.18)

$$\tilde{W}_{\rm NS} = -e^{{\rm i}\theta_+} C \int_{\mathcal{M}} e^{-\tilde{\phi}} \Big({\rm i}\ f\diamond J + \frac{1}{2}\, Q \diamond_r (J\wedge J) + \frac{\rm i}{3!}\, R \vee_r (J\wedge J\wedge J)\Big) \wedge {\rm Re}\,\Omega\ . \tag{6.2.19}$$

[11] The O5, O9, case is also sometimes discussed, in the literature on superpotentials, with heterotic, as the two are simply related by S-duality.

This formula agrees completely with the proposal of [86], up to fixing C. The same goes for the comparison to [99], up to a redefinition of Ω, and a conventional minus sign difference in the R-flux. Finally, our formula agrees with that of [98], up to C and numerical factors in the contractions.

The comparison of our proposed W and formulas of the literature implicitly considers that the ten-dimensional non-geometric fluxes of β-supergravity are the same as the four-dimensional ones. This has worked well so far, but in type IIA, we did not reach formulas with explicit moduli dependence.[12] Indeed, a derivation of the moduli formula of [27] does not seem to have been performed directly in the literature, its comparison to other expressions is usually rather done thanks to duality arguments. In [99], an oxidation is made in type IIA, instead of a reduction, and ends with a comparison and matching of the DFT Lagrangian of [50]. Since β-supergravity fluxes and the Lagrangian agree with the DFT ones, upon the strong constraint and setting $b = 0$ [101], we would conclude on a matching with [99]. In the latter however a difference is indicated between ten-dimensional and four-dimensional fluxes, on the contrary to what we have considered so far. This discrepancy might be related to the way four-dimensional scalar fields, loosely called here moduli, are defined. Following STU models, the authors of [99] include the fluctuation of the b-field in a modulus. In β-supergravity, we would obviously not get such a modulus when expanding our superpotential. It is unclear whether the b-field modulus would simply be traded for us into a β modulus, because there is actually no explicit dependence in β in (6.2.15), (6.2.16) and (6.2.19). An expansion of our superpotential may then only include the geometric moduli and dilaton, and the comparison should thus be done at that level. These points deserve in any case more study. Still, we conclude that the general formulas proposed here for the superpotential, depending on the Dirac operator, reproduces remarkably well expressions in the literature in terms of structure forms and non-geometric fluxes.

6.3 Geometrical characterization of the backgrounds

As discussed in the Introduction, the b-field and its counterpart β can be viewed as twists on $T\mathcal{M} \oplus T^*\mathcal{M}$ by looking at the generalized vielbein $\mathcal{E}$ or $\tilde{\mathcal{E}}$ (3.1.1) corresponding to each theory. We discuss this point in more details in the following, and argue how this is crucially related to the geometrical characterization of supersymmetric backgrounds of β-supergravity.

Conditions for preserving SUSY usually provide a geometrical characterization of the manifold $\mathcal{M}$. The prime example is here the CY condition. As mentioned in the Introduction, the formulation in terms of GCG [40, 41] has provided such a characterization in presence of background fluxes.[13] We analyze in this section the situation for β-supergravity. Let us first recall some terminology, and the results for standard SUGRA. To each pure

[12] Doing so would allow a comparison to [201, 202] where new non-geometric terms were obtained from M-theory. They are however unlikely to be reproduced here, due to the assumption of an $SU(3)$ structure.

[13] Reviews can be found in [26, 42].

spinor Φ corresponds a generalized complex structure (GCS). Then, if a pure spinor satisfies

$$\mathrm{d}\Phi = (v \vee + \xi \wedge)\Phi \ , \tag{6.3.1}$$

for some $(v + \xi) \in T\mathcal{M} \oplus T^*\mathcal{M}$ the GCS is integrable and $\mathcal{M}$ generalized complex. Furthermore, if Φ is closed, $\mathcal{M}$ is a GCY manifold. Finally, having a generalized Kähler manifold requires two distinct closed pure spinors. For standard SUGRA, a SUSY Mink background with $H = 0$ asks for $\mathcal{M}$ to be a GCY [97, 26]: the pure spinor $e^{2A-\phi}\Phi_1$ in (6.1.26) is closed for $\mu = 0$ and $H = 0$. In absence of RR fluxes, with a constant warp factor, the second condition (6.1.27) further constrains $\mathcal{M}$ to be generalized Kähler.[14] The GCY characterization was proven useful, leading for instance to an extensive search for solutions on six-dimensional nilmanifolds, as those are all GCY [206]. In presence of a closed H-flux, the corresponding b-field induces a twist and a pure spinor closed under $d - H\wedge$, as in (6.1.26) with $\mu = 0$, then characterizes a twisted GCY. The twist by the b-field can be seen through the rewriting $d - H\wedge = e^{b\wedge}\mathrm{d}e^{-b\wedge}$, or in the off-diagonal block of the generalized vielbein $\mathcal{E}$ (3.1.1), as discussed in the Introduction. It twists the local $T\mathcal{M} \oplus T^*\mathcal{M}$ into the, globally non-trivial, generalized tangent bundle E_T.

We now turn to β-supergravity and the pure spinor conditions (6.1.24) and (6.1.25), to study whether an analogous characterization can be obtained.[15] We focus on the Mink case $\mu = 0$ and the first condition (6.1.24). We thus look at (6.1.28), written in terms of the Dirac operator (1.4.23) as

$$\mathcal{D}\Phi_1 = -4\left(\mathrm{d}A \wedge + \check{\nabla} A \vee\right)\Phi_1 \ . \tag{6.3.2}$$

This equation is analogous to the case of a b-twisted integrable GCS, as in (6.3.1). As mentioned in section 6.1.3, whenever $\check{\nabla} A = 0$, the warp factor can be absorbed in the LHS, to get $e^{2A}\Phi_1$ closed under $\mathcal{D}$, precisely as for standard SUGRA. We now consider this case in more details, i.e.

$$\mathcal{D}\Phi = 0 \ . \tag{6.3.3}$$

This is the analogue to the b-twisted GCY condition. It is thus natural in β-supergravity to talk of a twist by β, and to consider (6.3.3) as a β-twisted GCY condition; for $\beta = 0$, we recover a GCY condition. If we can make sense of it, the geometrical characterization of $\mathcal{M}$ is then this β-twisted GCY. Furthermore, in absence of RR fluxes and with a constant warp factor[16], we get from (6.1.25) a second pure spinor closed under $\mathcal{D}$, analogously to a twisted generalized Kähler $\mathcal{M}$. It is traditionally considered that non-geometric backgrounds cannot be described by GCG and E_T, e.g. discussed in [207, 78, 39], but the construction just mentioned now seems to provide a description within GG, understanding this formalism in a slight extended sense though.

[14] Reviews on this case can be found in [203, 204, 205].

[15] For an $SU(3)$ structure, an alternative might be to study the conditions in terms of the $SU(3)$ torsion classes, and compare them to the fluxes, as e.g. in [12].

[16] A constant warp factor automatically follows for a SUSY solution from (6.1.16).

Chapter 7

Conclusion & Outlook

In this thesis, we developed an effective field theory, named β-supergravity, for the specific goal of investigating non-geometric backgrounds at the level of ten-dimensional supergravity (SUGRA). Our focus has been on the classification of this theory with regard to the frameworks of Generalized Geometry (GG) and Double Field Theory (DFT). Whereas earlier constructions relied on a specific field redefinition in order to directly reformulate the standard SUGRA action, we studied here a consistent set of tools and geometric structures heavily inspired by GG. This led us to the action of β-supergravity which features besides a new covariant derivative also non-geometric Q- and R-fluxes. The latter, up to then only present in lower-dimensional gauged SUGRAs and being an indication for non-geometric phenomena, were given in this way ten-dimensional expressions and explicitly appear in the action of β-supergravity in contrast to other approaches, e.g. DFT. Moreover, the action experiences invariance under standard diffeomorphisms and β gauge transformations which have been investigated with respect to being helpful in constructing new geometric background solutions within β-supergravity. These properties make this theory a promising framework for studying aspects of non-geometry directly in ten dimensions. However, it turned out that only in the presence of isometries consistent glueing transformations for geometric backgrounds of β-supergravity are provided in the form of β-transforms in contrast to the expected β gauge transformations. The construction of a well-defined generalized cotangent bundle E_{T*}, which was argued to be the correct bundle on which the generalized frames $\tilde{\mathcal{E}}(\beta)$ should live, is closely related to mentioned transformations. In particular, it would be interesting to have one concrete construction of E_{T*} at hand.

A major goal of this thesis has been the analysis of vacua of β-supergravity. Especially, the relation of geometric backgrounds in β-supergravity and non-geometric ones in standard SUGRA have been considered. In this process, a class of consistent geometric backgrounds of β-supergravity has been identified which however have been shown to lie on a geometric T-duality orbit and hence is not able to provide new physics. This is consistent with the analysis of the equations of motion of β-supergravity for a simple compactifications ansatz with regard to pure NSNS solutions. In particular, the usage of certain solvmanifolds for the internal space failed. Reductions from DFT to some SUGRA theories produced similar results [31]. Nevertheless, suggestions for how to find truely new vacua and physics have been made in [36, 146, 208, 58] and we proposed some possibilities to circumvent the lack in β-supergravity so far.

Apart from the sobering results, the reformulation of standard SUGRA offered a useful description of some backgrounds. First hints were previously found by studying the toroidal example and here we applied an extended analysis to the set of *NS*-branes. Detailed account has been given for the *NS*5-brane, the Kaluza-Klein monopole and the Q-brane in section 5.2. In particular, the latter experiences a clearer brane picture in terms of β-supergravity fields and the non-geometric fluxes lead to corresponding Bianchi identities (BIs) which get corrected by source terms. Having presented convenient descriptions at the level of SUGRA, it would be interesting to study this set of branes as stringy respectively M-theory objects. This might also give hints on a world-volume action for these *NS*-branes that then would lead to possible source contributions to the equations of motion and BIs. These could be compared to the results of the Q-brane derived within β-supergravity, as we have obtained corrections to the dilaton equation of motion (D.2.4), the Einstein equation (D.2.15) - (D.2.18) and the BIs. Interestingly, we did not observe the expected modification for the β equation of motion. Further thoughts concern the relation of the Q-brane and the D_7-brane as these are both codimension two objects. Since there exists a non-perturbative description of the latter within F-theory, a similar construction might be interesting for the Q-brane [92]. Finally, the possibility of an R-brane has been discussed.[1] Performing a further standard T-duality problematic for the lack of an isometry. Nevertheless, we derived a constant warp factor and consider the BI (1.4.18) to be a natural candidate corrected by this codimension 1 *NS*-brane. Hypothetically, we expected a codimension zero *NS*-brane being related to the last BI (1.4.19).

We close with an outlook on the complete supersymmetric framework of β-supergravity. The analysis of SUSY in this thesis has been restricted to SUSY conditions and the proposition of a superpotential for β-supergravity in terms of pure spinors. It would be interesting to derive the SUSY variations from a SUSY completion of the bosonic NSNS Lagrangian at hand. This raises the question whether the respective fermions and a RR sector for β-supergravity should also be obtained via field redefinitions of the standard ones. First, let us also mention that the $Spin(D,D) \times \mathbb{R}^+$ covariant derivative $\mathcal{D}$, used to derive the NSNS BIs, might play a role in determining a set of non-geometric RR fluxes F by acting on a standard polyform potential C, analogue to $F = (\mathrm{d} - H\wedge)C$ in standard SUGRA. This would introduce contractions among non-geometric fluxes and standard RR potentials which may lead to new types of RR fluxes. A second distinct possibility would be to introduce polyvectors instead of forms. Such a proposal was sketched in [209] and would clearly provide new types of fluxes. In the end consistency of a supersymmetric version of β-supergravity should be the decisive criterion for choosing the correct RR-fluxes. More generally, RR fluxes of β-supergravity may provide uplifts to some of the known four-dimensional non-geometric RR fluxes [98, 145, 210]. Finally, we remark that the pure spinor conditions despite from providing a classification of certain backgrounds are of practical use when looking for possible vacua of any given theory. For a standard $\mathcal{N} = 1$

[1] A different object was also named R-brane in [85].

Minkowski vacua without $NS5$-brane these conditions together with the BI for the fluxes are equivalent to solving the equations of motion [211, 212, 26, 195]. This provides massive technical simplification and it would be interesting to observe whether in β-supergravity an analogue statement holds.

Appendix A

Conventions

A.1 Space-Time and index conventions

In this thesis, we use a wide range of indices of different kind. To clarify our notation at various places, we give here an overview of the use of indices for space-time directions of certain dimensions. In particular, we distinguish the use of two sets of notations depending on whether we are dealing with a generic framework or theory or whether we take a certain compactification ansatz that allows us to split space-time directions into external and internal parts. In any given chapter or section, this distinction should be clear to the reader by the respective context.

We introduce a first set of indices to deal with quantities in a generic d-dimensional space-time

$$\begin{array}{ccll} \mathcal{A},\ldots,\mathcal{L} & \in & 1,\ldots,2d & \text{flat } O(d,d) \text{ indices} \\ \mathcal{M},\ldots,\mathcal{Z} & \in & 1,\ldots,2d & \text{curved } O(d,d) \text{ indices} \\ a,\ldots,l & \in & 1,\ldots,d & \text{flat (tangent space) indices} \\ m,\ldots,z & \in & 1,\ldots,d & \text{curved space-time indices} \end{array} \tag{A.1.1}$$

We use this set throughout most chapters of this thesis.

However, in section 6 we will encounter the situation that the above set is not sufficient to maintain a convenient notation and we proceed using a second set of indices

$$\begin{array}{ccll} \mathcal{A},\ldots,\mathcal{L} & \in & 1,\ldots,2d & \text{flat } O(D,D) \text{ indices} \\ \mathcal{M},\ldots,\mathcal{Z} & \in & 1,\ldots,2d & \text{curved } O(D,D) \text{ indices} \\ A,\ldots,L & \in & 1,\ldots,D & \text{flat (tangent space) indices} \\ M,\ldots,Z & \in & 1,\ldots,D & \text{curved space-time indices} \\ a,\ldots,l & \in & 1,\ldots,d & \text{flat internal indices} \\ m,\ldots,z & \in & 1,\ldots,\text{d} & \text{curved internal indices} \\ \alpha,\beta,\gamma,\delta,\epsilon,\zeta,\eta & \in & 1\ldots D-d & \text{flat external indices} \\ \lambda,\mu,\nu,\rho,\sigma,\xi & \in & 1\ldots D-d & \text{curved external indices} \end{array} \tag{A.1.2}$$

This set is more suited to the situation of compactification with respect to a prefered background where we need to distinguish between external space-time directions and internal directions. The starting point is now a certain D-dimensional background, that splits into d internal coordinates and $D-d$ external ones. In this way, we manage to lay the focus on the d-dimensional quantities and are able to present the results of the compactification with Latin indices, as before. Further, we refer the reader to appendix (E.1), where we layout the explicit compactification ansatz and make heavy use of the above set of indices.

A.2 Differential forms and Clifford algebra

A p-form A is given by

$$A = \frac{1}{p!} A_{m_1 \ldots m_p} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p} = \frac{1}{p!} A_{a_1 \ldots a_p} \tilde{e}^{a_1} \wedge \ldots \wedge \tilde{e}^{a_p} \ . \tag{A.2.1}$$

For a p-form A and a q-form B we deduce the coefficient

$$(A \wedge B)_{m_1 \ldots m_{p+q}} = \frac{(p+q)!}{p!q!} A_{[m_1 \ldots m_p} B_{m_{p+1} \ldots m_{p+q}]} \ . \tag{A.2.2}$$

Further, the contraction of a vector $V = V^m \partial_m = V^a \partial_a$ on A is defined by

$$V \vee A = \frac{1}{(p-1)!} V^{m_1} A_{m_1 \ldots m_p} \mathrm{d}x^{m_2} \wedge \ldots \wedge \mathrm{d}x^{m_p} \ . \tag{A.2.3}$$

It is also denoted by $\iota_a = \tilde{e}^m{}_a \iota_m$, that satisfies the following commutation relations

$$V \vee A = V^a \iota_a A \ , \qquad \{\tilde{e}^a, \iota_b\} = \delta^a_b \ , \quad \{\iota_a, \iota_b\} = 0 \ , \tag{A.2.4}$$

while a contraction on a scalar vanishes. Here we used the notation $\iota_m A = \iota_{dx^m} A$ which denotes the contraction of a k-form A by the one-form dx^m by use of the metric g. The full contraction of two k-forms A and B using a metric g is given by

$$A \cdot B = \frac{A_{m_1 \cdots m_k} B_{n_1 \cdots n_k} g^{m_1 n_1} \cdots g^{m_k n_k}}{k!} \ . \tag{A.2.5}$$

In the case of multiple contractions, such as $Q_c{}^{ab} \iota_a \iota_b$, one should be careful with their order, that may generate signs when acting on a form.

Finally, we introduce the totally antisymmetric quantity ϵ, given by $\epsilon_{m_1 \ldots m_n} = +1/-1$ for $(m_1 \ldots m_n)$ being an even/odd permutation of $(1 \ldots n)$, and 0 otherwise. The one with flat indices $\epsilon_{a_1 \ldots a_n}$ has the same value, i.e. ϵ is not a tensor. This can be seen by preserving the volume form.

Finally, we consider (constant) matrices γ^a with flat indices, satisfying the standard Clifford algebra and the following associated properties [213]

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \ , \ [\gamma^a, \gamma^b] = 2\gamma^{ab} \text{ with } \gamma^{a_1 a_2 \ldots a_p} \equiv \gamma^{[a_1} \gamma^{a_2} \ldots \gamma^{a_p]} \tag{A.2.6}$$

$$\gamma^a \gamma^b = \eta^{ab} + \gamma^{ab} \ , \ \gamma^a \gamma^{bc} = \gamma^{abc} + 2\eta^{a[b} \gamma^{c]} \ , \ \gamma^a \gamma^{bcd} = \gamma^{abcd} + 3\eta^{a[b} \gamma^{cd]} \tag{A.2.7}$$

$$[\gamma^{ab}, \gamma^{cd}] = -8\delta^{[c}_{[g} \gamma_{h]}{}^{d]} \eta^{ag} \eta^{bh}, \ \{\gamma^{ab}, \gamma^{cd}\} = 2\gamma^{abcd} - 4\eta^{c[a} \eta^{b]d} \tag{A.2.8}$$

$$\{\gamma^a, \gamma^{bcd}\} = 6\eta^{a[b} \gamma^{cd]} \ , \ \{\gamma^{abc}, \gamma^{def}\} = 18\delta^{[d}_{[g} \gamma_{hi]}{}^{ef]} \eta^{ag} \eta^{bh} \eta^{ci} - 12\delta^d_{[g} \delta^e_h \delta^f_{i]} \eta^{ag} \eta^{bh} \eta^{ci} \ . \tag{A.2.9}$$

A.3 Curvatures and fluxes

For a generic metric $\tilde{g}_{mn}$ with Levi-Civita connection, the connection coefficients, covariant derivative, and Ricci scalar, are given by

$$\Gamma^m_{np} = \frac{1}{2}\tilde{g}^{mq}\left(\partial_n \tilde{g}_{qp} + \partial_p \tilde{g}_{qn} - \partial_q \tilde{g}_{np}\right) \ , \tag{A.3.1}$$

$$\nabla_m V^n = \partial_m V^n + \Gamma^n_{mp} V^p \ , \ \nabla_m V_n = \partial_m V_n - \Gamma^p_{mn} V_p \ , \tag{A.3.2}$$

$$\nabla_p V^p = \frac{1}{\sqrt{|\tilde{g}|}}\ \partial_p \left(\sqrt{|\tilde{g}|}\ V^p\right) \ , \tag{A.3.3}$$

$$\mathcal{R}(\tilde{g}) = \tilde{g}^{mn}\mathcal{R}_{mn} \ , \ \mathcal{R}_{mn} = \partial_p \Gamma^p_{mn} - \partial_n \Gamma^q_{qm} + \Gamma^p_{mn}\Gamma^q_{qp} - \Gamma^p_{mq}\Gamma^q_{np} \ . \tag{A.3.4}$$

The Riemann tensor is generically given as follows; for the Levi-Civita connection, it satisfies the following properties

$$\begin{aligned} &\mathcal{R}^n{}_{rms} = \partial_m \Gamma^n_{sr} - \partial_s \Gamma^n_{mr} + \Gamma^q_{sr}\Gamma^n_{mq} - \Gamma^q_{mr}\Gamma^n_{sq} \ , \ \mathcal{R}_{rs} = \mathcal{R}^n{}_{rns} \ , \\ &\tilde{g}_{np}\mathcal{R}^n{}_{rms} = \tilde{g}_{nm}\mathcal{R}^n{}_{spr} = -\tilde{g}_{ns}\mathcal{R}^n{}_{mpr} = \tilde{g}_{ns}\mathcal{R}^n{}_{mrp} \ , \ \mathcal{R}^n{}_{[rms]} = 0 \ . \end{aligned} \tag{A.3.5}$$

The vielbein $\tilde{e}^a{}_m$ and its inverse $\tilde{e}^n{}_b$, associated to the metric $\tilde{g}$ by $\tilde{g}_{mn} = \tilde{e}^a{}_m \,\eta_{ab}\, \tilde{e}^b{}_n$, allow to go from curved to flat indices. Going to flat indices, we use the vielbein $\tilde{e}^a{}_m$ and its inverse $\tilde{e}^n{}_b$, associated to the metric $\tilde{g}_{mn} = \tilde{e}^a{}_m \tilde{e}^b{}_n \eta_{ab}$, with η_{ab} the components of the flat metric η_D. Tensors with flat indices are obtained after multiplication by the appropriate (inverse) vielbein(s), e.g. $\beta^{ab} = \tilde{e}^a{}_m \tilde{e}^b{}_n \beta^{mn}$, and we also denote $\partial_a = \tilde{e}^m{}_a \partial_m$. Going to matrix notation, one should be careful that the matrix product reproduces the correct index contraction. With the line index always on the left and the column on the right, whatever up or down, one then has sometimes to take the transpose. For instance, $\tilde{g} = \tilde{e}^T \eta_d \tilde{e}$, $\beta_{\text{flat}} = \tilde{e}\beta\tilde{e}^T$ and $b_{\text{flat}} = e^{-T} b e^{-1}$, where $(\tilde{e}^T)_m{}^a = (\tilde{e})^a{}_m = \tilde{e}^a{}_m$ and $(\tilde{e}^{-T})_b{}^n = (\tilde{e}^{-1})^n{}_b = \tilde{e}^n{}_b$. The above quantities lead to the definition of geometric and non-geometric fluxes

$$\begin{aligned} &H_{abc} = 3\partial_{[a} b_{bc]} \ , && f^a{}_{bc} = 2\tilde{e}^{a m}\partial_{[b}\tilde{e}^m{}_{c]} \ , \\ &Q_a{}^{bc} = \partial_a \beta^{bc} - 2\beta^{d[b} f^{c]}{}_{ad} \ , && R^{abc} = 3\beta^{d[a}\partial_d \beta^{bc]} - 3\beta^{d[a} f^b{}_{de}\beta^{c]e} \ . \end{aligned} \tag{A.3.6}$$

We further give some interesting rewritings of the R-flux

$$R^{abc} = 3\beta^{d[a}\nabla_d \beta^{bc]} = 3\beta^{d[a} Q_d{}^{bc]} + 3\beta^{d[a} f^b{}_{de}\beta^{c]e} \ . \tag{A.3.7}$$

The spin connection coefficient ω^a_{bc} are defined in analogy to the Levi-Civita connection, but for flat indices. The covariant derivative in flat indices on a vector V is then given by

$$\nabla_a V^b = \partial_a V^b + \omega^b_{ac} V^c \ , \ \nabla_a V_b = \partial_a V^b - \omega^c_{ab} V_c \ . \tag{A.3.8}$$

We can deduce from this the relation to the standard connection coefficients

$$\nabla_b(\partial_c) \equiv \omega^a_{bc}\partial_a \ , \ \omega^a_{bc} \equiv \tilde{e}^n{}_b \tilde{e}^a{}_m \left(\partial_n \tilde{e}^m{}_c + \tilde{e}^p{}_c \Gamma^m_{np}\right) = \tilde{e}^n{}_b \tilde{e}^a{}_m \nabla_n \tilde{e}^m{}_c \ , \tag{A.3.9}$$

The structure constant $f^a{}_{bc}$ (or so-called geometric flux) are defined as

$$f^a{}_{bc} = 2\tilde{e}^a{}_m \partial_{[b}\tilde{e}^m{}_{c]} = -2\tilde{e}^m{}_{[c}\partial_{b]}\tilde{e}^a{}_m \ , \ 2\partial_{[a}\partial_{b]} = f^c{}_{ab}\partial_c \ , \tag{A.3.10}$$

For the Levi-Civita connection, one has the relation between ω and f and the Ricci scalar in flat indices

$$\omega^a_{bc} = \frac{1}{2}\left(f^a{}_{bc} + \eta^{ad}\eta_{ce}f^e{}_{db} + \eta^{ad}\eta_{be}f^e{}_{dc}\right) \ , \tag{A.3.11}$$

$$\mathcal{R}(\tilde{g}) = 2\eta^{bc}\partial_a\omega^a_{bc} + \eta^{bc}\omega^a_{ad}\omega^d_{bc} - \eta^{bc}\omega^a_{db}\omega^d_{ac} \tag{A.3.12}$$

as in (A.3.24). In addition one has the following properties

$$\eta^{dc}\omega^a_{bc} = -\eta^{ac}\omega^d_{bc} \ , \ f^a{}_{bc} = 2\omega^a_{[bc]} \ , \ f^a{}_{ab} = \omega^a_{ab} \ . \tag{A.3.13}$$

The new covariant derivative on a (co)-vector V and the derived new curvature scalar are defined as

$$\check{\Gamma}^{mn}_p = \frac{1}{2}\tilde{g}_{pq}\left(-\beta^{mr}\partial_r\tilde{g}^{nq} - \beta^{nr}\partial_r\tilde{g}^{mq} + \beta^{qr}\partial_r\tilde{g}^{mn}\right) + \tilde{g}_{pq}\tilde{g}^{r(m}\partial_r\beta^{n)q} - \frac{1}{2}\partial_p\beta^{mn} \ , \tag{A.3.14}$$

$$\mathcal{T}^n \equiv \check{\Gamma}^{pn}_p = \partial_p\beta^{np} - \frac{1}{2}\beta^{nm}\tilde{g}_{pq}\partial_m\tilde{g}^{pq} = \nabla_p\beta^{np} \ , \tag{A.3.15}$$

$$\check{\nabla}^m V^p = -\beta^{mn}\partial_n V^p - \check{\Gamma}^{mp}_n V^n \ , \ \check{\nabla}^m V_p = -\beta^{mn}\partial_n V_p + \check{\Gamma}^{mn}_p V_n \ , \tag{A.3.16}$$

$$\check{\nabla}^p V_p = \frac{1}{\sqrt{|\tilde{g}|}}\,\partial_p\left(\sqrt{|\tilde{g}|}\ \beta^{pm}\ V_m\right) + 2\mathcal{T}^p V_p \ , \tag{A.3.17}$$

$$\check{\mathcal{R}} = \tilde{g}_{mn}\check{\mathcal{R}}^{mn} \ , \ \check{\mathcal{R}}^{mn} = -\beta^{pq}\partial_q\check{\Gamma}^{mn}_p + \beta^{mq}\partial_q\check{\Gamma}^{pn}_p + \check{\Gamma}^{mn}_p\check{\Gamma}^{qp}_q - \check{\Gamma}^{qm}_p\check{\Gamma}^{pn}_q \ . \tag{A.3.18}$$

We also use this covariant derivative $\check{\nabla}$ whose action on a vector V in flat indices is given by

$$\check{\nabla}^a V^b = -\beta^{ac}\partial_c V^b + \omega_{Q}{}^{ab}_{c} V^c \ , \ \check{\nabla}^a V_b = -\beta^{ac}\partial_c V_b - \omega_{Q}{}^{ac}_{b} V_c \ . \tag{A.3.19}$$

Proceeding similarly for the new $\check{\nabla}$ leads us to introduce ω_Q, (the opposite of) the flat connection associated to $\check{\Gamma}$

$$\tilde{e}^m{}_a\tilde{e}^b{}_n\check{\nabla}^n V_m = \check{\nabla}^b V_a \equiv -\beta^{bd}\partial_d V_a - \omega_{Q}{}^{bc}_{a} V_c \Leftrightarrow -\omega_{Q}{}^{bc}_{a} \equiv \tilde{e}^b{}_n\tilde{e}^m{}_a\left(-\beta^{nq}\partial_q\tilde{e}^c{}_m + \tilde{e}^c{}_p\check{\Gamma}^{np}_m\right) \tag{A.3.20}$$

From it, we can define a quantity $\mathcal{R}_Q$ analogous to the standard Ricci scalar $\mathcal{R}(\tilde{g})$

$$\omega_{Q}{}^{bc}_{a} = \frac{1}{2}\left(Q_a{}^{bc} + \eta_{ad}\eta^{ce}Q_e{}^{db} + \eta_{ad}\eta^{be}Q_e{}^{dc}\right) \ , \tag{A.3.21}$$

$$\mathcal{R}_Q \equiv 2\eta_{bc}\beta^{ad}\partial_d\omega_{Q}{}^{bc}_{a} + \eta_{bc}\omega_{Q}{}^{ad}_{a}\omega_{Q}{}^{bc}_{d} - \eta_{bc}\omega_{Q}{}^{db}_{a}\omega_{Q}{}^{ac}_{d} \ . \tag{A.3.22}$$

where (A.3.21) is obtained by using the definition (3.1.11) of $\check{\Gamma}$ and the Q-flux given by the proposal (A.3)! This ω_Q enjoys similar properties as those of (A.3.13)

$$\eta_{dc}\omega_{Q}{}^{bc}_{a} = -\eta_{ac}\omega_{Q}{}^{bc}_{d} \ , \ Q_a{}^{bc} = 2\omega_{Q}{}^{[bc]}_{a} \ , \ \omega_{Q}{}^{ad}_{a} = Q_a{}^{ad} \ , \ \eta_{bc}\omega_{Q}{}^{bc}_{a} = \eta_{ad}Q_b{}^{db} \ . \tag{A.3.23}$$

We then mimic the definition of the Ricci scalar in terms of ω or f and introduce the analogous quantity $\mathcal{R}_Q$

$$\begin{aligned}
\mathcal{R}(\tilde{g}) &= 2\eta^{bc}\partial_a\omega^a_{bc} + \eta^{bc}\omega^a_{ad}\omega^d_{bc} - \eta^{bc}\omega^a_{db}\omega^d_{ac} \qquad (A.3.24)\\
&= 2\eta^{ab}\partial_a f^c{}_{bc} - \eta^{cd}f^a{}_{ac}f^b{}_{bd} - \frac{1}{4}\left(2\eta^{cd}f^a{}_{bc}f^b{}_{ad} + \eta_{ad}\eta^{be}\eta^{cg}f^a{}_{bc}f^d{}_{eg}\right)\ ,\\
\mathcal{R}_Q &\equiv 2\eta_{bc}\beta^{ad}\partial_d{\omega_Q}^{bc}_a + \eta_{bc}{\omega_Q}^{ad}_a{\omega_Q}^{bc}_d - \eta_{bc}{\omega_Q}^{db}_a{\omega_Q}^{ac}_d \qquad (A.3.25)\\
&= 2\eta_{ab}\beta^{ad}\partial_d Q_c{}^{bc} - \eta_{cd}Q_a{}^{ac}Q_b{}^{bd} - \frac{1}{4}\left(2\eta_{cd}Q_a{}^{bc}Q_b{}^{ad} + \eta^{ad}\eta_{be}\eta_{cg}Q_a{}^{bc}Q_d{}^{eg}\right)\ .
\end{aligned}$$

Another curvature tensor related to the new covariant derivative $\check{\nabla}$ takes the form

$$\check{\mathcal{R}}^{ab} = \beta^{cd}\partial_d{\omega_Q}^{ab}_c - \beta^{ad}\partial_d{\omega_Q}^{cb}_c + {\omega_Q}^{ab}_c{\omega_Q}^{dc}_d - {\omega_Q}^{ca}_d{\omega_Q}^{db}_c - \frac{1}{2}R^{adc}f^b{}_{dc}\ , \qquad (A.3.26)$$

and one finds the following relation between $\mathcal{R}_Q$ and $\check{\mathcal{R}}$

$$\check{\mathcal{R}} = \mathcal{R}_Q - \frac{1}{2}R^{acd}f^b{}_{cd}\eta_{ab}\ . \qquad (A.3.27)$$

Finally, we list for completeness our conventions used in writing down the Lagrangians (3.1.5) and (3.1.9). $|g|$ and $|\tilde{g}|$ denote the absolute value of the determinant of the metrics g and respectively $\tilde{g}$. The squares introduced are defined as

$$\begin{aligned}
&(\partial\phi)^2 \equiv g^{mn}\partial_m\phi\ \partial_n\phi\ ,\ H^2 \equiv \frac{1}{3!}H_{mnp}H_{qrs}g^{mq}g^{nr}g^{ps}\ ,\ R^2 \equiv \frac{1}{3!}R^{mnp}R^{qrs}\tilde{g}_{mq}\tilde{g}_{nr}\tilde{g}_{ps}\ , \qquad (A.3.28)\\
&(\partial\tilde{\phi})^2 \equiv \tilde{g}^{mn}\partial_m\tilde{\phi}\ \partial_n\tilde{\phi}\ ,\ (\beta^{mp}\partial_p\tilde{\phi} - \mathcal{T}^m)^2 \equiv \tilde{g}_{mn}(\beta^{mp}\partial_p\tilde{\phi} - \mathcal{T}^m)(\beta^{nq}\partial_q\tilde{\phi} - \mathcal{T}^n)\ .
\end{aligned}$$

Appendix B

Graviton scattering and the DFT action

In the first part of this appendix we rederive the on-shell three graviton scattering amplitude for vertex operators which do explicitly depend on winding modes in addition to momenta, given in [119]. In the second part we are going to expand the Double Field Theory (DFT) action into third order in fluctuations and show that these interactions precisely match with the above string scattering amplitude. This computation is meant to provide evidence for the relevance of this T-duality invariant conformal field theory (CFT) for DFT.

B.1 3-Graviton scattering from CFT

Calculating an N-point function of insertions of graviton vertex operators (2.2.20) $\mathcal{E}_{p,w}(z,\bar{z})$ is combinatorially more involved than a tachyon amplitude. For taking care of that one conveniently defines

$$\mathcal{V}_i(z_i,\bar{z}_i) = :e^{\kappa_i\cdot\partial X(z_i)-\lambda_i\cdot\bar{\partial}\widetilde{X}(\bar{z}_i)}e^{ip_i\cdot X(z_i,\bar{z}_i)}\,e^{iw_i\cdot\widetilde{X}(z_i,\bar{z}_i)}: \tag{B.1.1}$$

with I labeling the winding and momenta and κ_i, λ_i auxiliary parameters. One can derive the vertex operators corresponding to the first excited states simply by acting on (B.1.1) with derivatives with respect to both κ_i and λ_i. This operator is related to a massless graviton vertex operator $\mathcal{E}_{p_i,w_i}$ by

$$\mathcal{E}_{p_i,w_i}(z_i,\bar{z}_i) = E_{iab}\frac{\partial}{\partial\kappa_{ia}}\frac{\partial}{\partial\lambda_{ib}}\,\mathcal{V}_i\Big|_{\kappa_i=\lambda_i=0}\,. \tag{B.1.2}$$

The N point correlation function can be written as

$$\begin{aligned}\Big\langle\prod_{i=1}^{N}\mathcal{V}_i(z_i,\bar{z}_i)\Big\rangle = \prod_{1\leqslant i<j\leqslant N}\big|z_i-z_j\big|^{\alpha'(p_i\cdot p_j+w_i\cdot w_j)}\left(\frac{z_i-z_j}{\bar{z}_i-\bar{z}_j}\right)^{\frac{\alpha'}{2}(p_i\cdot w_j+w_i\cdot p_j)}\\ \times F_{ij}(z_{ij},\bar{z}_{ij})\,\delta\Big(\sum p_i\Big)\delta\Big(\sum w_i\Big)\end{aligned} \tag{B.1.3}$$

with

$$F_{ij}(z_{ij},\bar z_{ij}) = \exp\Bigg(-\frac{\alpha'}{2}\Big[\frac{\kappa_i\cdot\kappa_j}{(z_i-z_j)^2}+2i\frac{(p_{[\underline{i}}+w_{[\underline{i}})\cdot\kappa_{\underline{j}]}}{z_i-z_j} \\ +\frac{\lambda_i\cdot\lambda_j}{(\bar z_i-\bar z_j)^2}+2i\frac{(p_{[\underline{i}}-w_{[\underline{i}})\cdot\lambda_{\underline{j}]}}{\bar z_i-\bar z_j}\Big]\Bigg)\,. \tag{B.1.4}$$

The full 3-graviton amplitude is then given by

$$\begin{aligned}\mathcal{A}_3(p_i,w_i,E_i) &= g_c^3\, C_{S^2}\,\Big\langle\prod_{i=1}^{3}(c\,\bar c\,\mathcal{E}_{p_i,w_i})\Big\rangle \\ &= g_c^3\, C_{S^2}\, A(\vec z,\vec{\bar z})\prod_{k=1}^{3}E_{kab}\frac{\partial}{\partial\kappa_{ka}}\frac{\partial}{\partial\lambda_{kb}}\prod_{1\leqslant i<j\leqslant 3}F_{ij}(z_{ij},\bar z_{ij})|_{\kappa_i=\lambda_i=0}\ ,\end{aligned} \tag{B.1.5}$$

where $A(\vec z,\vec{\bar z})$ collects the contractions of the remaining exponentials (2.2.29). Notice that we can treat the derivatives with respect to κ and the ones with respect to λ separately. Denoting $F(\vec z,\vec{\bar z}) := \prod_{1\leqslant i<j\leqslant 3}F_{ij}(z_{ij},\bar z_{ij})$ and taking three derivatives with respect to κ, we find

$$\begin{aligned}\prod_{k=1}^{3}\frac{\partial}{\partial\kappa_{ka}}\ F|_{\kappa_i=\lambda_i=0} &= \frac{\alpha'^2}{4}\,\frac{\eta^{ac}p_{1L}^b+\eta^{bc}p_{3L}^a+\eta^{ab}p_{2L}^c}{z_{12}z_{13}z_{23}} \\ &\quad+\frac{\alpha'}{2}\left(\frac{p_{1L}^a}{z_{12}}-\frac{p_{3L}^a}{z_{23}}\right)\left(\frac{p_{2L}^b}{z_{12}}+\frac{p_{3L}^b}{z_{13}}\right)\left(\frac{p_{1L}^c}{z_{13}}+\frac{p_{2L}^c}{z_{23}}\right)\ ,\end{aligned} \tag{B.1.6}$$

where we made use of momentum and winding conservation as well as the transverse polarization of E_{mn}. The λ-derivatives can be worked out analogously. We can now contract the two parts with the corresponding polarization tensors of the massless vertex operators to get the full 3-point amplitude. We restrict ourselves to second order in momentum and winding and we consider the correct normalization of the graviton vertex operator which makes it necessary to include a factor of $\frac{2}{\alpha'}$ in each $\mathcal{E}$. Then we find the 3-graviton scattering amplitude to be

$$\mathcal{A}_3(p_i,w_i,E_i) = 4\pi g_c\, E_{1ad}E_{2be}E_{3cf}t^{abc}\tilde t^{def}+\mathcal{O}(p^4,p^3w,\ldots,w^4)\ , \tag{B.1.7}$$

with

$$\begin{aligned} t^{abc} &= \eta^{ca}p_{1L}^b+\eta^{ba}p_{2L}^c+\eta^{cb}p_{3L}^a \\ \tilde t^{abc} &= \eta^{ca}p_{1R}^b+\eta^{ba}p_{2R}^c+\eta^{cb}p_{3R}^a\,. \end{aligned} \tag{B.1.8}$$

Here we used $C_{S^2} = \frac{8\pi}{\alpha' g_c^2}$ which can be determined from unitarity by factorizing the 4-point amplitude (2.2.37) over the tachyonic pole. This result was first presented in [119] and consistently reduces to the well-known 3-graviton scattering amplitude [214] for vanishing B-field and zero winding.

B.2 3-point interaction from DFT

For our purposes, it is convenient to consider DFT theory formulated in terms of the field $\mathcal{E}_{ij} = G_{ij} + B_{ij}$ and the dilaton field d [115]:

$$\begin{aligned} S = \int dx\, d\tilde{x}\, e^{-2d}\Big[&-\frac{1}{4}g^{ik}g^{jl}\mathcal{D}^p\mathcal{E}_{kl}\mathcal{D}^p\mathcal{E}_{ij} + \frac{1}{4}g^{kl}(\mathcal{D}^j\mathcal{E}_{ik}\mathcal{D}^i\mathcal{E}_{jl} + \bar{\mathcal{D}}^j\mathcal{E}_{ki}\bar{\mathcal{D}}^i\mathcal{E}_{lj}) \\ &+(\mathcal{D}^i d\bar{\mathcal{D}}^j\mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\mathcal{D}^j\mathcal{E}_{ij}) + 4\mathcal{D}^i d\mathcal{D}_i d\Big] \ . \end{aligned} \tag{B.2.1}$$

Despite the fact that T-duality is no longer a manifest symmetry, this description nicely covers momenta and winding modes in the derivatives $\mathcal{D}_i = \partial_i - \mathcal{E}_{ik}\tilde{\partial}^k$ and $\bar{\mathcal{D}}_i = \partial_i + \mathcal{E}_{ki}\tilde{\partial}^k$. The inverse metric g^{ij} is used to raise indices and we set $2\kappa_d^2 = 1$. The construction of this action from string field theory made use of a field redefinition establishing the link to the low-energy effective field theories [215]. As given in [37], at zeroth order in $\alpha^{'}$ this field redefinition is

$$\mathcal{E}_{ij} = E_{ij} + f_{ij}(e,d) \ , \quad f_{ij}(e,d) = e_{ij} + \frac{1}{2}e_i{}^k e_{kj} + \mathcal{O}(e^3) \ . \tag{B.2.2}$$

Using (B.2.2), we now expanding the action (B.2.1) around Minkowski space to cubic order in the fluctuation e_{ij} (see [115]). Here E_{ij} denotes the constant background, which for vanishing B-field reduces to the Minkowski metric η_{ij}. It is important to take the higher order fluctuation into account in the expansion of the different objects. The metric g_{ij} is simply given by $g_{ij} = \frac{1}{2}(\mathcal{E}_{ij} + \mathcal{E}_{ji})$ and hence, for example, the expansion of the inverse metric takes the following form

$$g^{ij} = \eta^{ij} - e^{(ij)} + \frac{1}{4}e^{ik}e^j{}_k + \frac{1}{4}e^{ki}e_k{}^j + \mathcal{O}(e^3) \ . \tag{B.2.3}$$

Then, up to a total derivative, the action to cubic order in the fluctuation reads

$$\begin{aligned} S = \int dx\, d\tilde{x}\, \Big[&\frac{1}{4}e_{ij}\Box e^{ij} + \frac{1}{4}(D^i e_{ij})^2 + \frac{1}{4}(\bar{D}^j e_{ij})^2 - 2dD^i\bar{D}^j e_{ij} - 4d\Box d \\ &+\frac{1}{4}e_{ij}\big((D^i e_{kl}(\bar{D}^j e^{kl}) - (D^i e_{kl})(\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl})\big) \\ &+\frac{1}{2}d\big((D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 + \frac{1}{2}(D^k e_{ij})^2 + \frac{1}{2}(\bar{D}^k e_{ij})^2 \\ &+2e^{ij}(D_i D^k e_{kj} + \bar{D}_j\bar{D}^k e_{ik})\big) + 4e_{ij}dD^i\bar{D}^j d + 4d^2\Box d\Big] \ , \end{aligned} \tag{B.2.4}$$

which was first derived in [37]. The derivatives are given by

$$\begin{aligned} D_i &= \partial_i - E_{ik}\tilde{\partial}^k \ , \\ \bar{D}_i &= \partial_i + E_{ki}\tilde{\partial}^k \ , \end{aligned} \quad \Box = \frac{1}{2}(D^i D_i + \bar{D}^j\bar{D}_j) \ . \tag{B.2.5}$$

In order to compare with the 3-point amplitude from the CFT side, we introduce κ_d by modifying the fluctuation to $2\kappa_d e_{ij}$. In this way we get a match with the expansion of the standard Einstein-Hilbert action to third order in the metric fluctuation h_{ij}. Then, from the second line in (B.2.4) and after a partial integration, we identify the interaction term for three e_{ij}'s to be

$$\begin{aligned}&\kappa_d e_{ij}\Big((D^i e_{kl}(\bar{D}^j e^{kl}) - (D^i e_{kl})(\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl})\Big)\\ =&-\kappa_d e_{ij}\Big(e^{kl} D^i \bar{D}^j e_{kl} + (D^i e_{kl})(\bar{D}^l e^{kj}) + (D^k e^{il})(\bar{D}^j e_{kl})\Big) + (\text{tot. der.})\ .\end{aligned} \tag{B.2.6}$$

The missing term from the partial integration vanishes because of $D^i e_{ij} = 0$, following from the polarization constraint as listed in table 2.1. Next, we can read off the value of the 3-graviton vertex in momentum space by using $\partial_i \to i p_i$ and $\tilde{\partial}^i \to i w^i$, which translates derivatives to momenta and winding modes. Moreover, we have to keep track of possible permutations and obtain

$$\begin{aligned}A^{eee} = 4\pi g_c\Big(&p^i_{3R} e_{1ij} p^j_{3L} e^{kl}_2 e_{3kl} + p^i_{3R} e_{1ij} (e^{kj}_2)^T e_{3kl} p^l_{2L} + p^k_{3R} e_{1kl} (e^{il}_3)^T e_{2ij} p^j_{1L}\\ &+ (\text{cyclic permutations})\Big)\ ,\end{aligned} \tag{B.2.7}$$

where $g_c = \frac{\kappa_d}{2\pi}$. This result nicely matches with the string scattering amplitude (B.1.7).The slight difference in the left- and right-moving momenta can be cured by switching the sign of the B-field.

Appendix C

GG derivation of $\tilde{\mathcal{L}}_\beta$ and its equation of motion

In this appendix, we provide details on computations mentioned in section 3.2.2. These allow eventually to derive the Lagrangian $\tilde{\mathcal{L}}_\beta$ given in (1.4.9) using the Generalized Geometry (GG) formalism. We also detail the claim that the field redefinition is an $O(d-1,1)\times O(1,d-1)$ transformation.

C.1 Determination of the $O(d-1,1)\times O(1,d-1)$ derivative

In section 3.2.2, we explain how preserving an $O(d-1,1)\times O(1,d-1)$ structure leads generically to the derivative (3.2.38). We determine here the various pieces of this derivative for the frame (3.2.7) and derivative (3.2.28), following the procedure described in that section. To start with, the derivatives $\partial_{\mathcal{A}}$ in the unbarred/barred notation can be read after a simple rotation (3.2.33) from the up/down one

$$\partial_{\mathcal{A}} = \begin{cases} \partial_a = \partial_a + \eta_{ab}\beta^{bc}\partial_c \\ \partial_{\overline{a}} = \partial_{\overline{a}} - \overline{\eta_{ab}}\beta^{\overline{bc}}\partial_{\overline{c}} \end{cases} \quad , \tag{C.1.1}$$

where in the right-hand sides we do not write the δ's and use the alignment of vielbeins. We now consider the connection. $\hat{\Omega}$ is made of the $O(d,d)$ piece Ω, and a piece due to the conformal weight; let us start with Ω alone. Its fully unbarred component is given by a rotation from up/down components, as follows

$$\begin{aligned} \Omega_a{}^b{}_c &= P_a{}^{\mathcal{D}} P_c{}^{\mathcal{F}} (P^{-T})^b{}_{\mathcal{E}}\ \Omega_{(u/d)\mathcal{D}}{}^{\mathcal{E}}{}_{\mathcal{F}} \\ &= \frac{1}{2}\left(\delta_a^d(\Omega_d{}^b{}_c + \eta^{be}\eta_{cf}\Omega_{de}{}^f) + \eta_{ad}(\Omega^{db}{}_c + \eta^{be}\eta_{cf}\Omega^d{}_e{}^f) + \eta_{ad}\eta_{cf}\delta_e^b\Omega^{def}\right) \ , \end{aligned} \tag{C.1.2}$$

where we used the fixing discussed in section 3.2.1 $\Omega_d{}^{ef} = \Omega^d{}_{ef} = \Omega_{def} = 0$, that lead eventually to the derivative (3.2.28). We also identified there

$$\Omega_a{}^b{}_c = \omega^b_{ac} \ , \ \Omega^a{}_b{}^c = \omega_Q{}^{ac}_b \ , \ \Omega^{abc} = \Omega^{[abc]} = \frac{1}{3} R^{abc} \ . \tag{C.1.3}$$

We recall as well that $\Omega_{\mathcal{D}e}{}^f = -\Omega_{\mathcal{D}}{}^f{}_e$. Using these results and the antisymmetry properties of ω and ω_Q, we conclude

$$\Omega_a{}^b{}_c = \omega^b_{ac} - \eta_{ad}\omega_Q{}^{db}_c + \frac{1}{6}\eta_{ad}\eta_{cf}\delta^b_e R^{def} \ . \tag{C.1.4}$$

One proceeds similarly for the other unbarred - barred components of Ω. A subtlety occurs for the mixed components, because of the projection to the $O(d-1,1)\times O(1,d-1)$ structure. For instance, in $\Omega_{\overline{a}}{}^b{}_c$, one obtains a piece given by $-\frac{1}{2}\overline{\eta_{ag}}\eta_{ch}\delta^{\overline{g}}_d\delta^b_e\delta^h_f\Omega^{def}$ (we write all δ's to clarify the discussion). Ω^{def} has been identified in section 3.2.1 with its fully antisymmetrized part related to the R-flux. However, because of the projection, one should be careful in the placement of unbarred and barred indices. As discussed above (3.2.38), the two indices on the right should be of the same type. Therefore, out of the decomposition $\Omega^{[def]} = 3(\Omega^{d[ef]} + \Omega^{f[de]} + \Omega^{e[fd]})$, one should only keep the contribution of the first term. This leads to $-\frac{1}{2}\overline{\eta_{ag}}\eta_{ch}\delta^{\overline{g}}_d\delta^b_e\delta^h_f R^{def}$.

Finally, let us consider the other piece of $\hat{\Omega}$, namely the contribution to be added due to the conformal weight. In [39], it is changed from (3.2.10) in the up/down notation to the following in the unbarred/barred[1]

$$\hat{\Omega}_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} = \Omega_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} - \frac{1}{9}(\delta^{\mathcal{B}}_{\mathcal{A}}\Lambda_{\mathcal{C}} - \eta_{\mathcal{A}\mathcal{C}}\eta^{\mathcal{B}\mathcal{E}}\Lambda_{\mathcal{E}}) \ , \tag{C.1.5}$$

where we believe that the normalization factor 9 can be understood as $\delta^a_a - 1 = \delta^{\overline{a}}_{\overline{a}} - 1$. The trace of the above remains the same as that of (3.2.10), i.e. given by

$$\hat{\Omega}_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{C}} = \Omega_{\mathcal{D}}{}^{\mathcal{D}}{}_{\mathcal{C}} - \Lambda_{\mathcal{C}} \ . \tag{C.1.6}$$

This implies that the identification of Λ (3.2.24) made thanks to the torsion-free condition is in any case valid. So we follow here the same prescription (C.1.5), and should only define the unbarred/barred components of Λ from the up/down ones (3.2.20). This is done again by a rotation

$$\Lambda_{\mathcal{C}} = \begin{cases} \Lambda_c = \lambda_c + \eta_{cd}\xi^d \\ \Lambda_{\overline{c}} = \lambda_{\overline{c}} - \overline{\eta_{cd}}\xi^{\overline{d}} \end{cases} \ . \tag{C.1.7}$$

Combining all these contributions to (3.2.38), we obtain eventually the $O(d-1,1)\times O(1,d-1)$ derivative as given in (3.2.39).

[1] It is worth noting that this term $\hat{\Omega}_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} - \Omega_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}}$ in (C.1.5) is automatically compatible with the metric $\eta_{\mathcal{A}\mathcal{B}}$, on the contrary to the one in (3.2.10). This is consistent with the fact that $\hat{\Omega}$ is the $O(d-1,1)\times O(1,d-1)$ connection, and that there is no conformal factor in the structure group anymore.

C.2 The field redefinition is an $O(d-1,1)\times O(1,d-1)$ transformation

We make here a short digression to comment on the transformation relating the generalized vielbeins $\mathcal{E}$ and $\tilde{\mathcal{E}}$ in (3.1.1). Let us first consider formally a $2d\times 2d$ matrix K given in terms of generic $d\times d$ matrices O_1 and O_2 or the combinations $O_\pm$ as

$$K=\begin{pmatrix} O_1 & O_2\ \eta_d^{-1} \\ \eta_d\ O_2 & \eta_d\ O_1\ \eta_d^{-1}\end{pmatrix}\ ,\ O_\pm = O_1 \pm O_2\ . \tag{C.2.1}$$

Then, one can show the equivalence between the four following sets of conditions

$$\left.\begin{array}{c} K\in O(2d-2,2) \\ K^T\ \mathbb{I}\ K=\mathbb{I}\end{array}\right| \Leftrightarrow \left|\begin{array}{c} K\in O(d,d) \\ K^T\ \eta_{(u/d)}\ K=\eta_{(u/d)}\end{array}\right| \Leftrightarrow \left|\begin{array}{c} O_+\in O(d-1,1)\ ,\ O_-\in O(1,d-1) \\ O_\pm^T\ (\pm\eta_d)\ O_\pm=\pm\eta_d\end{array}\right.$$

$$\Leftrightarrow \left|\begin{array}{l} O_1^T\ \eta_d\ O_1+O_2^T\ \eta_d\ O_2=\eta_d \\ O_1^T\ \eta_d\ O_2+O_2^T\ \eta_d\ O_1=0\end{array}\right.\ , \tag{C.2.2}$$

with $\mathbb{I}$ defined in (3.1.1) and $\eta_{(u/d)}$ in (3.2.1).

Let us now show that such a matrix K is the one allowing to transform one generalized vielbein into the other

$$\mathcal{E}=K\tilde{\mathcal{E}}\Leftrightarrow K=\mathcal{E}\tilde{\mathcal{E}}^{-1}=\begin{pmatrix} e\tilde{e}^{-1} & -e\beta\tilde{e}^T \\ e^{-T}b\tilde{e}^{-1} & e^{-T}\tilde{e}^T-e^{-T}b\beta\tilde{e}^T\end{pmatrix}\ . \tag{C.2.3}$$

To do so, we need the information that the fields in (C.2.3) are not independent but related by the field redefinition (3.1.3). We rewrite the latter in a more convenient way[2]

$$\left.\begin{array}{r} e^T\eta_d e=\tilde{e}^T F^{-T}\eta_d F^{-1}\tilde{e} \\ b=-\tilde{e}^T F^{-T}\eta_d\tilde{e}\beta\tilde{e}^T\eta_d F^{-1}\tilde{e}\end{array}\right|\ ,\quad \text{with } F=\mathbb{1}+\tilde{e}\beta\tilde{e}^T\eta_d$$
$$\Leftrightarrow\ e=kF^{-1}\tilde{e}\ ,\ e^{-T}be^{-1}=-k^{-T}\eta_d\tilde{e}\beta\tilde{e}^T\eta_d k^{-1}\ ,\quad \text{with } k^T\eta_d k=\eta_d\ . \tag{C.2.5}$$

A little algebra then allows to show that K defined in (C.2.3) can be written as in (C.2.1), with

$$O_1=kF^{-1}\ ,\ O_2=k(F^{-1}-\mathbb{1})\ . \tag{C.2.6}$$

Interestingly, the field redefinition that we used to obtain this result is equivalent to having $K\in O(2d-2,2)$. Therefore, the properties (C.2.2) should be automatically satisfied with (C.2.6). It is indeed the case: when using (C.2.6), they boil down to the condition

$$2F^{-T}\eta_d F^{-1}=\eta_d F^{-1}+F^{-T}\eta_d\ \Leftrightarrow\ 2\eta_d=F^T\eta_d+\eta_d F\ , \tag{C.2.7}$$

[2]The starting point to get (C.2.5) is to rewrite (3.1.3) as

$$g=(\tilde{g}^{-1}-\beta)^{-1}\tilde{g}^{-1}\tilde{g}\tilde{g}^{-1}(\tilde{g}^{-1}+\beta)^{-1}\ ,\ b=-(\tilde{g}^{-1}-\beta)^{-1}\beta(\tilde{g}^{-1}+\beta)^{-1}\ , \tag{C.2.4}$$

where the change of sign in front of β in the brackets with respect to (3.1.3) is actually allowed: this sign can be chosen freely without affecting the field redefinition [84].

which is true given the definition of F. To conclude, we have shown that the transformation taking us from $\mathcal{E}$ to $\tilde{\mathcal{E}}$ and realizing the field redefinition is given by the matrix K in (C.2.1) with the entries (C.2.6), and it satisfies the properties (C.2.2).

The fact that $K \in O(d,d)$ is also important as it acts on the $O(d,d)$ index of the generalized vielbeins. As such, it can then be rotated as described in (3.2.33). One obtains the simple result

$$P^{-T} K P^{T} = \begin{pmatrix} O_+ & 0 \\ 0 & O_- \end{pmatrix} \ . \tag{C.2.8}$$

This result makes it obvious that this transformation is an $O(d-1,1)\times O(1,d-1)$ [78], thanks to the equivalence (C.2.2). Additionally, it coincides with the $O(d-1,1)\times O(1,d-1)$ structure we want to preserve in section 3.2.2. As a side remark, note though that it does not survive the alignment of vielbeins we impose there, as $O_+ \neq O_-$ a priori. This is expected because this transformation does not even preserve the form of the generalized vielbeins (by definition), i.e. one has $K \notin G_{\text{split}}$ for either of the two frames (3.2.6) and (3.2.7). More precisely for this particular K,

$$O_+ = O_- \ \Leftrightarrow \ F = \mathbb{1} \ \Leftrightarrow \ \beta = 0 \ \Leftrightarrow \ b = 0 \ , \tag{C.2.9}$$

which is indeed the only case where the form of the generalized vielbeins is preserved (they are actually the same, up to K).

C.3 Computation of S

In this appendix, we compute explicitly the quantity S as given in (3.2.54), using the definitions of section 3.2.2, analogously to [69]. As explained below (3.2.54), the first three lines of this expression should vanish: let us first detail the verification of this point. To start with, we compute, using (A.3.10), (A.2.6), and the alignment of vielbeins

$$\begin{aligned}
&\gamma^a\gamma^b(\partial_a + \eta_{ad}\beta^{de}\partial_e)(\partial_b + \eta_{bc}\beta^{cf}\partial_f) - \overline{\eta^{ab}}(\partial_{\overline{a}} - \overline{\eta_{ad}}\beta^{\overline{de}}\partial_{\overline{e}})(\partial_{\overline{b}} - \overline{\eta_{bc}}\beta^{\overline{cf}}\partial_{\overline{f}}) \qquad \text{(C.3.1)}\\
=&\gamma^{ab}\left(\frac{1}{2}f^f{}_{ab} + \eta_{ad}\beta^{de}f^f{}_{eb} - \eta_{ad}\partial_b(\beta^{df}) + \eta_{ad}\eta_{bc}\beta^{de}\left(\partial_e(\beta^{cf}) + \frac{1}{2}\beta^{cg}f^f{}_{eg}\right)\right)\partial_f + 2\partial_c(\beta^{cf})\partial_f \ .
\end{aligned}$$

One should then verify that

$$0 = 2\partial_c(\beta^{cf})\partial_f + 2\eta^{ac}X_c(\partial_a + \eta_{ad}\beta^{de}\partial_e) - Z_{\overline{a}}\overline{\eta^{ab}}(\partial_{\overline{b}} - \overline{\eta_{bc}}\beta^{\overline{cf}}\partial_{\overline{f}}) \ , \tag{C.3.2}$$

$$\begin{aligned}
0 = \gamma^{ab}\Bigg(&\frac{1}{2}f^f{}_{ab}\partial_f + \eta_{ad}\beta^{de}f^f{}_{eb}\partial_f - \eta_{ad}\partial_b(\beta^{df})\partial_f + \eta_{ad}\eta_{bc}\beta^{de}\left(\partial_e(\beta^{cf}) + \frac{1}{2}\beta^{cg}f^f{}_{eg}\right)\partial_f \\
&+ 6\eta^{ce}X_{[eab]}(\partial_c + \eta_{cd}\beta^{df}\partial_f) - 2\overline{\eta^{de}}Y_{\overline{d}ab}(\partial_{\overline{e}} - \overline{\eta_{ec}}\beta^{\overline{cf}}\partial_{\overline{f}})\Bigg) \ .
\end{aligned} \tag{C.3.3}$$

To prove (C.3.2), it is useful to recall that Λ was given in (3.2.40), and that one can rewrite ξ from (3.2.26) as

$$\xi^d = \beta^{de}\lambda_e - 2\mathcal{T}^d \ . \tag{C.3.4}$$

Using (A.3.23), (3.2.25), and (A.3.13), one then verifies (C.3.2). To prove (C.3.3), one can decompose it into the terms having no, one, two, or three β, and show that they vanish separately. The antisymmetry of the a, b indices and the properties of ω and ω_Q are useful, together with the alignment of vielbeins and (A.3.7), to prove the cancellation in (C.3.3).

We are then left with the last three lines of (3.2.54). Using the identities (A.2.7), (A.2.9) and (A.2.8), one can rewrite these lines, and therefore S, as

$$\begin{aligned}
&-\frac{1}{4}S\epsilon^+ \qquad\qquad (\text{C.3.5})\\
&=\Bigg[(\gamma^{abcf} + 3\eta^{a[b}\gamma^{cf]})(\partial_a + \eta_{ad}\beta^{de}\partial_e)(X_{bcf}) + \gamma^a\gamma^c(\partial_a + \eta_{ad}\beta^{de}\partial_e)(X_c)\\
&\quad + \frac{1}{2}X_{ade}X_{bcf}(18\delta^{[b}_{[g}\gamma_{hi]}{}^{cf]}\eta^{ag}\eta^{dh}\eta^{ei} - 12\delta^b_{[g}\delta^c_h\delta^f_{i]}\eta^{ag}\eta^{dh}\eta^{ei}) + X_{ade}X_c(6\eta^{c[a}\gamma^{de]}) + X_aX_c\gamma^a\gamma^c\\
&\quad - \overline{\eta^{ab}}(\partial_{\overline{a}} - \overline{\eta_{ad}}\beta^{\overline{de}}\partial_{\overline{e}})(Y_{\overline{b}cf})\gamma^{cf} - \frac{1}{2}\overline{\eta^{ab}}Y_{\overline{a}de}Y_{\overline{b}cf}(2\gamma^{decf} - 4\eta^{c[d}\eta^{e]f}) - Z_{\overline{a}}\overline{\eta^{ab}}Y_{\overline{b}cf}\gamma^{cf}\Bigg]\epsilon^+ \ .
\end{aligned}$$

To compute this expression, we decompose it into the various orders of antisymmetric products of γ matrices. The zeroth order will give the scalar of interest, while the higher orders (two and four γ's) will vanish. This is consistent with the idea of S being a scalar. The following identities will be helpful to show the vanishing of the terms at order γ^{ab}, and γ^{abcd}

$$\begin{aligned}
\partial_{[a}f^e{}_{bf]} &= f^e{}_{d[a}f^d{}_{bf]} \ , && (\text{C.3.6})\\
\partial_{[a}Q_{f]}{}^{de} - \beta^{g[d}\partial_g f^{e]}{}_{af} &= \frac{1}{2}Q_g{}^{de}f^g{}_{af} - 2Q_{[a}{}^{g[d}f^{e]}{}_{f]g} \ , && (\text{C.3.7})\\
\partial_a R^{ghi} - 3\beta^{d[g}\partial_d Q_a{}^{hi]} &= -3R^{d[gh}f^{i]}{}_{ad} + 3Q_a{}^{d[g}Q_d{}^{hi]} \ , && (\text{C.3.8})\\
\beta^{g[d}\partial_g R^{abc]} &= -\frac{3}{2}R^{g[da}Q_g{}^{bc]} \ . && (\text{C.3.9})
\end{aligned}$$

Let us start with the terms at order γ^{ab}. We proceed by using the explicit expressions for X_{ade}, X_a, $Y_{\overline{b}cf}$ and $Z_{\overline{a}}$, the alignment of vielbeins, and computing separately the terms at each order in β. At zeroth order in β the following equation holds thanks to (C.3.6) with two indices contracted

$$\begin{aligned}
0 =& \frac{3}{4}\eta^{a[b}\gamma^{cf]}\eta_{ce}\partial_a\omega^e_{bf} + \frac{1}{2}\gamma^{ac}\partial_a(\omega^g_{gc} - \lambda_c) + \frac{3}{4}\eta_{dh}\omega^h_{ae}(\omega^g_{gc} - \lambda_c)\eta^{c[a}\gamma^{de]}\\
&- \frac{1}{4}\overline{\eta^{ab}}\eta_{ce}\partial_{\overline{a}}\omega^e_{\overline{b}f}\gamma^{cf} - \frac{1}{4}(\omega^{\overline{g}}_{\overline{g}\overline{a}} - \lambda_{\overline{a}})\overline{\eta^{ab}}\eta_{ce}\omega^e_{\overline{b}f}\gamma^{cf} \ .
\end{aligned} \tag{C.3.10}$$

At first order in β, we make use of (C.3.6) and (C.3.7) with two indices contracted to show

$$\begin{aligned}0 =& \frac{3}{4}\eta^{a[b}\gamma^{cf]}(-\eta_{ce}\eta_{bh}\partial_a\omega_{Q}{}_f^{he} + \eta_{ad}\eta_{cg}\beta^{de}\partial_e\omega_{bf}^{g}) \\ &+ \frac{1}{2}\gamma^{ac}(\eta_{ce}\partial_a(\omega_{Q}{}_d^{de} - \xi^e) + \eta_{ad}\beta^{de}\partial_e(\omega_{gc}^{g} - \lambda_c)) \\ &+ \frac{3}{4}(\eta_{dh}\omega_{ae}^{h}\eta_{cg}(\omega_{Q}{}_d^{dg} - \xi^g) - \eta_{dg}\eta_{ah}\omega_{Q}{}_e^{hg}(\omega_{gc}^{g} - \lambda_c))\eta^{c[a}\gamma^{de]} \\ &- \frac{1}{4}(\eta_{ce}\partial_{\overline{a}}\omega_{Q}{}_f^{\overline{a}e} - \eta_{ce}\beta^{\overline{be}}\partial_{\overline{e}}\omega_{\overline{b}f}^{e})\gamma^{cf} \\ &- \frac{1}{4}(\omega_{\overline{ga}}^{\overline{g}} - \lambda_{\overline{a}})\eta_{ce}\omega_{Q}{}_f^{\overline{a}e}\gamma^{cf} + \frac{1}{4}(\omega_{Q}{}_{\overline{d}}^{\overline{db}} - \xi^{\overline{b}})\eta_{ce}\omega_{\overline{b}f}^{e}\gamma^{cf} \ .\end{aligned} \tag{C.3.11}$$

At second order in β, we verify using (C.3.7) and (C.3.8) with two indices contracted

$$\begin{aligned}0 =& \frac{3}{24}\eta^{a[b}\gamma^{cf]}(\eta_{be}\eta_{cg}\eta_{fh}\partial_a R^{egh} - 6\eta_{cg}\eta_{bh}\eta_{ad}\beta^{de}\partial_e\omega_{Q}{}_f^{hg}) \\ &+ \frac{1}{2}\gamma^{ac}\eta_{ad}\beta^{de}\partial_e\eta_{cg}(\omega_{Q}{}_d^{dg} - \xi^g) \\ &- \frac{1}{8}(6\eta_{dg}\eta_{ah}\omega_{Q}{}_e^{hg}\eta_{cf}(\omega_{Q}{}_b^{bf} - \xi^f) - \eta_{af}\eta_{dg}\eta_{eh}R^{fgh}(\omega_{bc}^{b} - \lambda_c))\eta^{c[a}\gamma^{de]} \\ &+ \frac{1}{8}(\eta_{cg}\eta_{fh}\partial_{\overline{a}}R^{\overline{a}gh} + 2\eta_{cg}\overline{\eta_{ad}}\beta^{\overline{de}}\partial_{\overline{e}}\omega_{Q}{}_f^{\overline{a}g})\gamma^{cf} \\ &+ \frac{1}{8}(\omega_{\overline{da}}^{\overline{d}} - \lambda_{\overline{a}})\eta_{cg}\eta_{fh}R^{\overline{a}gh}\gamma^{cf} + \frac{1}{4}\overline{\eta_{ag}}(\omega_{Q}{}_{\overline{d}}^{\overline{dg}} - \xi^{\overline{g}})\eta_{ce}\omega_{Q}{}_f^{\overline{a}e}\gamma^{cf} \ .\end{aligned} \tag{C.3.12}$$

The terms at third order in β vanish without using any of the above identities

$$\begin{aligned}0 =& \frac{3}{24}\eta^{a[b}\gamma^{cf]}\eta_{bg}\eta_{ch}\eta_{fi}\eta_{ad}\beta^{de}\partial_e R^{ghi} + \frac{1}{8}\eta_{af}\eta_{dg}\eta_{eh}R^{fgh}\eta_{ci}(\omega_{Q}{}_b^{bi} - \xi^i)\eta^{c[a}\gamma^{de]} \\ &- \frac{1}{8}\eta_{ch}\eta_{fi}\overline{\eta_{ad}}\beta^{\overline{de}}\partial_{\overline{e}}R^{\overline{a}hi}\gamma^{cf} - \frac{1}{8}\overline{\eta_{ag}}(\omega_{Q}{}_{\overline{d}}^{\overline{dg}} - \xi^{\overline{g}})\eta_{cg}\eta_{fh}R^{\overline{a}gh}\gamma^{cf} \ ,\end{aligned} \tag{C.3.13}$$

which concludes our verification that all terms in γ^{ab} vanish.

We now turn to the terms coming with an antisymmetric product of four γ matrices. For these, we first use

$$X_{abc}X_{def}9\delta^{[d}_{[g}\gamma_{hi]}{}^{ef]}\eta^{ag}\eta^{bh}\eta^{ci} = X_{abc}X_{def}(\eta^{ad}\gamma^{bcef} + 4\eta^{ae}\gamma^{bcfd} + 4\eta^{be}\gamma^{cafd}) \ , \tag{C.3.14}$$

and the explicit expressions of X_{bcf} and $Y_{\overline{b}cf}$. We then show that the resulting expression vanishes order by order in β, using the alignment of vielbeins. Starting at zeroth order in β, we have to prove

$$\begin{aligned}0 =& \frac{1}{4}\eta_{ce}\partial_a\omega_{bf}^{e}\gamma^{abcf} \\ &+ \frac{1}{16}\eta_{bg}\omega_{ac}^{g}\eta_{eh}\omega_{df}^{h}(\eta^{ad}\gamma^{bcef} + 4\eta^{ae}\gamma^{bcfd} + 4\eta^{be}\gamma^{cafd}) - \frac{1}{16}\overline{\eta^{ad}}\eta_{bg}\omega_{\overline{a}c}^{g}\eta_{eh}\omega_{\overline{d}f}^{h}\gamma^{bcef} \ .\end{aligned} \tag{C.3.15}$$

This can be verified, thanks to (C.3.6). At first order in β, we use (C.3.7) to show that

$$0 = -\frac{1}{4}\eta_{ce}(\eta_{bg}\partial_a\omega_{Q}{}^{ge}_{f} - \eta_{ad}\beta^{dg}\partial_g\omega^{e}_{bf})\gamma^{abcf} - \frac{1}{16}(\eta_{bg}\omega^{g}_{\overline{a}c}\eta_{eh}\omega_{Q}{}^{\overline{d}h}_{f} + \eta_{bh}\omega_{Q}{}^{\overline{a}h}_{c}\eta_{eg}\omega^{g}_{\overline{a}f})\gamma^{bcef}$$
$$- \frac{1}{16}(\eta_{bg}\omega^{g}_{ac}\eta_{eh}\eta_{di}\omega_{Q}{}^{ih}_{f} + \eta_{bh}\eta_{ai}\omega_{Q}{}^{ih}_{c}\eta_{eg}\omega^{g}_{df})(\eta^{ad}\gamma^{bcef} + 4\eta^{ae}\gamma^{bcfd} + 4\eta^{be}\gamma^{cafd}) \ . \quad \text{(C.3.16)}$$

At second order in β, we verify using (C.3.8)

$$0 = \frac{1}{24}(\eta_{bg}\eta_{ch}\eta_{fi}\partial_a R^{ghi} - 6\eta_{ch}\eta_{bg}\eta_{ad}\beta^{de}\partial_e\omega_{Q}{}^{gh}_{f})\gamma^{abcf} \quad \text{(C.3.17)}$$
$$+ \frac{1}{96}\Big(6\eta_{bh}\eta_{ag}\omega_{Q}{}^{gh}_{c}\eta_{ej}\eta_{di}\omega_{Q}{}^{ij}_{f}$$
$$+ \eta_{bg}\omega^{g}_{ac}\eta_{dh}\eta_{ei}\eta_{fj}R^{hij} + \eta_{ag}\eta_{bh}\eta_{ci}R^{ghi}\eta_{ej}\omega^{j}_{df}\Big)(\eta^{ad}\gamma^{bcef} + 4\eta^{ae}\gamma^{bcfd} + 4\eta^{be}\gamma^{cafd})$$
$$- \frac{1}{32}(2\overline{\eta_{ad}}\eta_{bg}\omega_{Q}{}^{\overline{a}g}_{c}\eta_{eh}\omega_{Q}{}^{\overline{d}h}_{f} - \eta_{bg}\omega^{g}_{\overline{a}c}\eta_{eh}\eta_{fi}R^{\overline{a}hi} - \eta_{bg}\eta_{ch}R^{\overline{a}gh}\eta_{ei}\omega^{i}_{\overline{a}f})\gamma^{bcef} \ .$$

At third order in β, we show using (C.3.9) that

$$0 = \frac{1}{24}\eta_{be}\eta_{ci}\eta_{fh}\eta_{ad}\beta^{dg}\partial_g R^{eih}\gamma^{abcf} \quad \text{(C.3.18)}$$
$$- \frac{1}{96}(\eta_{bh}\eta_{ag}\omega_{Q}{}^{gh}_{c}\eta_{di}\eta_{ej}\eta_{fk}R^{ijk} + \eta_{ag}\eta_{bh}\eta_{ck}R^{ghk}\eta_{ej}\eta_{di}\omega_{Q}{}^{ij}_{f})(\eta^{ad}\gamma^{bcef} + 4\eta^{ae}\gamma^{bcfd} + 4\eta^{be}\gamma^{cafd})$$
$$+ \frac{1}{32}\overline{\eta_{ad}}(\eta_{bh}\omega_{Q}{}^{\overline{a}h}_{c}\eta_{ej}\eta_{fg}R^{\overline{d}jg} + \eta_{bh}\eta_{cg}R^{\overline{a}hg}\eta_{ej}\omega_{Q}{}^{\overline{d}j}_{f})\gamma^{bcef} \ .$$

Finally, the forth order in β vanishes as follows

$$0 = \frac{1}{576}\eta_{ag}\eta_{bh}\eta_{ci}R^{ghi}\eta_{dj}\eta_{ek}\eta_{fl}R^{jkl}(\eta^{ad}\gamma^{bcef} + 4\eta^{ae}\gamma^{bcfd} + 4\eta^{be}\gamma^{cafd}) \quad \text{(C.3.19)}$$
$$- \frac{1}{64}\overline{\eta_{ad}}\eta_{bh}\eta_{ci}R^{\overline{a}hi}\eta_{eg}\eta_{fj}R^{\overline{d}gj}\gamma^{bcef} \ .$$

We thus have shown that all terms in γ^{abcd} vanish. From (C.3.5), we are then left only with terms without any γ. We compute them and finally get

$$-\frac{1}{4}S\epsilon^{+} = \Bigg[\frac{1}{2}\eta^{ac}\partial_a\omega^{g}_{gc} + \frac{1}{4}\eta^{ac}\omega^{d}_{da}\omega^{g}_{gc} - \frac{1}{4}\eta^{eb}\omega^{h}_{ae}\omega^{a}_{bh} - \frac{1}{2}\eta^{ac}\nabla_a\lambda_c + \frac{1}{4}\eta^{ac}\lambda_a\lambda_c \quad \text{(C.3.20)}$$
$$+ \frac{1}{2}\eta_{ac}\beta^{ae}\partial_e\omega_{Q}{}^{gc}_{g} + \frac{1}{4}\eta_{hg}\omega_{Q}{}^{fh}_{e}\omega_{Q}{}^{eg}_{f} + \frac{1}{4}\eta_{cg}\omega_{Q}{}^{dc}_{d}\omega_{Q}{}^{fg}_{f} + \frac{1}{4}\eta_{fg}\omega^{f}_{ae}R^{aeg}$$
$$+ \frac{1}{2}\eta_{ab}\check{\nabla}^{a}\xi^{b} + \frac{1}{4}\eta_{ab}\xi^{a}\xi^{b} - \frac{1}{48}\eta_{ec}\eta_{bh}\eta_{fg}R^{bfc}R^{heg}\Bigg]\epsilon^{+}$$
$$= -\frac{1}{4}\Bigg(\mathcal{R}(\tilde{g}) + \mathcal{R}_{Q} - \frac{1}{2}R^{acd}f^{b}{}_{cd}\eta_{ab} - \frac{1}{2}R^{2}$$
$$- 4(\partial\tilde{\phi})^{2} + 4\nabla^{2}\tilde{\phi} - 4(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^{a})^{2} - 4\eta_{ab}\check{\nabla}^{a}(\beta^{bc}\partial_c\tilde{\phi} - \mathcal{T}^{b})\Bigg)\epsilon^{+} \ ,$$

where the last line, given also in (3.2.55), is obtained using (A.3.24), (A.3.25) and (3.2.24).

C.4 Computation of $R_{a\bar{b}}$

We explain at the end of section 3.2.2 the main procedure to derive the equations of motion in flat indices from the GG formalism. Here, we give some details on the computation of the generalized Ricci tensor (3.2.58). We start from its expression (3.2.60). We observe that all derivatives acting on the spinor ϵ^+ should vanish, since the generalized Ricci tensor only acts on the spinor via a multiplication by a γ-matrix. One can therefore verify that

$$\begin{aligned}
&\Big(\gamma^a\partial_a\partial_b - \gamma^a\overline{\eta_{bg}}\partial_a\beta^{\overline{ge}}\partial_{\overline{e}} - \gamma^a\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_a\partial_{\overline{e}} + \gamma^a\gamma^{gh}Y_{\overline{b}gh}\partial_a \\
&+ \gamma^a\eta_{ad}\beta^{dc}\partial_c\partial_{\overline{b}} - \gamma^a\eta_{ad}\overline{\eta_{bg}}\beta^{dc}\partial_c\beta^{\overline{ge}}\partial_{\overline{e}} - \gamma^a\eta_{ad}\overline{\eta_{bg}}\beta^{dc}\partial_c\beta^{\overline{ge}}\partial_c\partial_{\overline{e}} + \gamma^a\gamma^{gh}Y_{\overline{b}gh}\eta_{ad}\beta^{dc}\partial_c \\
&+ X_{acd}\gamma^{acd}\partial_{\overline{b}} - X_{acd}\gamma^{acd}\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}} + \gamma^a X_a\partial_{\overline{b}} - \gamma^a X_a\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}} \\
&- \gamma^a\omega^{\overline{c}}_{a\overline{b}}\partial_{\overline{c}} + \gamma^a\omega^{\overline{c}}_{a\overline{b}}\overline{\eta_{cg}}\beta^{\overline{ge}}\partial_{\overline{e}} + \gamma^a\eta_{ad}\omega_Q{}^{d\overline{c}}{}_{\overline{b}}\partial_{\overline{c}} - \gamma^a\eta_{ad}\omega_Q{}^{d\overline{c}}{}_{\overline{b}}\overline{\eta_{cg}}\beta^{\overline{ge}}\partial_{\overline{e}} \\
&- \frac{1}{2}\gamma^a\eta_{ad}\overline{\eta_{bf}}R^{d\overline{fc}}\partial_{\overline{c}} + \frac{1}{2}\gamma^a\eta_{ad}\overline{\eta_{bf}}R^{d\overline{fc}}\overline{\eta_{cg}}\beta^{\overline{ge}}\partial_{\overline{e}} \\
&- \gamma^a\partial_{\overline{b}}\partial_a - \gamma^a\eta_{ad}\partial_{\overline{b}}\beta^{dc}\partial_c - \gamma^a\eta_{ad}\beta^{dc}\partial_{\overline{b}}\partial_c - \gamma^{acd}X_{acd}\partial_{\overline{b}} - \gamma^a X_a\partial_{\overline{b}} \\
&+ \gamma^a\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}}\partial_a + \gamma^a\eta_{ad}\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}}\beta^{dc}\partial_c + \gamma^a\eta_{ad}\overline{\eta_{bg}}\beta^{\overline{ge}}\beta^{dc}\partial_{\overline{e}}\partial_c + \gamma^{acd}X_{acd}\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}} + \gamma^a X_a\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}} \\
&- \gamma^{gh}\gamma^a Y_{\overline{b}gh}\partial_a - \gamma^{gh}\gamma^a Y_{\overline{b}gh}\eta_{ad}\beta^{dc}\partial_c\Big)\epsilon^+ = 0\ .
\end{aligned} \tag{C.4.1}$$

We are then left with γ-matrices acting on ϵ^+. Using several identities on γ-matrices listed in the appendix of [101], we obtain

$$\begin{aligned}
\frac{1}{2}R_{a\overline{b}}\gamma^a\epsilon^+ = \Big(&(\gamma^{agh} + 2\eta^{a[g}\gamma^{h]})\partial_a Y_{\overline{b}gh} + (\gamma^{agh} + 2\eta^{a[g}\gamma^{h]})\eta_{ad}\beta^{dc}\partial_c Y_{\overline{b}gh} \\
&+ [\gamma^{acd},\gamma^{gh}]X_{acd}Y_{\overline{b}gh} + [\gamma^a,\gamma^{gh}]X_a Y_{\overline{b}gh} \\
&- (\gamma^{agh} + 2\eta^{a[g}\gamma^{h]})\omega^{\overline{c}}_{a\overline{b}}Y_{\overline{c}gh} + (\gamma^{agh} + 2\eta^{a[g}\gamma^{h]})\eta_{ad}\omega_Q{}^{d\overline{c}}{}_{\overline{b}}Y_{\overline{c}gh} \\
&- \frac{1}{2}(\gamma^{agh} + 2\eta^{a[g}\gamma^{h]})\eta_{ad}\overline{\eta_{bf}}R^{d\overline{fc}}Y_{\overline{c}gh} \\
&- \gamma^{acd}\partial_{\overline{b}}X_{acd} - \gamma^a\partial_{\overline{b}}X_a + \gamma^{acd}\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}}X_{acd} + \gamma^a\overline{\eta_{bg}}\beta^{\overline{ge}}\partial_{\overline{e}}X_a\Big)\ \epsilon^+
\end{aligned} \tag{C.4.2}$$

Similarly to the calculation of the scalar S in [101], we should then distinguish the different orders in γ-matrices. Here, we only consider the lowest order in γ^a, and assume that all higher orders vanish: this would be analogous to the computation of S, where the BI (C.3.6) - (C.3.9) played an important role; we expect the same here. In addition, the lowest order will be enough to obtain the equations of motion. Then at first order in γ^a,

$\frac{1}{2}R_{a\bar{b}}\gamma^a$ gives

$$
\begin{aligned}
&\Bigg(\frac{1}{2}\mathcal{R}_{\bar{b}a} - \frac{1}{2}\eta_{ae}\overline{\eta_{bg}}\check{\mathcal{R}}^{\bar{g}e} + \frac{1}{8}\eta_{ae}\overline{\eta_{bg}}\eta_{if}\overline{\eta_{cd}}R^{i\overline{gc}}R^{\bar{d}fe} - \frac{1}{4}\eta_{ae}\overline{\eta_{bg}}e^{2\tilde{\phi}}\nabla_d(e^{-2\tilde{\phi}}R^{\bar{g}de}) \\
&+ \nabla_{\bar{b}}\nabla_a\tilde{\phi} - \eta_{ae}\overline{\eta_{bg}}\check{\nabla}^{\bar{g}}(\check{\nabla}^e\tilde{\phi}) - \eta_{ae}\overline{\eta_{bg}}\check{\nabla}^{\bar{g}}\mathcal{T}^e \\
&+ \frac{1}{4}\overline{\eta_{bg}}\partial_d Q_a{}^{\bar{g}d} + \frac{1}{4}\eta_{ae}\overline{\eta_{bg}}\partial_d Q_f{}^{e\bar{g}}\eta^{df} + \frac{1}{4}\eta_{ae}\overline{\eta_{bg}}\partial_d Q_{\bar{f}}{}^{ed}\overline{\eta^{gf}} - \frac{1}{2}\eta_{ae}\partial_{\bar{b}}Q_d{}^{de} \\
&- \frac{1}{4}\eta_{ae}\beta^{gc}\partial_c f^e{}_{\bar{b}g} - \frac{1}{4}\beta^{gc}\partial_c f^e{}_{a\bar{b}}\eta_{ge} - \frac{1}{4}\beta^{gc}\partial_c f^{\bar{e}}{}_{ag}\overline{\eta_{be}} + \frac{1}{2}\overline{\eta_{bg}}\beta^{\overline{gc}}\partial_{\bar{c}}f^d{}_{da} \\
&+ \frac{1}{4}\overline{\eta_{bg}}f^d{}_{dc}Q_a{}^{\bar{g}c} + \frac{1}{4}\eta_{ae}f^d{}_{dc}Q_{\bar{b}}{}^{ec} + \frac{1}{4}\overline{\eta_{bg}}\eta_{ae}\eta^{ch}f^d{}_{dc}Q_h{}^{e\bar{g}} \\
&- \frac{1}{4}\eta_{ae}Q_d{}^{dc}f^e{}_{\bar{b}c} - \frac{1}{4}\overline{\eta_{bh}}Q_d{}^{dc}f^{\bar{h}}{}_{ac} - \frac{1}{4}\eta_{ch}Q_d{}^{dc}f^h{}_{a\bar{b}} \\
&+ \frac{1}{8}\overline{\eta_{bg}}f^{\bar{g}}{}_{c\bar{d}}Q_a{}^{\bar{d}c} + \frac{1}{8}\eta_{ch}f^h{}_{\overline{bd}}Q_a{}^{\bar{d}c} + \frac{1}{8}\overline{\eta_{dh}}f^{\bar{h}}{}_{\bar{b}c}Q_a{}^{\bar{d}c} \\
&+ \frac{1}{8}\eta_{ae}\overline{\eta_{bg}}\eta^{cf}f^{\bar{g}}{}_{c\bar{d}}Q_f{}^{e\bar{d}} + \frac{1}{8}\eta_{ae}f^h{}_{\overline{gd}}Q_h{}^{e\bar{d}} + \frac{1}{8}\eta_{ae}\overline{\eta_{dh}}\eta^{ci}f^{\bar{h}}{}_{\bar{b}c}Q_i{}^{e\bar{d}} \\
&+ \frac{1}{8}\eta_{ae}\overline{\eta_{bg}}\overline{\eta^{df}}f^{\bar{g}}{}_{c\bar{d}}Q_{\bar{f}}{}^{ec} + \frac{1}{8}\eta_{ae}\eta_{ch}\overline{\eta^{di}}f^h{}_{\overline{bd}}Q_{\bar{i}}{}^{ec} + \frac{1}{8}\eta_{ae}f^{\bar{h}}{}_{\bar{b}c}Q_{\bar{h}}{}^{ec} \\
&- \frac{1}{8}\eta_{ae}f^e{}_{\bar{c}d}Q_{\bar{b}}{}^{d\bar{c}} - \frac{1}{8}\eta_{ae}\overline{\eta_{bg}}\eta^{dh}f^e{}_{\bar{c}d}Q_h{}^{\overline{gc}} - \frac{1}{8}\eta_{ae}\overline{\eta_{bg}}\overline{\eta^{ch}}f^e{}_{\bar{c}d}Q_{\bar{h}}{}^{\bar{g}d} \\
&- \frac{1}{8}\eta_{de}f^e{}_{a\bar{c}}Q_{\bar{b}}{}^{d\bar{c}} - \frac{1}{8}\overline{\eta_{bg}}f^e{}_{a\bar{c}}Q_e{}^{\overline{gc}} - \frac{1}{8}\overline{\eta_{bg}}\overline{\eta^{ch}}\eta_{de}f^e{}_{a\bar{c}}Q_{\bar{h}}{}^{\bar{g}d} \\
&- \frac{1}{8}\overline{\eta_{ce}}f^{\bar{e}}{}_{ad}Q_{\bar{b}}{}^{d\bar{c}} - \frac{1}{8}\overline{\eta_{bg}}\eta^{dh}\overline{\eta_{ce}}f^{\bar{e}}{}_{ad}Q_h{}^{\overline{gc}} - \frac{1}{8}\overline{\eta_{bg}}f^{\bar{e}}{}_{ad}Q_{\bar{e}}{}^{\bar{g}d} \\
&- \eta_{ae}\nabla_{\bar{b}}(\check{\nabla}^e\tilde{\phi}) - \eta_{ae}\nabla_{\bar{b}}\mathcal{T}^e + \overline{\eta_{bg}}\check{\nabla}^{\bar{g}}\nabla_a\tilde{\phi} \\
&- \frac{1}{2}\eta_{ae}\overline{\eta_{bg}}\eta_{fc}R^{\bar{g}fe}\mathcal{T}^c + \frac{1}{4}\eta_{ae}\overline{\eta_{bg}}\eta_{df}e^{2\tilde{\phi}}\check{\nabla}^d(e^{-2\tilde{\phi}}R^{\bar{g}fe})\Bigg)\gamma^a\ .
\end{aligned}
\tag{C.4.3}
$$

By considering aligned vielbeins, the previous expression reduces to

$$\begin{aligned}
&\bigg(\frac{1}{2}\mathcal{R}_{ba} - \frac{1}{2}\eta_{ae}\eta_{bg}\check{\mathcal{R}}^{ge} + \frac{1}{8}\eta_{ae}\eta_{bg}\eta_{if}\eta_{cd}R^{igc}R^{dfe} - \frac{1}{4}\eta_{ae}\eta_{bg}e^{2\tilde{\phi}}\nabla_d(e^{-2\tilde{\phi}}R^{gde}) \\
&+ \nabla_b\nabla_a\tilde{\phi} - \eta_{ae}\eta_{bg}\check{\nabla}^g(\check{\nabla}^e\tilde{\phi}) - \eta_{ae}\eta_{bg}\check{\nabla}^g\mathcal{T}^e \\
&+ \frac{1}{2}\partial_d Q_{(a}{}^{gd}\eta_{b)g} + \frac{1}{4}\eta_{ae}\eta_{bg}\eta^{df}\partial_d Q_f{}^{eg} - \frac{1}{2}\eta_{ae}\partial_b Q_d{}^{de} \\
&- \frac{1}{4}\beta^{gc}\partial_c f^e{}_{ab}\eta_{ge} + \frac{1}{2}\beta^{gc}\partial_c f^e{}_{g(a}\eta_{b)e} + \frac{1}{2}\eta_{bg}\beta^{gc}\partial_c f^d{}_{da} \\
&+ \frac{1}{2}f^d{}_{dc}Q_{(a}{}^{gc}\eta_{b)g} + \frac{1}{4}\eta_{bg}\eta_{ae}\eta^{ch}f^d{}_{dc}Q_h{}^{eg} + \frac{1}{2}Q_d{}^{dc}f^e{}_{c(a}\eta_{b)e} - \frac{1}{4}\eta_{ch}Q_d{}^{dc}f^h{}_{ab} \\
&+ \frac{1}{4}f^g{}_{cd}Q_{[a}{}^{dc}\eta_{b]g} + \frac{1}{2}\eta_{e[a}f^h{}_{b]d}Q_i{}^{ec}\eta_{ch}\eta^{di} + \frac{1}{2}\eta_{e[a}f^h{}_{b]c}Q_h{}^{ec} \\
&- \eta_{ae}\nabla_b(\check{\nabla}^e\tilde{\phi}) - \eta_{ae}\nabla_b\mathcal{T}^e + \eta_{bg}\check{\nabla}^g\nabla_a\tilde{\phi} \\
&- \frac{1}{2}\eta_{ae}\eta_{bg}\eta_{fc}R^{gfe}\mathcal{T}^c + \frac{1}{4}\eta_{ae}\eta_{bg}\eta_{df}e^{2\tilde{\phi}}\check{\nabla}^d(e^{-2\tilde{\phi}}R^{gfe})\bigg)\gamma^a \ .
\end{aligned} \tag{C.4.4}$$

We can further simplify the above using the following identities. First, one can show

$$\eta_{g(a}\check{\nabla}^g\nabla_{b)}\tilde{\phi} - \eta_{g(a}\nabla_{b)}(\check{\nabla}^g\tilde{\phi}) = 0 \ , \quad -\eta_{e[a}\eta_{b]g}\check{\nabla}^g(\check{\nabla}^e\tilde{\phi}) = \frac{1}{2}\eta_{e[a}\eta_{b]g}R^{ged}\nabla_d\tilde{\phi} \ , \tag{C.4.5}$$

where the second one cancels the term coming from $-\frac{1}{4}\eta_{ae}\eta_{bg}e^{2\tilde{\phi}}\nabla_d(e^{-2\tilde{\phi}}R^{gde})$. In addition, three terms antisymmetric in (a,b) at second order in β vanish thanks to the following identity using (C.3.7) and (C.3.8)[3]

$$-\frac{1}{2}\eta_{e[a}\eta_{b]g}\check{\mathcal{R}}^{ge} - \eta_{e[a}\eta_{b]g}\check{\nabla}^g\mathcal{T}^e - \frac{1}{4}\eta_{ae}\eta_{bg}\nabla_d R^{gde} = 0 \ , \tag{C.4.6}$$

and the seven terms symmetric in (a,b) at linear order in β cancel using (C.3.6) and (C.3.7)

$$\begin{aligned}
&\frac{1}{2}\partial_d Q_{(a}{}^{gd}\eta_{b)g} - \frac{1}{2}\eta_{e(a}\partial_{b)}Q_d{}^{de} + \frac{1}{2}\beta^{gc}\partial_c f^e{}_{g(a}\eta_{b)e} + \frac{1}{2}\beta^{gc}\partial_c f^d{}_{d(a}\eta_{b)g} \\
&-\eta_{e(a}\nabla_{b)}\mathcal{T}^e + \frac{1}{2}f^d{}_{dc}Q_{(a}{}^{gc}\eta_{b)g} + \frac{1}{2}Q_d{}^{dc}f^e{}_{c(a}\eta_{b)e} = 0 \ .
\end{aligned} \tag{C.4.7}$$

[3]One also has the identity $2\check{\mathcal{R}}^{[ab]} = -\nabla_c R^{cab}$ [82], related to (C.3.8).

Using all those, we are finally left with the following expression for $\frac{1}{2}R_{ab}\gamma^a$ at first order in γ-matrices, that we give also in (3.2.61)

$$\begin{aligned}&\Big(\frac{1}{2}\mathcal{R}_{ba} - \frac{1}{2}\eta_{e(a}\eta_{b)g}\check{\mathcal{R}}^{ge} + \frac{1}{8}\eta_{ae}\eta_{bg}\eta_{if}\eta_{cd}R^{igc}R^{dfe} \\ &+ \nabla_b\nabla_a\tilde{\phi} - \eta_{e(a}\eta_{b)g}\check{\nabla}^g(\check{\nabla}^e\tilde{\phi}) - \eta_{e(a}\eta_{b)g}\check{\nabla}^g\mathcal{T}^e \\ &+ \frac{1}{4}\eta_{ae}\eta_{bg}\eta^{df}\partial_d Q_f{}^{eg} - \frac{1}{2}\eta_{e[a}\partial_{b]}Q_d{}^{de} - \frac{1}{4}\beta^{gc}\partial_c f^e{}_{ab}\eta_{ge} + \frac{1}{2}\beta^{gc}\partial_c f^d{}_{d[a}\eta_{b]g} \\ &+ \frac{1}{4}\eta_{bg}\eta_{ae}\eta^{ch}f^d{}_{dc}Q_h{}^{eg} - \frac{1}{4}\eta_{ch}Q_d{}^{dc}f^h{}_{ab} \\ &+ \frac{1}{4}f^g{}_{cd}Q_{[a}{}^{dc}\eta_{b]g} + \frac{1}{2}\eta_{e[a}f^h{}_{b]d}Q_i{}^{ec}\eta_{ch}\eta^{di} + \frac{1}{2}\eta_{e[a}f^h{}_{b]c}Q_h{}^{ec} \\ &- \eta_{e[a}\nabla_{b]}(\check{\nabla}^e\tilde{\phi}) - \eta_{e[a}\nabla_{b]}\mathcal{T}^e + \eta_{g[b}\check{\nabla}^g\nabla_{a]}\tilde{\phi} \\ &- \frac{1}{2}\eta_{ae}\eta_{bg}\eta_{fc}R^{gfe}\mathcal{T}^c + \frac{1}{4}\eta_{ae}\eta_{bg}\eta_{df}e^{2\tilde{\phi}}\check{\nabla}^d(e^{-2\tilde{\phi}}R^{gfe})\Big)\gamma^a \ .\end{aligned} \tag{C.4.8}$$

C.5 Relation to the subcase with the simplifying assumption

A simplifying assumption was considered in [84], given by the conditions $\beta^{mn}\partial_n\cdot = 0$, where the dot stands for any field, and $\partial_p\beta^{np} = 0$. This provided a simple Lagrangian, corresponding to a subcase of β-supergravity: one can reduce $\tilde{\mathcal{L}}_\beta$ to the former upon the assumption. Let us study here the simplification of the equations of motion. First, the assumption leads to $R^{abc} = 0$ and $\mathcal{T}^a = 0$. In addition, the Q-flux gets reduced as in (5.2.43), implying that $Q_a{}^{ab} = 0$ and $Q_c{}^{ha}f^b{}_{ha} = 0$. The dilaton equation of motion (1.4.12) and the Einstein equation (1.4.13), boil down to

$$\frac{1}{4}\Big(\mathcal{R}(\tilde{g}) + \check{\mathcal{R}}(\tilde{g})\Big) - (\partial\tilde{\phi})^2 + \nabla^2\tilde{\phi} = 0 \ , \tag{C.5.1}$$

$$\mathcal{R}_{ab} - \eta_{c(a}\eta_{b)d}\check{\mathcal{R}}^{cd} + 2\nabla_a\nabla_b\tilde{\phi} = 0 \ , \tag{C.5.2}$$

where $\check{\mathcal{R}}$ and $\check{\mathcal{R}}^{ab}$ can be further simplified using (A.3.27) and (D.2.8). The β equation of motion (1.4.14) becomes

$$\begin{aligned}&Q_a{}^{gf}f^a{}_{g[c}\eta_{e]f} + \frac{1}{2}\eta_{ef}\eta_{cd}\eta^{gk}Q_g{}^{fd}f^a{}_{ak} + \eta_{gi}\eta^{ab}Q_a{}^{dg}f^i{}_{b[e}\eta_{c]d} \\ &= -\frac{1}{2}\eta_{ef}\eta_{cd}\eta^{ab}\partial_a Q_b{}^{fd} + \eta^{ab}\eta_{cd}\eta_{ef}\nabla_b\beta^{fd}\ \partial_a\tilde{\phi} + 2\beta^{ab}\eta_{a[c}\nabla_{e]}\partial_b\tilde{\phi} \ ,\end{aligned} \tag{C.5.3}$$

where the last term does not vanish due to the connection terms. Using for the penultimate term (**??**) and for the last term the different definitions, one can show that all explicit dependence on β vanishes with the assumption, leaving the β equation of motion as

$$\begin{aligned}&\eta_{ef}\eta_{cd}\eta^{gk}Q_g{}^{fd}f^a{}_{ak} + 2\eta_{gi}\eta^{ab}Q_a{}^{dg}f^i{}_{b[e}\eta_{c]d} + e^{2\tilde{\phi}}\eta_{ef}\eta_{cd}\eta^{ab}\partial_a(e^{-2\tilde{\phi}}Q_b{}^{fd}) \\ &+ 2Q_a{}^{gf}f^a{}_{g[c}\eta_{e]f} = 0 \ .\end{aligned} \tag{C.5.4}$$

The last term can be simplified further by the assumption towards $2Q_a{}^{gf}\ \tilde{e}^a{}_m \eta_{f[c}\partial_{e]}\tilde{e}^m{}_g$. It is interesting to compare this equation (C.5.4) to the one obtained in [84]:

$$\partial_m (e^{-2\tilde{\phi}} \sqrt{|\tilde{g}|}\ \tilde{g}^{mn} \tilde{g}_{pq} \tilde{g}_{rs} \partial_n \beta^{qs}) = 0\ . \tag{C.5.5}$$

This comparison was initiated in curved indices in [84]. Here, we turn (C.5.5) into flat indices and get, using the assumption,

$$\begin{aligned} &\eta_{ef}\eta_{cd}\eta^{gk} Q_g{}^{fd} f^a{}_{ak} + 2\eta_{gi}\eta^{ab} Q_a{}^{dg} f^i{}_{b[e}\eta_{c]d} + e^{2\tilde{\phi}} \eta_{ef}\eta_{cd}\eta^{ab}\partial_a (e^{-2\tilde{\phi}} Q_b{}^{fd}) \\ &+ 2Q_a{}^{gf}\eta_{gd}\eta^{ab}\ \tilde{e}^d{}_m \eta_{f[e}\partial_{c]}\tilde{e}^m{}_b = 0\ . \end{aligned} \tag{C.5.6}$$

We see that (C.5.4) and (C.5.6) do not match: they differ by their second rows, i.e. their last term. This fact can be understood as follows: applying the simplifying assumption to the Lagrangian and deriving the β equation of motion do not commute. This can be seen for instance on a Lagrangian term like $\beta^{mn}\partial_n \tilde{g}^{pq}\partial_q \tilde{g}_{mp}$, that would contribute to (C.5.4) but not to (C.5.6). This problem does not affect the other equations of motion (one can verify directly the matching) because the assumption does not involve the other fields. So to conclude, the correct β equation of motion for field configurations satisfying the simplifying assumption of [84] is (C.5.4) and not (C.5.5). Note though that for the toroidal example and the Q-brane, the two differing terms vanish.

Appendix D

Bianchi identities and *NS*-branes

D.1 Derivation of BI from the $Spin(d,d)\times\mathbb{R}^+$ covariant derivative

In section 5.1.2, we introduced a $Spin(d,d)\times\mathbb{R}^+$ derivative and its associated Dirac operator in (5.1.17). Before studying its nilpotency condition (5.1.31), let us first give some details on how to compute a piece of it, namely $\mathcal{D}_2$. This piece is given by

$$\mathcal{D}_2 = \frac{1}{4}\Omega_{\mathcal{ABC}}\Gamma^{\mathcal{ABC}} = \frac{1}{4}\Omega_{[\mathcal{ABC}]}\Gamma^{\mathcal{A}}\Gamma^{\mathcal{B}}\Gamma^{\mathcal{C}} \ , \tag{D.1.1}$$

where the index $_{\mathcal{B}}$ is lowered by an $O(d,d)$ metric. To compute this antisymmetry, we use

$$\Omega_{\mathcal{ABC}}\Gamma^{\mathcal{B}} \equiv \Omega_{\mathcal{A}}{}^{\mathcal{D}}{}_{\mathcal{C}}\ \eta_{\mathcal{DB}}\Gamma^{\mathcal{B}} = \frac{1}{2}\left(\Omega_{\mathcal{A}}{}^{b}{}_{\mathcal{C}}\Gamma_b + \Omega_{\mathcal{A}b\mathcal{C}}\Gamma^b\right) \ . \tag{D.1.2}$$

One then obtains for instance

$$(\Omega_{\mathcal{ABC}} - \Omega_{\mathcal{ACB}})\Gamma^{\mathcal{A}}\Gamma^{\mathcal{B}}\Gamma^{\mathcal{C}} = \Gamma^{\mathcal{A}}(\Omega_{\mathcal{A}bc}\Gamma^b\Gamma^c + \Omega_{\mathcal{A}}{}^{bc}\Gamma_b\Gamma_c + \Omega_{\mathcal{A}}{}^{b}{}_{c}\Gamma_b\Gamma^c - \Omega_{\mathcal{A}}{}^{c}{}_{b}\Gamma^b\Gamma_c) \ , \tag{D.1.3}$$

using the antisymmetry properties of the connection coefficient [101]. The six terms from $\Omega_{[\mathcal{ABC}]}$ can be grouped two by two to use the above formula, and further combinations give

$$\begin{aligned}\mathcal{D}_2 = \frac{8}{24}\Big(&3\Omega_{[abc]}\tilde{e}^a\wedge\tilde{e}^b\wedge\tilde{e}^c\wedge \\ &+2\Omega_{[a}{}^{b}{}_{c]}\tilde{e}^a\wedge\iota_b\,\tilde{e}^c\wedge +2\Omega_{[b}{}^{c}{}_{a]}\tilde{e}^a\wedge\tilde{e}^b\wedge\iota_c + 2\Omega_{[c}{}^{a}{}_{b]}\iota_a\,\tilde{e}^b\wedge\tilde{e}^c\wedge \\ &+2\Omega^{[a}{}_{b}{}^{c]}\iota_a\,\tilde{e}^b\wedge\iota_c + 2\Omega^{[b}{}_{c}{}^{a]}\iota_a\,\iota_b\,\tilde{e}^c\wedge +2\Omega^{[c}{}_{a}{}^{b]}\tilde{e}^a\wedge\iota_b\,\iota_c \\ &+3\Omega^{[abc]}\iota_a\,\iota_b\,\iota_c\Big) \ ,\end{aligned} \tag{D.1.4}$$

where we also set some connection coefficients to zero following [101], and the Γ-matrices have been rewritten with the Clifford map of section 5.1.2. Using the commutation properties of forms and contractions, and the value of the connection coefficients derived in [101], one obtains eventually the two $\mathcal{D}_2$ given in section 5.1.2.

We now turn to the derivation of the Bianchi identities using the nilpotency condition (5.1.31) on the Dirac operator $\mathcal{D}$ (5.1.17). We focus only on the β-supergravity case, and use the expressions for the three parts $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ given in section 5.1.2. We start with $\mathcal{D}_2$, that we showed to be related to the derivative $\mathcal{D}_\sharp$ of [86]. As mentioned in (5.1.13), the vanishing square of this last derivative is known to reproduce the BI for constant fluxes, together with an additional constraint. So this piece should be a good starting point. That square, acting on a p-form A, was computed explicitly in [86] and can be translated here as follows (we use conventions of appendix A.1)

$$\begin{aligned}\frac{1}{4}\mathcal{D}_2^2 A = \mathcal{D}_\sharp^2 A = &+ \frac{1}{4} f^g{}_{gd} f^d{}_{ab} \tilde{e}^a \wedge \tilde{e}^b \wedge A \\ &+ \frac{1}{2} f^d{}_{ga} f^g{}_{bc} \tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^c \wedge \iota_d A \\ &+ \frac{1}{4} f^g{}_{gd} Q_a{}^{da} A \\ &- \frac{1}{2}(f^b{}_{cd} Q_a{}^{cd} + f^c{}_{cd} Q_a{}^{db} + f^b{}_{da} Q_c{}^{cd}) \tilde{e}^a \wedge \iota_b A \\ &+ \frac{1}{4}\Big(4 f^c{}_{ga} Q_b{}^{gd} + f^g{}_{ab} Q_g{}^{cd}\Big) \tilde{e}^a \wedge \tilde{e}^b \wedge \iota_c \iota_d A \\ &- \frac{1}{2}\Big(f^a{}_{cd} R^{cdb} + \frac{1}{2} f^c{}_{cd} R^{dab} + \frac{1}{2} Q_c{}^{cd} Q_d{}^{ab}\Big) \iota_a \iota_b A \\ &- \frac{1}{2}\Big(f^d{}_{ga} R^{gbc} + Q_g{}^{bc} Q_a{}^{gd}\Big) \tilde{e}^a \wedge \iota_b \iota_c \iota_d A \\ &- \frac{1}{4} Q_g{}^{ab} R^{gcd} \iota_a \iota_b \iota_c \iota_d A \ .\end{aligned} \tag{D.1.5}$$

Let us now add to $\mathcal{D}_2$ the derivative part $\mathcal{D}_1$

$$\begin{aligned}\frac{1}{4}\Big(\mathcal{D}_1^2 + \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1\Big) A = &- \frac{1}{2}\partial_a f^d{}_{db} \tilde{e}^a \wedge \tilde{e}^b \wedge A - \frac{1}{2}\partial_a f^d{}_{bc} \tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^c \wedge \iota_d A \\ &+ \frac{1}{2}\Big(\beta^{ac} f^d{}_{ca} \partial_d - \beta^{de} f^g{}_{gd} \partial_e + Q_d{}^{db} \partial_b - \beta^{de} \partial_e f^g{}_{gd}\Big) A \\ &+ \Big(- \beta^{de} \partial_e f^b{}_{da} + \frac{1}{2}(\partial_a Q_d{}^{db} + \beta^{be} \partial_e f^d{}_{da})\Big) \tilde{e}^a \wedge \iota_b A \\ &- \frac{1}{2}\Big(\partial_a Q_b{}^{cd} - \beta^{gc} \partial_g f^d{}_{ab}\Big) \tilde{e}^a \wedge \tilde{e}^b \wedge \iota_c \iota_d A \\ &+ \frac{1}{6}\Big(- 3\beta^{dc} \partial_c Q_d{}^{ab} + 3\beta^{ac} \partial_c Q_d{}^{db}\Big) \iota_a \iota_b A \\ &+ \frac{1}{6}\Big(\partial_a R^{bcd} - 3\beta^{eb} \partial_e Q_a{}^{cd}\Big) \tilde{e}^a \wedge \iota_b \iota_c \iota_d A \\ &- \frac{1}{6}\beta^{ga} \partial_g R^{bcd} \iota_a \iota_b \iota_c \iota_d A \ .\end{aligned} \tag{D.1.6}$$

Bringing indices in the right order and writing out antisymmetries, we obtain a set of identities by adding the above to $\frac{1}{4}\mathcal{D}_2^2$. Among those are already present our four BI (5.1.1) - (5.1.4). However the additional identities are independent and non-trivial; they contain in particular derivatives acting on A. To get rid of those, the missing part $\mathcal{D}_3$ of the Dirac operator is then necessary. Note that this last part contains terms that include the dilaton. So the additional terms to the square are

$$
\begin{aligned}
&\frac{1}{4}\Big(\mathcal{D}_1\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_1 + \mathcal{D}_2\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_2 + \mathcal{D}_3^2\Big)A \\
&= \Big(-\frac{1}{4}f^g{}_{gd}f^d{}_{ab} + \frac{1}{2}f^c{}_{ab}\partial_c\tilde{\phi} + \frac{1}{2}\partial_a f^d{}_{db} - \partial_a\partial_b\tilde{\phi}\Big)\tilde{e}^a \wedge \tilde{e}^b \wedge A \\
&+ \Big(\frac{1}{4}Q_d{}^{da}f^g{}_{ga} - \frac{1}{2}f^d{}_{da}(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a) - \frac{1}{2}Q_d{}^{da}\partial_a\tilde{\phi} + \partial_a\tilde{\phi}(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a) + \frac{1}{2}Q_d{}^{da}\partial_a \\
&\quad + \mathcal{T}^a\partial_a + \frac{1}{2}\beta^{ac}\partial_c f^d{}_{da} + \frac{1}{2}\beta^{ac}f^d{}_{da}\partial_c - \beta^{ac}\partial_c\partial_a\tilde{\phi} + \frac{1}{2}f^g{}_{gd}(\beta^{dc}\partial_c\tilde{\phi} - \mathcal{T}^d) - \frac{1}{2}Q_d{}^{da}\partial_a\tilde{\phi}\Big)A \\
&+ \Big(\frac{1}{2}\partial_a Q_d{}^{db} - \partial_a(\beta^{bc}\partial_c\tilde{\phi} - \mathcal{T}^b) - \frac{1}{2}\beta^{bc}\partial_c f^d{}_{da} - \beta^{bc}\partial_c\partial_a\tilde{\phi} \\
&\quad + f^b{}_{da}(\beta^{dc}\partial_c\tilde{\phi} - \mathcal{T}^d) + Q_a{}^{bc}\partial_c\tilde{\phi} + \frac{1}{2}f^b{}_{ad}Q_g{}^{gd} - \frac{1}{2}f^g{}_{gc}Q_a{}^{bc}\Big)\tilde{e}^a \wedge \iota_b A \\
&+ \frac{1}{2}\Big(\beta^{ac}\partial_c Q_d{}^{db} - 2\beta^{ac}\partial_c(\beta^{bd}\partial_d\tilde{\phi} - \mathcal{T}^b) \\
&\quad + \frac{1}{2}f^g{}_{gd}R^{abd} - R^{abd}\partial_d\tilde{\phi} - \frac{1}{2}Q_d{}^{ab}Q_g{}^{gd} + Q_d{}^{ab}(\beta^{dc}\partial_c\tilde{\phi} - \mathcal{T}^d)\Big)\iota_a\iota_b A \, .
\end{aligned}
\tag{D.1.7}
$$

All these contributions add-up to the following identities

$$\frac{1}{2}\partial_{[a}f^d{}_{b]d} + \frac{1}{4}f^g{}_{gd}f^d{}_{ab} - \frac{1}{4}f^g{}_{gd}f^d{}_{ab} + \frac{1}{2}f^c{}_{ab}\partial_c\tilde{\phi} - \frac{1}{2}\partial_{[a}f^d{}_{b]d} - \partial_{[a}\partial_{b]}\tilde{\phi} = 0 \tag{D.1.8}$$

$$-\frac{1}{2}\partial_{[a}f^d{}_{bc]} + \frac{1}{2}f^d{}_{g[a}f^g{}_{bc]} = 0 \tag{D.1.9}$$

$$\begin{aligned}
&\frac{1}{2}(\beta^{ac}f^d{}_{ca}\partial_d - \beta^{de}f^g{}_{gd}\partial_e + Q_d{}^{db}\partial_b - \beta^{de}\partial_e f^g{}_{gd}) + \frac{1}{4}f^g{}_{gd}Q_a{}^{da} + \frac{1}{4}Q_d{}^{da}f^g{}_{ga}\\
&-\frac{1}{2}f^d{}_{da}(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a) - \frac{1}{2}Q_d{}^{da}\partial_a\tilde{\phi} + \partial_a\tilde{\phi}(\beta^{ab}\partial_b\tilde{\phi} - \mathcal{T}^a) + \frac{1}{2}Q_d{}^{da}\partial_a + \mathcal{T}^a\partial_a\\
&+\frac{1}{2}\beta^{ac}\partial_c f^d{}_{da} + \frac{1}{2}\beta^{ac}f^d{}_{da}\partial_c + \beta^{ac}\partial_c\partial_a\tilde{\phi} + \frac{1}{2}f^g{}_{gd}(\beta^{dc}\partial_c\tilde{\phi} - \mathcal{T}^d) - \frac{1}{2}Q_d{}^{da}\partial_a\tilde{\phi} = 0
\end{aligned} \tag{D.1.10}$$

$$\begin{aligned}
&-\beta^{de}\partial_e f^b{}_{da} + \frac{1}{2}(\partial_a Q_d{}^{db} + \beta^{be}\partial_e f^d{}_{da}) - \frac{1}{2}(f^b{}_{cd}Q_a{}^{cd} + f^c{}_{cd}Q_a{}^{db} + f^b{}_{da}Q_c{}^{cd})\\
&\qquad + \frac{1}{2}\partial_a Q_d{}^{db} - \partial_a(\beta^{bc}\partial_c\tilde{\phi} - \mathcal{T}^b) - \frac{1}{2}\beta^{bc}\partial_c f^d{}_{da} - \beta^{bc}\partial_c\partial_a\tilde{\phi}\\
&\qquad + f^b{}_{da}(\beta^{dc}\partial_c\tilde{\phi} - \mathcal{T}^d) + Q_a{}^{bc}\partial_c\tilde{\phi} + \frac{1}{2}f^b{}_{ad}Q_g{}^{gd} - \frac{1}{2}f^g{}_{gc}Q_a{}^{bc} = 0
\end{aligned} \tag{D.1.11}$$

$$-\frac{1}{2}(\partial_{[a}Q_{c]}{}^{de} - \beta^{g[d}\partial_g f^{e]}{}_{ac}) + \frac{1}{4}(-4f^{[d}{}_{g[a}Q_{c]}{}^{e]g} + f^g{}_{ac}Q_g{}^{de}) = 0 \tag{D.1.12}$$

$$\begin{aligned}
&\frac{1}{6}(-3\beta^{dc}\partial_c Q_d{}^{ab} + 3\beta^{c[a}\partial_c Q_d{}^{b]d}) - \frac{1}{2}(f^{[a}{}_{cd}R^{b]cd} + \frac{1}{2}f^c{}_{cd}R^{dab} + \frac{1}{2}Q_c{}^{cd}Q_d{}^{ab})\\
&\qquad + \frac{1}{2}(\beta^{ac}\partial_c Q_d{}^{db} - 2\beta^{ac}\partial_c(\beta^{bd}\partial_d\tilde{\phi} - \mathcal{T}^b)\\
&\qquad + \frac{1}{2}f^g{}_{gd}R^{abd} - R^{abd}\partial_d\tilde{\phi} - \frac{1}{2}Q_d{}^{ab}Q_g{}^{gd} + Q_d{}^{ab}(\beta^{dc}\partial_c\tilde{\phi} - \mathcal{T}^d)) = 0
\end{aligned} \tag{D.1.13}$$

$$\frac{1}{6}(\partial_a R^{bcd} - 3\beta^{e[b}\partial_e Q_a{}^{cd]}) - \frac{1}{2}(-R^{g[bc}f^d{}_{a]g} + Q_a{}^{g[d}Q_g{}^{bc]}) = 0 \tag{D.1.14}$$

$$-\frac{1}{6}\beta^{g[a}\partial_g R^{bcd]} - \frac{1}{4}Q_g{}^{[ab}R^{cd]g} = 0\ . \tag{D.1.15}$$

Using in particular the expression of $\mathcal{T}^a$ in terms of the other fluxes, (D.1.10), (D.1.11) and (D.1.13) can be simplified respectively to

$$-\frac{1}{2}Q_d{}^{da}f^g{}_{ga} = 0 \tag{D.1.16}$$

$$-\frac{3}{2}\beta^{de}\partial_{[e}f^b{}_{da]} + \frac{3}{2}\beta^{de}f^b{}_{h[a}f^h{}_{ed]} = 0 \tag{D.1.17}$$

$$-\frac{1}{2}\beta^{dc}\partial_c Q_d{}^{ab} - \frac{1}{2}\beta^{cd}\beta^{g[a}\partial_g f^{b]}{}_{cd} - \beta^{dc}Q_c{}^{g[a}f^{b]}{}_{dg} + \frac{1}{4}\beta^{dc}Q_g{}^{ab}f^g{}_{cd} = 0\ . \tag{D.1.18}$$

In addition, (D.1.8) simply vanishes. We are then left with seven identities, namely (D.1.9), (D.1.16), (D.1.17), (D.1.12), (D.1.18), (D.1.14) and (D.1.15), that we respectively give in (5.1.32) - (5.1.38). As we show there, only five of those are independent and give our four BI (5.1.1) - (5.1.4) together with the expected scalar condition.

D.2 The Q-brane background

The *NS*5-brane and the *KK* monopole are known vacua of standard supergravity (SUGRA). We verify explicitly in this appendix that the Q-brane, given in sections 5.2.1 and 5.2.3, satisfies the equations of motion of β-supergravity. We recall that this makes the Q-brane a vacuum of standard SUGRA as well. As discussed in section 3.1 and appendix C.5, for a field configuration satisfying $\beta^{mn}\partial_n\cdot = 0$ and $\partial_p\beta^{np} = 0$, β-supergravity gets simplified to the theory worked out in [84]. These two conditions turn out to be verified by the Q-brane, even at the singularity. Using this property, the Q-brane was verified in [85] to solve the simple equations of motion of [84]. We show however in appendix C.5 that the β equation of motion of [84] is a priori not correct. In addition, the warp factor was considered in [85] to be harmonic, which only holds away from the singularity. Here we will get some new information at the singularity. So we start with the full equations of motion of β-supergravity, obtained in this paper in flat indices. Using the two above conditions, the three equations of motion have been simplified towards (C.5.1), (C.5.2), and (C.5.4).

For the Q-brane, given the non-zero components of the fluxes, each term of the β equation of motion (C.5.4) simply vanishes because of the indices contractions: it is trivially satisfied. So let us turn to the dilaton equation of motion (C.5.1). One computes

$$\mathcal{R} = -\frac{5}{2}f^{-3}(\partial_\rho f)^2 + f^{-2}\Delta_2 f \ , \ \check{\mathcal{R}} = -\frac{1}{2}f^{-3}(\partial_\rho f)^2 \ , \tag{D.2.1}$$

$$(\partial\tilde{\phi})^2 = \frac{1}{4}f^{-3}(\partial_\rho f)^2 \ , \ \nabla^2\tilde{\phi} = f^{-3}(\partial_\rho f)^2 - \frac{1}{2}f^{-2}\Delta_2 f \ . \tag{D.2.2}$$

Note that in these expressions and the following ones, the LHS is given in flat indices, whereas the RHS involves derivatives in curved indices. One way to compute $\nabla^2\tilde{\phi}$ is to use

$$\eta^{ab}\nabla_a V_b = \eta^{ab}\partial_a V_b + \eta^{cd} f^b{}_{bc} V_d \ . \tag{D.2.3}$$

This leads to

$$\frac{1}{4}\Big(\mathcal{R}(\tilde{g}) + \check{\mathcal{R}}(\tilde{g})\Big) - (\partial\tilde{\phi})^2 + \nabla^2\tilde{\phi} = -\frac{1}{4}f^{-2}\Delta_2 f \ . \tag{D.2.4}$$

So away from the singularity, (C.5.1) is satisfied, since $\Delta_2 f = 0$ for $\rho > 0$. At the singularity, we get a δ, which is expected. Indeed, one should in principle add a source action to the bulk action, and the former would contribute to the equations of motion by a δ within the energy-momentum tensor. This is what we obtain here.

Finally, we focus on the simplified Einstein equation (C.5.2). The only non-zero components of the Ricci tensor in flat indices are

$$\mathcal{R}_{xx} = \mathcal{R}_{yy} = -f^{-3}(\partial_\rho f)^2 + \frac{1}{2}f^{-2}\Delta f \tag{D.2.5}$$

$$\mathcal{R}_{\rho\rho} = -\frac{3}{2}f^{-3}(\partial_\rho f)^2 + \frac{1}{2}f^{-2}\partial_\rho^2 f - \frac{1}{2}f^{-2}\rho^{-1}\partial_\rho f \tag{D.2.6}$$

$$\mathcal{R}_{\varphi\varphi} = f^{-3}(\partial_\rho f)^2 - \frac{1}{2}f^{-2}\partial_\rho^2 f + \frac{1}{2}f^{-2}\rho^{-1}\partial_\rho f \ . \tag{D.2.7}$$

The other curvature tensor takes the form

$$\breve{\mathcal{R}}^{ab} = \beta^{cd}\partial_d \omega_{Q}{}_c^{ab} - \beta^{ad}\partial_d \omega_{Q}{}_c^{cb} + \omega_{Q}{}_c^{ab}\omega_{Q}{}_d^{dc} - \omega_{Q}{}_d^{ca}\omega_{Q}{}_c^{db} - \frac{1}{2}R^{adc}f^b{}_{dc} \simeq -\omega_{Q}{}_d^{ca}\omega_{Q}{}_c^{db} \ , \tag{D.2.8}$$

where the last equality is obtained thanks to the aforementioned simplifications verified by the Q-brane. The non-zero components are

$$\breve{\mathcal{R}}^{xx} = \breve{\mathcal{R}}^{yy} = -\frac{1}{2}(Q_\varphi{}^{yx})^2 = -\frac{1}{2}f^{-3}(\partial_\rho f)^2 \tag{D.2.9}$$

$$\breve{\mathcal{R}}^{\varphi\varphi} = \frac{1}{2}f^{-3}(\partial_\rho f)^2 \ , \quad \breve{\mathcal{R}}^{\rho\rho} = 0 \ . \tag{D.2.10}$$

In addition the dilaton terms in flat indices yield

$$\nabla_x\nabla_x\tilde{\phi} = -\omega^\rho_{xx}f^{-\frac{1}{2}}\partial_\rho\tilde{\phi} = \frac{1}{4}f^{-3}(\partial_\rho f)^2 \tag{D.2.11}$$

$$\nabla_y\nabla_y\tilde{\phi} = -\omega^\rho_{yy}f^{-\frac{1}{2}}\partial_\rho\tilde{\phi} = \frac{1}{4}f^{-3}(\partial_\rho f)^2 \tag{D.2.12}$$

$$\nabla_\rho\nabla_\rho\tilde{\phi} = f^{-\frac{1}{2}}\partial_\rho(f^{-\frac{1}{2}}\partial_\rho\tilde{\phi}) = -\frac{1}{2}f^{-\frac{1}{2}}\partial_\rho(f^{-\frac{3}{2}}\partial_\rho f) = \frac{3}{4}f^{-3}(\partial_\rho f)^2 - \frac{1}{2}f^{-2}\partial_\rho^2 f \tag{D.2.13}$$

$$\nabla_\varphi\nabla_\varphi\tilde{\phi} = -\omega^\rho_{\varphi\varphi}f^{-\frac{1}{2}}\partial_\rho\tilde{\phi} = -\frac{1}{4}f^{-3}(\partial_\rho f)^2 - \frac{1}{2}f^{-2}\rho^{-1}\partial_\rho f \ , \tag{D.2.14}$$

from which we eventually deduce

$$\mathcal{R}_{xx} - \breve{\mathcal{R}}^{xx} + 2\nabla_x\nabla_x\tilde{\phi} = \frac{1}{2}f^{-2}\Delta f \tag{D.2.15}$$

$$\mathcal{R}_{yy} - \breve{\mathcal{R}}^{yy} + 2\nabla_y\nabla_y\tilde{\phi} = \frac{1}{2}f^{-2}\Delta f \tag{D.2.16}$$

$$\mathcal{R}_{\rho\rho} - \breve{\mathcal{R}}^{\rho\rho} + 2\nabla_\rho\nabla_\rho\tilde{\phi} = -\frac{1}{2}f^{-2}\Delta f \tag{D.2.17}$$

$$\mathcal{R}_{\varphi\varphi} - \breve{\mathcal{R}}^{\varphi\varphi} + 2\nabla_\varphi\nabla_\varphi\tilde{\phi} = -\frac{1}{2}f^{-2}\Delta f \ . \tag{D.2.18}$$

As explained for the dilaton equation of motion (D.2.4), the above equations vanish away from the singularity as (C.5.2), and receive at the singularity an energy-momentum tensor contribution in the form of a δ, due to the Q-brane action to be added.

Appendix E

Compactification and pure spinors conditions

E.1 Consequences of the compactification ansatz

From the compactification ansatz of the ten-dimensional fields given in section 6.1.2, we compute here the various components of the fluxes; we recall they are defined as

$$f^A{}_{BC} = 2\tilde{e}^A{}_M \partial_{[B}\tilde{e}^M{}_{C]} \ , \ Q_A{}^{BC} = \partial_A \beta^{BC} - 2\beta^{D[B} f^{C]}{}_{AD} \ , \ R^{ABC} = 3\tilde{e}^A{}_M \tilde{e}^B{}_N \tilde{e}^C{}_P \beta^{Q[M}\partial_Q \beta^{NP]} \ .$$

We get

$$\begin{aligned}
& f^\alpha{}_{\beta\gamma} = 2\tilde{e}^\alpha{}_\mu \partial_{[\beta}\tilde{e}^\mu{}_{\gamma]} \ , \ f^a{}_{bc} = 2\tilde{e}^a{}_m \partial_{[b}\tilde{e}^m{}_{c]} \ , \ f^\alpha{}_{b\gamma} = -\delta^\alpha_\gamma \partial_b A \ , \ f^\alpha{}_{\beta c} = \delta^\alpha_\beta \partial_c A \ , && \text{(E.1.1)} \\
& f^a{}_{\beta\gamma} = f^a{}_{\beta c} = f^a{}_{b\gamma} = f^\alpha{}_{bc} = 0 \ , \\
& Q_a{}^{bc} = \partial_a \beta^{bc} - 2\beta^{d[b} f^{c]}{}_{ad} \ , \ Q_\alpha{}^{b\gamma} = -\delta^\gamma_\alpha \beta^{db}\partial_d A \ , \ Q_\alpha{}^{\beta c} = \delta^\beta_\alpha \beta^{dc}\partial_d A \ , && \text{(E.1.2)} \\
& Q_\alpha{}^{\beta\gamma} = Q_a{}^{\beta\gamma} = Q_a{}^{\beta c} = Q_a{}^{b\gamma} = Q_\alpha{}^{bc} = 0 \ , \\
& R^{abc} = 3\beta^{d[a}\nabla_d \beta^{bc]} \ , \ \text{any other component of } R^{ABC} = 0 \ . && \text{(E.1.3)}
\end{aligned}$$

From those we obtain the components of the two ten-dimensional spin connections (in flat indices)

$$\omega^B_{AC}\ \eta_{BD} = \frac{1}{2}(\eta_{BD} f^B{}_{AC} + \eta_{CE} f^E{}_{DA} + \eta_{AE} f^E{}_{DC}) \ , \quad \text{(E.1.4)}$$

$$\begin{aligned}
\omega^\beta_{\alpha\gamma}\ \eta_{\beta\delta} &= \frac{1}{2}(\eta_{\beta\delta} f^\beta{}_{\alpha\gamma} + \eta_{\gamma\epsilon} f^\epsilon{}_{\delta\alpha} + \eta_{\alpha\epsilon} f^\epsilon{}_{\delta\gamma}) \ , && \text{(E.1.5)} \\
\omega^b_{ac}\ \eta_{bd} &= \frac{1}{2}(\eta_{bd} f^b{}_{ac} + \eta_{ce} f^e{}_{da} + \eta_{ae} f^e{}_{dc}) \ , \\
\omega^b_{\alpha\gamma}\ \eta_{bd} &= -\eta_{\alpha\gamma}\partial_d A \ , \ \omega_\alpha{}^\beta{}_c\ \eta_{\beta\delta} = \eta_{\alpha\delta}\partial_c A \ , \\
\omega^b_{\alpha c} &= \omega^\beta_{a\gamma} = \omega^b_{a\gamma} = \omega^\beta_{ac} = 0 \ ,
\end{aligned}$$

$$\omega_{Q\,B}^{\ \ AD}\ \eta_{DC} = \frac{1}{2}(\eta_{CD} Q_B{}^{AD} + \eta_{BD} Q_C{}^{DA} + \eta_{BD}\eta_{CE}\eta^{AF} Q_F{}^{DE}) \ , \tag{E.1.6}$$

$$\omega_{Q\,b}^{\ \ ad}\ \eta_{dc} = \frac{1}{2}(\eta_{cd} Q_b{}^{ad} + \eta_{bd} Q_c{}^{da} + \eta_{bd}\eta_{ce}\eta^{af} Q_f{}^{de}) \ , \tag{E.1.7}$$

$$\omega_{Q\,b}^{\ \ \alpha\delta}\ \eta_{\delta\gamma} = -\delta^{\alpha}_{\gamma}\eta_{bd}\beta^{ed}\partial_e A \ , \ \omega^{\alpha}_{Q\beta}{}^{d}\ \eta_{dc} = \delta^{\alpha}_{\beta}\eta_{cd}\beta^{ed}\partial_e A \ ,$$
$$\omega_{Q\,\beta}^{\ \ \alpha\gamma} = \omega_{Q\,b}^{\ \ \alpha c} = \omega_{Q\,b}^{\ \ a\gamma} = \omega_{Q\,\beta}^{\ \ ac} = \omega_{Q\,\beta}^{\ \ a\gamma} = 0 \ .$$

As discussed in section 6.1.2, the ten-dimensional Γ-matrices satisfying the Clifford algebra $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ are decomposed as follows

$$\Gamma^A = \begin{cases} \Gamma^\alpha = \gamma^\alpha \otimes \mathbb{1}, & \alpha = 0, \dots, 3 \\ \Gamma^a = \gamma_{(4)} \otimes \gamma^a, & a = 4, \dots, 9 \end{cases} \ . \tag{E.1.8}$$

The six-dimensional γ^a and four-dimensional γ^α satisfy as well the Clifford algebra, and are constant. In addition, the γ^a are purely imaginary and Hermitian: $\gamma^{a\dagger} = \gamma^a$. The chirality operators are given by $\gamma_{(6)} = -\mathrm{i}\gamma^{4\dots9}$ and $\gamma_{(4)} = \mathrm{i}\gamma^{0\dots3}$; they square to the identity and anticommute with the other γ-matrices. $\gamma_{(6)}$ is also Hermitian.

Given this decomposition, we compute the following combinations

$$\begin{aligned} \Gamma^{\alpha\beta} =& \gamma^{\alpha\beta} \otimes \mathbb{1} \ , \ \Gamma^{ab} = \mathbb{1} \otimes \gamma^{ab} \ , \\ \Gamma^{a\beta} =& \frac{1}{2}\big((\gamma_{(4)} \otimes \gamma^a)(\gamma^\beta \otimes \mathbb{1}) - (\gamma^\beta \otimes \mathbb{1})(\gamma_{(4)} \otimes \gamma^a)\big) = -\gamma^\beta\gamma_{(4)} \otimes \gamma^a \ . \end{aligned} \tag{E.1.9}$$

Together with the above components of spin connections, this leads to the following components of the spinorial covariant derivatives

$$\nabla_{A=a} = \mathbb{1} \otimes \nabla_a \ , \ \check{\nabla}^{A=a} = \mathbb{1} \otimes \check{\nabla}^a \ , \tag{E.1.10}$$

$$\nabla_{A=\alpha} = \nabla_\alpha \otimes \mathbb{1} + \frac{1}{2}\omega_\alpha{}^b{}_\gamma \eta_{bd}\Gamma^{d\gamma} = \nabla_\alpha \otimes \mathbb{1} + \frac{1}{2}\eta_{\alpha\beta}\gamma^\beta\gamma_{(4)} \otimes \gamma^d\partial_d A \ , \tag{E.1.11}$$

$$\check{\nabla}^{A=\alpha} = \frac{1}{2}\omega^{\alpha}_{Qc}{}^{\beta}\eta_{\beta\delta}\Gamma^{\delta c} = \frac{1}{2}\gamma^\alpha\gamma_{(4)} \otimes \gamma^c\eta_{cd}\beta^{de}\partial_e A \ . \tag{E.1.12}$$

These are used in the SUSY variations in section 6.1.2.

E.2 Reformulation of the supersymmetry conditions with pure spinors

We introduce in section 6.1.3 the pure spinors $\Phi_\pm$ in terms of which we want to reformulate the supersymmetry (SUSY) conditions (6.1.16) - (6.1.18). To do so, we will use the Clifford map (6.1.22), and some of its properties that we first detail here, before starting the reformulation. For a k-form A_k, the Clifford map gives the following rules

$$\gamma^a \not{A}_k = \cancel{(\tilde{e}^a \wedge +\eta^{ab}\iota_b) A_k} \ , \quad \not{A}_k \gamma^a = (-1)^k \cancel{(\tilde{e}^a \wedge -\eta^{ab}\iota_b) A_k} \ , \tag{E.2.1}$$

that come from identities on γ-matrices, such as

$$\{\gamma^b, \gamma^{a_1\ldots a_k}\} = 2\gamma^{ba_1\ldots a_k} \text{ for } k \text{ even} , \quad [\gamma^b, \gamma^{a_1\ldots a_k}] = 2\gamma^{ba_1\ldots a_k} \text{ for } k \text{ odd} . \tag{E.2.2}$$

One subtlety in the Clifford map is due to the fact that the $\tilde{e}^a$ are real while the γ^a are purely imaginary. This makes a difference when considering a complex conjugation on forms of odd degree: one has

$$\eta^1_- \otimes \eta^{2\dagger}_- = \overline{\slashed{\Phi}_+} = \slashed{\overline{\Phi}_+} , \quad \eta^1_- \otimes \eta^{2\dagger}_+ = \overline{\slashed{\Phi}_-} = -\slashed{\overline{\Phi}_-} , \tag{E.2.3}$$

implying

$$\mathrm{Re}(\mu \overline{\slashed{\Phi}_-}) = \mathrm{i} \ \slashed{\mathrm{Im}(\overline{\mu}\Phi_-)} . \tag{E.2.4}$$

We now use these properties, as well as the hermitian conjugation of γ-matrices

$$\gamma^{a\dagger} = \gamma^a , \ \left(\gamma^{ab}\right)^\dagger = -\gamma^{ab} , \ \left(\gamma^{abc}\right)^\dagger = -\gamma^{abc} , \tag{E.2.5}$$

to compute the exterior derivative on the pure spinors (6.1.23). We also use the bispinor expressions, and the SUSY conditions (6.1.17) and (6.1.18) in type IIB. We obtain

$$\begin{aligned}
2\slashed{d\Phi_+} =& \{\gamma^a, \nabla_a \slashed{\Phi}_+\} \qquad\qquad (E.2.6)\\
=& \slashed{\nabla}\eta^1_+ \otimes \eta^{2\dagger}_+ + \gamma^a \eta^1_+ \otimes \left(\nabla_a \eta^2_+\right)^\dagger + \nabla_a \eta^1_+ \otimes \eta^{2\dagger}_+ \gamma^a + \eta^1_+ \otimes \left(\slashed{\nabla}\eta^2_+\right)^\dagger \\
=& \left(\left(\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} - \slashed{\partial}\left(2A - \tilde{\phi}\right) + \slashed{\beta}_\partial \left(2A + \tilde{\phi}\right) - \slashed{\mathcal{T}}\right)\eta^1_+ - 2e^{-A}\mu\eta^1_-\right) \otimes \eta^{2\dagger}_+ \\
&+ \gamma^a \eta^1_+ \otimes \left(\eta_{ab}\check{\nabla}^b \eta^2_+ + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf} R^{def} \gamma^{bc} \eta^2_+\right)^\dagger \\
&+ \left(-\eta_{ab}\check{\nabla}^b \eta^1_+ + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf} R^{def}\gamma^{bc}\eta^1_+\right) \otimes \eta^{2\dagger}_+ \gamma^a \\
&+ \eta^1_+ \otimes \left(\left(-\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} - \slashed{\partial}\left(2A - \tilde{\phi}\right) - \slashed{\beta}_\partial\left(2A + \tilde{\phi}\right) + \slashed{\mathcal{T}}\right)\eta^2_+ - 2e^{-A}\mu\eta^2_-\right)^\dagger \\
=& \check{\slashed{\nabla}}\eta^1_+ \otimes \eta^{2\dagger}_+ + \eta_{ab}\gamma^a \eta^1_+ \otimes \left(\check{\nabla}^b \eta^2_+\right)^\dagger - \eta_{ab}\check{\nabla}^b \eta^1_+ \otimes \eta^{2\dagger}_+ \gamma^a - \eta^1_+ \otimes \left(\check{\slashed{\nabla}}\eta^2_+\right)^\dagger \\
&- \frac{1}{4}\slashed{R}\eta^1_+ \otimes \eta^{2\dagger}_+ + \frac{1}{4}\eta^1_+ \otimes \eta^{2\dagger}_+ \slashed{R} - \frac{1}{8}\eta_{ad}\gamma^a \eta^1_+ \otimes \eta^{2\dagger}_+ \gamma^{bc}\eta_{be}\eta_{cf}R^{def} + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^1_+ \otimes \eta^{2\dagger}_+ \gamma^a \\
&- \{\slashed{\partial}\left(2A - \tilde{\phi}\right), \slashed{\Phi}_+\} + [\slashed{\beta}_\partial\left(2A + \tilde{\phi}\right) - \slashed{\mathcal{T}}, \slashed{\Phi}_+] - 4e^{-A}\,\mathrm{Re}(\mu\, \overline{\slashed{\Phi}_-}) .
\end{aligned}$$

We rewrite the R-flux terms via the above rules: denoting $\tilde{e}^a \wedge \pm\eta^{ab}\iota_b$ by $\tilde{e} \pm \iota$, we get

$$\begin{aligned}
&- \frac{1}{4}[\slashed{R}, \slashed{\Phi}_+] - \frac{1}{8}\eta_{ad}\gamma^a \tilde{\slashed{\Phi}}_+ \gamma^{bc}\eta_{be}\eta_{cf}R^{def} + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\slashed{\Phi}_+\gamma^a \qquad\qquad (E.2.7)\\
=& - \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\left(\frac{1}{3}\left((\tilde{e} + \iota)^3 - (\tilde{e} - \iota)^3\right) + (\tilde{e} + \iota)(\tilde{e} - \iota)^2 - (\tilde{e} + \iota)^2(\tilde{e} - \iota)\right)^{abc} \Phi_+ \\
=& - \frac{1}{3}R^{abc}\iota_a\iota_b\iota_c\Phi_+ ,
\end{aligned}$$

where the last two lines should be overall slashed. $\tilde{e}$ and ι with different indices anticommute, and the R-flux is antisymmetric, so maintaining the indices a, b, c fixed allows to commute $\tilde{e}$ and ι in the second line. With notations of the Introduction and appendix A.2, we finally obtain

$$\begin{aligned} 2\mathrm{d}\slashed{\Phi}_+ =&[\gamma^a, \check{\nabla}_a \slashed{\Phi}_+] - \{\slashed{\partial}\left(2A-\tilde{\phi}\right), \slashed{\Phi}_+\} + [\slashed{\beta}_\partial \left(2A+\tilde{\phi}\right) - \slashed{\mathcal{T}}, \slashed{\Phi}_+] - 2\, \cancel{R \vee \Phi_+} - 4e^{-A}\ \mathrm{i}\ \cancel{\mathrm{Im}(\overline{\mu}\Phi_-)} \\ =&2\ \left(\check{\nabla}^a \cdot \iota_a - \partial_a \left(2A-\tilde{\phi}\right)\tilde{e}^a \wedge + \left(\beta^{ab}\partial_b\left(2A+\tilde{\phi}\right) - \mathcal{T}^a\right)\iota_a - R\vee\right)\Phi_+ - 4e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_-)\ , \end{aligned}$$

where the last line should be overall slashed. We compute similarly

$$\begin{aligned} 2\mathrm{d}\slashed{\Phi}_- =&[\gamma^a, \nabla_a \slashed{\Phi}_-] \\ =&\slashed{\nabla}\eta^1_+ \otimes \eta^{2\dagger}_- + \gamma^a\eta^1_+ \otimes \left(\nabla_a \eta^2_-\right)^\dagger - \nabla_a\eta^1_+ \otimes \eta^{2\dagger}_-\gamma^a - \eta^1_+ \otimes \left(\slashed{\nabla}\eta^2_-\right)^\dagger \\ =&\left(\left(\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} - \slashed{\partial}\left(2A-\tilde{\phi}\right) + \slashed{\beta}_\partial\left(2A+\tilde{\phi}\right) - \slashed{\mathcal{T}}\right)\eta^1_+ - 2e^{-A}\mu\eta^1_-\right) \otimes \eta^{2\dagger}_- \\ &+\gamma^a\eta^1_+ \otimes \left(\eta_{ab}\check{\nabla}^b\eta^2_- + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^2_-\right)^\dagger \\ &-\left(-\eta_{ab}\check{\nabla}^b\eta^1_+ + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^1_+\right) \otimes \eta^{2\dagger}_-\gamma^a \\ &-\eta^1_+ \otimes \left(\left(-\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} - \slashed{\partial}\left(2A-\tilde{\phi}\right) - \slashed{\beta}_\partial\left(2A+\tilde{\phi}\right) + \slashed{\mathcal{T}}\right)\eta^2_- + 2e^{-A}\overline{\mu}\eta^2_+\right)^\dagger \\ =&\{\gamma^a, \check{\nabla}_a\slashed{\Phi}_-\} - \frac{1}{8}\left(2\{\slashed{R}, \slashed{\Phi}_-\} + \eta_{ad}\gamma^a\slashed{\Phi}_-\gamma^{bc}\eta_{be}\eta_{cf}R^{def} + \eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\slashed{\Phi}_-\gamma^a\right) \\ &-[\slashed{\partial}\left(2A-\tilde{\phi}\right), \slashed{\Phi}_-] + \{\slashed{\beta}_\partial\left(2A+\tilde{\phi}\right) - \slashed{\mathcal{T}}, \slashed{\Phi}_-\} - 4e^{-A}\mu\,\mathrm{Re}\left(\slashed{\Phi}_+\right)\ , \end{aligned} \tag{E.2.8}$$

and as above

$$\begin{aligned} &-\frac{1}{8}\left(2\{\slashed{R}, \slashed{\Phi}_-\} + \eta_{ad}\gamma^a\slashed{\Phi}_-\gamma^{bc}\eta_{be}\eta_{cf}R^{def} + \eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\slashed{\Phi}_-\gamma^a\right) \\ =&-\frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\left(\frac{1}{3}\left((\tilde{e}+\iota)^3 - (\tilde{e}-\iota)^3\right) + (\tilde{e}+\iota)(\tilde{e}-\iota)^2 - (\tilde{e}+\iota)^2(\tilde{e}-\iota)\right)^{abc}\Phi_- \\ =&-\frac{1}{3}R^{abc}\iota_a\iota_b\iota_c\Phi_-\ , \end{aligned} \tag{E.2.9}$$

where the last two lines should be slashed, as well as the following resulting one

$$2\mathrm{d}\Phi_- = 2\left(\check{\nabla}^a \cdot \iota_a - \partial_a\left(2A-\tilde{\phi}\right)\tilde{e}^a \wedge + \left(\beta^{ab}\partial_b\left(2A+\tilde{\phi}\right) - \mathcal{T}^a\right)\iota_a - R\vee\right)\Phi_- - 4e^{-A}\mu\,\mathrm{Re}\left(\Phi_+\right)\ .$$

Using the Clifford map backwards, we finally get for type IIB two equations on forms

$$e^{\tilde{\phi}}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde{\phi}}\Phi_+\right) + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_+ = -2e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_-) \tag{E.2.10}$$

$$e^{\tilde{\phi}}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde{\phi}}\Phi_-\right) + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_- = -2e^{-A}\mu\,\mathrm{Re}(\Phi_+)\ . \tag{E.2.11}$$

This calculation can be done as well in type IIA: the difference comes from the SUSY conditions (6.1.17) and (6.1.18) where one has to change the chirality of η^2. The above computation can be reproduced almost identically considering $\mathrm{d}\Phi_+$ in place of $\mathrm{d}\Phi_-$ and vice versa: this replaces $\eta^2_\pm$ into one another, and the type IIA SUSY conditions can then be used, leading simply to an exchange of Φ_+ and Φ_- in the computation. Doing so, commutators and anti-commutators get exchanged because of the even/odd degree change, but this goes through without issue; in particular we get eventually the same R-flux term, since (E.2.7) and (E.2.9) give the same resulting action on the pure spinors. The only difference in the process may appear in the μ-terms, because of (E.2.3), and in the signs induced by the (anti)-commutators. In the end, we obtain in type IIA

$$e^{\tilde{\phi}}\left(\mathrm{d}-\check{\nabla}^a\cdot\iota_a+\mathcal{T}\vee+R\vee\right)\left(e^{-\tilde{\phi}}\Phi_-\right)+e^{-2A}\left(\mathrm{d}+\check{\nabla}^a\cdot\iota_a\right)(e^{2A})\Phi_-=2e^{-A}\,\mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_+) \tag{E.2.12}$$

$$e^{\tilde{\phi}}\left(\mathrm{d}-\check{\nabla}^a\cdot\iota_a+\mathcal{T}\vee+R\vee\right)\left(e^{-\tilde{\phi}}\Phi_+\right)+e^{-2A}\left(\mathrm{d}+\check{\nabla}^a\cdot\iota_a\right)(e^{2A})\Phi_+=2e^{-A}\mu\ \mathrm{Re}(\Phi_-)\ . \tag{E.2.13}$$

The above computation should in principle be completed by RR contributions, that we do not have here; we would still like to obtain the result as if they were present. We thus follow closely the analogous computation done for standard SUGRA with RR fluxes in [26] (the result is even specified there not to hold without RR), and we perform an additional step, which in absence of RR may not look required. It involves the SUSY condition (6.1.16), that has not been used so far. From that condition in type IIB, we obtain

$$0=\eta^1_-\otimes\left(\mu\ \eta^2_-+e^A\left(\partial\!\!\!/ A-\beta\!\!\!/_\partial A\right)\eta^2_+\right)^{\dagger *}=\mu\overline{\Phi\!\!\!/_-}-e^A\ \overline{\Phi\!\!\!/_+}\left(\partial\!\!\!/ A-\beta\!\!\!/_\partial A\right) \tag{E.2.14}$$

$$0=\left(\mu\ \eta^1_-+e^A\left(\partial\!\!\!/ A+\beta\!\!\!/_\partial A\right)\eta^1_+\right)^*\otimes\eta^{2\dagger}_-=\overline{\mu}\Phi\!\!\!/_--e^A\left(\partial\!\!\!/ A+\beta\!\!\!/_\partial A\right)\overline{\Phi\!\!\!/_+}\ , \tag{E.2.15}$$

from which we deduce

$$0=2\,\mathrm{Re}(\overline{\mu}\Phi\!\!\!/_-)-e^A\{\partial\!\!\!/ A,\overline{\Phi\!\!\!/_+}\}-e^A[\beta\!\!\!/_\partial A,\overline{\Phi\!\!\!/_+}] \tag{E.2.16}$$

$$\longleftrightarrow\quad 0=e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_-)-e^{-A}\left(\mathrm{d}-\check{\nabla}^a\cdot\iota_a\right)(e^A)\overline{\Phi_+}\ . \tag{E.2.17}$$

We subtract this quantity on the RHS of (E.2.10) and get

$$\begin{aligned}e^{\tilde{\phi}}\left(\mathrm{d}-\check{\nabla}^a\cdot\iota_a+\mathcal{T}\vee+R\vee\right)\left(e^{-\tilde{\phi}}\Phi_+\right)&+e^{-2A}\left(\mathrm{d}+\check{\nabla}^a\cdot\iota_a\right)(e^{2A})\Phi_+\\&=-3e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_-)+e^{-A}\left(\mathrm{d}-\check{\nabla}^a\cdot\iota_a\right)(e^A)\overline{\Phi_+}\ .\end{aligned} \tag{E.2.18}$$

In type IIA, we proceed similarly with the SUSY condition (6.1.16)

$$0=\eta^1_-\otimes\left(\mu\ \eta^2_++e^A\left(\partial\!\!\!/ A-\beta\!\!\!/_\partial A\right)\eta^2_-\right)^{\dagger *}=\mu\overline{\Phi\!\!\!/_+}-e^A\ \overline{\Phi\!\!\!/_-}\left(\partial\!\!\!/ A-\beta\!\!\!/_\partial A\right) \tag{E.2.19}$$

$$0=\left(\mu\ \eta^1_-+e^A\left(\partial\!\!\!/ A+\beta\!\!\!/_\partial A\right)\eta^1_+\right)^*\otimes\eta^{2\dagger}_+=\overline{\mu}\Phi\!\!\!/_+-e^A\left(\partial\!\!\!/ A+\beta\!\!\!/_\partial A\right)\overline{\Phi\!\!\!/_-}\ , \tag{E.2.20}$$

to get

$$0 = 2\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\not{\Phi}_+) - e^{A}[\not{\partial}A, \overline{\not{\Phi}_-}] - e^{A}\{\not{\beta}_{\partial}A, \overline{\not{\Phi}_-}\} \qquad \text{(E.2.21)}$$

$$\longleftrightarrow \quad 0 = e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_+) + e^{-A}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a\right)(e^{A})\overline{\Phi_-}\ . \qquad \text{(E.2.22)}$$

We add this quantity to the RHS of (E.2.12) to obtain

$$\begin{aligned} e^{\tilde{\phi}}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a + \mathcal{T}\vee + R\vee\right)\left(e^{-\tilde{\phi}}\Phi_-\right) + e^{-2A}\left(\mathrm{d} + \check{\nabla}^a \cdot \iota_a\right)(e^{2A})\Phi_- \qquad \text{(E.2.23)} \\ = 3e^{-A}\ \mathrm{i}\,\mathrm{Im}(\overline{\mu}\Phi_+) + e^{-A}\left(\mathrm{d} - \check{\nabla}^a \cdot \iota_a\right)(e^{A})\overline{\Phi_-}\ . \end{aligned}$$

There is a priori no RR contribution to the other pure spinor condition, so we do not modify (E.2.11) or (E.2.13). Our final pure spinors conditions are then given by (E.2.11) and (E.2.18) in type IIB, and (E.2.13) and (E.2.23) in type IIA, as summarized in (6.1.24) and (6.1.25).

E.3 On the sufficiency of the pure spinors conditions

In section 6.1.3 and appendix E.2, we have derived the pure spinors conditions (6.1.24) and (6.1.25) using the SUSY conditions (6.1.16), (6.1.17) and (6.1.18); in other words, we have shown that (6.1.24) and (6.1.25) are necessary for the backgrounds of interest to preserve SUSY. We study here whether these two conditions are also sufficient. Following [26], this amounts to considering a generic expansion of $\nabla_a \eta^i_+$ and of further quantities appearing in (6.1.16), (6.1.17) and (6.1.18) on a complete basis of six-dimensional spinors. One then checks whether the coefficients in these expansions are determined by the pure spinors conditions to be those of the SUSY conditions. It will turn out not to be the case, implying that the conditions (6.1.24) and (6.1.25) are not sufficient. We argue that this is due to the absence of RR fluxes.

We start by expanding the following combinations on a complete basis of spinors $\{\eta^{1,2}$, $\gamma^a\eta^{1,2}$, $\gamma_{(6)}\eta^{1,2}\}$. Taking chiralities into account, we get in type IIB

$$\begin{aligned} \left(\gamma^a\left(\nabla_a \mp \eta_{ad}\check{\nabla}^d\right) + \frac{1}{24}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{abc}\right)\eta^{1,2}_+ &= \left(T^{1,2} + \mathrm{i}U^{1,2}\gamma_{(6)}\right)\eta^{1,2}_- + V^{1,2}_a\gamma^a\eta^{1,2}_+ \qquad \text{(E.3.1)} \\ \left(\nabla_a \pm \eta_{ad}\check{\nabla}^d - \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\right)\eta^{1,2}_+ &= \left(P^{1,2}_a + \mathrm{i}Q^{1,2}_a\gamma_{(6)}\right)\eta^{1,2}_+ + \mathrm{i}S^{1,2}_{ad}\gamma^d\eta^{1,2}_-\ , \end{aligned}$$

where the coefficients $V^{1,2}_a$, $P^{1,2}_a$ and $Q^{1,2}_a$ must be real. A more generic situation would be to consider only ∇ on the internal spinors. However (6.1.24) and (6.1.25) impose without ambiguity these particular combinations of ∇, $\check{\nabla}$ and R-flux to act on the spinors, so there is actually no restriction here. From these generic expansions, we compute the exterior derivative of the pure spinors as in (E.2.6) and (E.2.8)

$$
\begin{aligned}
2\mathrm{d}\slashed{\Phi}_+ =& \{\gamma^a, \nabla_a \slashed{\Phi}_+\} \\
=& \slashed{\nabla}\eta^1_+ \otimes \eta^{2\dagger}_+ + \gamma^a \eta^1_+ \otimes \left(\nabla_a \eta^2_+\right)^\dagger + \nabla_a \eta^1_+ \otimes \eta^{2\dagger}_+ \gamma^a + \eta^1_+ \otimes \left(\slashed{\nabla}\eta^2_+\right)^\dagger \\
=& \left(\left(\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} + \slashed{V}^1\right)\eta^1_+ + \left(T^1 + \mathrm{i}U^1\gamma_{(6)}\right)\eta^1_-\right) \otimes \eta^{2\dagger}_+ \\
&+\gamma^a \eta^1_+ \otimes \left(\eta_{ab}\check{\nabla}^b \eta^2_+ + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^2_+ + \left(P^2_a + \mathrm{i}Q^2_a\gamma_{(6)}\right)\eta^2_+ + \mathrm{i}S^2_{ad}\gamma^d\eta^2_-\right)^\dagger \\
&+ \left(-\eta_{ab}\check{\nabla}^b \eta^1_+ + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^1_+ + \left(P^1_a + \mathrm{i}Q^1_a\gamma_{(6)}\right)\eta^1_+ + \mathrm{i}S^1_{ad}\gamma^d\eta^1_-\right) \otimes \eta^{2\dagger}_+\gamma^a \\
&+\eta^1_+ \otimes \left(\left(-\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} + \slashed{V}^2\right)\eta^2_+ + \left(T^2 + \mathrm{i}U^2\gamma_{(6)}\right)\eta^2_-\right)^\dagger \\
=& [\gamma^a, \check{\nabla}_a \slashed{\Phi}_+] - 2\,\cancel{R \vee \Phi_+} \\
&+\gamma^a \slashed{\Phi}_+\left(P^2_a - \mathrm{i}Q^2_a\right) - \mathrm{i}\overline{S^2_{ad}}\gamma^a\slashed{\Phi}_-\gamma^d + \left(P^1_a + \mathrm{i}Q^1_a\right)\slashed{\Phi}_+\gamma^a + \mathrm{i}S^1_{ad}\gamma^d\overline{\slashed{\Phi}_-}\gamma^a \\
&+\slashed{V}^1\slashed{\Phi}_+ + \slashed{\Phi}_+\slashed{V}^2 + \left(T^1 - \mathrm{i}U^1\right)\overline{\slashed{\Phi}_-} + \slashed{\Phi}_-\left(\overline{T^2} + \mathrm{i}\overline{U^2}\right)\ ,
\end{aligned}
\tag{E.3.2}
$$

$$
\begin{aligned}
2\mathrm{d}\slashed{\Phi}_- =& [\gamma^a, \nabla_a \slashed{\Phi}_-] \\
=& \slashed{\nabla}\eta^1_+ \otimes \eta^{2\dagger}_- + \gamma^a \eta^1_+ \otimes \left(\nabla_a \eta^2_-\right)^\dagger - \nabla_a \eta^1_+ \otimes \eta^{2\dagger}_- \gamma^a - \eta^1_+ \otimes \left(\slashed{\nabla}\eta^2_-\right)^\dagger \\
=& \left(\left(\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} + \slashed{V}^1\right)\eta^1_+ + \left(T^1 + \mathrm{i}U^1\gamma_{(6)}\right)\eta^1_-\right) \otimes \eta^{2\dagger}_- \\
&+\gamma^a \eta^1_+ \otimes \left(\eta_{ab}\check{\nabla}^b \eta^2_- + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^2_- + \left(P^2_a + \mathrm{i}Q^2_a\gamma_{(6)}\right)\eta^2_- + \mathrm{i}\overline{S^2_{ad}}\gamma^d\eta^2_+\right)^\dagger \\
&- \left(-\eta_{ab}\check{\nabla}^b \eta^1_+ + \frac{1}{8}\eta_{ad}\eta_{be}\eta_{cf}R^{def}\gamma^{bc}\eta^1_+ + \left(P^1_a + \mathrm{i}Q^1_a\gamma_{(6)}\right)\eta^1_+ + \mathrm{i}S^1_{ad}\gamma^d\eta^1_-\right) \otimes \eta^{2\dagger}_-\gamma^a \\
&-\eta^1_+ \otimes \left(\left(-\check{\slashed{\nabla}} - \frac{1}{4}\slashed{R} + \slashed{V}^2\right)\eta^2_- - \left(\overline{T^2} + \mathrm{i}\overline{U^2}\gamma_{(6)}\right)\eta^2_+\right)^\dagger
\end{aligned}
\tag{E.3.3}
$$

$$
\begin{aligned}
2\mathrm{d}\slashed{\Phi}_- =& \{\gamma^a, \check{\nabla}_a \slashed{\Phi}_-\} - 2\,\cancel{R \vee \Phi_-} \\
&+\gamma^a \slashed{\Phi}_-\left(P^2_a + \mathrm{i}Q^2_a\right) - \mathrm{i}S^2_{ad}\gamma^a\slashed{\Phi}_+\gamma^d - \left(P^1_a + \mathrm{i}Q^1_a\right)\slashed{\Phi}_-\gamma^a - \mathrm{i}S^1_{ad}\gamma^d\overline{\slashed{\Phi}_+}\gamma^a \\
&+\slashed{V}^1\slashed{\Phi}_- - \slashed{\Phi}_-\slashed{V}^2 + \left(T^1 - \mathrm{i}U^1\right)\overline{\slashed{\Phi}_+} + \slashed{\Phi}_+\left(T^2 - \mathrm{i}U^2\right)\ .
\end{aligned}
$$

We then use the Clifford map on these equations. We first compare the result from (E.3.3) to (6.1.24) and deduce

$$
\begin{aligned}
& S^1_{ad} = S^2_{ad} = 0,\ Q^1_a = Q^2_a = 0 \\
& P^2_a + V^1_a = -\partial_a \left(2A - \tilde{\phi}\right) + \eta_{ab} \left(\beta^{bd}\partial_d \left(2A + \tilde{\phi}\right) - \mathcal{T}^b\right) \\
& P^1_a + V^2_a = -\partial_a \left(2A - \tilde{\phi}\right) - \eta_{ab} \left(\beta^{bd}\partial_d \left(2A + \tilde{\phi}\right) - \mathcal{T}^b\right) \\
& T^1 - \mathrm{i}U^1 = T^2 - \mathrm{i}U^2 = -2e^{-A}\mu \ .
\end{aligned}
\tag{E.3.4}
$$

Fixing this way the coefficients reproduces (6.1.17) and (6.1.18), provided one sets $P^1_a = P^2_a = 0$; we will come back to that point. We turn to (E.3.2): comparing it to (6.1.25), taking into account the identifications (E.3.4), one obtains precisely (E.2.17) as a constraint. The latter should allow to reproduce the remaining SUSY condition (6.1.16). To verify this, we introduce a generic expansion of the following quantity

$$
\partial_a A \gamma^a \eta^{1,2}_+ = \gamma^a (R^{1,2}_a + \mathrm{i}W^{1,2}_a \gamma_{(6)})\eta^{1,2}_+ - X^{1,2}\eta^{1,2}_- \ , \tag{E.3.5}
$$

where $R^{1,2}_a$ and $W^{1,2}_a$ are real. Then, we consider the sum

$$
\begin{aligned}
0 =& \eta^1_- \otimes (X^2\eta^2_- + \partial_a A\gamma^a\eta^2_+ - \gamma^a(R^2_a + \mathrm{i}W^2_a\gamma_{(6)})\eta^2_+)^{\dagger *} \\
& + (X^1\eta^1_- + \partial_a A\gamma^a\eta^1_+ - \gamma^a(R^1_a + \mathrm{i}W^1_a\gamma_{(6)})\eta^1_+)^* \otimes \eta^{2\dagger}_- \\
=& -\{\partial\!\!\!/ A, \overline{\Phi\!\!\!/_+}\} + (R^2_a + \mathrm{i}W^2_a)\overline{\Phi\!\!\!/_+}\gamma^a + (R^1_a - \mathrm{i}W^1_a)\gamma^a\overline{\Phi\!\!\!/_+} + \overline{\Phi\!\!\!/_-}X^2 + \Phi\!\!\!/_-\overline{X^1} \ .
\end{aligned}
\tag{E.3.6}
$$

Using the Clifford map on the last equation, and comparing the result to the obtained constraint (E.2.17), we get

$$
W^1_a = W^2_a = 0 \ , \ R^1_a = -R^2_a = -\beta^{ab}\partial_b A \ , \ X^1 = X^2 = \mu e^{-A} \ . \tag{E.3.7}
$$

This reproduces precisely the SUSY condition (6.1.16).

To conclude, the SUSY conditions (6.1.16), (6.1.17) and (6.1.18) are reproduced starting from the pure spinors conditions (6.1.24) and (6.1.25) in type IIB, provided one fixes $P^1_a = P^2_a = 0$. The ambiguity or freedom in the $P^{1,2}_a$ is in our opinion related to the absence of RR fluxes: those would otherwise bring more constraints. The $P^{1,2}_a$ could also be related to the norms of the internal spinors, so far not needed. These norms are fixed in [26] thanks to the RR contributions; this may explain the ambiguity we get here. We conclude that the pure spinors conditions (6.1.24) and (6.1.25) are not sufficient, but the remaining ambiguity should be fixed by considering the RR sector. We expect the same situation in type IIA.

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