Formal Methods in the Theories of Rings and Domains

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Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
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Abstract

In recent years, Hilbert's Programme has been resumed within the framework of constructive mathematics. This undertaking has already shown its feasability for a considerable part of commutative algebra. In particular, point-free methods have been playing a primary role, emerging as the appropriate language for expressing the interplay between real and ideal in mathematics.

This dissertation is written within this tradition and has Sambin's notion of formal topology at its core. We start by developing general tools, in order to make this notion more immediate for algebraic application. We revise the Zariski spectrum as an inductively generated basic topology, and we analyse the constructive status of the corresponding principles of spatiality and reducibility. Through a series of examples, we show how the principle of spatiality is recurrent in the mathematical practice.

The tools developed before are applied to specific problems in constructive algebra. In particular, we find an elementary characterization of the notion of codimension for ideals of a commutative ring, by means of which a constructive version of Krull's principal ideal theorem can be stated and proved. We prove a formal version of the projective Eisenbud-Evans-Storch theorem. Finally, guided by the algebraic intuition, we present an application in constructive domain theory, by proving a finite version of Kleene-Kreisel density theorem for non-flat information systems.

Zusammenfassung

In den vergangenen Jahren wurde das Hilbertsche Programm im Rahmen der konstruktiven Mathematik wiederaufgenommen. Diese Unternehmung hat sich vor allem in der kommutativen Algebra als praktikabel erwiesen. Insbesondere spielen punktfreie Methoden eine wesentliche Rolle: sie haben sich als die angemessene Sprache herausgestellt, um das Zwischenspiel von "real" und "ideal" in der Mathematik auszudrücken.

Die vorliegende Dissertation steht in dieser Tradition; zentral ist Sambins Begriff der formalen Topologie. Zunächst entwickeln wir ein allgemeines Instrumentarium, das geeignet ist, diesen Begriff seinen algebraischen Anwendungen näherzubringen. Sodann arbeiten wir das Zariski-Spektrum in eine induktiv erzeugte "basic topology" um und analysieren den konstruktiven Status der einschlägigen Varianten von Spatialität und Reduzibilität. Durch Angabe einer Reihe von Instanzen zeigen wir, wie häufig das Prinzip der Spatialität in der mathematischen Praxis vorkommt.

Die eigens entwickelten Werkzeuge werden schließlich auf spezifische Probleme aus der konstruktiven Algebra angewandt. Insbesondere geben wir eine elementare Charakterisierung der Kodimension eines Ideals in einem kommutativen Ring an, mit der eine konstruktive Fassung des Krullschen Hauptidealsatzes formuliert und bewiesen werden kann. Ferner beweisen wir eine formale Fassung des Satzes von Eisenbud-Evans-Storch im projektiven Fall. Geleitet von der algebraischen Intuition stellen wir zuletzt eine Anwendung in der konstruktiven Bereichstheorie vor, indem wir eine finite Variante des Dichtheitssatzes von Kleene und Kreisel für nicht-flache Informationssysteme beweisen.

Publications Included in the Thesis

This thesis contains material which has already been published or is submitted for publication. The chapter subdivision partially reflects the following list of publications:

1. Giovanni Sambin, Davide Rinaldi and Peter Schuster. *The Basic Zariski Topology*. Submitted.

2. Davide Rinaldi. *A constructive notion of codimension*. Journal of Algebra, 383:178-196, 2013.

3. Davide Rinaldi. *A formal proof of the projective Eisenbud-Evans-Storch theorem*, Archiv der Mathematik, 99:9–24, 2012.

4. Davide Rinaldi and Peter Schuster. *A Universal Krull-Lindenbaum Theorem*. Submitted.

Introduction

In the early 1920s, Hilbert collected his philosophical views in a foundational proposal, which became known as *Hilbert's Programme*. The realisation of such ambitious programme should have given the ultimate response to the foundational crisis of mathematics. In Hilbert's view, there is a privileged part of mathematics, which we could call finitistic or "real". He essentially identifies this part with elementary number theory, which stands before logic inasmuch as it leans on the intuition of "concrete signs"; on the other hand, there is a conceptual and "ideal" side of mathematics, which has emerged with set theory, formal induction principles and the principle of excluded middle. The only method to provide a secure ground for classical mathematics is to reduce also the latter to finitary reasoning. More precisely, such a method should consist in giving every mathematical theory a finite and complete axiomatization, together with a proof of its consistency.

A few years later, Gödel's incompleteness theorems showed bluntly the unfeasibility of such a project for the whole of mathematics. Although almost all of Hilbert's main goals were dashed by Gödel's work, the methods and the ideas originated in that pursuit had a great impact on the development of mathematical logic of the 20th century and evolved naturally into modern proof theory. Beyond the field of logic, the failure of Hilbert's programme unfortunately favoured a somewhat nihilistic approach in classical mathematics, still dominant nowadays: if finitary mathematics cannot fulfill Hilbert's foundational promise, then there is no reason to regard it differently from ideal mathematics[1]. In plain terms, a real number is then as real (or ideal) as a natural number is. And indeed this freedom let classical mathematics successfully achieve an unexpected degree of complexity and solve time-honoured problems, giving further motivation to such a standpoint.

Several tentatives to adjust and relativise the original Hilbert programme were undertaken in the following years. In particular, as Bernays noticed, Hilbert's programme could be revived by allowing more general constructive arguments instead of merely finitist methods in the consistency proofs. This remark originated the so called *relativised Hilbert's programme*, proposed by Kreisel and brought forward by Feferman. In this formulation, real mathemat-

[1] This was exactly the approach that Hilbert wanted to justify, so, if not theoretically, his point of view was practically realised.

ics is identified with the constructive ones and opposed to the abstract classical ones; the goal is to provide methods and isolate conditions for reducing locally the latter to the former.

Constructivism initiated at the beginning of the last century, as Brouwer's critical reaction to Hilbert's formalist view of mathematics. The overwhelming success of ideal methods, upon which the new branches of measure theory and topology were exclusively grounded, casted shadow on Brouwer's intuitionistic programme from its very birth. In spite of this, it slowly began to attain more systematic forms, both with Heyting's studies on intuitionistic logic and with russian recursive mathematics. More recently, in his 1967 book *Foundations of constructive analysis*, Bishop showed that a large part of classical analysis could be carried out by constructive means, bringing back dignity and interest to Brouwer's ideas. A few years later, Martin-Löf put forward his intuitionistic system of types, which shed light on the connections between constructive mathematics and the growing field of theoretical computer science.

The diverse approaches to constructive mathematics differs with respect to degree of formalism and foundational restriction, sometimes to the extent of being incompatible. For example, a certain tradition, which we will not follow in this dissertation, regards as constructive the mathematics that is formalizable in a topos, since the natural associated logic is intuitionistic and the axiom of choice does not generally hold. Yet in this setting, the power-set axiom is valid, which is not the case for Martin-Löf type theory or Bishop's style, where instead the existence of a choice function is justified by the stronger meaning of the quantifiers.

Probably the most intriguing fact about constructive mathematics, and which attracted the attention in recent years, is that it can be naturally given a computational content, reflecting the Brouwer-Heyting-Kolmogorov interpretation, or, in more precise terms, a form of *realizability*. As a consequence, from a constructive proof of a statement A we can canonically extract a program, correct with respect to the specification of A. In classical mathematics, the use of excluded middle or of certain transfinite methods turns out in destroying such a clean correspondence. It is worth to mention that the growing effort for giving a formalization of existing mathematics on a computer machine, and the implementation of many proof assistants (Coq, Minlog, etc..), have found in constructive mathematics the proper environment.

Bishop's commitment to constructive analysis encouraged Richman and others to set a similar plan for abstract algebra [MRR88]. The need for a constructive basis was here arguably stronger since algebra was the field that originally dealt with computations or at least cared more about this aspect. And in fact elementary algebra is essentially constructive, while non-constructive assumptions undermined already the very basis of analysis. Because of this, Hilbert's proof of the basis theorem, published in 1890, raised long controversies before being accepted as mainstream mathematics. The development of abstract algebra in the last century, fostered by its connection with geometry and topology, is extraordinary. The drawback of this success is the strong dependence of the new elegant tools on transfinite principles, such as the use of Zorn's Lemma.

Although nowadays the use of transfinite methods in algebra is widely and acritically acknowledged, there still is a certain healthy reluctance about their application.

The recent course of constructive algebra, in particular through the work of Coquand, Lombardi and others [CL06, Coq05, Coq09, CLS09, Lom06, LQ12], focuses especially on commutative algebra in the spirit of a "partial realisation of Hilbert's programme". Far from denying the importance of topological techniques in this field, which is in fact guided by its interaction with geometry, the goal is that of understanding and analysing effectively the reasons of their success. The work undertaken by now has shown that many concrete statements, the proofs of which rely on the use of ideal objects in the classical tradition, can be given a purely elementary proof. The elimination of the ideal objects is usually performed by substitution by finite approximations, or by introducing a lower-level logical description of the topological spaces involved (e.g. the Zariski spectrum). By proceding this way, we restore "the feeling that commutative algebra can be seen computationally as a machine that produces algebraic identities" [CL02].

As already hinted at above, topological spaces behave badly from a constructive perspective, and even more in a non-algebraic context. Moreover, the notion of topological space is completely conceptual; for instance, it is not of the same kind as the one of a commutative ring, which is just the abstraction of objects for which we have concrete instances at hand. The uneasiness with the axiom of choice first gave rise to point-free topology, with the notion of *locale*, which is in fact an algebraic description of the lattice of open subsets. The theory of locales [Joh82] was not yet enough for a proper formalization within Martin-Löf Type Theory [ML84], despite, in its elegance, it already suggested the right path. As a solution to this issue, in the mid 1980s Sambin's notion of *formal topology* appeared [Sam87]: while points can be hard to grasp constructively (e.g. real numbers as infinite sequences, prime ideals over a general commutative ring), a base for the corresponding topological space is often available constructively (e.g. rational intervals, elements of the ring itself). It is worth mentioning the forerunners of this idea, of which formal topology can be seen as common generalization: Scott's notions of entailment relation [Sco74] and information system [Sco82], and Fourman and Grayson's notion of formal space [FG82].

This dissertation fits in this tradition and has the notion of formal topology at its core. The formal topologies presented in this text arise in an algebraic context, and for this reason share some special finiteness (or compactness) property. This, in particular, allows for an equivalent description by means of entailment relations or distributive lattices, as is more common in the aforementioned new course of constructive algebra. Nevertheless we decided to prefer the approach of formal topologies, and the reason is two-fold: on the one hand the classical topological intuition is closer and can be recovered at will through the notion of formal point; such an intuition could still be helpful for the common mathematician and should not, in the author's opinion, be squeezed into a fully logical one. On the other hand formal topologies can be seen as a careful rephrasing of natural inductive processes on the underlying structures (e.g. the generation of ideals in a ring). This gives also further significance and explanation to topology itself, reducing the conceptual gap mentioned above.

In this work we put flesh on the idea of formal topology as a language for expressing the interaction between real and ideal in mathematics. On each *basic topology* (i.e. a weak formal topology) one can canonically consider ideal points and describe their interaction with the basis through the non-constructive principle of *spatiality*. As paradigmatic example, one can codify the Zariski spectrum on a commutative ring A as an inductively defined formal topology: in this setting, formal points correspond to prime ideals, while spatiality is an instance of Krull's Lemma

$$a \ \epsilon \ \bigcap_{U \subseteq \mathfrak{p}} \mathfrak{p} \to \exists n(a^n \ \epsilon \ (U)).$$

Let us have a closer look on this: on the left-hand, we have an ideal statement, raising in addition impredicativity issues; on the right-hand, we instead have a concrete statement, postulating an equational witness for $a^n \ \epsilon \ (U)$. In the literature, the existence of such a witness is obtained indirectly by Zorn's Lemma, without giving any clue of its construction. Many concrete statements are proved by this or analogous lemmas (e.g. local-global principles). In view of the revised Hilbert's programme in commutative algebra, we want to abandon its use, so to find a constructive and elementary proof.

This kind of "completeness" results find its natural generalization in the language of formal topologies. This language needs still a development and a refinement of its basic concepts and definitions; we made a certain effort in this direction, in order to make it more suited for constructive algebra. A recently proposed formulation of formal topology [CMS13], which generalizes the existing ones, turned out very appropriate for our needs and gave rise to the structure of *finitary basic topology with finitary operation*. This structure is very natural for commutative algebra, since it instantiates constructive versions of the Zariski and projective spectra and the space of valuations. Conversely, it shares an intuition with commutative rings: we can, for instance, define quotients, localisations, boundary ideals and Krull dimension. Incidentally, such intuition proved its applicability in a completely different area, that is, constructive domain theory. Although the links between formal topology and Scott's information systems had already been studied, reapproaching this subject, with the intuition from commutative algebra in mind, allowed us to generalise existing results in this field.

Structure and Content of the Thesis

This thesis is divided into six chapters, each preceded by a short introduction. We briefly summarize their content, while for appropriate references we refer to the corresponding chapter.

Chapter 1. We give a short overview of formal topology within the framework of the so-called basic picture, including a description of the inductive/coinductive generation of basic topologies. We further reformulate this process in terms of generating relations, which is tailored made for the algebraic examples later under study. Following the most recent formulation, we assume the convergence operation associated to a basic topology as a primary object.

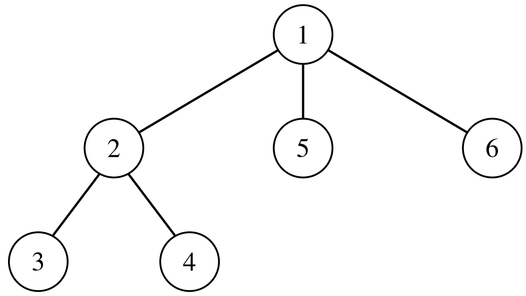
Finitary basic topologies with finitary operation deserve a closer study. In particular, we isolate an axiom, *Weak-Right*, under which these structures satisfy a completeness theorem, or in different terms, they are spatial. In this setting, it is possible to define in a natural way both quotients and localisations. We conclude this chapter suggesting a constructive definition of Krull dimension for finitary formal topologies with finitary operation, based upon the elementary characterization of the Krull dimension for distributive lattices given in earlier works by Coquand, Lombardi, and Roy.

Chapter 2. We present the first application of this topological machinery: we describe the Zariski spectrum as an inductively generated basic topology. Inasmuch we can thus get by without considering powers and radicals, this simplifies the presentation as a formal topology initiated by Sigstam. The topological operations of quotient and localisation reflect their ring-theoretic version. The notions of spatiality and reducibility are characterized for the class of Zariski formal topologies, and their nonconstructive content is pointed out: while spatiality implies classical logic, reducibility correspond to a fragment of the Axiom of Choice in the form of Russell's Multiplicative Axiom. This chapter corresponds to the submitted paper [RSS13].

Chapter 3. We use the constructive description of the Zariski spectrum presented in Chapter 2 and we are able to give a point-free and constructively meaningful characterization of the notion of codimension for ideals of a commutative Noetherian ring, which with classical logic and the axiom of choice is equivalent to the customary one. This characterization is obtained by means of an inductively generated modification of the formal Zariski topology, and is based upon the characterization of the Krull dimension for finitary formal topologies. As an application we prove a constructive version of Krull's Principal Ideal Theorem. This chapter corresponds to the published paper [Rin13].

Chapter 4. By a convenient modification of the Zariski formal topology, the projective spectrum associated to a graded ring can also be described as a finitary formal topology. We develop this idea, and extend a constructive proof of the Eisenbud-Evans-Storch theorem, developed in a previous work by Coquand, Schuster and Lombardi, from the affine to the projective case. This chapter corresponds to the published paper [Rin12].

Chapter 5. We formulate a natural common generalisation of Krull's theorem on prime ideals and of Lindenbaum's lemma on complete consistent theories. This theorem clearly has instantiations in algebra, such as the Artin-Schreier theorem, but also in logic, where it comprises Henkin's approach to the Gödel completeness theorem. Inspired by the notion of spatiality, we put the Krull-Lindenbaum theorem in universal rather than existential form, which move allows us to give a relatively direct proof with Raoult's Open Induction in place



Dependence among chapters.

of Zorn's Lemma. This chapter corresponds to the submitted paper [RS14].

Chapter 6. In this last part, we analyse an application of the notions introduced in the first chapter to constructive domain theory. We put forward a structure, based on Sambin's basic picture, which generalizes Scott's notion of information system and for which the usual categorical constructions can easily be performed. Finally, we prove a finite version of Kleene-Kreisel-Berger's density theorem for these structures and a compact version of it under suitable hypothesis. To this end, several algebraic ideas and intuition from the previous chapters will be of use, in particular when dealing with generators of boundary ideals. A study of the constructive content of this statement is finally undertaken.

Chapter 1

Preliminaries and Results in Formal Topology

1.1 Foundation and Terminology

As already stressed in the introduction, formal topology provides an elegant and canonical language for analysing the interplay between real and ideal in mathematics. In this thesis we want to provide evidences supporting this and, to this end, we need a notation flexible enough to describe together the real side, that one approaches from a fully constructive standpoint, and the ideal side, where at times impredicative definitions and classical reasoning are admitted.

We follow essentially the style of Bishop [BB85], as also usual in the tradition of constructive algebra [MRR88, LQ12]. We make nevertheless a few distinctions which belong to the established tradition and style of formal topology. More precisely, the constructive mathematics here presented is meant to be compatible within Sambin & Maietti's minimalist foundation [MS05] and in particular formalizable into Martin-Löf intuitionistic type theory [ML84]. As a consequence, every proof can be regarded as a program or an effective procedure.

We prefer to keep informal and not to give here a systematic foundational account, but rather list the basic concepts and notational peculiarities. The notion of *construction* is fundamental and left undefined. Once agreed upon that, one defines a *set* X by giving a finite number of rules to produce its elements. To our needs, we must assume at least the set of natural numbers $\mathbb{N}$ to be definable in such setting. Each set X is provided with an *equality*, that is, an equivalence relation $=$ between its elements. If such relation is decidable, we will say that the set X is *discrete*. Set operations can be defined as usual.

There are plenty of objects which cannot be constructed by such finite means, as for instance $\mathcal{P}(\mathbb{N})$. One will then say that they form a *collection*. In particular, we will refer to any unjustified use of the Power-Set Axiom (PSA) as *impredicative*.

We do not fix a formal language, which can always be done thereafter, and we leave the notion of *proposition* as an open concept. This may also depend on some variables ranging on a given set. Then one can build new proposition by means of the logical connectives &, $\vee$ and $\rightarrow$, and of the quantifiers $\forall$ or $\exists$ over a fixed set. The meaning of connectives and quantifiers is provided by Brouwer's interpretation. In particular, we will adopt intuitionistic logic.

It is sometimes convenient to use the following notation

$$\frac{\psi}{\varphi} \quad \frac{\psi \quad \eta}{\varphi} \quad \frac{\psi}{\overline{\overline{\varphi}}}$$

to denote, respectively, $\psi \rightarrow \varphi$, $\psi \ \& \ \eta \rightarrow \varphi$ and $\psi \leftrightarrow \varphi$.

A *subset* $U \subseteq X$ is intended to be a proposition $U(a)$ depending on one argument a in X. In plain terms, $a \epsilon U$ means $U(a)$. Subset operations can be defined as usual. We say that a subset $U \subseteq X$ is *complemented* if $x \epsilon U \vee x \not\epsilon U$.

We will keep then three different membership symbols, $\in$, ϵ, and $\in_c$, to indicate, respectively, membership to a set, membership to a subset and membership to a collection [Samng]. The symbol $\equiv$ denotes definitional equality. These distinctions can be ignored at first reading.

The *overlap* symbol $≬$, introduced by Sambin, is used to denote that two subsets have positive intersection. More precisely, given $U, V \subseteq X$, one defines

$$U ≬ V \equiv \exists x(x \epsilon U \ \& \ x \epsilon V).$$

If classical logic holds, then $U ≬ V$ is equivalent to $U \cap V \neq \emptyset$.

A *finite set* is a possibly empty list $\{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$. Such a set is also called *finitely enumerable* in the literature. Given a list $\{x_1, \ldots, x_n\}$ $(n \in \mathbb{N})$ finite list of elements of a set X, we can define a subset

$$x \epsilon U_{\{x_1,\ldots,x_n\}} \equiv x = x_1 \vee \cdots \vee x = x_n.$$

A subset $U_{\{x_1,\ldots,x_n\}}$ defined in this way is called *finite* and we say that $n \equiv |U_{\{x_1,\ldots,x_n\}}|$ is its *formal cardinality*. The finite subsets of a given set X do not form explicitly a set, but we can give a constructive meaning to a quantification over them, formally identifying the finite subsets with the finite lists over X (see [CS08]).

A *binary relation* between two sets X and Y, also represented by $X \stackrel{r}{\rightarrow} Y$, is a subset $r \subseteq X \times Y$. We use the infix notation xry to mean $(x, y) \epsilon r$. Such a relation is said to be *total* if $(\forall x \in X)(\exists y \in Y)(xry)$, and *single-valued* if

$$(\forall x, x' \in X)(\forall y, y' \in Y)(xry \ \& \ x'ry' \ \& \ x = x' \rightarrow y = y').$$

Given relations $X \stackrel{r}{\rightarrow} Y$ and $Y \stackrel{r}{\rightarrow} Z$ we define their composition $X \stackrel{s \circ r}{\rightarrow} Z$ as

$$x(s \circ r)z \equiv (\exists y \in Y)(xry \ \& \ ysz)$$

for al $x \in X$ and $z \in Z$. A total and single-valued relation is called *functional*. A *function* $f : X \rightarrow Y$ is instead a procedure which takes as input an element $x \in X$ and a proof that $x \in X$ and gives as output an element $y \in Y$ and a proof that $y \in Y$. A function f is called *extensional* if the procedure respects the equality on the underlying sets.

When reasoning instead about ideal objects, we will have to abandon temporarily the foregoing constructive foundation (see also [Mai09, Sam12]). At times we will identify sets and collections (e.g. by applying PSA), employ the law of excluded middle (CL), or use the *axiom of choice* (AC) to extract a *choice function* from a total relation. These assumptions will cause a loss in the effective content and their use will be always made explicit.

1.2 Basic Pairs and Concrete Spaces

The Basic Picture, introduced by Gebellato & Sambin in [SG99], provides a minimal constructive framework to deal with basic topological concepts such as open and closed subsets, and continuous functions. Unlike most of the other constructive approaches, the basic picture tries also to provide an independent justification of topology, not based on the classical established intuition, but arising from symmetry and logical duality.

Instead of topological spaces, we start here by revising binary relations:

Definition 1.1. A *basic pair* is a structure $(X, \Vdash, S)$, where X and S are sets and $\Vdash: X \to S$ is a relation between them.

Let $(X, \Vdash, S)$ be a basic pair. We can think of S as a set of indices for an open basis $\{B_a\}_{a \in S}$ of a topological space X, and $\Vdash$ defined by $x \Vdash a \equiv x \;\epsilon\; B_a$. In these terms, one could express closure and interior as follows:

$$x \;\epsilon\; \mathsf{cl}_{\Vdash}(D) \equiv \forall a \in S(x \Vdash a \to \exists y \in X(y \Vdash a \;\&\; y \;\epsilon\; D)),$$

$$x \;\epsilon\; \mathsf{int}_{\Vdash}(D) \equiv \exists a \in S(x \Vdash a \;\&\; \forall y \in X(y \Vdash a \to y \;\epsilon\; D)),$$

for all $x \in X$ and $D \subseteq X$. Notice how the notions of interior and closure are not given as mutual complements, but related by the logical duality[1] which exchanges $\exists$ with $\forall$, and $\&$ with $\to$.

We will always drop the index $_{\Vdash}$, when no ambiguity occurs. The operators cl and int, defined as above for a general $\Vdash$, are, respectively, a closure and interior operator on X, that is,

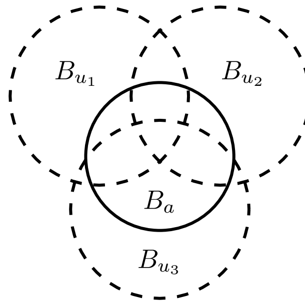
$$\frac{D \subseteq \mathsf{cl}(E)}{\mathsf{cl}(D) \subseteq \mathsf{cl}(E)} \qquad\qquad \frac{\mathsf{int}(D) \subseteq E}{\mathsf{int}(D) \subseteq \mathsf{int}(E)}$$

for all $D, E \subseteq X$. By analogy with the classical setting, we will call closed (resp. open) subsets the fixed point of cl (resp. int). If we invert formally the roles of X and S, we obtain two relations between elements a and subsets U of S,

$$a \lhd_{\Vdash} U \equiv \forall x \in X(x \Vdash a \to \exists b \in S(x \Vdash b \;\&\; b \;\epsilon\; U)), \tag{1.1}$$

$$a \ltimes_{\Vdash} U \equiv \exists x \in X(x \Vdash a \;\&\; \forall b \in S(x \Vdash b \to b \;\epsilon\; U)), \tag{1.2}$$

[1] Of course, this distinction is only visible, and meaningful, in a constructive setting.



Topological intuition behind $a \lhd_{\Vdash} U$, with $u_1, u_2, u_3 \,\epsilon\, U$.

for all $a \in S$ and $U \subseteq S$. It is often useful to denote $\lhd_{\Vdash}$ and $\ltimes_{\Vdash}$ as operator on subsets:

$$a \,\epsilon\, \mathscr{A}(U) \equiv a \lhd_{\Vdash} U, \qquad a \,\epsilon\, \mathscr{I}(U) \equiv a \ltimes_{\Vdash} U, \tag{1.3}$$

for all $a \in S$ and $U \subseteq S$. These are a closure and an interior operator on the subsets of S, that is,

$$\frac{U \subseteq \mathscr{A}(V)}{\mathscr{A}(U) \subseteq \mathscr{A}(V)} \qquad \frac{\mathscr{I}(U) \subseteq V}{\mathscr{I}(U) \subseteq \mathscr{I}(V)}$$

for all $U, V \subseteq S$. In plain words, the primitive concepts of closure and interior are reflected by symmetry on the set of the basis indices S, which assumes a role equal to that of the space of points X.

Most of the spaces treated in this thesis do not allow a constructive description by means of basic pairs. However, basic pairs will supply the motivation for the more general definitions of basic topology and formal topology, which are basically an axiomatization of the operators $\lhd_{\Vdash}$ and $\ltimes_{\Vdash}$ above.

There is an further connection between cl and int, and between $\mathscr{A}$ and $\mathscr{I}$, that express in a positive way the fact that they are the complements of each other:

$$\frac{\mathsf{int}(D) \between E}{\mathsf{int}(D) \between \mathsf{cl}(E),} \qquad \frac{\mathscr{I}(U) \between V}{\mathscr{I}(U) \between \mathscr{A}(V),} \tag{1.4}$$

for all $D, E \subseteq X$ and $U, V \subseteq S$.

Like opens and closed subsets are the fixed points of the interior and closure operators, one calls a subset $U \subseteq S$ a *formal open* if it is a fixed point of $\mathscr{A}_{\Vdash}$, that is, if $\mathscr{A}_{\Vdash}(U) = U$, and a *formal closed* if it is a fixed point of $\mathscr{I}_{\Vdash}$, i.e. $\mathscr{I}_{\Vdash}(U) = U$.

One introduces the subsets

$$x \,\epsilon\, \mathsf{ext}_a \equiv a \Vdash x \qquad a \,\epsilon\, \Box_x \equiv a \Vdash x,$$

for all $a \in S, x \in X$. This generates four operators between subsets of S and X,

$$\mathcal{P}(S) \underset{\mathsf{Rest}}{\overset{\mathsf{Ext}}{\rightrightarrows}} \mathcal{P}(X) \qquad\qquad \mathcal{P}(X) \underset{\Diamond}{\overset{\Box}{\rightrightarrows}} \mathcal{P}(S),$$

defined as follows

$$\begin{aligned} x \;\epsilon\; \mathsf{Ext}(U) &\equiv \Box_x \between U \equiv & a \;\epsilon\; \Box(D) &\equiv \mathsf{ext}_a \between D \equiv \\ &\equiv \exists a(a \Vdash x \;\&\; a \;\epsilon\; U) & &\equiv \exists x(a \Vdash x \;\&\; x \;\epsilon\; D) \\ x \;\epsilon\; \mathsf{Rest}(U) &\equiv \Box_x \subseteq U \equiv & a \;\epsilon\; \Diamond(D) &\equiv \mathsf{ext}_a \subseteq D \equiv \\ &\equiv \forall a(a \Vdash x \to a \;\epsilon\; U) & &\equiv \forall x(a \Vdash x \to x \;\epsilon\; D), \end{aligned}$$

and in particular, one checks that

$$x \;\epsilon\; \mathsf{cl}(D) \equiv x \;\epsilon\; \mathsf{Rest}(\Box(D)) \qquad a \;\epsilon\; \mathscr{A}(U) \equiv a \;\epsilon\; \Diamond(\mathsf{Ext}(U)) \tag{1.5}$$

$$x \;\epsilon\; \mathsf{int}(D) \equiv a \;\epsilon\; \mathsf{Ext}(\Diamond(D)) \qquad a \;\epsilon\; \mathscr{I}(U) \equiv a \;\epsilon\; \Box(\mathsf{Rest}(U)) \tag{1.6}$$

and also

$$x \;\epsilon\; \Box(\mathsf{Rest}(\Box(D))) \leftrightarrow x \;\epsilon\; \Box(D) \qquad a \;\epsilon\; \mathsf{Ext}(\Diamond(\mathsf{Ext}(U))) \leftrightarrow a \;\epsilon\; \mathsf{Ext}(U) \tag{1.7}$$

$$x \;\epsilon\; \Diamond(\mathsf{Ext}(\Diamond(D))) \leftrightarrow x \;\epsilon\; \Box(D) \qquad a \;\epsilon\; \mathsf{Rest}(\Box(\mathsf{Rest}(U))) \leftrightarrow a \;\epsilon\; \mathsf{Rest}(U) \tag{1.8}$$

for all $a \in S, x \in X$, $U \subseteq S$, $D \subseteq X$.

We follow [Samng, SB06] and we give the following constructive notion of *suplattice* and *complete* lattice:

Definition 1.2. A *suplattice* $L = (L, \leqslant, \bigvee)$ is a partially ordered set $(L, \leqslant)$ such that, for every set-indexed family of objects $p_i \in L$ $(i \in I)$, there is an object $\vee_{i\in I} p_i \in L$, the *supremum*, such that, for all $q \in L$, $\vee_{i\in I} p_i \leqslant q$ holds if and only if, for every $i \in I$, $p_i \leqslant q$.

A suplattice is called a *complete lattice* if for every set-indexed family of objects $p_i \;\epsilon\; L$ $(i \in I)$ there also is an element $\wedge_{i\in I} p_i \in L$, the *infimum*, such that $q \leqslant \wedge_{i\in I} p_i$ if and only if, for every $i \in I$, $q \leqslant p_i$.

Remark 1. If we allow impredicative reasoning, every suplattice is a complete lattice, since we can define the infimum of a family $\{p_i\}_{i\in I}$ as

$$\wedge_{i\in I} p_i \equiv \vee\{q : \forall i(q \leqslant p_i)\}.$$

In our setting, the right member is general not well-defined, because the family on which the supremum is taken is not set-indexed in general.

The collection of fixed points of a closure and interior operator, such as $\mathscr{A}$ and $\mathscr{I}$, form a complete lattice. In fact, given $\{U_i\}_{i\in I}$ (resp. $\{V_j\}_{j\in J}$) is a set-indexed family of formal opens (resp. formal closed), one can define

$$\vee_{i\in I}U_i \equiv \mathscr{A}(\bigcup_{i\in I} U_i), \qquad \wedge_{i\in I}V_i \equiv \bigcap_{i\in I} V_i,$$
$$\vee_{i\in I}U_i \equiv \bigcup_{i\in I} U_i, \qquad \wedge_{i\in I}V_i \equiv \mathscr{I}(\bigcap_{i\in I} V_i),$$

and similarly can act for cl and int. There is an isomorphism between the complete lattice $Open(X)$ of open subsets and that of formal opens $FOpen(S)$, given by the correspondences

$$FOpen(S) \underset{\mathsf{Ext}}{\overset{\Diamond}{\leftrightarrows}} Open(X).$$

The fact that Ext and $\Diamond$ are inverse of each other on the subcollections $FOpen(S)$ and $Open(X)$ is direct consequence of the equations (1.5,1.6) and (1.7,1.8). The correspondences also respect the supremum and the infimum:

$$x \;\epsilon\; \mathsf{Ext}(\vee_{i\in I}U_i) \equiv x \;\epsilon\; \mathsf{Ext}(\mathscr{A}(\bigcup_{i\in I} U_i)) \leftrightarrow x \;\epsilon\; \bigcup_{i\in I}\mathsf{int}(\mathsf{Ext}(U_i)),$$
$$x \;\epsilon\; \mathsf{Ext}(\wedge_{i\in I}U_i) \equiv x \;\epsilon\; \mathsf{Ext}(\Diamond(\bigcap_{i\in I}\mathsf{Ext}(U_i))) \leftrightarrow x \;\epsilon\; \mathsf{int}(\bigcap_{i\in I}\mathsf{Ext}(U_i)).$$

for all $U_i \subseteq S$. Dually one can show the same for $\Diamond$.

This discussion should convince the reader that we can give up talking about the open subsets of a topological space (and later on, also the notion of points), since the same structure can be reproduced symmetrically and isomorphically on (the set of indices of) a basis of the space.

Reversing the argumentation above, we discuss in this setting which properties must the family $\{\mathsf{ext}_a\}_{a\in S}$ possess, in order to be a base of neighborhoods for a topological space structure on X. This amounts to the following two conditions:

$$\mathsf{ext}_a \cap \mathsf{ext}_b \subseteq \bigcup_{c \;\epsilon\; a\downarrow b} \mathsf{ext}_c, \qquad (\forall x \in X)(\exists a \in S)(x \Vdash a),$$

where, for all $a, b \in S$,

$$\begin{aligned} c \;\epsilon\; a \downarrow b &\equiv c \lhd a \;\&\; c \lhd b \equiv \\ &\equiv \mathsf{ext}_c \subseteq \mathsf{ext}_a \cap \mathsf{ext}_b \equiv \\ &\equiv \forall x (x \Vdash c \to x \Vdash a \;\&\; x \Vdash b). \end{aligned} \tag{1.9}$$

It is then motivated the introduction of the following definition [Sam03, Samng]:

Definition 1.3. A *concrete space* is a basic pair $(X, \Vdash, S)$ satisfying the two extra conditions:

$$\frac{x \Vdash a \quad x \Vdash b}{x \Vdash a \downarrow b}, \qquad (\forall x \in X)(\exists a \in S)(x \Vdash a), \tag{1.10}$$

for all $x \;\epsilon\; X$, and $a, b \in S$, where $x \Vdash a \downarrow b \equiv \exists c (x \Vdash c \;\&\; c \;\epsilon\; a \downarrow b)$.

In other words, the concrete spaces are the basic pairs corresponding to topological spaces, or, more precisely, inducing a topological space structure on X.

In presence of a concrete space, the complete lattice of formal open subsets $FOpen(S)$ is a *locale* [Joh82, Samng], i.e. infinite suprema commutes with finite meets. In fact, notice that if U and V are formal opens, then so is $U \downarrow V$ $(*)$. We have then

$$\begin{aligned}(\vee_{i\in I} U_i) \wedge V &= \mathscr{A}(\bigcup U_i) \cap V \overset{(1.10)}{=} \\ &= \mathscr{A}((\bigcup U_i) \downarrow V) = \\ &= \mathscr{A}(\bigcup_{i\in I}(U_i \downarrow V)) \overset{(*)}{=} \\ &= \vee_{i\in I}(U_i \wedge V).\end{aligned} \tag{1.11}$$

In other words, $FOpen(S)$ (and therefore its isomorphic $Open(S)$) is a *locale* [Joh82, Samng].

A topological space arising in this way from a concrete space $(X, \Vdash, S)$ is not sober in general, that is, in terms of the corresponding topological space, not every irreducible closed subset[2] is the closure of exactly one point x of X [Joh82]. In other words, not everything that looks as a point from S corresponds to a point of X. There is a canonical impredicative way to add the "missing points", and obtain the *sobrification* of the space.

One defines points as particular subsets of S, whose corresponding base opens form a completely prime filter [Sam03, Samng]:

Definition 1.4. Let $(X, \Vdash, S)$ be a concrete space. A point is a subset $\alpha \subseteq S$ such that:

1. α is inhabited;

2. α is a formal closed, that is, $\mathscr{I}_{\Vdash}(\alpha) = \alpha$;

3. α is convergent: $\forall a, b(a, b \ \epsilon\ \alpha \to a \downarrow b \between \alpha)$.

One denotes by $\mathscr{P}t_{\Vdash}(S)$ the collection of formal points of $(X, \Vdash, S)$.

One can then define the sobrification of $(X, \Vdash, S)$ as the following concrete space:

$$\mathscr{P}t_{\Vdash}(S) \xrightarrow{\Vdash_{\ni}} S$$

where $\alpha \Vdash_{\ni} a \equiv a \ \epsilon\ \alpha$. Notice that the collection $\mathscr{P}t_{\Vdash}(S)$ do not form a set in general, so that the sobrification is not always constructively well-defined[3].

[2] We will say something more about irreducible subsets in section 1.6.1. The main reference is still the book [Samng].

[3] And very seldom is the case.

1.2.1 Morphisms of Basic Pairs and Concrete Spaces

One defines morphisms between basic pairs and concrete spaces, in order to imitate formally the behaviour of continuous functions:

Definition 1.5. [Sam03, Samng] Let $(X, \Vdash, S)$ and $(X', \Vdash', S')$ be two basic pairs. A *continuous morphism* consists of a pair of relations (r, s) making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\Vdash} & S \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X' & \xrightarrow{\Vdash'} & S' \end{array}$$

If moreover the two basic pairs are concrete spaces, one says that (r, s) is *convergent* if:

(C1) $s^{-}a \downarrow s^{-}b \subseteq \mathscr{A}_{\Vdash}(s^{-}(a \downarrow' b))$, for all $a, b \in S'$;

(C2) $S = \mathscr{A}_{\Vdash}(s^{-}S')$.

The continuity condition ensures that the inverse image through r of an open subset is an open subset, or, dually, that the direct image through s of a formal closed is a formal closed. The essence of continuity is therefore enclosed in a commutative diagram, that provides also, as we will see, a useful combinatorial notation for continuous morphisms.

The convergence condition $(C1)$ says that the direct image of a convergent and formal closed subset through s is a convergent subset. The convergence condition $(C2)$ implies that the direct image of an inhabited closed subset is inhabited. As a consequence, the direct image through s of a formal point is still a formal point. In this sense, a continuous and convergent morphism (r, s) as above induces a function

$$\begin{array}{rcl} f_{(r,s)} : \mathscr{P}t_{\Vdash}(S) & \longrightarrow & \mathscr{P}t_{\Vdash'}(S') \\ \alpha & \mapsto & s\alpha. \end{array}$$

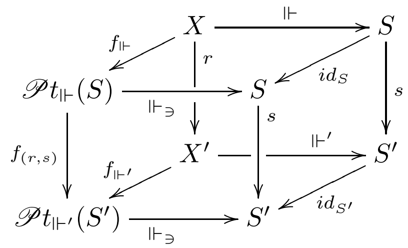
continuous in the usual sense.

Example 1.1. Let $(X, \Vdash, S)$ be a concrete space. We have a canonical morphism to its sobrification

$$\begin{array}{ccc} X & \xrightarrow{\Vdash} & S \\ {\scriptstyle f_{\Vdash}}\downarrow & & \downarrow{\scriptstyle id_S} \\ \mathscr{P}t_{\Vdash}(S) & \xrightarrow{\Vdash_{\ni}} & S \end{array}$$

where $f_{\Vdash}(x) \equiv \{a : x \Vdash a\}$ is a well-defined continuous function.

Example 1.2. Let (r, s) be a continuous and convergent morphism between the concrete spaces $(X, \Vdash, S)$ and $(X', \Vdash', S')$. The function $f_{(r,s)}$ defined above makes the following diagram commutative:



The composition $(r, s) \circ (r', s')$ of two continuous (resp. continuous and convergent) morphisms (r, s) and (r', s') (between suitable concrete spaces) is defined as $(r \circ r', s \circ s')$ in a natural way. In particular, the pair (id_X, id_S), where id_X and id_S are the identity relations on X and S, act as an identity morphism.

With these composition and identity, the basic pairs (resp. concrete spaces) form a category BPair (resp. CSpa).

Different continuous morphisms can induce the same maps between the open subsets, and therefore, if they are convergent, the same function between points. This property induces an equivalence relation $\cong$ on the collection of continuous morphisms between two basic pairs $(X, \Vdash, S)$ and $(X', \Vdash', S')$, that we can luckily handle on: if (r, s) is a continuous morphism, its *saturation* $(\overline{r}, \overline{s})$ is defined as

$$x\overline{r}y \equiv y \;\epsilon\; \mathsf{cl}_{\Vdash'}(rx), \qquad\qquad a\overline{s}a' \equiv a \;\epsilon\; \mathscr{A}_{\Vdash}(s^{-}a').$$

One says then that two relation are *equivalent* if they have the same saturation. In particular, the saturation gives a canonical representative (the maximal, with respect to inclusion) for each equivalent class. The equivalence relation $\cong$ on the homset respects composition. We will denote by $\mathrm{CSpa}_{\cong}$ the category BTop where the homsets are quotiented by $\cong$.

The correspondence $(r, s) \mapsto f_{(r,s)}$ defines a functor

$$\mathscr{P}t : \mathrm{CSpa}_{\cong} \to \mathrm{Top}$$

where Top is the usual category of topological spaces. Impredicatively, every topological space can be represented as a concrete space, this functor gives an equivalence. We have the usual categorical notion of isomorphism:

Definition 1.6. An *isomorphism* between two basic pairs (resp. concrete spaces) $(X, \Vdash, S)$ and $(X', \Vdash', S')$ consists of two continuous (resp. continuous and convergent) morphisms (r, s), (r', s') such that

$$(r, s) \circ (r', s') \cong (id_{X'}, id_{S'}), \qquad\qquad (r', s') \circ (r, s) \cong (id_X, id_S).$$

The relations (r, s) and (r', s') are said to be *inverse* of each other.

In particular, if a continuous and convergent morphism (r, s) is an isomorphism, the corresponding $f_{(r,s)}$ between points is a homeomorphism.

In the concrete case, one can prove [Samng] that two continuous morphisms (r, s) and (r', s') are inverse of each other if the following hold

$$\mathscr{A}(s^- s'^- a) = \mathscr{A}(a), \quad \mathscr{A}' s'^- s^- b = \mathscr{A}' b \tag{1.12}$$

for all $a \,\epsilon\, S$ and $b \,\epsilon\, S'$. In the following, to prove that a pair of morphisms is an isomoprhism, we will always make use of (1.12).

1.3 Basic Topologies with Convergence Operation

The notion of basic topology results from the axiomatization of $\lhd_{\Vdash}$ (or $\mathscr{A}_{\Vdash}$) (1.1,1.2) in a basic pair. In this way, any reference to the existence of a space of points X is avoided.

Definition 1.7. A *basic topology* is a structure $(S, \lhd, \ltimes)$ where S is a set, and $\lhd$, $\ltimes$ are relations between elements and subsets of S satisfying:

$$\frac{a \,\epsilon\, U}{a \lhd U}\ \textit{Reflexivity} \qquad \frac{a \lhd U \quad U \lhd V}{a \lhd V}\ \textit{Transitivity}$$

$$\frac{a \ltimes U}{a \,\epsilon\, U}\ \textit{Coreflexivity} \qquad \frac{a \ltimes U \quad \forall b(b \ltimes U \rightarrow b \,\epsilon\, V)}{a \ltimes V}\ \textit{Cotransitivity}$$

$$\frac{a \lhd U \quad a \ltimes V}{U \ltimes V}\ \textit{Compatibility}$$

where

$$U \lhd V \equiv (\forall u \,\epsilon\, U)(u \lhd V), \qquad U \ltimes V \equiv (\exists u \,\epsilon\, U)(u \ltimes V),$$

for all $a, b \in S$ and $U, V \subseteq S$. The relations $\lhd$ and $\ltimes$ are called *cover* and *positivity* respectively.

Instead of $\lhd$ and $\ltimes$, we could have introduced a closure and an interior operator $\mathscr{A}$ and $\mathscr{I}$, as in (1.3). Then *Reflexivity* and *Transitivity* say that $\mathscr{A}$ is a closure operator, and, *Coreflexivity* and *Cotransitivity* that $\mathscr{A}$ is an interior operator. Finally, *Compatibility* axiomatises (1.4).

In particular, the collection $FOpen(S)$ of fixed points of $\mathscr{A}$ is a complete lattice. However, we prefer to identify here $FOpen(S)$ with the powerset $\mathcal{P}(S)$ quotiented by the equivalence relation

$$U =_{\lhd} V \equiv U \lhd V \;\&\; V \lhd U.$$

Remark 2. The relation $\ltimes$ is a generalization of the unary positivity predicate Pos, associated with the old definition of formal topology. We can in fact define $Pos(a) \equiv a \ltimes S$ for all $a \in S$.

One extends a basic topology $(S, \lhd, \ltimes)$ by adding a *convergence operation* $*$, that to every $a, b \in S$ returns a subset $a * b$. This convergence operation should axiomatise the operation $\downarrow$ introduced in (1.9), that is, it represents a sort of formal intersection between basis open subsets.

By tuning the properties which relate $*$ with the cover $\triangleleft$, we obtain different algebraic structures on the lattice of formal opens. For an extended description of convergence operations we refer to [CMS13]. As basic requirement, the operation $*$ has to be associative and commutative.

Notice that to any basic topology $(S, \triangleleft, \ltimes)$ we can canonically add the $\downarrow$ operation

$$c \ \epsilon \ a \downarrow b \equiv c \triangleleft a \ \& \ c \triangleleft b \tag{1.13}$$

for all $a, b \in S$.

The choice of having a primary and independent convergence operation $*$ is justfied by the concrete applications, where such an operation is often immediately available[4].

A convergence operation $*$ is existentially generalized to subsets as

$$c \ \epsilon \ U * V \equiv (\exists u, v)(u \ \epsilon \ U \ \& \ v \ \epsilon \ V \ \& \ c \ \epsilon \ u * v)$$

for all $U, V \subseteq S$. When U is a singleton, say $U \equiv \{u\}$, we will write $u * V$ instead of $\{u\} * V$, and the same for V. The product of n elements $u_1, \ldots, u_n$ (respectively, of n subsets $U_1, \ldots, U_n$) will be often denoted as $\prod_{i=1}^n u_i$ (resp. as $\prod_{i=1}^n U_i$).

We now list and analyse some basic properties of the convergence operation, that will emerge naturally in this dissertation. Most of them can be interpreted as logical rules, regarding the operation $*$ as a sort of disjunction, and this intuition is helpful for their understanding. As first property for $*$, we may require it to be well-defined on $FOpen(S)$. In other words,

$$U * V =_{\triangleleft} \mathscr{A}(U) * \mathscr{A}(V)$$

for all $U, V \subseteq S$. Since $U * V \subseteq \mathscr{A}(U) * \mathscr{A}(V)$, it is enough to ask that the right side is covered by the left side. This is equivalent to the following property, called *Localisation*:

$$\frac{a \triangleleft U \quad b \triangleleft V}{a * b \triangleleft U * V} \ Loc. \tag{1.14}$$

A simplified formulation is the following, also called *Stability*,

$$\frac{a \triangleleft U}{a * b \triangleleft U * b} \ Stab, \tag{1.15}$$

for all $a, b \in S$, $U \subseteq S$. Further natural properties, inspired by the notion of a formal intersection, are *Contraction* and *Left* (or *Weakening*):

$$a \triangleleft a * a \quad Con, \qquad\qquad a * b \triangleleft a \quad Left, \tag{1.16}$$

[4] As we will see, an independent convergence operation can satisfy additional properties, like having $a * b$ finite for all $a, b \in S$. This feature turns out to be essential in algebraic contexts.

for all $a, b \in S$; the first states that an intersection of a basic neighborhood with itself contains itself, while the second formalizes that the basis neighborhoods of $a * b$ must be contained in a.

The following relative versions of Con and $Left$, equivalent thanks to transitivity, can be useful in later applications:

$$\frac{a * a \lhd U}{a \lhd U}\ Con, \qquad\qquad \frac{a \lhd U}{a * b \lhd U}\ Left,$$

for all $a, b \in S$, $U \subseteq S$. With Loc (with $b = a$) and Con, the property called $Right$ can be derived:

$$\frac{a \lhd U \quad a \lhd V}{a \lhd U * V}\ Right. \tag{1.17}$$

for all $a \in S$, $U, V \subseteq S$. Vice versa, from $Right$ follows Con, by taking $U = V = \{a\}$, and from $Right$ and $Left$ follows Loc, as the following derivation shows:

$$\frac{\dfrac{a \lhd U}{a * b \lhd U}\ Left \quad \dfrac{b \lhd V}{a * b \lhd V}\ Left}{a * b \lhd U * V}\ Right.$$

In other words, the set of properties $Left + Right$, and $Loc + Left + Con$ are equivalent. In terms of the operator $\mathscr{A}$ we have

$$U * V \lhd \mathscr{A}(U) \cap \mathscr{A}(V) \quad (Left), \qquad \mathscr{A}(U) \cap \mathscr{A}(V) \lhd U * V \quad (Right),$$

that is, together,

$$\mathscr{A}(U * V) =_{\lhd} \mathscr{A}(U) \cap \mathscr{A}(V) \quad (Left + Right), \tag{1.18}$$

for all $U, V \subseteq S$. Similarly, one proves

$$\mathscr{A}(U) * \mathscr{A}(V) = \mathscr{A}(U * V).$$

These are exactly the properties of $\downarrow$ we used in (1.11); by repeating the same argument for $*$, it follows that $FOpen(S)$ has structure of locale. This discussion should justify the following definition:

Definition 1.8. A *formal topology* is a structure $(S, \lhd, \ltimes, *)$ where $(S, \lhd, \ltimes)$ is a basic topology and $*$ is a *convergence operation* satisfying Left, Contraction and Stability.

As just noticed, $FOpen(S)$ is a locale for all formal topology $(S, \lhd, \ltimes, *)$. By dropping, for example, Con, the operation $*$ will give $FOpen(S)$ structure of commutative quantale [CMS13]. Such quantale will be a *unital quantale* if moreover

$$a \lhd a * S, \qquad (Unit) \tag{1.19}$$

for all $a \in S$.

Definition 1.9. We say that two convergence operations $*$ and $*'$ on a basic topology $(S, \lhd, \ltimes)$ are *topologically equivalent* if $a * b =_{\lhd} a *' b$ for all $a, b \in S$.

To a basic topology $(S, \lhd, \ltimes, *)$ with a convergence operation[5] one can asso-

[5] Not necessarily a formal topology.

ciate a notion of point, and, impredicatively, a *space of points* $\mathscr{P}t(S)$, exactly as in the case of concrete space (Def. 1.4). A point is a formal closed, inhabited and convergent[6] subset $\alpha \subseteq S$. We will reserve lowercase greek letters $\alpha, \beta, \gamma, \ldots$ to denote points.

The membership relation defines, impredicatively, a basic pair

$$\mathscr{P}t_{\Vdash}(S) \xrightarrow{\Vdash_{\ni}} S$$

where $\alpha \Vdash_{\ni} a \equiv a \;\epsilon\; \alpha$. The machinery of basic pairs produces a closure operator cl and an interior operator int on $\mathscr{P}t_{\Vdash}(S)$, and a new cover $\lhd_{\mathscr{P}t}$ and positivity $\ltimes_{\mathscr{P}t}$ on S:

$$a \lhd_{\mathscr{P}t} U \equiv \forall\alpha(a \;\epsilon\; \alpha \to \exists u(u \;\epsilon\; \alpha \;\&\; u \;\epsilon\; U)) \equiv \forall\alpha(a \;\epsilon\; \alpha \to U \between \alpha), \quad (1.20)$$

$$a \ltimes_{\mathscr{P}t} U \equiv \exists\alpha(a \;\epsilon\; \alpha \;\&\; \forall u(u \;\epsilon\; \alpha \to u \;\epsilon\; U)) \equiv \exists\alpha(a \;\epsilon\; \alpha \;\&\; U \subseteq \alpha). \quad (1.21)$$

If $a \lhd U$ and $a \;\epsilon\; \alpha$, then $a \ltimes \alpha$, because α is formal closed, and then $U \between \alpha$ follows from compatibility. This proves that $a \lhd U$ implies $a \lhd_{\mathscr{P}t} U$. Similarly, we can prove that $a \ltimes_{\mathscr{P}t} U$ implies $a \ltimes U$. The converse implications do not hold in general, and one isolates them as further properties:

Definition 1.10. A basic topology $(S, \lhd, \ltimes, *)$ with convergence operation is said to be *spatial*, or, to have enough points, if $a \lhd_{\mathscr{P}t} U$ implies $a \lhd U$ and *reducible* if $a \ltimes U$ implies $a \ltimes_{\mathscr{P}t} U$ for all $a \in S$, $U \subseteq S$.

Remark 3. This definition is motivated by the locale-theoretic counterpart: if $(S, \lhd, \ltimes, *)$ is a spatial formal topology, then the corresponding locale $FOpen(S)$ of formal opens is spatial [Joh82]. Impredicatively, also the converse can be shown.

Proposition 1.3. *Let $(S, \lhd, \ltimes, *)$ be a basic topology with convergence operation satisfying Loc and $Left$, and $\mathscr{P}t(S)$ the corresponding collection of points. The operator* int *defined impredicatively by the basic pair*

$$\Vdash_{\ni} : \mathscr{P}t(S) \to S$$

is a topological interior operator on $\mathscr{P}t(S)$. In other words

$$\mathsf{int}\, D \cap \mathsf{int} E \;= \mathsf{int}\,(D \cap E)$$

for all D, E subcollections of $\mathscr{P}t(S)$.

Proof. We have explicitly

$$\alpha \in_c \mathsf{int} D \equiv \exists a(a \;\epsilon\; \alpha \;\&\; \forall\beta(a \;\epsilon\; \beta \to \beta \in_c D)).$$

Since int is an interior operator, $\subseteq$ is the only non-trivial inclusion. Suppose that $\alpha \in_c \mathsf{int} D \cap \mathsf{int} E$, that is, $\alpha \in_c \mathsf{int}\; D$ and $\alpha \in_c \mathsf{int}\; E$; by definition

[6] Convergence is here relativized to the operation $*$:$\forall a, b(a, b \;\epsilon\; \alpha \to a * b \between \alpha)$.

there exist $a, a' \epsilon \alpha$ such that $a \epsilon \beta \to \beta \in_c D$ and $a' \epsilon \beta \to \beta \in_c E$ for all $\beta \in_c \mathscr{P}t(S)$. We must produce $b \epsilon \alpha$ such that $b \epsilon \beta \to \beta \in_c D \cap E$ for all $\beta \in_c \mathscr{P}t(S)$. Since α is convergent, we have $\alpha \between a * a'$ and we choose b witnessing it. Since $b \lhd a$ (Left), if $b \epsilon \beta$, then $a \epsilon \beta$ for all $\beta \in_c \mathscr{P}t(S)$, and then $\beta \in_c D$. Symmetrically, for all $\beta \in_c \mathscr{P}t(S)$, if $b \in_c \beta$, then $\beta \in_c E$. Thus $b \epsilon \beta \to \beta \in_c D \cap E$. □

In particular, the last proposition applies to every formal topology.

1.3.1 Morphisms of Basic Topologies with Operation

Before introducing the notion of morphism between basic and formal topologies, we need to add a few more lines about binary relations: if $r \subseteq S \times T$, four operators between $\mathcal{P}(S)$ and $\mathcal{P}(T)$ are naturally given [Samng, CMS13]:

$$\mathcal{P}(S) \underset{r^{-*}}{\overset{r}{\rightrightarrows}} \mathcal{P}(T) \qquad\qquad \mathcal{P}(T) \underset{r^{-}}{\overset{r^*}{\rightrightarrows}} \mathcal{P}(S),$$

as follows:

$$\begin{aligned} &\textit{Direct existential image:} && t \epsilon rU \equiv r^- t \between U, \\ &\textit{Direct universal image:} && t \epsilon r^{-*}U \equiv r^- t \subseteq U, \\ &\textit{Inverse existential image:} && s \epsilon r^- V \equiv rs \between V, \\ &\textit{Inverse universal image:} && s \epsilon r^* V \equiv rs \subseteq V, \end{aligned}$$

where $s \epsilon r^- t \equiv srt$, $U \subseteq S$, $V \subseteq T$, $s \in S$ and $t \in T$.

Remark 4. If r is a functional relation, then r^- and r^* coincide with the usual reverse image and r with the direct image.

These operators respect composition, i.e. one has

$$(s \circ r)^- = r^- \circ s^-, \qquad (s \circ r)^* = r^* \circ s^*, \qquad (s \circ r)^{-*} = s^{-*} \circ r^{-*},$$

for every relation $r : S \to T$ and $s : T \to R$.

Here follows the notion of morphism of basic topologies:

Definition 1.11. Let $(S, \lhd_S, \ltimes_S)$ and $(T, \lhd_T, \ltimes_T)$ be two basic topologies. A relation $s : S \to T$ is *continuous* if the following two conditions hold

$$\frac{b \lhd_T U}{s^- b \lhd_S s^- U}\ Cont_\lhd \qquad\qquad \frac{a \ltimes_S s^* U}{sa \ltimes_T U}\ Cont_\ltimes \tag{1.22}$$

for all $a \in S, b \in T$ and $U \subseteq T$.

In terms of closure and interior operators, the conditions (1.22) rewrite as

$$\frac{U = \mathscr{A}_S(U)}{s^{-*}U = \mathscr{A}_T(s^{-*}U)}\ Cont_{\mathscr{A}}, \qquad\qquad \frac{U = \mathscr{I}_S(U)}{sU = \mathscr{I}_T(sU)}\ Cont_{\mathscr{I}},$$

for all $U \subseteq S$. From $Con_{\lhd}$ one deduces that s^- respects the equality $=_{\lhd_T}$ on $FOpen(T)$ and $=_{\lhd_S}$ on $FOpen(S)$, so that s^- can be qutiented to a morphism of complete lattices from $FOpen(T)$ to $FOpen(S)$. From $Cont_{\mathscr{I}}$ it follows that the direct existential image sF of a formal closed $F \subseteq S$ is a formal closed.

Different continuous morphisms can induce the same operator between the formal opens. This property induces an equivalence relation $\cong$ on the collection of continuous morphisms between two basic pairs $(S, \lhd_S, \ltimes_S)$ and $(T, \lhd_T, \ltimes_T)$: if s is a continuous morphism, its *saturation* $\overline{s}$ is defined as

$$a \overline{s} a' \equiv a \;\epsilon\; \mathscr{A}_{\Vdash}(s^- a'). \tag{1.23}$$

One says then that two relation are *equivalent* if they have the same saturation.

One verifies that the identity relation is continuous, and the composition of two continuous relations is a continuous relation [Samng]. With these notions of identity and composition, the basic topologies with continuous relations form a category, which one denotes by BTop. The equivalence relation $\cong$ on the homset respects composition. We will denote by $\mathrm{BTop}_{\cong}$ the category BTop where the homsets are quotiented by $\cong$.

Definition 1.12. Let $(S, \lhd_S, \ltimes_S, *_S)$ and $(T, \lhd_T, \ltimes_T, *_T)$ be basic topologies with operation. A continuous relation $r : S \to T$ is *convergent*[7] if it satisfies the conditions

(C1) $r^-(a) *_S r^-(b) \lhd_{\mathcal{S}} r^-(a *_T b)$

(C2) $S \lhd_{\mathcal{S}} r^- T$

for all $a, b \;\epsilon\; A$.

Remark 5. The condition C1 can be reformulated by saying that the direct existential image of a convergent reduced subset $F \subseteq S$, through a relation $r : S \to T$, is convergent. In fact, let $a, b \;\epsilon\; rF$; equivalently $r^- a \between F$ and $r^- b \between F$, from which we get $r^- a *_S r^- b \between F$ because F is convergent. Since F is also reduced, from the first convergence condition, by compatibility we obtain $r^-(a *_T b) \between F$ and finally $a *_T b \between rF$.

The property C2 entails that the direct image of a reduced inhabited subset $F \subseteq S$ is inhabited. Suppose that $F \between S$; since F is reduced, we have $F \between r^- T$ and therefore $rF \between T$.

The identity relation is trivially convergent, and the composition of two continuous and convergent operations is again continuous and convergent [Samng]. We denote by BTopO the category of basic topologies with continuous and convergent relations, and FTop the full subcategory of formal topologies. Let also $\mathrm{BTopO}_{\cong}$ and $\mathrm{FTop}_{\cong}$ be obtained by quotienting the homsets by $\cong$.

[7] In [CMS13], the most recent version of formal topology, the continuous relations satisfying just C1 are called convergent, while they are called *unital* convergent if they satisfy both C1 and C2. For the cases we are going to study, this difference is not relevant.

In particular, the direct existential image of a continuous and convergent morphism r from $(S, \lhd_S, \ltimes_S, *_S)$ to $(T, \lhd_T, \ltimes_T, *_T)$ determines a map

$$\begin{array}{rcl} \mathscr{P}t(r) : \mathscr{P}t(S) & \to & \mathscr{P}t(T). \\ \alpha & \mapsto & r\alpha \end{array}$$

If S and T are formal topologies, $\mathscr{P}t(r)$ is continuous in the classical sense between the topological space structures on $\mathscr{P}t(S)$ and $\mathscr{P}t(T)$ (see Proposition 1.3). This correspondence, being nothing but the direct image, defines an equivalence of categories

$$\mathscr{P}t : \mathrm{FTop}_{\cong} \to \mathrm{Top}.$$

All the reasoning involving the collection of points $\mathscr{P}t(S)$ must be considered in general impredicative, and therefore will be carefully avoided while proving constructive results. However it is useful, since it provides, in most cases, the link with the classical intuition.

1.4 Generation of Basic Topologies

As pointed out by Sambin and Per-Martin Löf, "the most typical technical contribution of formal topology is the introduction of inductive methods in topology". In [CMS13, MLSng], a uniform method is provided for generating inductively complete lattices, commutative quantales and locales. We present it briefly, simplifying it a bit for our purposes.

We start from a set of generators S. The relations between the generators are given in the form of an axiom-set I, C:

$$I(a) \text{ set} : (a \in S),$$
$$C(a,i) \subseteq S : (i \in I(a), a \in S).$$

With every axiom set I, C we can associate a cover $\lhd_{I,C}$ satisfying:

1. $a \lhd_{I,C} C(a,i)$ for all $a \in S$ and $i \in I(a)$;

2. $\lhd_{I,C}$ is minimal for such property: if $\lhd'$ is another cover satisfying $a \lhd C(a,i)$ for all $a \in S$ and $i \in I(a)$, then $a \lhd_{I,C} U \to a \lhd U$ for all $a \in S$, $U \subseteq S$.

One shows [MLSng] that $\lhd_{I,C}$ is the unique relation between elements and subsets satisfying

$$\frac{a \;\epsilon\; U}{a \lhd_{I,C} U}\ \mathit{Reflexivity}, \qquad \frac{i \in I(a) \quad C(a,i) \lhd_{I,C} U}{a \lhd_{I,C} U}\ \mathit{Generation}, \tag{1.24}$$

$$\frac{a \lhd_{I,C} U \quad U \subseteq P \qquad \begin{array}{c}[i \in I(b), \quad C(b,i) \subseteq P] \\ \vdots \\ b \;\epsilon\; P\end{array}}{a \;\epsilon\; P}\ \mathit{Induction}, \tag{1.25}$$

for every $a, b \in S$ $U, P \subseteq S$.

As reported in [CMS13] and explained in [CSSV03], this kind of inductive definition can be formalized in a constructive framework such as Martin-Löf type theory. In practice, proving $a \lhd U \to a \;\epsilon\; P$ by induction on $a \lhd U$, we have to check that $a \;\epsilon\; P$ holds in either of the two cases: the assumption $a \;\epsilon\; U$ and the inductive hypothesis $C(a,i) \subseteq P$ for some $i \in I(a)$ [CMS13].

We assume form now on by convention that $C(a,i) \equiv \{a\}$ for some $i \in I(a)$ is inhabited for all $a \;\epsilon\; S$. This is not restrictive, since we can otherwise redefine $I'(a) \equiv I(a) \cup \{*\}$ and $C(a,*) \equiv \{a\}$.

One can define dually, by coinduction, a positivity $\ltimes_{I,C}$ satisfying:

1. $(S, \lhd_{I,C}, \ltimes_{I,C})$ is a basic topology, i.e. $\ltimes_{I,C}$ is compatible with $\lhd_{I,C}$;
2. $\lhd_{I,C}$ is maximal for such property: if $\ltimes'$ is a positivity compatible to $\lhd_{I,C}$ then $a \ltimes U \to a \ltimes_{I,C} U$ for all $a \in S$, $U \subseteq S$.

One shows [MLSng] that $\ltimes_{I,C}$ is the unique relation between elements and subsets satisfying

$$\frac{a \ltimes_{I,C} U}{a \;\epsilon\; U}\ Coreflexivity, \qquad \frac{i \in I(a) \quad a \ltimes_{I,C} U}{C(a,i) \ltimes_{I,C} U}\ Cogeneration, \tag{1.26}$$

$$\frac{a \;\epsilon\; P \quad P \subseteq U \quad \begin{array}{c}[i \in I(b),\quad b \;\epsilon\; P]\\ \vdots \\ C(b,i) \between P\end{array}}{a \ltimes_{I,C} U}\ Coinduction, \tag{1.27}$$

for every $a, b \in S$ $U, P \subseteq S$. In practice, proving $a \;\epsilon\; P \to a \ltimes U$ by coinduction on $a \ltimes U$ means checking that $a \;\epsilon\; P$ implies $a \;\epsilon\; U$ and $C(a,i) \between U$ for all $i \in I(a)$.

For a generated basic topology $(S, \lhd_{I,C}, \ltimes_{I,C})$, we can explicitly characterize the formal open subsets U and the formal closed subsets V as those satisfying

$$\frac{C(a,i) \subseteq U}{a \;\epsilon\; U}, \qquad \frac{a \;\epsilon\; V}{C(a,i) \between V}, \tag{1.28}$$

for all $a \in S$ and $i \in I(a)$. In particular, a subset V in a generated topology is formal closed if and only if it *splits the cover*, i.e., if $a \;\epsilon\; V$ and $a \lhd U$, then $U \between V$.

If we admit classical logic, the relation between $\lhd_{I,C}$ and $\ltimes_{I,C}$ is given by complementation:

Proposition 1.4. *(CL) Let $\mathscr{S} = (S, \lhd_{I,C}, \ltimes_{I,C})$ be the basic topology generated by the axiom-set (I, C). Admitting classical logic, one has*

$$a \lhd_{I,C} -U \leftrightarrow \neg(a \ltimes_{I,C} U) \tag{1.29}$$

or, equivalently, $a \ltimes_{I,C} U \leftrightarrow \neg(a \lhd_{I,C} -U)$, for every $a \in S$ and $U \subseteq S$. As a consequence, a subset $U \subseteq S$ is a formal open if and only if its complement $-U$ is a formal closed.

Proof. ($\rightarrow$) We apply induction on $a \lhd_{I,C} -U$, with $a \,\epsilon\, P \equiv \neg(a \ltimes_{I,C} U)$: if $a \notin U$ and $a \ltimes_{I,C} U$, then by reflexivity we also have $a \,\epsilon\, U$, a contradiction; suppose now $i \in I(b)$, $C(b,i) \subseteq P$, and $b \ltimes_{I,C} U$. Then, by Cogeneration, $C(b,i) \ltimes_{I,C} U$, so that $P \ltimes_{I,C} U$, again a contradiction.
($\leftarrow$) We show $\neg(a \lhd_{I,C} U) \rightarrow a \ltimes_{I,C} -U$ by coinduction on $a \ltimes_{I,C} -U$: if $\neg(b \lhd_{I,C} U)$ then $b \notin U$. If $i \in I(b)$ and $\neg(b \lhd_{I,C} U)$, then $\neg(C(b,i) \subseteq U)$, that is, $C(b,i) \between -U$. □

Remark 6. With classical logic, we can prove that a generated topology

$$(S, \lhd, \ltimes, *)$$

is spatial if and only if it is reducible. We have in fact the following chain of equivalences, starting from

$$\begin{array}{rll} \text{(Reducibility)} & a \ltimes -U \rightarrow \exists\alpha(a \,\epsilon\, \alpha \,\&\, \alpha \subseteq -U), & \Leftrightarrow \\ & \neg\exists\alpha(a \,\epsilon\, \alpha \,\&\, \alpha \subseteq -U) \rightarrow \neg(a \ltimes -U), & \Leftrightarrow \\ & \forall\alpha(a \,\epsilon\, \alpha \rightarrow \neg(\alpha \subseteq -U)) \rightarrow \neg a \ltimes U, & \Leftrightarrow \; (Prop.1.29) \\ \text{(Spatiality)} & \forall\alpha(a \,\epsilon\, \alpha \rightarrow \alpha \between U) \rightarrow a \lhd U, & \end{array}$$

for all $a \in S$, $U \subseteq S$.

Suppose now $(S, \lhd_{I,C}, \ltimes_{I,C})$ be a basic topology generated by I, C, and $*$ an operation on S. We would like to generate inductively a basic topology with operation $(S, \lhd^*_{I,C}, \ltimes^*_{I,C}, *)$ such that:

1. $a \lhd_{I,C} U \rightarrow a \lhd^*_{I,C} U$ and $a \ltimes^*_{I,C} U \rightarrow a \ltimes_{I,C} U$;
2. $(S, \lhd^*_{I,C}, \ltimes^*_{I,C}, *)$ satisfies Loc, $Left$, or Con;
3. $\lhd^*_{I,C}$, $\ltimes^*_{I,C}$ are respectively the minimal cover and the maximal positivity satisfying 1. and 2.

The solution is obtained modifying the axiom set accordingly to the properties we want. If we want Loc, we just need to add to I, C the following

$$I_{Loc}(a) \equiv I(a) \times \{(b,c) : a \,\epsilon\, b * c\} : (a \in S),$$
$$C_{Loc}(a, (i, (b,c))) \equiv C_{Loc}(a,i) * b : ((i,(b,c)) \in I_{Loc}(a), a \in S).$$

The new axiom set $I \cup I_{Loc}, C \cup C_{Loc}$ generates the required $(S, \lhd^*_{I,C}, \ltimes^*_{I,C}, *)$ satisfying Loc. Similarly, we can obtain Con by adding

$$I_{Con}(a) \equiv \{*\} : (a \in S),$$
$$C_{Con}(a, *) \equiv a * a : (a \in S).$$

and $Left$ by adding

$$I_{Left}(a) \equiv \{b : \exists c(a \,\epsilon\, b * c)\} : (a \in S),$$
$$C_{Left}(a,b) \equiv \{b\} : ((i,b) \in I_{Left}(a), a \in S).$$

to the axiom set. The proofs can be easily carried out by inductive/coinductive arguments [CMS13]. Summing up, by adding the suitable generating axioms, we obtain the required property. On the other side, if we do not choose a meaningful operation, we will obtain only uninteresting basic topologies.

Between generated basic topologies, the notion of continuous relation can be simplified [MLSng]:

Proposition 1.5. *Let* $(S, \lhd_{I,C}, \ltimes_{I,C})$ *and* $(T, \lhd_{J,D}, \ltimes_{J,D})$ *be basic topologies, generated by the axiom-sets* I, C *and* J, D*. Then, a relation* $s : S \to T$ *is continuous if and only if it* respects the axioms*, that is, the following holds:*

$$s^- b \lhd_S s^- D(b, j) \quad \text{for all } b, \in T, j \ \epsilon \ J(b). \tag{1.30}$$

In particular, no condition is needed on the positivities. This reflects the fact that these are not independently defined, but determined by the cover.

1.4.1 Finitary Basic Topologies

A particular case of generated topology is given by finitary basic topologies [CS08].

Definition 1.13. A basic topology $(S, \lhd, \ltimes)$ is called *finitary* if and only if there is an axiom set I, C such that:

1. $(S, \lhd, \ltimes)$ is generated by I, C, i.e. $a \lhd U \leftrightarrow a \lhd_{I,C} U$ and $a \ltimes U \leftrightarrow a \ltimes_{I,C} U$;
2. I, C is a *finitary* axiom set, i.e. $C(a, i) \subseteq S$ is finite for all $a \in S$ and $i \in I(a)$.

An operation $*$ on S is called *finitary* if $a * b$ is finite for all $a, b \in S$.

Remark 7. Notice that, if $*$ is a finite operation for a finitary basic topology $(S, \lhd, \ltimes)$, the generation of the basic topologies with operation satisfying Loc, Con are $Left$, as treated in the last section, is performed within the finitary formal topologies. More precisely, if I, C is a finitary axiom set, and $*$ is a finitary operation, then also I_{Loc}, C_{Loc}, I_{Con}, C_{Con} and I_{Left}, C_{Left} are finitary.

Except for trivial cases, the operation $\downarrow$ is not finitary, even if a topologically equivalent and finitary operation $*$ might exist[8]. This is, in author's opinion, one of the main justification for the use of a primary notion of convergence operation. Most of the basic topologies appearing later on in the thesis are finitary and have indeed a finitary operation.

If $(S, \lhd, \ltimes)$ is a finitary basic topology, then $\lhd$ is a *finitary cover*, that is, the following holds

$$a \lhd U \leftrightarrow (\exists U_0 \subseteq_\omega U)(a \lhd U_0), \tag{1.31}$$

[8] In the same fashion, if $(S, \lhd, \ltimes)$ is finitary, then not every axiom set generating it is finitary.

for all $a \in S$ and $U \subseteq S$. Vice versa, if $\lhd$ is a *finitary cover*, we can find a finitary axiom set I, C such that $a \lhd U \leftrightarrow a \lhd_{I,C} U$:

$$I_{Fin}(a) \equiv \{U_0 : U_0 \text{ finite } \ \& \ a \lhd U_0\} : (a \in S),$$
$$C_{Fin}(a, U_0) \equiv U_0 : (U_0 \in I_{fin}(a), a \in S).$$

A detailed proof can be found in [CS08].

We present now an alternative and equivalent way of defining finitary formal covers, more natural in algebraic contexts. Instead of a set of axiom, we start from a family R of relations

$$R_i : S^{\delta(i)} \to S$$

where i is ranging over a set I, $\delta : I \to \mathbb{N}$ (where we set by convention $S^0 = \{*\}$). In particular, $(a_1, \ldots, a_n)$ will denote $*$ when $n = 0$. One can generate inductively a minimal basic cover $\lhd$ on S such that

$$\frac{(a_1, \ldots, a_{\delta(i)}) R_i b}{b \lhd_R \{a_1, \ldots, a_{\delta(i)}\}} \tag{1.32}$$

for all $i \in I$, $a_1, \ldots, a_{\delta(i)}, b \in S$. This is defined uniquely by the following rules and induction axiom:

$$\frac{b \ \epsilon \ U}{b \lhd_R U} \ Reflexivity, \quad \frac{a_1 \lhd U \quad \cdots \quad a_{\delta(i)} \lhd_R U \quad (a_1, \ldots, a_{\delta(i)}) R_i b}{b \lhd_R U} \ Gen_{R_i}, \tag{1.33}$$

$$\frac{a \lhd_R U \quad U \subseteq P \qquad \begin{matrix} [(a_1, \ldots, a_{\delta(i)}) R_i b, \{a_1, \ldots, a_{\delta(i)}\} \subseteq P] \\ \vdots \\ b \ \epsilon \ P \end{matrix}}{a \ \epsilon \ P} \ Induction \quad , \tag{1.34}$$

where $i \in I$, $U, P \subseteq S$ and $a_1, \ldots, a_{\delta(i)}, b \in S$.

Remark 8. For $\delta(i) = 0$, the relation R_i can be identified with the subset $V_i \equiv R_i *$. The corresponding axiom Gen_{R_i} says then $V_i \lhd_R U$ for all $U \subseteq S$.

We can in fact provide a suitable axiom-set I, C, such that $\lhd_R$ coincide with $\lhd_{I,C}$. Let $\Sigma \equiv \bigcup_{n \in \mathbb{N}} S^n$ and we denote with $\sigma : \Sigma \to S$ the natural relation defined by

$$(a_1, \ldots, a_n) \sigma b \equiv b = a_1 \vee \cdots \vee b = a_n$$

for all $b \in S$. In plain terms, $\sigma(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}$ and in particular $\sigma * = \emptyset$. We define then I, C as follows:

$$I_R(a) \equiv \bigcup_{i \in I} R_i^- a : (a \in S),$$
$$C_R(a, i) \equiv \sigma i : (i \in I(a), a \in S).$$

One explicitly checks that with this choice of I, C, the generation axioms coincide with (1.33) and (1.34). In particular, the corresponding positivity relation $\ltimes_R$ is the unique relation between elements and subsets satisfying

$$\frac{b \ltimes_R V}{b \;\epsilon\; V}\; Coreflexivity, \qquad \frac{(a_1, \ldots, a_{\delta(i)}) R_i b \quad b \ltimes_R V}{\{a_1, \ldots, a_{\delta(i)}\} \ltimes_R V}\; Cogen_{R_i},$$

$$\frac{a \;\epsilon\; Q \quad Q \subseteq V \qquad \begin{array}{c}[b \;\epsilon\; Q, (a_1, \ldots, a_{\delta(i)}) R_i b] \\ \vdots \\ \{a_1, \ldots, a_{\delta(i)}\} \between Q\end{array}}{a \ltimes_R V}\; Coinduction,$$

where $i \in I$, $U, Q \subseteq S$ and $a_1, \ldots, a_{\delta(i)}, b \in S$.

A basic topology $(S, \lhd_R, \ltimes_R)$ obtained as above from a family of relations $R \equiv \{R_i\}_{i \in I}$ will be called *generated by the relations* R_i. The axiom set I, C obtained as before from the relations R_i will be called *canonical*.

A basic topology generated by relations is finitary, because the canonical axiom set is finitary. Vice versa, let $(S, \lhd, \ltimes)$ be a finitary cover on S generated by the axiom set (I, C). We define, for every $n \in \mathbb{N}$, the relations $R_n : S^n \to S$ as follows

$$(a_1, \ldots, a_n) R_n b \equiv \exists i (i \in I(b) \;\&\; \sigma(a_1, \ldots, a_n) = C(b, i))$$

for all $(a_1, \ldots, a_n) \in S^n$ and $b \in S$. It is straight to verify that the cover generated by the relations $\{R_n\}_{n \in \mathbb{N}}$ coincide with the one generated by I, C. In particular, the rule

$$\frac{i \in I(b) \quad C(b, i) \lhd U}{b \lhd U}$$

is contained in Gen_{R_n}, with $n = |C(b, i)|$, which is finite by assumption. We collect the previous considerations, extended directly to the positivity relation $\ltimes$, in the following proposition:

Proposition 1.6. *The notion of finitary basic topology and of basic topology generated by relations are equivalent. More precisely, we have:*

1. *For every basic topology $(S, \lhd_R, \ltimes_R)$ generated by the family of relations $R \equiv \{R_i\}_{i \in I}$, there exists a finitary axiom set (I_R, C_R) generating $\lhd_R$ and $\ltimes_R$.*
2. *For every basic topology $(S, \lhd_{I,C}, \ltimes_{I,C})$ generated by a finitary axiom set (I, C), there exists a family of relations $R_{I,C} \equiv \{R_n : S^n \to S\}_{n \in \mathbb{N}}$ generating $\lhd_{I,C}$ and $\ltimes_{I,C}$.*

As a consequence, if $(S, \lhd, \ltimes)$ is generated by relations, then $a \lhd U$ if and only if there exists $U_0 \subseteq U$ finite such that $U \subseteq S$.

If $(S, \lhd_R, \ltimes_R)$ is generated by $R \equiv \{R_i\}_{i \in I}$, we can explicitly characterize the formal open subsets U and the formal closed subsets V as those satisfying, for all $i \in I$:

$$\frac{a_1 \ \epsilon \ U, \ldots, a_{\delta(i)} \ \epsilon \ U \quad (a_1, \ldots, a_{\delta(i)}) R_i b}{b \ \epsilon \ U}, \quad \frac{b \ \epsilon \ V \quad (a_1, \ldots, a_{\delta(i)}) R_i b}{a_1 \ \epsilon \ P \vee \cdots \vee a_{\delta(i)} \ \epsilon \ V}, \quad (1.35)$$

for all $a \in S$ and $i \in I(a)$.

Let $(S, \lhd_S, \ltimes_S)$ and $(T, \lhd_T, \ltimes_T)$ be finitary basic topologies, where the latter is generated by the relations $\{R_i : T^{\delta(i)} \to T\}_{i \in I}$ and (J, D) the associated canonical finitary axiom set. Then, a relation $r : S \to T$ is continuous if and only if it respects the axioms (1.30). In terms of relations, this amounts to

$$s^- b \lhd_S s^- \{a_1, \ldots, a_{\delta(i)}\} \quad (1.36)$$

for all $b \in T$ such that $(a_1, \ldots, a_{\delta(i)}) R_i b$ and $i \in I$.

Definition 1.14. Let $(S, \lhd_S, \ltimes_S)$ and $(T, \lhd_T, \ltimes_T)$ be basic topologies, generated, respectively, by the relations $\{H_i\}_{i \in I}$ and $\{K_j\}_{j \in J}$. A function $f : T \to S$ *respects the relations* $\{K_j\}_{j \in J}$ and $\{H_i\}_{i \in I}$ if

$$(a_1, \ldots, a_{\delta(i)}) K_i b \to (f(a_1), \ldots, f(a_{\delta(i)})) H_i f(b), \quad (1.37)$$

for all $a_1, \ldots, a_{\delta(i)}, b \in T$ and $i \in I$.

Proposition 1.7. *Let $(S, \lhd_S, \ltimes_S)$ and $(T, \lhd_T, \ltimes_T)$ be basic topologies generated by the relations $\{H_i\}_{i \in I}$ and $\{K_j\}_{j \in J}$ and $f : T \to S$ a function which respects the corresponding relations. The the inverse relation $f^- : S \to T$ is continuous between the corresponding topologies.*

Proof. We show directly the condition (1.36), that is, since $(f^-)^- = f$,

$$f(b) \lhd_S \{f(a_1), \ldots, f(a_{\delta(i)})\}$$

for all $b \in T$ such that $(a_1, \ldots, a_{\delta(i)}) K_i b$ and $i \in I$. Thanks to (1.37), if $(a_1, \ldots, a_{\delta(i)}) K_i b$ then $(f(a_1), \ldots, f(a_{\delta(i)})) H_i f(b)$. Because of (1.32), this implies $f(b) \lhd_S \{f(a_1), \ldots, f(a_{\delta(i)})\}$. □

Remark 9. The request of having a function, instead of a general relation, is, in the last proposition, rather restrictive. However, it is employed in this form in the applications.

1.4.2 Further Properties of the Convergence Operation

Let $(S, \lhd, \ltimes, *)$ be a basic topology with operation. In Section 1.3, we have introduced the properties *Loc* (or *Stab*), *Left*, *Con* and *Right*, their reciprocal relations and their effects on the basic topology structure. More precisely, *Con* follows from *Right*, and the set of properties *Loc*, *Left* and *Con* is equivalent to *Left* and *Right* (and both means that S is a formal topology).

The operation is what distinguish a basic topology from a simple revisitation of an inductive/coinductive generation process. We collect here some further observations and properties, that we will encounter naturally in the concrete instances. These properties, as before, relates the basic cover $\lhd$ to the operation $*$.

Since the positivity relation $\ltimes$ plays here a secondary role (and this is even more the case for generated topologies, where the positivity is completely determined by the cover) we will drop $\ltimes$ in the notation. In particular, a subset will be called formal closed if it splits the cover. The same convention will be used further in this thesis.

At first, if the cover $\lhd$ is finitary, then *Loc* (or *Stab*) and *Right* are equivalent, respectively, to their finite versions

$$\frac{a \lhd U_0 \quad b \lhd V_0}{a * b \lhd U_0 * V_0}\ Loc_{fin} \qquad\qquad \frac{a \lhd U_0 \quad a \lhd V_0}{a \lhd U_0 * V_0}\ Right_{fin} \tag{1.38}$$

for all $a \, \epsilon \, S$ and $U_0, V_0 \subseteq S$ finite. In fact, *Loc* implies trivially Loc_{fin}; vice versa, if $a \lhd U$ and $b \lhd V$, then we can find $U_0 \subseteq U$ and $V_0 \subseteq V$ such that $a \lhd U_0$ and $b \lhd V_0$, because $\lhd$ is fnitary. If also Loc_{fin} holds, then $a * b \lhd U_0 * V_0$. By transitivity, we finally get $a * b \lhd U * V$. A similiar argument proves the equivalence between *Right* and $Right_{fin}$.

We say that the basic topology with operation $(S, \lhd, *)$ satisfies:

1. *Cut* if, for all $a, b \, \epsilon \, S$ and $U \subseteq S$,

$$\frac{a \lhd U \cup \{b\} \quad a * b \lhd U}{a \lhd U}\ Cut \tag{1.39}$$

2. *Weak-Right*, or WR if, for all $b, b' \, \epsilon \, S$ and $U \subseteq S$,

$$\frac{a \lhd U \cup \{b\} \quad a \lhd U \cup \{b'\}}{a \lhd U \cup b * b'}\ WR. \tag{1.40}$$

Both these two properties can be proved to be equivalent to their finite versions, if the cover $\lhd$ is finitary:

$$\frac{a \lhd U_0 \cup \{b\} \quad a * b \lhd U_0}{a \lhd U_0}\ Cut_{fin} \quad \frac{a \lhd U_0 \cup \{b\} \quad a \lhd U_0 \cup \{b'\}}{a \lhd U_0 \cup b * b'}\ WR. \tag{1.41}$$

for all $U_0 \subseteq S$ finite. The property WR implies Cut:

$$\cfrac{\cfrac{\cfrac{}{a \lhd U \cup \{b\}}\ hyp \quad \cfrac{a \, \epsilon \, U \cup \{a\}}{a \lhd U \cup \{a\}}\ Refl.}{a \lhd U \cup a * b}\ WR \quad \cfrac{}{a * b \cup U}\ hyp}{a \lhd U}\ Trans. \tag{1.42}$$

for all $U \subseteq S$ and $a, b \, \epsilon \, S$.

Proposition 1.8. *A formal topology with operation* $(S, \lhd, *)$ *satisfies* WR.

Proof. Suppose $a \lhd U \cup \{b\}$ and $a \lhd U \cup \{b'\}$. Since *Right* holds, we have

$$a \lhd U * U \cup \{b'\} * U \cup U * \{b\} \cup b * b'.$$

Since *Left* holds, the subsets $U * U$, $\{b'\} * U$ and $U * \{b\}$ are covered by U. By transitivity, we have then $a \lhd U \cup b * b'$. □

In particular, a formal topology satisfies *Cut*.

We can prove that for all finitary basic topologies $(A, \lhd, *)$, *Weak-Right* implies *Right*. The proof is elementary, but we need to introduce an intermediate property: $(S, \lhd, *)$ satisfies *extended*-Weak-Right eWR if

$$\frac{a \lhd U_0 \cup V_0 \quad a \lhd U_0 \cup V_0'}{a \lhd U_0 \cup V_0 * V_0'}\ eWR$$

for all $a \,\epsilon\, S$, and $U_0, V_0, V_0' \subseteq S$ finite. It is clear that eWR implies $Right_{fin}$, for $U_0 \equiv \emptyset$, and therefore *Right*. We prove now that WR implies eWR.

Proposition 1.9. *Let $(S, \lhd, *)$ be a finitary basic topology. If $(S, \lhd, *)$ satisfies WR then it satisfies eWR.*

Proof. We prove it by induction on the formal cardinality of U_0, V_0, V_0'. Let's write $eWR(n, m, m')$ to say that eWR holds for the subsets U_0, V_0, V_0' of formal cardinalities n, m, m' respectively. We prove $P(m, m') \equiv \forall n(eWR(n, m, m'))$ by induction on $p = m + m'$. The case $p \leqslant 2$ is either obvious or consequence of WR. Suppose now P to hold up to $p - 1$ with $p > 2$. Suppose moreover V_0 with $m > 1$ (take V_0' instead) and let $V_0 = V_0'' \cup \{v\}$. We have the following derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{a \lhd U_0 \cup V_0}{a \lhd U_0 \cup \{v\} \cup V_0''} \quad \cfrac{a \lhd U_0 \cup V_0'}{a \lhd U_0 \cup \{v\} \cup V_0'}}{a \lhd U_0 \cup V_0'' * V_0' \cup \{v\}}\ P(m-1, m') \quad \cfrac{a \lhd U_0 \cup V_0'}{a \lhd U_0 \cup V_0'' * V_0' \cup V_0'}}{a \lhd U_0 \cup V_0'' * V_0' \cup \{v\} * V_0'}\ P(1, m')}{a \lhd U_0 \cup V_0 * V_0'}$$

that proves $P(m, m')$ from $P(m - 1, m')$ and $P(1, m')$, which both hold by inductive hypothesis. □

We have proved the following:

Corollary 1.10. *Let $(S, \lhd, *)$ be a finitary formal topology. If $(S, \lhd, *)$ satisfies WR then it satisfies Right.*

At the end of the next section, we will point out a semantic version of Corollary 1.10, which relies on non-constructive methods. Its simplicity, compared to the more tricky Proposition 1.9, is however a paradigmatic and probably the most simple example of the power of point-theoretic methods.

1.5 Entailment Relations and Completeness Results

Scott introduced the notion of entailment relation in [Sco74], as a generalization of Tarski's notion of *consequence relation* [TC83]. Also Sambin's notion of formal topology can be seen as a development of Tarski's ideas. In fact, a consequence relation does not differ in practice and intuition from a finitary basic cover: it is a monotone operator **Cn** between sets of sentences. Starting from a set of "sentences" S, **Cn** induces a finitary basic cover $\mathscr{A}_{\mathbf{Cn}}$ (or $\lhd_{\mathbf{Cn}}$) as follows:

$$a \ \epsilon \ \mathscr{A}_{\mathbf{Cn}}(U) \equiv (\exists U_0 \subseteq_\omega U)(a \ \epsilon \ \mathbf{Cn}(U_0))$$

for all $a \in S$ and $U \subseteq S$. Vice versa, given a finitary basic cover $\mathscr{A}$, we can define a consequence relation as

$$a \ \epsilon \ \mathbf{Cn}_{\mathscr{A}}(U_0) \equiv a \ \epsilon \ \mathscr{A}(U_0)$$

for all finite $U_0 \subseteq S$. These correspondences are clearly inverse of each other. In Gentzen style, one can denote a consequence relation as:

$$a_1, \ldots, a_n \ \vdash \ b \quad \text{iff} \quad b \ \epsilon \ \mathbf{Cn}(\{a_1, \ldots, a_n\}).$$

In analogy with Gentzen, who introduced multiple entries on the right hand side, Scott formalized a notion of consequence relation

$$a_1, \ldots, a_n \ \vdash \ b_1, \ldots, b_m$$

with $a_1, \ldots, a_n, b_1, \ldots, b_m \ \epsilon \ S$, "meaning roughly, that the *conjunction* of the sentences a_i has the *disjunction* of the b_j as a logical consequence" [Sco74].

More precisely, an *entailment relation* on an arbitrary set S is a relation $\vdash$ between finite subsets of S, satisfying

$$\frac{U_0 \between V_0}{U_0 \vdash V_0}\ R_\vdash, \qquad \frac{U_0 \vdash V_0}{U_0, U_0' \vdash V_0, V_0'}\ M_\vdash, \qquad \frac{U_0 \vdash V_0, a \quad U_0, a \vdash V_0}{U_0 \vdash V_0}\ T_\vdash, \tag{1.43}$$

for all finite subsets U_0, U_0', V_0, V_0' and $a \in S$ (where U_0, a is a shorthand for $U_0 \cup \{a\}$). The first thing to notice is that the axiom T (where T stands for transitivity) is a form of cut rule. As Scott states, "in many formalizations a great deal of effort is expended to eliminate cut as a primitive rule; but it has to be proved as a derived rule".

As just done for consequence relations, given an entailment relation $\vdash$ on S, we can define a cover $\lhd_\vdash$ on S as

$$a \lhd_\vdash V \equiv (\exists V_0 \subseteq_\omega V)(V_0 \vdash a).$$

Vice versa, to any finitary cover $\lhd$ on S, we can associate a minimal $\vdash_{min}$ and a maximal $\vdash_{max}$ entailment relation such that $\lhd_{\vdash_{min}} = \lhd = \lhd_{\vdash_{max}}$ [Sco74]. They are defined as follows:

$$U_0 \vdash_{min} V_0 \equiv (\exists a \ \epsilon \ V_0)(a \lhd U_0),$$

$$U_0 \vdash_{max} V_0 \equiv (\forall U'_0 \supseteq_\omega U_0)(\forall c)((\forall b \ \epsilon \ V_0)(c \lhd U'_0 \cup \{b\}) \to c \lhd U'_0).$$

If $\lhd_\vdash = \lhd$, then $\vdash_{min} \subseteq \vdash \subseteq \vdash_{max}$. A proof of this is given in [Sco74, Thm 1.2].

The operation $*$ on a basic cover $\lhd$ plays similarly the role of a disjunction. The following proposition establish the link between finitary basic topologies with finitary operation and entailment relations and it is related to a result by Coquand and Lombardi [CL02], the formal Hilbert Nullstellensatz.

Proposition 1.11. *Let $(S, \lhd, *)$ be a finitary basic topology with finitary operation. We define a relation $\vdash_*$ between finite subsets as follows:*

$$U_0 \ \vdash_* \ V_0 \quad \textit{iff} \quad \prod V_0 \lhd U_0, \tag{1.44}$$

where $\prod V_0 \equiv \prod_{b \ \epsilon \ V_0} b$. Then $\vdash_*$ is an entailment relation if and only $(S, \lhd, *)$ satisfies $Left$ and Cut. Moreover, if $(S, \lhd, *)$ is a formal topology, we have $\vdash_* \equiv \vdash_{max}$.

Proof. Suppose $(S, \lhd, *)$ to satisfy $Left$ and Cut. We prove that $\vdash_*$ satisfies (R), (M) and (T):

(R) If $U_0 \between V_0$ hold and is witnessed by b, we have by reflexivity $b \lhd U_0$, and, by $Left$, $\prod V_0 \prod V_0 \lhd U_0$, that is, $U_0 \ \vdash_* \ V_0$.

(M) Let now $U_0 \vdash_* V_0$, that is, $\prod V_0 \lhd U_0$, and U'_0, V'_0 finite. We have $U_0 \lhd U_0 \cup U'_0$ by reflexivity and $\prod(V_0 \cup V'_0) \lhd \prod V_0$ by $Left$. Then by transitivity $\prod(V_0 \cup V'_0) \lhd U_0 \cup U'_0$, that is, $U_0 \cup U'_0 \vdash_* V_0 \cup V'_0$.

(T) Let $U_0 \vdash_* V_0, a$ and $U_0, a \vdash_* V_0$, that is, $a * \prod V_0 \lhd U_0$ and $\prod V_0 \lhd U_0 \cup \{a\}$. By Cut we get immediately $\prod V_0 \lhd U_0$.

Suppose now $\vdash_*$ to be an entailment relation. We prove that $\lhd$ satisfies $Left$ and Cut:

$(Left)$ If $U_0 \ \vdash_* \ a$ holds, then by (M) $U_0 \ \vdash_* \ a, b$ for all $b \in S$. Explicitly, if $a \lhd U_0$ than $a * b \lhd U_0$.

(Cut) If $U_0 \ \vdash_* \ a, b$ and $U_0, b \vdash_* \ a$ then by (T) we get $U_0 \ \vdash_* \ a$. Explicitly, if $a * b \lhd U_0$ and $a \lhd U_0 \cup \{b\}$ then $a \lhd U_0$.

Finally, if $(S, \lhd, *)$ is a formal topology, then

$$(\forall b \ \epsilon \ V_0)(c \lhd U'_0 \cup \{b\}) \quad \text{iff} \quad c \lhd U'_0 \cup \prod V_0.$$

The right hand side follows from the left hand side with WR (that holds by Prop. 1.8); vice versa, $Left$ implies $(\forall b' \ \epsilon \ V_0)(\prod V_0 \lhd b')$ and then $(\forall b \ \epsilon \ V_0)(c \lhd U'_0 \cup \{b\})$ by transitivity. Hence, we get

$$\underbrace{(\forall U'_0 \supseteq_\omega U_0)(\forall c)(c \lhd U'_0 \cup \prod V_0 \to c \lhd U'_0)}_{U_0 \vdash_{max} V_0} \quad \text{iff} \quad \underbrace{\prod V_0 \lhd U_0}_{U_0 \vdash_* V_0}.$$

since the left hand side hold for $c \ \epsilon \ \prod V_0$ and U_0, and that suffices by transitivity. □

In this sense, to a basic topology with operation satisfying $Left$ and Cut corresponds an entailment relation. There are nevertheless entailment relations not definable naively in this way, as, for instance, the space of valuations [CP01]. On the other hand, we will see meaningful examples of basic cover with operation in which WR holds, but not $Left$, and which therefore are not directly definable in form of an entailment relation.

A formal topology proves $Left$ and Cut, and hence it generates an entailment relation. This means in particular that the properties of formal topology give a canonical proof of cut for the corresponding entailment relation, and here lies one of the main differences between the two approaches.

Entailment relations fulfill, classically, a completeness result, in the form of a Lindenbaum Theorem. One says that an entailment relation $\Vdash$ on S is *complete* if, for all $a \,\epsilon\, S$, either $a \Vdash$ or $\Vdash a$. It is called *consistent* if $\Vdash a$ does not hold for all $a \,\epsilon\, S$. One then proves that any entailment relation $\Vdash$ is the intersection of the complete and consistent relations above it [Sco74]. As Scott points out, to prove this fact it is not needed to introduce anything at all about sentential connectives, being all the required sentential calculus "absorbed into the properties" (R), (M) and (T).

We get to a similar result for basic formal topologies $(S, \lhd, *)$ satisfying first $Left$ and Cut, and then only WR. In the proofs we will use, instead of Zorn's Lemma, Raoult's principle of Open Induction, that will be defined and treated more accurately in Section 5.1.

Basic Topologies satisfying $Left$ and Cut. We call a finitary basic topology with operation $(S, \lhd_f, *)$ satisfying $Left$ and Cut *complete* if, for all $a \,\epsilon\, S$, either $a \lhd_f \emptyset$ or $S \lhd_f a$, and *consistent* if $\neg(a \lhd_f \emptyset)$ for some $a \,\epsilon\, S$. Let us also denote with $\mathcal{C}(S)$ the collection of complete and consistent finitary formal topologies on S.

Proposition 1.12 (OI)**.** *Let $(S, \lhd, *)$ be a finitary basic topology with finitary operation satisfying $Left$ and Cut. Then*

$$a \lhd U \leftrightarrow (\forall \lhd_f \in_c \mathcal{C}(S))(\lhd \subseteq \lhd_f \to a \lhd_f U)$$

holds for all $a \,\epsilon\, S$ and $U \subseteq S$.

Proof. The implication $(\leftarrow)$ is the non-trivial one. Suppose $a \lhd_f U$ for all $\lhd_f \supseteq \lhd$ in $\mathcal{C}(S)$. Consider the partial order $\mathcal{X}$ of all the consistent and finitary $\lhd' \supseteq \lhd$, and the subcollection $\mathcal{S}$ of those such that $a \lhd' U$. This subcollection is open because the covers in $\mathcal{X}$ are finitary. It is also progressive: suppose in fact $a \lhd'' U$ for all $\lhd'' \supsetneq \lhd'$; then two cases are possible: either $\lhd' \in_c \mathcal{C}(S)$, and then $a \lhd' U$ by hypothesis, or there exists $b \,\epsilon\, S$ such that neither $b \lhd' \emptyset$ nor $S \lhd' b$. Then we can define two new basic covers

$$a \lhd'^b V \equiv a \lhd' V \cup \{b\},$$
$$a \lhd'_b V \equiv a * b \lhd' V,$$

for all $a, b \in S$ and $V \subseteq S$. One checks easily that they are well-defined and finitary (for $\lhd'_b$, one needs the operation $*$ to be finitary). We have then $\lhd'^b \supsetneq \lhd'$ and $\lhd'_b \supsetneq \lhd'$ and therefore $a \lhd'^b U$ and $a \lhd'_b U$. This means, $a \lhd' U \cup \{b\}$ and $a * b \lhd' U$; by Cut this implies $a \lhd' U$. By Open Induction, we deduce that $\mathcal{S} = \mathcal{X}$ and in particular $a \lhd U$. □

Note that every $\lhd \in_c \mathcal{C}(S)$ is completely determined, with classical logic, by the subset

$$b \ \epsilon \ \alpha_\lhd \equiv S \lhd b;$$

more precisely, it holds

$$a \lhd U \leftrightarrow (a \ \epsilon \ \alpha_\lhd \rightarrow U \between \alpha_\lhd).$$

for all $a \ \epsilon \ S$ and $U \subseteq S$. In fact, if $a \lhd U$ and $S \lhd a$, then by transitivity $S \lhd U$; for all $u \ \epsilon \ U$, we have either $S \lhd u$ or $u \lhd \emptyset$. But if $u \lhd \emptyset$ for all $u \ \epsilon \ U$, then $S \lhd U \lhd \emptyset$, against the consistency. Hence $U \between \alpha_\lhd$.

Vice versa, we have $a \lhd \emptyset$ or $S \lhd a$; in the first case, $a \lhd U$ holds trivially, otherwise, from $S \between \alpha_\lhd$ we deduce $a \ \epsilon \ \alpha_\lhd$ and therefore $U \between \alpha_\lhd$. This means, $S \lhd u$ for some $u \ \epsilon \ U$ and in particular $a \lhd u \lhd U$.

The implication $(\rightarrow)$ shows that $\alpha_\lhd$ splits the cover $\lhd$, that is, it is formal closed with respect to the generated positivity. In particular, α split the cover $\lhd'$ for all $\lhd' \subseteq \lhd$. Moreover, α is inhabited, because by consistency we can find $a \ \epsilon \ S$ with $\neg(a \lhd \emptyset)$, and therefore $S \lhd a$.

Given a finitary basic topology with operation $(S, \lhd, *)$ satisfying *Left* and *Cut*, and an inhabited formal closed $\alpha \subseteq S$ we can define a complete and consistent cover $\lhd_\alpha \supseteq \lhd$ as follows

$$a \lhd_\alpha U \equiv a \ \epsilon \ \alpha \rightarrow U \between \alpha.$$

Such cover satisfies $Left$, as consequence of the fact that α is formal closed. In order $\lhd_\alpha$ to satisfy Cut, the following condition must hold:

$$\frac{a \ \epsilon \ \alpha \quad a \ \epsilon \ \alpha \rightarrow U \between \alpha \vee b \ \epsilon \ \alpha \quad a * b \between \alpha \rightarrow U \between \alpha}{U \between \alpha} \tag{1.45}$$

for all $a, b \ \epsilon \ S$ and $U \subseteq S$. Then Proposition 1.12 offers us the following equivalence:

$$a \lhd U \leftrightarrow \forall \alpha (\underbrace{a \ \epsilon \ \alpha \rightarrow U \between \alpha}_{a \lhd_\alpha U}) \tag{1.46}$$

for all $\lhd$ finitary basic cover with operation satisfying $Left$ and Cut, where α ranges over all the inhabited formal closed subsets satisfying (1.45).

Notice that, if α is convergent, and hence a formal point, then (1.45) is satisfied: if $a \ \epsilon \ \alpha$ then $U \between \alpha$ or $b \ \epsilon \ \alpha$; in this last case, we have also $a * b \between \alpha$ and then again $U \between \alpha$. Summarizing that, if α is a formal point, then $\lhd_\alpha$ satisfies $Left$ and Cut. Moreover, $\lhd_\alpha$ satisfies also $Right$, and then it is a formal cover: if $a \lhd_\alpha U$, $a \lhd_\alpha V$ and $a \ \epsilon \ \alpha$, then $U \between \alpha$ and $V \between \alpha$ so that $U * V \between \alpha$.

The equivalence (1.46) is nevertheless weaker than spatiality since α varies on more general subsets than formal points. Moreover, if (1.46) holds, then $\lhd$ must be a formal cover because it is the intersection of the formal covers $\lhd_\alpha$ on the right. As we are going to prove in the next section, also the converse holds.

Basic Topologies satisfying WR. Let $(S, \lhd, *)$ be a finitary basic topology satisfying *Weak-Right*. In this setting, we will produce a completeness theorem directly in the form of the equivalence (1.46), where $\alpha \in_c \mathscr{P}t(S)$ ranges over the formal points of S.

Theorem 1.13 (OI+CL). *Every finitary basic topology* $(S, \lhd, \ltimes, *)$ *satisfying* WR *is spatial, that is,*

$$a \lhd U \leftrightarrow (\forall \alpha \in_c \mathscr{P}t(S))(\underbrace{a \,\epsilon\, \alpha \to U \between \alpha}_{a \lhd_\alpha U})$$

holds for all $a \,\epsilon\, S$ *and* $U \subseteq S$.

Proof. The implication $(\leftarrow)$ is the non-trivial one. Suppose $a \lhd_\alpha U$ for all $\forall \alpha \in_c \mathscr{P}t(S)$. Consider the partial order $\mathcal{X}$ of all the proper subsets $V \supseteq U$, and the subcollection $\mathcal{S}$ of those such that $a \lhd V$. This subcollection is open because the cover $\lhd$ is finitary. It is also progressive, in fact, suppose $a \lhd V'$ for all $V' \supsetneq V$; then two cases are possible (CL):

1. for all b, b' such that $b * b' \subseteq V$ we have $b \,\epsilon\, V$ or $b' \,\epsilon\, V$. Then the complement $-V$ is a formal point, and we have by hypothesis $a \lhd_{-V} V$. This means, $a \,\epsilon\, -V \to V \between -V$, that is, $a \,\epsilon\, V$.
2. there exists $b, b' \,\epsilon\, S$ such that $b * b' \subseteq V$, but neither $b \,\epsilon\, V$ nor $b' \,\epsilon\, V$. Then we have by hypothesis $a \lhd V \cup \{b\}$ and $a \lhd V \cup \{b'\}$. By WR, $a \lhd V \cup b * b'$ follows and, since $b * b' \subseteq V$, we have by transitivity $a \lhd V$.

By open induction, we deduce that $\mathcal{S} = \mathcal{X}$ and in particular $a \lhd \mathscr{A}(U)$, and in particular $a \lhd U$. □

We will revise the same proof in Section 5.1. In particular, this proof will look more natural by expressing it in terms of complements of the formal points, that we will call, by analogy with commutative algebra, *prime ideals*. Formal points, instead, turn out to be more direct in the applications of topological nature.

Since, by Remark 6, with classical logic, spatiality and reducibility are equivalent concepts, we have:

Corollary 1.14 (OI+CL). *Every finitary basic topology* $(S, \lhd, \ltimes, *)$ *satisfying* WR *is reducible. In other words:*

$$a \ltimes U \leftrightarrow (\exists \alpha \,\epsilon\, \mathscr{P}t(S))(a \,\epsilon\, \alpha \;\&\; \alpha \subseteq U)$$

for all $a \,\epsilon\, S$ *and* $U \subseteq S$.

Thanks to Proposition 1.8, the following corollary follows:

Corollary 1.15 (OI+CL). *Every finitary formal topology* $(S, \lhd, \ltimes, *)$ *is spatial and reducible.*

The latter result will be repeatedly used in this thesis to prove the (classical) equivalence between certain common ideal statements and their concrete counterpart.

As already pointed out, formal topologies correspond to locales. More precisely, finitely generated formal topologies correspond to coherent locales, and spatial formal topologies correspond to spatial locales [Samng, CS08]. Corollary 1.6.1 corresponds then to a well-known fact in locale theory, that is, every coherent locale is spatial.

Reasoning with points by using Theorem 1.13 can lead to short and elegant proof, although non-constructive, and this is the main reason for their widespread application in classical mathematics. As a first example, one can prove directly Corollary 1.10, which states that Weak-Right implies Right for a finitary basic topology. In fact, if $(S, \lhd, \ltimes, *)$ satisfies WR and is finitary, then it is spatial. Therefore, if $a \lhd U$, $a \lhd V$ and α is a formal point such that $a \;\epsilon\; \alpha$, then $\alpha \between U$ and $\alpha \between V$, so that $\alpha \between U{*}V$. In other words, $\forall \alpha (a \;\epsilon\; \alpha \to U{*}V \between \alpha)$. By spatiality, this implies $a \lhd U * V$.

The proof we gave involved the effective but lengthy trick of Proposition 1.9 and we invite the reader to compare the two approaches.

1.6 Quotient and Localisation of a Basic Topology

In this section, we show how to obtain the quotient and the localisation of a generated basic topology with operation on a given subset. These constructions should be thought in analogy with the corresponding and common ones on commutative rings.

In particular, properties like being finitary and being a formal topology are preserved. Even if quotients and localisations could be defined for more general kinds of basic topologies, we restrict here to the generated case, because it fits better to our needs.

Quotients. Let $(S, \lhd, \ltimes)$ be a basic topology generated by the axiom set I, C, and $U \subseteq S$. The *quotient* of S in U is the basic topology $(S, \lhd^U, \ltimes^U)$ generated by the axiom set $I \cup I', C \cup C'$, where

$$I'(u) \equiv \{*\} : (u \;\epsilon\; U),$$
$$C'(u, *) \equiv \emptyset : (u \;\epsilon\; U).$$

In other words, we added the conditions for having $U \lhd \emptyset$. Since I', C' is finitary, if the basic topology $(S, \lhd, \ltimes)$ is finitary and I, C is a finitary axiom set, then so is its quotient $(S, \lhd^U, \ltimes^U)$, generated by the finitary axiom set $I \cup I', C \cup C'$. One characterizes the cover $\lhd^U$ as follows:

Proposition 1.16. *Let $(S, \lhd, \ltimes)$ be a basic topology, generated by the axiom set I, C, and $U \subseteq S$. Then*

$$a \lhd^U V \leftrightarrow a \lhd V \cup U \tag{1.47}$$

holds for all $V \subseteq S$ and $a \,\epsilon\, S$.

Proof. We prove it by induction in both directions: $(\rightarrow)$ If $a \,\epsilon\, V$, then $a \lhd V \cup U$; if $C(a,i) \lhd V \cup U$, then $a \lhd V \cup U$; if $u \,\epsilon\, U$, then again $a \lhd V \cup U$. $(\leftarrow)$ Conversely, if $a \,\epsilon\, V \cup U$, then $a \,\epsilon\, V$ or $a \,\epsilon\, U$, and in both cases we directly get $a \lhd^U V$. If $C(a,i) \lhd^U V$, then again $a \lhd^U V$. □

In particular, the quotient in U is equivalent to the quotient in $\mathscr{A}(U)$.

Suppose now $(S, \lhd, \ltimes)$ endowed with an operation $*$. The equivalence (1.47) makes easy to verify that if $(S, \lhd, \ltimes, *)$ satisfies $Left$ (resp. Cut, Con, WR) then also its quotient $(S, \lhd^U, \ltimes^U, *)$ does, for all $U \subseteq S$. If moreover $(S, \lhd, \ltimes, *)$ satisfies both $Left$ and Loc, then also its quotient satisfies $Left$ and Loc: if $a \lhd^U V$ and $b \lhd^U V'$, then $a \lhd V \cup U$ and $a \lhd V' \cup U$. By Loc, we get

$$a * b \lhd V * V' \cup U * U \cup U * V \cup U * V',$$

and by $Left$, $U*U \cup U*V \cup U*V' \lhd U$ holds; finally, by transitivity, $a*b \lhd V*V' \cup U$, that is, $a * b \lhd^U V * V'$. As a consequence, the quotient of a formal topology in a subset U is a formal topology.

The formal open subsets of $(S, \lhd^U, \ltimes^U)$ are the formal open subsets of $(S, \lhd, \ltimes)$ containing U. Dually, the formal closed subsets of $(S, \lhd^U, \ltimes^U)$ are the formal closed subsets of $(S, \lhd^U, \ltimes^U)$ contained in $-U$. In particular, denoted by $\mathscr{P}t(S^U)$ the collection of points of the quotient topology, $\alpha \,\epsilon\, \mathscr{P}t(S^U)$ if $\alpha \,\epsilon\, \mathscr{P}t(S)$ and $\alpha \subseteq -U$, or, in other terms, $\mathscr{P}t(S^U) \equiv Rest(-U)$. Hence, the quotient describes a closed subspace.

Notice that the identity relation from $(S, \lhd^V, \ltimes^V, *)$ to $(S, \lhd, \ltimes, *)$ is continuous and convergent. In fact, $\lhd^U$ is obtained from $\lhd$ by adding new axioms, so that $a \lhd V$ implies $a \lhd^U V$.

We close this section with the following proposition, which shows that $Right$ implies[9] WR, under suitable conditions :

Proposition 1.17. *Let $\mathcal{S}$ be a family of finitary basic topologies with operation satisfying Right and closed by quotients over finite subsets. Then $Weak-Right$ holds, for all the members of the family.*

Proof. Let $(S, \lhd, \ltimes, *)$ in $\mathcal{S}$, and $a \lhd U_0 \cup \{b\}$ and $a \lhd U_0 \cup \{b'\}$ for $U_0 \subseteq_\omega S$ and $a, b, b' \in S$. Then $a \lhd^{U_0} b$ and $a \lhd^{U_0} b'$, so that $a \lhd^{U_0} b * b'$ because WR holds for the quotient of S in U_0. It follows $a \lhd U_0 \cup b * b'$. □

Localisations. Dually to quotients, we introduce the notion of localisation of a basic topology with operation. The presentation is here more general then

[9] It is therefore equivalent to it, after Prop. 1.10.

what required in the thesis, where the formal topologies involved always satisfy $Left$ and Loc. There are nevertheless natural applications, not covered here, which explicitly require this level of generality and on which the definitions that follows are shaped.

Let $(S, \lhd, \ltimes, *)$ be a basic topology with operation generated by the axiom set I, C, and $F \subseteq S$ inhabited and convergent subset. The *localisation* of S in F is the basic topology $(S, \lhd_F, \ltimes_F, *)$ generated by the axiom set $I \cup I', C \cup C'$, where

$$I'(a) \equiv F : (a \in S),$$
$$C'(a, s) \equiv a * s : (i \in I(a), s \;\epsilon\; F, a \in S).$$

In other terms, we add axioms in order to have

$$\frac{s \;\epsilon\; F \quad a * s \lhd_F U}{a \lhd_F U},$$

for all $a \in S$ and $U \subseteq S$. If the operation $*$ is finitary, then so is I', C'. In particular, if the basic topology $(S, \lhd, \ltimes, *)$ is finitary with finitary operation and I, C is a finitary axiom set, then the localisation $(S, \lhd_F, \ltimes_F, *)$ is finitary.

The notion of localisation is easier to handle, under suitable assumptions:

Proposition 1.18. *Let $(S, \lhd, \ltimes, *)$ be a finitary basic topology with finitary operation , generated by the axiom set I, C and $F \subseteq S$ convergent and inhabited. We suppose, moreover, that S satisfies $Left$ and Loc with respect to F, that is*

$$\frac{a \lhd U}{a * s \lhd U}\; Left_F, \qquad \frac{a \lhd U}{a * s \lhd U * s}\; Loc_F, \tag{1.48}$$

hold, for all $s \;\epsilon\; F$, $a \in S$ and $U \subseteq S$. Then

$$a \lhd_F U \leftrightarrow (\exists s \;\epsilon\; F)(a * s \lhd U) \tag{1.49}$$

holds for all $U \subseteq S$ and $a \;\epsilon\; S$.

Proof. We prove $(\rightarrow)$ by induction: If $a \;\epsilon\; U$, then $a \lhd U$ and then $a * s \lhd U$ for any $s \;\epsilon\; F$; if for all $b \;\epsilon\; C(a, i)$ there exists $s_b \;\epsilon\; F$ such that $b * s_b \lhd U$, we can pick

$$s \;\epsilon\; (\prod_{b \;\epsilon\; C(a,i)} s_b) \cap F$$

to obtain $C(a, i) * s \lhd U$, since $Left_F$ holds and $C(a, i)$ is finite; in particular, since $a \lhd C(a, i)$, by Loc_F we have $a * s \lhd C(a, i) * s \lhd U$. Similarly, if for all $b \;\epsilon\; a * s$ we can find $s_b \;\epsilon\; F$ such that $b * s_b \lhd U$, we can first pick $s' \;\epsilon\; (\prod_{b \;\epsilon\; a*s} s_b) \cap F$ (here we need $*$ to be finitary), for which $a * s * s' \lhd U$, and then $s'' \;\epsilon\; (s * s') \cap F$, for which $a * s'' \lhd U$. Conversely, if $a * s \lhd U$ with $s \;\epsilon\; F$, then $a * s \lhd_F U$ and therefore $a \lhd_F U$. ∎

Let $F \subseteq S$ convergent, inhabited and $(S, \lhd, \ltimes, *)$ finitary with finitary operation. Notice that if $Left$ and Loc hold, then also $Left_F$ and Loc_F are automatically satisfied. The equivalence (1.49) make easy to verify that if $(S, \lhd, \ltimes, *)$ satisfies $Left$ and Loc, and has finitary operation, then also its localisation $(S, \lhd_F, \ltimes_F, *)$ does. It is also easy to recognize that Con, if present, is also satisfied. As a consequence, if $(S, \lhd, \ltimes, *)$ is a formal topology, also its localisation $(S, \lhd_F, \ltimes_F, *)$ in F is a formal topology.

Suppose that $(S, \lhd, \ltimes, *)$ satisfies WR, in addition to $Left_S$ and Loc_S, and we show that also $(S, \lhd_F, \ltimes_F, *)$ does; in fact, if $a*s \lhd U \cup \{b\}$ and $a*s' \lhd U \cup \{b'\}$, then, by $Left_F$ and Loc_F, we find $s'' \epsilon\ s * s'$ such that $a * s'' \lhd U \cup \{b\}$ and $a * s'' \lhd U \cup \{b'\}$; finally, by WR, we obtain $a * s'' \lhd U \cup b * b'$, i.e., $a \lhd_F U \cup b * b'$.

The formal closed subsets of $(S, \lhd_F, \ltimes_F)$ are the formal closed subsets U of $(S, \lhd, \ltimes)$ which are F-*convergent*, that is,

$$\frac{s\ \epsilon\ S \quad a\ \epsilon\ U}{a * s \between U}$$

for all $a \in S$, and $U \subseteq S$. In particular, denoted by $\mathscr{P}t(S_F)$ the collection of points of the localisation, $\alpha\ \epsilon\ \mathscr{P}t(S_F)$ if $\alpha\ \epsilon\ \mathscr{P}t(S)$ and α is F-convergent. Notice that, if $F \subseteq \alpha$, then α is already F-convergent.

Remark 10. In practice, the condition (1.48) will be often realized by a unit element $e\ \epsilon\ F$, such that $a * e =_{\lhd} a$ for all $a\ \epsilon\ A$. If such an element exists, and $Left_F$ holds then $S \lhd e$: by $Left_F$ we have in fact $a \lhd a * e \lhd e$ for all $a\ \epsilon\ S$. Moreover, we have $S \lhd_F e * s \lhd_F s$ for all $s\ \epsilon\ F$. This implies that $F \subseteq \alpha$ for all $\alpha\ \epsilon\ \mathscr{P}t(S_F)$ and $s\ \epsilon\ F$. In plain words, the formal points of the localisation coincide with the formal points containing S.

The identity relation is continuous and convergent from $(S, \lhd_F, \ltimes_F, *)$ to $(S, \lhd, \ltimes, *)$. In fact, $\lhd_F$ is obtained from $\lhd$ by adding new axioms, so that $a \lhd U$ implies $a \lhd_F U$.

1.6.1 Krull Dimension of a Formal Topology

Several works of Joyal [Joy76], Español [En82], Coquand, Lombardi & Roy [CLR05] have shown possible a predicative and equivalent definition of Krull dimension for a distributive lattice, and in particular for commutative rings. After recalling the basics, we show how to adapt such notion for a finitary basic topology with finitary operation.

An inhabited closed subset $D \subseteq X$ of a topological space is called *irreducible* if whenever $D \subseteq D' \cup D''$, with $D', D'' \subseteq D$ closed, then $D \subseteq D'$ or $D \subseteq D''$. Equivalently, an inhabited closed subset D is irreducible, when for all open subsets E, E', if $D \between E$ and $D \between E'$ then $D \between E \cap E'$.

Definition 1.15. Let X be a topological space. We say that X has *Krull dimension* smaller or equal to $n \in \mathbb{N} \cup \{-1\}$ if every chain

$$D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n$$

of irreducible closed subsets is not strict, i.e. there exist $0 \leqslant i, j \leqslant n$ such that $D_i = D_j$.

Suppose now that X is the space of points of a formal topology with operation $(S, \lhd, \ltimes, *)$, and $D \equiv \mathsf{Rest}(V)$ an inhabited closed subset, with $V \subseteq S$ formal closed subset. Then D is irreducible if and only if

$$D \between \mathsf{Ext}(U) \ \& \ D \between \mathsf{Ext}(U') \to D \between \mathsf{Ext}(U * U'),$$

for all $U, U' \subseteq S$. Equivalently, if for all $a, b \ \epsilon \ S$,

$$D \between \mathsf{ext}_a \ \& \ D \between \mathsf{ext}_b \to D \between \mathsf{Ext}(a * b),$$

that is,

$$a \ \epsilon \ \Diamond D \ \& \ b \ \epsilon \ \Diamond D \to a * b \between \Diamond D;$$

Since $\Diamond D = V$, this is equivalent to say that V is a formal point. In other words, an inhabited closed subset D is irreducible if and only if $D = \mathsf{Rest}(\alpha)$ for some $\alpha \in_c \mathscr{P}t(S)$.

Moreover, we have $\mathsf{Rest}(\alpha) \subseteq \mathsf{Rest}(\alpha')$ if and only if $\alpha \subseteq \alpha'$. We get then:

Proposition 1.19. *Let $(S, \lhd, \ltimes, *)$ be a formal topology with operation. The space $\mathscr{P}t(S)$ has* Krull dimension *smaller or equal to $n \in \mathbb{N} \cup \{-1\}$ if every chain*

$$\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_n$$

of formal points is not strict.

If the formal topology $(S, \lhd, \ltimes, *)$ is finitary and with finitary operation, then we can characterize the Krull dimension of $\mathscr{P}t(S)$ in a predicative way, without involving quantification over the collection of formal points.

We start by defining for all $a \in S$ the *boundary formal closed* $N(a)$ as

$$b \ \epsilon \ N(a) \equiv b \lhd \{a\} \cup a \to \emptyset$$

where

$$b \ \epsilon \ a \to \emptyset \equiv a * b \lhd \emptyset.$$

Since S is a formal topology, then $a \to \emptyset$ is a formal closed: if $b \lhd a \to \emptyset$, by $Stab$ we have $a * b \lhd a * (a \to \emptyset) \lhd \emptyset$, that is, $b \ \epsilon \ a \to \emptyset$. In terms of points, $\mathsf{Ext}(a \to \emptyset)$ is precisely the pseudocomplement of ext_a. The latter in fact, is the union of the ext_b such that $\neg(\mathsf{ext}_a \between \mathsf{ext}_b)$, that is, $\forall \alpha (b \ \epsilon \ \alpha \to a \notin \alpha)$. This can be rewritten as $\forall \alpha (a \ \epsilon \ \alpha \ \& \ b \ \epsilon \ \alpha \to \bot)$. Since α is convergent, if $a \ \epsilon \ \alpha \ \& \ b \ \epsilon \ \alpha$ then $a * b \between \alpha$, and, since $Left$ holds, then also the converse implication holds. Hence b is in the pseudocomplement of ext_a if and only if

$$\forall \alpha (a * b \between \alpha \to \emptyset \between \alpha). \tag{1.50}$$

Since the formal topology S is finitary, then spatiality holds, and (1.50) is equivalent to $a * b \lhd \emptyset$, that is $b \ \epsilon \ a \to \emptyset$.

Remark 11. More generally, if $U, V \subseteq S$, one defines

$$a \;\epsilon\; U \to V \equiv a * U \lhd V \equiv (\forall b \;\epsilon\; U)(a * b \lhd V). \tag{1.51}$$

Then $U \to V$ is a formal open and, with this definition of implication, the lattice $FOpen(S)$ of formal opens has structure of Heyting algebra [CMS13].

As a result, $\mathsf{Ext}(N(a))$ is precisely the complement of the boundary δ_a of the basic neighborhood ext_a. The quotient topology $(S^{N(a)}, \lhd^{N(a)}, \ltimes^{N(a)}, *)$ of S in $N(a)$ corresponds in particular to the closed subspace determined by the boundary[10] δ_a.

We state now the predicative definition of Krull dimension for all formal topologies with operation.

Definition 1.16. Let $(S, \lhd, \ltimes, *)$ be a formal topology with operation. We define the inequality Kdim $S \leqslant n$ by induction on $n \in \mathbb{N} \cup \{-1\}$ as follows:

$(n = -1)$ Kdim $S \leqslant -1$ holds if $S \lhd \emptyset$;

$(n \geqslant 0)$ Kdim $S \leqslant n$ holds if Kdim $S^{N(a)} \leqslant n - 1$ for all $a \;\epsilon\; S$, where $(S^{N(a)}, \lhd^{N(a)}, \ltimes^{N(a)}, *)$ is the quotient of S in $N(a)$.

By using the characterization contained in Proposition 1.19, we are now able to prove that, if S is finitary and with finitary operation, Kdim $S \leqslant n$ holds if and only if the the space of points $\mathscr{P}t(S)$ has *Krull dimension* smaller or equal to n. Definition 1.16 covers also the non-finitary context, and a reason for this will be made clear at the end of the section.

Proposition 1.20 (OI+CL)**.** *Let $(S, \lhd, \ltimes, *)$ be a finitary formal topology with finitary operation. Then* Kdim $S \leqslant n$ *if and only if every chain*

$$\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_n$$

of formal points in $\mathscr{P}t(S)$ is not strict.

Proof. $(\to)$ If $S \lhd \emptyset$, then there are no formal points. Suppose now that the statement holds for $n-1$, and we derive a contradiction, by supposing

$$\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \overset{a_n}{\subsetneq} \alpha_n$$

to be a strict chain of formal points in S, with $a_n \;\epsilon\; \alpha_n$ but $a_n \notin \alpha_{n-1}$. Consider the quotient of S in $N(a_n)$: $\alpha_n \between N(a_n)$, and also $\alpha_{n-1} \between N(a_n)$ must hold, since Kdim $S^{N(a)} \leqslant n-1$ and by inductive hypothesis $\alpha_0, \dots, \alpha_{n-1}$ cannot form a chain of points of S. Therefore $a_n \;\epsilon\; \alpha_{n-1}$ or $a_n \to \emptyset \between \alpha_{n-1}$. The first case cannot occur by hypothesis. Hence, there is $b \;\epsilon\; \alpha_{n-1}$ such that $a_n * b \lhd \emptyset$. However, this implies $b \;\epsilon\; \alpha_n$, so that $a_n * b \between \alpha_n$ and $\alpha_n \between \emptyset$, again a contradiction.

[10] In fact, the points of $S^{N(a)}$ are the points α of S which do not meet $N(a)$.

($\leftarrow$) If there is no formal point, this means that $S \lhd \emptyset$. In fact, if $a \ltimes S$ for some $a \,\epsilon\, S$, we could find by reducibility a formal point α containing a. Suppose now that the statement holds for $n-1$, and that Kdim $S^{N(a)} \leqslant n-1$ does not hold for a given $a \,\epsilon\, S$. By hypothesis, there exists a strict chain

$$\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_{n-1}$$

of formal points such that $\neg(\alpha_{n-1} \between N(a))$. Consider now the localisation $(S, \lhd_{\alpha_{n-1}}, \ltimes_{\alpha_{n-1}}, *)$: since $\neg(\alpha_{n-1} \between a \to \emptyset)$ we have[11] $\neg(a \lhd_{\alpha_{n-1}} \emptyset)$, and then $a \ltimes_{\alpha_{n-1}} S$. We can find by Corollary a formal point α_n such that $b \,\epsilon\, \alpha_n$ and $\alpha_{n-1} \subseteq \alpha_n$. We obtain in this way a chain $\alpha_0 \subsetneq \cdots \subsetneq \alpha_n$ of $n+1$ distinct prime ideals in S, a contradiction. □

Example 1.21. A finitary formal topology $(S, \lhd, \ltimes, *)$ is *zero-dimensional* if Kdim $S \leqslant 0$ This means that for all $a \in A$ exists $U \subseteq S$, such that

$$S \lhd \{a\} \cup U, \quad a * U \lhd \emptyset. \tag{1.52}$$

In particular, if there is a *convincing* element e such that $S \lhd e$, we can replace S with e in 1.52 and, therefore, U can be taken finite.

The notion of Krull dimension, initially given in the context of commutative rings, is related to and originated from another more general version of topological dimension, the Brouwer-Menger-Urysohn notion of (small) *inductive dimension* of a topological space. We recall the definition commonly found in the literature [Eng78]:

Definition 1.17. Let X be a regular topological space. One defines ind $X \leqslant n$ by induction on $n \in \mathbb{N} \cup \{-1\}$ as follows:

$(n = -1)$ $X = \emptyset$;

$(n \geqslant 0)$ For all $x \in X$, and $D \subseteq X$ open with $x \,\epsilon\, D$, there is an open subset E_x such that $x \,\epsilon\, E_x$, $E_x \subseteq D$ and ind $\delta E_x \leqslant n-1$,where δE_x is the boundary of E_x.

Let X be a regular space. Then, one has that ind $X \leqslant n$ if and only if X has a base $\mathscr{B}$ (obtained by collecting the open subsets E_x) such that ind $\delta B \leqslant n-1$ for all $B \in_c \mathscr{B}$. In particular, if $X = \mathscr{P}t(S)$ is the space of points of a formal topology with operation $(S, \lhd, \ltimes, *)$ and is regular, then Kdim $S \leqslant n$ states exactly that this holds for the base $\mathscr{B} \equiv \{\mathsf{ext}_a\}_{a \in S}$. In plain words, Kdim $S \leqslant n$ implies ind $X \leqslant n$.

In the light of this consideration, the definition of Krull dimension of a formal topology can be meaningful also in a non-finitary context and deserves to be studied as an independent notion. In the following, however, we will just encounter examples of finitary formal topologies, where the definition of Krull dimension agrees, at least classically, with the customary one.

[11] In fact, by Prop. 1.18, $a \lhd_{\alpha_{n-1}} \emptyset$ if and only if there is $a \,\epsilon\, \alpha_{n-1}$ such that $a * b \lhd \emptyset$, that is, $\alpha_{n-1} \between a \to \emptyset$.

Chapter 2

The Zariski Spectrum as a Basic Topology

Note. This chapter is based on the following submitted article [RSS13]: Giovanni Sambin, Davide Rinaldi and Peter Schuster. *The Basic Zariski Topology.*

Introduction

The most typical example of space in commutative algebra is the *Zariski spectrum* $\mathfrak{Spec}(A)$ associated to a commutative ring A. This consists of the collection of its prime ideals $\mathfrak{p}$, endowed with the *Zariski topology*: the topology generated from the basis of opens $\{D(a)\}_{a \in A}$ where

$$D(a) = \{\mathfrak{p} \in \mathfrak{Spec}(A) : a \notin \mathfrak{p}\} \tag{2.1}$$

for every $a \in A$. This topological space was one of the starting points for modern algebraic geometry, and its impredicative nature determines the apparent non-constructive character of large parts of the subsequent theory. Moreover, the existence of a prime ideal in general depends on Zorn's Lemma or other forms of AC.[1]

The Zariski spectrum lends itself naturally to a point-free description, both in terms of locales [Joh82, Vic89] and formal topologies [Sch06a, Sch08, Sig95]. We develop this second approach within the framework of the basic picture and the tools introduced in the first chapter. In particular, instead of the collection of prime ideals, we work directly on the index set for the basis (2.1): the ring A.

In Section 1.4, a strategy was described to generate basic topologies by induction and coinduction. By means of this, we can equip every ring A with

[1] Banaschewski [Ban83] has proved that the existence of a prime ideal in a non-trivial ring is equivalent to the Boolean Ultrafilter Theorem. Joyal [Joy76] has built, inside a topos, a ring without prime ideals.

a basic topology, starting from the inductive generation of ideals. The novelty with respect to [Sch06a, Sch08, Sig95] is that all the topological definitions, and all the related proofs, are explicitly of inductive/coinductive sort. Therefore, regardless of foundational issues, an effective implementation of these concepts is direct. Moreover, we can get by with ideals rather than radical ideals.

In this setting, the multiplication of the ring A is a natural operation for this basic topology, and the induced notion of formal point matches classically with that of (the complement of) a prime ideal of A, which is the usual notion of point of the prime spectrum. In other words, this basic topology corresponds precisely to the customary Zariski topology. The correspondence which to each ring assigns a basic topology with operation is then extended to a functor, as in the classical case.

The *formal* Zariski topology [Sch06a, Sch08, Sig95] is obtained from the *basic* Zariski topology by adding a further generation rule, and every property of the latter extends canonically to the former. One can thus return to radical ideals as occasion demands.

The last part of this chapter is devoted to the analysis of two impredicative principles associated to the formal Zariski topology, *spatiality* and *reducibility*. Assuming classical logic, each of these two principles is equivalent to Krull's Lemma, but with intuitionistic logic they must be kept apart. While spatiality corresponds to the spatiality of the locale of radical ideals and is a completeness principle, reducibility affirms the existence of a formal point—that is, an appropriate sort of model—and so is a satisfiability principle. We will show, strengthening some results from [GS07, Neg02], that these principles are constructively untenable for the class of formal Zariski topologies as a whole.

Another constructive and predicative approach to the Zariski spectrum, developed in [Joy71, Joy76] and used e.g. in [CP01, Coq09, CLS07], is by way of *distributive lattices*. In contrast to this, the avenue via the basic picture is closer to the customary treatment, and allows us to consider simultaneously the notions of closed and open subsets. Last but not least, the potential presence of points makes working on the formal side more intuitive. In Chapter 3, we will employ the tools here developed to obtain an elementary characterization of the height of an ideal in a commutative ring.

2.1 The Basic Zariski topology

Let us fix a commutative ring with unit $(A, +, \cdot, 0, 1)$. We define a finitary basic topology $\mathrm{Zar}(A)$, called the *basic Zariski topology*, using generation by relations, as explained in Section 1.4.1. In more explicit terms, we define a cover $\lhd$ on the ring A which satisfies the axioms

$$\frac{a \;\epsilon\; U}{a \lhd U}\ \mathrm{Refl} \qquad \frac{\top}{0 \lhd U}\ 0 \qquad \frac{a \lhd U \quad b \lhd U}{a + b \lhd U}\ \Sigma \qquad \frac{a \lhd U \quad \lambda \in A}{\lambda \cdot a \lhd U}\ \Pi \tag{2.2}$$

and is the least relation which satisfies this property, that is, the *induction* axiom

$$\dfrac{a \lhd U \quad U \subseteq P \quad 0 \mathrel{\epsilon} P \quad \begin{matrix}[a \mathrel{\epsilon} P, b \mathrel{\epsilon} P] \\ \vdots \\ a+b \mathrel{\epsilon} P\end{matrix} \quad \begin{matrix}[a \mathrel{\epsilon} P, \lambda \in A] \\ \vdots \\ \lambda \cdot a \mathrel{\epsilon} P\end{matrix}}{a \mathrel{\epsilon} P} \ \lhd\text{–}\mathit{induction}$$

holds for all $U \subseteq A$ and $a \in A$. We can describe the cover in this way: one has $a \lhd U$ if and only if $a = 0$ or if there exists a derivation tree which uses just the rules Refl, Σ and Π, and has leaves of the form $c \mathrel{\epsilon} U$ and $a \lhd U$ as root. Here is a brief analysis of the derivation trees:

1. Since the product is associative, if in a proof one applies the product rule twice consecutively, then one application is sufficient:

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots \\ a \lhd U\end{matrix}}{\lambda \cdot a \lhd U}\ \Pi}{\lambda' \cdot (\lambda \cdot a) \lhd U}\ \Pi \quad \mapsto \quad \dfrac{\begin{matrix}\pi \\ \vdots \\ a \lhd U\end{matrix}}{(\lambda' \cdot \lambda) \cdot a \lhd U}\ \Pi$$

2. Since the product distributes over the sum, if in a derivation tree the product rule follows the sum rule, we can swap the two operations. More precisely:

$$\dfrac{\dfrac{\begin{matrix}\pi_a \\ \vdots \\ a \lhd U\end{matrix} \quad \begin{matrix}\pi_b \\ \vdots \\ b \lhd U\end{matrix}}{a + b \lhd U}\ \Sigma}{\lambda \cdot (a + b) \lhd U}\ \Pi \quad \mapsto \quad \dfrac{\dfrac{\begin{matrix}\pi_a \\ \vdots \\ a \lhd U\end{matrix}}{\lambda \cdot a \lhd U}\ \Pi \quad \dfrac{\begin{matrix}\pi_b \\ \vdots \\ b \lhd U\end{matrix}}{\lambda \cdot b \lhd U}\ \Pi}{\lambda \cdot a + \lambda \cdot b \lhd U}\ \Sigma$$

Therefore, given a derivation tree π, by applying these transformations we obtain a normalized derivation tree: each leaf is followed by an invocation of Refl, then by one of Π and eventually by Σ a finite number of times.

Hence we have shown

$$a \lhd U \leftrightarrow a = 0 \vee (\exists n \in \mathbb{N})(\exists u_1, \ldots, u_n \mathrel{\epsilon} U)(\exists \lambda_1, \ldots, \lambda_n \in A)(a = \sum_{i=1}^{n} \lambda_i \cdot u_i) \tag{2.3}$$

for all $a \in A$ and all $U \subseteq A$. The case $a = 0$ can be included as combination of zero coefficients, so that $a \lhd U$ if and only if a belongs to

$$I(U) = \{a \in A : (\exists n \in \mathbb{N})(\exists u_1, \ldots, u_n \mathrel{\epsilon} U)(\exists \lambda_1, \ldots, \lambda_n \in A)(a = \sum_{i=1}^{n} \lambda_i \cdot u_i)\},$$

the ideal generated by U.

Given $a, b \in A$, one has $a \lhd b$ if and only if there exists $c \in A$ such that $a = b \cdot c$. In particular b is invertible if and only if $1 \lhd b$.

It follows from characterization (2.3) that the cover is finitary, that is

$$a \lhd U \leftrightarrow \exists U_0 \in \mathcal{P}_\omega(U)(a \lhd U_0).$$

where $\mathcal{P}_\omega(U)$ is the set of finite subsets of U. This can also be denoted for short by (U).

In addition to the cover $\lhd$, it is generated by *coinduction* a positivity $\ltimes$ by means of the axioms

$$\frac{a \ltimes F}{a \,\epsilon\, F} \qquad \frac{0 \ltimes F}{\bot} \qquad \frac{a + b \ltimes F}{a \ltimes F \vee b \ltimes F} \qquad \frac{a \cdot b \ltimes F}{a \ltimes F}$$

closed coinductively by the rule

$$\frac{a \,\epsilon\, F \quad F \subseteq G \quad \neg(0 \,\epsilon\, F) \quad \begin{array}{c}[a + b \,\epsilon\, F]\\ \vdots \\ a \,\epsilon\, F \vee b \,\epsilon\, F\end{array} \quad \begin{array}{c}[\lambda \cdot a \,\epsilon\, F, \lambda \in A]\\ \vdots \\ a \,\epsilon\, F\end{array}}{a \ltimes G} \ltimes\text{–}coinduction.$$

The general theory [MLSng] states that $\ltimes$ is precisely the greatest positivity compatible with the cover $\lhd$. We denote by $\mathrm{Zar}(A)$ the basic topology $(A, \lhd, \ltimes)$ just defined.

As cover and positivity encode closure and interior operator, the saturated and reduced subsets correspond on the basis to open and closed subsets. Hence it is worthwhile to give an explicit characterization of these two concepts for $\mathrm{Zar}(A) = (A, \lhd, \ltimes)$. The formal open subsets are the $U \subseteq A$ satisfying

$$\frac{\top}{0 \,\epsilon\, U} \qquad \frac{a \,\epsilon\, U \quad b \,\epsilon\, U}{a + b \,\epsilon\, U} \qquad \frac{a \,\epsilon\, U \quad \lambda \in A}{\lambda \cdot a \,\epsilon\, U}$$

for all $a, b \in A$, and thus are the *ideals* of A. This is easy to see: one direction is the Reflexivity axiom, the reverse one is obtained from the $\lhd$-*induction* axiom, setting $P = U$. Symmetrically, a subset $F \subseteq A$ is formal closed if and only if

$$\frac{0 \,\epsilon\, F}{\bot} \qquad \frac{a + b \,\epsilon\, F}{a \,\epsilon\, F \vee b \,\epsilon\, F} \qquad \frac{a \cdot b \,\epsilon\, F}{a \,\epsilon\, F}$$

or, in other words, F is a *coideal* of the ring A. The coideals do not appear in the usual theory of rings, because with classical logic F is a coideal if and only if $\neg F$ is an ideal, and the two notions are interchangeable.[2] The relation $a \ltimes F$ asserts that a belongs to the greatest coideal contained in F. Constructively, the link between ideals and coideals is richer, and it is contained in the compatibility condition:

$$I(U) \between F \to U \between F$$

[2] Incidentally, the same criterion does not apply in general topology: open and closed subsets do coexist and play distinct roles, though these notions are equally interchangeable by complementation.

for all $U \subseteq A$ and F reduced. Explicitly, F is reduced if and only if

$$\frac{\lambda_1 \cdot a_1 + \cdots + \lambda_n \cdot a_n \;\epsilon\; F}{a_1 \;\epsilon\; F \vee a_2 \;\epsilon\; F \vee \cdots \vee a_n \;\epsilon\; F}$$

for all $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_n \in A$. In particular, we notice that a reduced subset F is inhabited if and only if it contains 1, because $(1) = A$ and $(1) \between F$ implies $\{1\} \in F$.

In the following section we will show that, in the case of the basic Zariski topology, the lattice of saturated subsets is endowed with a further operation that provides it with a structure of *quantale*.

2.1.1 The Product as Convergence Operation

In this section we will show how the basic topology $\mathrm{Zar}(A)$ is linked to its classical counterpart. It is worthwhile to point out its naturalness within the algebraic context, being just an accurate revision of the inductive generation process of ideals. We take the product $\cdot$ as operation for the ring A:

Proposition 2.1. *The basic topology with operation $(A, \lhd, \ltimes, \cdot)$ satisfies Loc, Unit and $Left$.*

Proof. We first prove *Loc*. Let π_a and π_b be two derivation trees for $a \lhd U$ and $b \lhd V$ respectively, and let us prove $a \cdot b \lhd U \cdot V$ by induction on π_a and π_b. If the two derivation trees consist of just one use of rule Refl, we proceed as follows:

$$\dfrac{a \;\epsilon\; U}{a \lhd U}\,\mathrm{Refl}, \quad \dfrac{b \;\epsilon\; V}{b \lhd V}\,\mathrm{Refl} \quad \mapsto \quad \dfrac{\dfrac{a \;\epsilon\; U \quad b \;\epsilon\; V}{a \cdot b \;\epsilon\; U \cdot V}}{a \cdot b \lhd U \cdot V}\,\mathrm{Refl}$$

If the last rule used in π_a is Σ where $a = a_1 + a_2$, we apply the following transformation:

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ a_1 \lhd U\end{matrix} \quad \begin{matrix}\pi_2 \\ \vdots \\ a_2 \lhd U\end{matrix}}{a \lhd U}\,\Sigma, \quad \begin{matrix}\pi_b \\ \vdots \\ b \lhd V\end{matrix} \quad \mapsto \quad \dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ a_1 \lhd U\end{matrix} \quad \begin{matrix}\pi_b \\ \vdots \\ b \lhd V\end{matrix}}{a_1 \cdot b \lhd U \cdot V}\,Loc \quad \dfrac{\begin{matrix}\pi_2 \\ \vdots \\ a_2 \lhd U\end{matrix} \quad \begin{matrix}\pi_b \\ \vdots \\ b \lhd V\end{matrix}}{a_2 \cdot b \lhd U \cdot V}\,Loc}{a \cdot b \lhd U \cdot V}\,\Sigma$$

Suppose instead that the last rule is Π, where $a = \lambda \cdot a_1$. Hence:

$$\dfrac{\begin{matrix}\pi_a \\ \vdots \\ a_1 \lhd U\end{matrix} \quad \lambda \in A}{a \lhd U}\,\Pi, \quad \begin{matrix}\pi_b \\ \vdots \\ b \lhd V\end{matrix} \quad \mapsto \quad \dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ a_1 \lhd U\end{matrix} \quad \begin{matrix}\pi_b \\ \vdots \\ b \lhd V\end{matrix}}{a_1 \cdot b \lhd U \cdot V}\,Loc \quad \lambda \in A}{a \cdot b \lhd U \cdot V}\,\Pi$$

By way of these modifications, we get the required deduction tree.

To prove $Unit$, it is sufficient to notice that $a = a \cdot 1 \;\epsilon\; a \cdot A$ and then to use Reflexivity. Finally, $Left$ is just a special case of the rule Π. $\square$

As first consequence, the product is a well-defined operation on the lattice of ideals $FOpen(A)$. Explicitly, for all subsets $U, V \subseteq A$, we have $I(U) \cdot I(V) = I(U \cdot V)$. It follows that this operation distributes with set-indexed joins and $\bigvee$:

$$V \cdot \bigvee_{i \in I} U_i \overset{Loc}{=_\lhd} V \cdot \bigcup_{i \in I} U_i =_\lhd \bigcup_{i \in I} V \cdot U_i = \bigvee_{i \in I} V \cdot U_i$$

In addition, there exists an element $A = (1)$ of $FOpen(A)$ such that, for all subsets $U \subseteq A$,

$$U \cdot A =_\lhd U.$$

In all, the lattice $(FOpen(A), \bigvee, \cdot, A)$ is a commutative unital quantale. The $Left$ property implies also $U * V \lhd U \wedge V$ for all $U, V \subseteq A$.

As already stressed, the intuition behind a basic topology is that of a set of basis indices for an ideal space of points. One can retrieve a notion of formal point as a subset of indices that behaves as a neighbourhood filter of an imaginary point. In the case of $\mathrm{Zar}(A)$, to say that α is a formal point amounts to say that α is a coideal, (i.e. *splits the cover* $\lhd$) satisfying $1 \in \alpha$ and $a, b \,\epsilon\, \alpha \to a{\cdot}b \,\epsilon\, \alpha$ for all $a, b \in A$. A subset with all these properties is called *prime coideal* or, more commonly, *prime filter*.

We regain moreover the link with the usual notion in algebraic geometry, where the points of the prime spectrum are the prime ideals of the ring. We recall that a *prime ideal* is a subset $\mathfrak{p} \subseteq A$ such that:

1. $\neg(1 \,\epsilon\, \mathfrak{p})$ (or, equivalently, $\mathfrak{p} \neq A$),
2. $a \cdot b \,\epsilon\, \mathfrak{p} \to a \,\epsilon\, \mathfrak{p} \vee b \,\epsilon\, \mathfrak{p}$, for all $a, b \in A$,
3. $\mathfrak{p}$ is an ideal.

With classical logic, $\mathfrak{p}$ is a prime ideal if and only if its complement $-\mathfrak{p}$ is a prime coideal, or formal point. We have therefore a bijective correspondence

$$\begin{array}{rcl} - : \mathfrak{Spec}(A) & \longrightarrow & \mathscr{P}t(A) \\ \mathfrak{p} & \mapsto & -\mathfrak{p} \end{array}$$

between the prime spectrum and the collection of formal points of A.

Let A be a set and $\mathcal{S}$ a sub-collection of the collection $\mathcal{P}(A)$. In general, reasoning impredicatively, the space of formal points $\mathscr{P}t(A)$ of a basic topology defines a basic pair

$$\Vdash : \mathscr{P}t(A) \longrightarrow A \tag{2.4}$$

where

$$\alpha \Vdash x \equiv x \ \epsilon \ \alpha$$

for all $x \in A$ and $\alpha \in_c \mathscr{P}t(A)$. The topology, in the usual sense, is precisely the one induced by the basic opens of the form ext_a, defined by

$$\alpha \in_c \mathsf{ext}_a \equiv a \ \epsilon \ \alpha$$

where $\alpha \in_c \mathscr{P}t(A)$ and $a \in \alpha$. One defines as usual the operators

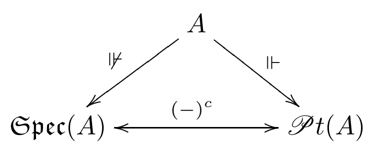
$$\alpha \in_c \mathsf{Ext}(U) \equiv \alpha \between U \qquad \alpha \in_c \mathsf{Rest}(F) \equiv \alpha \subseteq F$$

where $U, F \subseteq A$. The sub-collections of the form $\mathsf{Ext}(U)$ or $\mathsf{Rest}(U)$ are, respectively, the fixed points of the operators int and cl, that is, the open and closed subsets of the new topology. Notice that these concepts are not deducible one from another through complementation.

Similarly, if we define the relation $\nVdash \in_c \mathfrak{Spec}(A) \to A$ as

$$\mathfrak{p} \nVdash a \equiv \neg(a \in \mathfrak{p}),$$

then we have, impredicatively, a basic pair, and a topology on $\mathfrak{Spec}(A)$ the basis of which is precisely $\{D(a)\}_{a \in A}$; in other words, the Zariski topology. The classical complementation $(-)^c$ extends to an isomorphism of basic pairs



and, in particular, to an isomorphism of topological spaces $\mathfrak{Spec}(A) \cong \mathscr{P}t(A)$. In this sense, assuming classical logic, our approach is equivalent to the usual one. To summarize, the Zariski topology has been obtained by applying the machinery of basic topologies to the inductive generation of ideals.

We recall that the impredicative basic pair (2.4) determines a new basic topology structure $\mathrm{Zar}_{\mathscr{P}t}(A) = (A, \lhd_{\mathscr{P}t}, \ltimes_{\mathscr{P}t}, \cdot)$ on A. Moreover, by applying Prop. 1.3, we know that $\mathrm{Zar}_{\mathscr{P}t}(A)$ is a formal topology. In particular, this says that $\mathrm{Zar}_{\mathscr{P}t}(A)$ must differ from $\mathrm{Zar}(A)$. In more precise terms:

Proposition 2.2. *The basic Zariski topology* $\mathrm{Zar}(A) = (A, \lhd, \ltimes, \cdot)$ *on a commutative ring* A *is neither spatial nor reducible.*

Proof. Let $A = \mathbb{Z}/4\mathbb{Z}$ and suppose $\mathrm{Zar}(A)$ to be spatial. Let $\alpha \in \mathscr{P}t(A)$ be such that $2 \in \alpha$; since α is filtering, $0 = 2{\cdot}2 \in \alpha$ follows. Hence $\forall \alpha (2 \,\epsilon\, \alpha \to \alpha \between \{0\})$, which is to say that $2 \lhd_{\mathscr{P}t} 0$. By spatiality, it follows $2 \lhd 0$, a contradiction.

Analogously, suppose $\mathrm{Zar}(A)$ to be reducible and consider the coideal $F = \{1, 2, 3\} = A \setminus \{0\}$, for which $2 \ltimes F$. By reducibility, we get $2 \ltimes_{\mathscr{P}t} F$, that is, there exists a formal point α such that $2 \in \alpha$ and $\alpha \subseteq F$. So $0 = 2 \cdot 2 \in \alpha \subseteq F$, again a contradiction. □

We will overcome this issue in the following pages, by introducing the Zariski formal topology.

2.1.2 Ring Homomorphisms and Continuous Relations

Let A and B be commutative rings with unit. One of the most fundamental properties of the prime spectrum lies in the fact that the inverse image of a prime ideal $\mathfrak{q} : \mathfrak{Spec}(B)$ through a ring homomorphism $f : A \to B$ is a prime ideal $f^- \mathfrak{q} : \mathfrak{Spec}(A)$. In other words, f induces a map $f^- : \mathfrak{Spec}(B) \to \mathfrak{Spec}(A)$. If the two spectra are equipped with the Zariski topology, then f^- is also continuous; more generally, this correspondence extends to a contravariant functor

$$\mathfrak{Spec} : \text{CRings} \to \text{Top}$$

from the category of commutative rings to that of topological spaces.

These classical observations find their constructive counterpart in the framework of the basic picture. Let

$$\text{Zar}(A) = (A, \lhd_A, \ltimes_A, \cdot), \qquad \text{Zar}(B) = (B, \lhd_B, \ltimes_B, \cdot)$$

be the basic Zariski topologies on A and B respectively.

A relation $r : \text{Zar}(B) \to \text{Zar}(A)$ is continuous if and only if it *respects the relations*, that is,

$$r^- 0_A \lhd_B \emptyset \qquad r^-(a + a') \lhd_B r^-\{a, a'\} \qquad r^-(a \cdot a') \lhd_B r^- a$$

for all $a, a' \in A$. Thanks to Proposition 1.7, if $f : A \to B$ is a ring homomorphism, then the inverse relation $\hat{f} : B \to A$, defined by $b \hat{f} a \equiv a f b$, is a continuous morphism of basic Zariski topologies. It is also convergent, in fact, the two convergence properties can be rewritten as

$$f(a) \cdot f(a') =_{\lhd_B} f(a \cdot a'), \qquad\qquad B \lhd_B f(A).$$

The first one follows from the fact that f respects the product, the second one follows from $f(1_A) = 1_B$.

Since $\widehat{g \circ f} = \hat{f} \circ \hat{g}$ and $\hat{id_A} = id_A$, the correspondence that assigns to each ring homomorphism a continuous and convergent relation defines a contravariant functor

$$\text{Zar} : \text{CRings} \to \text{BTopO}$$

from the category of commutative rings to the one of basic topologies with operation.

As a further consequence, the direct existential image through $\hat{f}$ of a formal point $\alpha \in \mathscr{P}t(B)$ is again a formal point $\hat{f}\alpha \in \mathscr{P}t(A)$, and the map

$$\mathscr{P}t(\hat{f}) : \mathscr{P}t(B) \to \mathscr{P}t(A)$$

is continuous with respect to the induced topologies. In the light of the classical link between formal points and prime ideals, the description above matches perfectly with the usual treatment.

Finally, the relation $\hat{f}$ induces a morphism

$$F(\hat{f}) : FOpen(A) \to FOpen(B)$$

of commutative unital quantales, defined by $U \to I(f(U))$. To $F(\hat{f})$, corresponds to the morphism $\mathscr{P}t(\hat{f})^-$ between the frames of opens associated with the spectra.

More generally, we have a functor

$$FOpen : \text{CRings} \to \text{CQuantU}$$

from the category of commutative rings to that of commutative unital quantales.

2.1.3 Quotients and Localisations of Zar(A)

In this section, we will show that the class of basic Zariski topologies is closed under the construction of quotients and localisation in a submonoid. Let A be a commutative ring with unit A and let

$$\text{Zar}(A) = (A, \lhd, \ltimes, \cdot)$$

be the corresponding basic Zariski topology.

Quotients. The quotient of $\text{Zar}(A)$ on a subset $U \subseteq A$ is the basic topology $(A, \lhd^U, \ltimes^U, \cdot)$ is characterized as follows (see (1.47))

$$a \lhd^U V \leftrightarrow a \lhd V \cup U$$

for all $a \in A$ and $V \subseteq A$. Since the cover $\lhd^U$ was obtained by adding a rule, we get

$$a \lhd V \to a \lhd^U V$$

for all $a \in A$ and $U \subseteq A$. In other words, the identity relation $id_A : A \to A$ is a continuous morphism from $(A, \lhd^U, \ltimes^U)$ to $(A, \lhd, \ltimes)$.

Thanks to (1.47), one easily verifies that the product $\cdot$ is an operation for $(A, \lhd^U, \ltimes^U)$ satisfying, as in $\text{Zar}(A)$, $Left$, Loc and $Unit$.

The points $\alpha : \mathscr{P}t_U(A)$ of the quotient $(A, \lhd^U, \ltimes^U, \cdot)$ are the prime coideals $\alpha : \mathscr{P}t(A)$ that split the extra axiom C_U, which is to say

$$\frac{U \between \alpha}{\bot}.$$

In all, $\mathscr{P}t_U(A)$ can be identified with the closed subspace $\mathsf{Rest}(-U) \subseteq \mathscr{P}t(A)$.

Let us now denote by $A_{/U}$ the set A equipped with the equality predicate

$$x =^U y \equiv x - y \lhd U$$

for all $x, y \in A$. This is nothing but the quotient $A/I(U)$ and therefore inherits a ring structure from A. The identity function is a well-defined ring homomorphisms $\pi_U : A \to A_{/U}$ and therefore, as described in the previous section, we have a continuous and convergent morphism $\pi_U^- : \text{Zar}(A_{/U}) \to \text{Zar}(A)$ between the corresponding basic Zariski topologies. This morphism, in the light of the equivalence (1.47), restricts to an isomorphism between $\text{Zar}(A_{/U})$ and the quotient $\text{Zar}(A)$ defined by U.

Proposition 2.3. *Let A be a commutative ring, $(A_{/U}, \lhd_{/(U)}, \ltimes_{/(U)}, \cdot)$ the basic Zariski topology on the quotient ring $A_{/U}$, and $(A, \lhd^U, \ltimes^U, \cdot)$ the quotient of the basic Zariski topology on A defined by $U \subseteq A$. The relations*

$$\pi_U^- : A_{/U} \to A \ \ \text{and} \ \ id_A : A \to A_{/U}$$

form an isomorphism between the basic topologies under considerations.

In conclusion, the quotients and corresponding morphisms are represented by quotient rings and projection homomorphisms.

Localisations. Let now $S \subseteq A$ be a multiplicative subset. In particular, S is convergent for $Zar(A)$ and we can consider the *localisation* $(A, \lhd_S, \ltimes_S)$ of $\mathrm{Zar}(A)$ in S. Since $Zar(A)$ satisfies $Left$ and Loc, Prop. 1.18 applies and the cover $\lhd_S$ can be characterized as follows

$$a \lhd_S U \leftrightarrow (\exists s \in S)(a \cdot s \lhd U), \tag{2.5}$$

for all $a \in A$ and $U \subseteq A$.

Starting from (2.5), it is easy to verify that the product $\cdot$ is a convergence operation for the localized basic topology, and moreover satisfies, as for $\mathrm{Zar}(A)$, Loc, $Left$ and $Unit$. We finally denote with $\mathrm{Zar}_S(A)$ the basic topology $(A, \lhd_S, \ltimes_S, \cdot)$ obtained in this way.

The formal open subsets of $\mathrm{Zar}_S(A)$ coincide with the ideals I of A which satisfy

$$\frac{s \ \epsilon \ S \quad a \cdot s \ \epsilon \ I}{a \ \epsilon \ I}$$

for all $a \in A$. Analogously, the formal closed subsets of A are the coideals P such that

$$\frac{s \ \epsilon \ S \quad a \ \epsilon \ P}{a \cdot s \ \epsilon \ P}. \tag{2.6}$$

for all $a \in A$; these subsets are called S-*convergent*.

In particular, a formal point of $\mathrm{Zar}_S(A)$ is nothing but an inhabited prime coideal α such that $S \subseteq \alpha$. In fact, for any such α the condition (2.6) is a particular instance of the filtering property:

$$\frac{\dfrac{s \ \epsilon \ S}{s \ \epsilon \ \alpha} \quad a \ \epsilon \ \alpha}{a \cdot s \ \epsilon \ \alpha}.$$

Vice versa, a formal point contains 1 and thence, as a particular instance of (2.6),

$$\frac{s \ \epsilon \ S \quad 1 \ \epsilon \ \alpha}{s \ \epsilon \ \alpha},$$

so that $S \subseteq \alpha$. In particular, if S is generated by a single element $a \ \epsilon \ A$, one has

$$\alpha \ \epsilon \ \mathscr{P}t(\mathrm{Zar}_S(A)) \leftrightarrow \alpha \ \epsilon \ \mathsf{ext}_a \leftrightarrow a \ \epsilon \ \alpha.$$

We briefly recall that the localisation of a ring A in a monoid S is the ring of fractions $(A_S, +, \cdot, 0, 1)$. More precisely, this is the set

$$A_S = \{\frac{x}{s} : x \in A, s \in S\}$$

of formal fractions together with the equality

$$\frac{x}{s} =_S \frac{y}{t} \equiv (\exists r \in S)(r \cdot t \cdot x = r \cdot s \cdot y),$$

and the operations and constants

$$\frac{x}{s} + \frac{y}{t} =_S \frac{x \cdot t + y \cdot s}{s \cdot t}, \qquad \frac{x}{s} \cdot \frac{y}{t} =_S \frac{x \cdot y}{s \cdot t}, \quad 0 =_S \frac{0}{1}, \qquad 1 =_S \frac{1}{1},$$

for all $x, y \,\epsilon\, A$ and $s, t \,\epsilon\, S$.

An element $\frac{x}{s}$ is invertible if and only if so is $\frac{x}{1}$, i.e., if there exists $r \in A$ such that $r \cdot x \,\epsilon\, S$. Moreover, the function $\varphi_S : A \to A_S$ which maps x to $\frac{x}{1}$ is a ring homomorphism. Corresponding to φ_S we have, as before, a continuous and convergent morphism $\varphi_S^- : A_S \to A$ from $\mathrm{Zar}(A_S)$ to $\mathrm{Zar}(A)$. In particular $\varphi_S(A^*) \subseteq A_S^*$, and each $\frac{x}{s} \in A_S$ is associated to an element of $\varphi_S(A)$ because $\frac{x}{s} = \frac{x}{1} \cdot \frac{1}{s}$.

The following proposition establish the link between the localisation of rings and the localisation of the corresponding basic Zariski topologies:

Proposition 2.4. *Let $(A_S, \lhd^S, \ltimes^S, \cdot)$ be the basic Zariski topology on the localisation A_S of the ring A in S, and $(A, \lhd_S, \ltimes_S, \cdot)$ the localisation in S of the basic Zariski topology on A. The pair of relations*

$$\varphi_S^- : A_S \to A \;\; \textit{and} \;\; \psi_S : A \to A_S,$$

where $a \psi_S \frac{x}{s} \equiv a \lhd_S x$, constitute an isomorphism of basic topologies with operation.

Proof. We first verify that ψ_S is well defined on A_S, that is,

$$a \lhd_S x \;\&\; \left(\frac{x}{s} =_S \frac{y}{t}\right) \to a \lhd_S y$$

for all $a, x, y \in A$ and $s, t \,\epsilon\, S$. By using (2.5), we have

$$a \cdot r \lhd x \;\&\; (x \cdot t \cdot r' = y \cdot s \cdot r') \to (\exists r'' \,\epsilon\, S)(a \cdot r'' \lhd y)$$

for some $r, r' \,\epsilon\, S$. From $a \cdot r \lhd x$ and $t \cdot r' \lhd t \cdot r'$, from *Loc* follows that $a \cdot r \cdot t \cdot r' \lhd x \cdot t \cdot r'$, that is, $a \cdot r \cdot t \cdot r' \lhd y \cdot s \cdot r'$. Since $y \cdot s \cdot r' \lhd y$, it is enough to take $r'' = r \cdot t \cdot r'$.

Secondly, we check that φ_S^- and ψ_S are both continuous and convergent relations. For φ_S^-, this follows from Section 2.1.2, since φ_S^- is the inverse relation of an homomorphism. To verify that ψ_S is continuous, we check that it respects the generating axioms:

(0) It respects the 0-axiom, namely $a\psi_S\frac{0}{1} \to a \lhd_S 0$; in fact $a\psi_S\frac{0}{1} \equiv a \lhd_S 0$.

(Σ) It respects the sum axiom, that is,

$$a\psi_S\frac{x\cdot t + x'\cdot s}{s\cdot t} \to a \lhd_S \psi_S^-(\frac{x}{s}) \cup \psi_S^-(\frac{x'}{t}).$$

Since in general $x \,\epsilon\, \psi_S^-\left(\frac{x}{s}\right)$ holds and $x\cdot t + x'\cdot s \lhd_S \{x, x'\}$, from $a \lhd_S x\cdot t + x'\cdot s$ we get $a \lhd_S \{x, x'\}$ and finally $a \lhd_S \psi_S^-\left(\frac{x}{s}\right) \cup \psi_S^-\left(\frac{x'}{t}\right)$.

(Π) It respects the product axiom, viz.

$$a\psi_S\frac{x\cdot\lambda}{s\cdot t} \to a \lhd_S \psi_S^-\left(\frac{x}{s}\right).$$

By hypothesis $a\lhd_S x\cdot\lambda$,and since $x\cdot\lambda\lhd_S x$ and $x \,\epsilon\, \psi_S^-\left(\frac{x}{s}\right)$, the conclusion follows.

The relation ψ_S is convergent: in fact, condition $C2$ is trivially satisfied, since $1 \,\epsilon\, \psi_S^-\left(\frac{1}{1}\right)$, and therefore $1 \lhd \psi_S^-(A_S)$. The first convergence condition $C1$ can be stated explicitly as

$$a\psi_S\frac{x}{s} \;\&\; a'\psi_S\frac{x'}{t} \to a\cdot a' \lhd_S \psi_S^-(\frac{x\cdot x'}{s\cdot t}).$$

By hypothesis $a \lhd_S x$ and $a' \lhd_S x'$, hence, by *Loc*, $a\cdot a' \lhd_S x\cdot x'$ so that $x\cdot x' \,\epsilon\, \psi_S^-\left(\frac{x\cdot x'}{s\cdot t}\right)$.

We leave to the reader the proof that the pair (φ_S^-, ψ_S) of convergent continuous relations is an isomorphism. This amounts to show $a =_{\lhd_S} \psi_S^-\varphi_S a$ and $\frac{x}{s} =_{\lhd^S} \varphi_S\psi_S^-\frac{x}{s}$ for all $a, x \,\epsilon\, A$ and $s \,\epsilon\, S$. □

In all, the localisation in a convergent subset of the basic Zariski topology and the corresponding inclusion morphism are represented by localized rings and localisation homomorphisms. The proof here above is lengthy but almost naive and we decided not to explicit similar arguments in the following.

2.2 The Formal Zariski topology $Zar_f(A)$

The counterexample in Proposition 2.2 relies on the fact that the basic Zariski topology on the ring $\mathbb{Z}/4\mathbb{Z}$ does not satisfy the axiom of contraction. We generate inductively the least basic topology $\lhd_c \supseteq \lhd$, over the basic Zariski topology, satisfying contraction: it is enough to add to the rules 0, Σ and Π the following generation rule:

$$\frac{a^2 \lhd_c U}{a \lhd_c U}\; Sq$$

In this way, we can generate $\mathrm{Zar}_f(A) = (A, \lhd_c, \ltimes_c, \cdot)$, the *formal Zariski topology*. We will show, in a few lines, that $\mathrm{Zar}_f(A)$ actually is a formal topology with operation.

In the presence of the rules S and P, the rule Sq is equivalent to the rule

$$\frac{a^n \lhd_c U \quad n \in \mathbb{N}}{a \lhd_c U}\ N$$

One direction is trivial, for $n = 2$. Vice versa, if n is an even number, then the property Sq allows to divide n by 2; if n is an odd number, by the rule P we can get from $a^n \lhd_c U$ to $a^{n+1} \lhd_c U$ where $n + 1$ is even. In this way, after a finite number of steps, we obtain $a \lhd_c U$ starting from $a^n \lhd_c U$.

One verifies directly that the rule N commutes with Σ and Π, and that two applications of the same rule can be collected into one. In other words, one obtains

$$a \lhd_c U \leftrightarrow \exists n(a^n \lhd U)$$

for all $a \in A$ and $U \subseteq A$. Recalling that $a \lhd U \leftrightarrow a \;\epsilon\; I(U)$, one thus has

$$a \lhd_c U \leftrightarrow a \;\epsilon\; R(U) \tag{2.7}$$

where

$$a \;\epsilon\; R(U) \equiv (\exists m \in \mathbb{N})(\exists u_1, \ldots, u_n \,\epsilon\, U)(\exists \lambda_1, \ldots, \lambda_n \in A)(a^m = \sum_{i=1}^{n} \lambda_i \cdot u_i)\}$$

is the radical ideal generated by U. This is also commonly denoted by $\sqrt{(U)}$.

Besides the cover $\lhd_c$, we generate by conduction a compatible positivity $\ltimes_c$ starting from the axioms of $\ltimes$ together with

$$\frac{a \ltimes_c U}{a^2 \ltimes_c U}$$

for all $a \in A$ and $U \subseteq A$. We indicate with $\mathrm{Zar}_f(A) = (A, \lhd_c, \ltimes_c, \cdot)$ the basic topology generated in this way, and endowed with the product operation.

Since the cover was obtained adding a rule, we have

$$a \lhd U \rightarrow a \lhd_c U$$

for all $a \in A$ and $U \subseteq A$. The reverse implication does not hold in general: in $\mathbb{Z}/4\mathbb{Z}$, one has $2 \lhd_c 0$ but not $2 \lhd 0$.

Proposition 2.5. *The basic topology* $\mathrm{Zar}_f(A) = (A, \lhd_c, \ltimes_c, \cdot)$ *is a formal topology with convergence operation, for every ring A. In particular,* $\lhd_c$ *satisfies*

$$\frac{a \lhd_c U \quad a \lhd_c V}{a \lhd_c U \cdot V}\ \textit{Convergence}$$

for all $a \;\epsilon\; A$ *and* $U, V \subseteq A$.

Proof. The $Left$ property follows directly from $\lhd \subseteq \lhd_c$. For Loc, one acts as in the proof of Proposition 2.1, that is, one builds by induction, starting from two deduction trees for $a \lhd_c U$ and $b \lhd_c V$, a deduction tree for $a \cdot b \lhd_c U \cdot V$.

For the sake of completeness, we will show that *Loc* lifts up with respect to an application of rule N in a deduction tree:

$$\dfrac{\dfrac{\begin{matrix}\pi_a\\ \vdots\\ a^n \lhd_c U\end{matrix}}{a \lhd_c U}\,N \qquad \begin{matrix}\pi_b\\ \vdots\\ b \lhd_c V\end{matrix}}{a \cdot b \lhd_c U \cdot V}\,Loc \quad \mapsto \quad \dfrac{\dfrac{\begin{matrix}\pi_a\\ \vdots\\ a^n \lhd_c U\end{matrix} \qquad \dfrac{\begin{matrix}\pi_b\\ \vdots\\ b \lhd_c V\end{matrix} \quad b^{n-1} \in A}{b^n \lhd_c V}\,\Pi}{(a \cdot b)^n \lhd_c U \cdot V}\,Loc}{a \cdot b \lhd_c U \cdot V}\,N$$

The property Contraction coincides precisely with the rule Sq. Finally, we have

$$\dfrac{\dfrac{a^2 \lhd a^2}{a \lhd_c a^2}\,Sq \qquad \dfrac{a \lhd^c U \quad a \lhd^c V}{a^2 \lhd^c U \cdot V}\,Loc}{a \lhd_c U \cdot V}\,Transitivity,$$

so that convergence follows from *Loc*. □

As already stressed, the lattice $\mathrm{Zar}_f(A)$ of saturated subsets can be identified with the lattice of radical ideals of A, and has the structure of a locale. The equality $U \wedge V =_{\lhd} U \circ V$ can be restated explicitly as $R(I) \cap R(J) = R(I \cdot J)$ for all ideals $I, J \subseteq A$.

The reduced subsets split the axioms, and therefore they can be identified with the coideals $P \subseteq A$ satisfying the further condition

$$\frac{a \ \epsilon \ P}{a^n \ \epsilon \ P}$$

for all $a \in A$, $n \in \mathbb{N}$. We call these subsets *radical coideals* [Sch06a]. The positivity relation can be characterized as follows:

$$\frac{b^m = \lambda_1 \cdot a_1 + \cdots + \lambda_n \cdot a_n \qquad b \ltimes_c F}{a_1 \ltimes_c F \vee a_2 \ltimes_c F \vee \cdots \vee a_n \ltimes_c F}.$$

for all $n, m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_n \in A$. Hence $a \ltimes_c F \to \forall n(a^n \ltimes F)$ for all $a \in A$ and $F \subseteq A$.

Every formal point of the basic topology $\mathrm{Zar}(A)$ is filtering and it is therefore a radical coideal and a formal point for $\mathrm{Zar}_f(A)$. Hence the points of the formal Zariski topology coincide with the points of the basic Zariski topology.

In the realm of formal topologies, the points can be identified with the continuous and convergent morphisms to the initial object of FTop, the category of formal topologies and continuous and convergent morphisms [Samng].

The results in sections 2.1.2 and 2.1.3 can easily be generalized to the formal Zariski topology.

Proposition 2.6. *Let A and B be commutative rings and $\varphi : A \to B$ a ring homomorphism. The relation $\varphi^- : B \to A$ is a continuous and convergent relation between the corresponding formal Zariski topologies $(B, \lhd_c, \ltimes_c, \cdot)$ and $(A, \lhd_c, \ltimes_c, \cdot)$. Moreover, any such correspondence is a functor from the category* CRings *of commutative rings to the category* FTop *of formal topologies.*

Also in the context of the Zariski formal topologies, we can talk about quotients and localisations, and the generation strategy remains the same. The characterizations take the following form:

$$a \lhd_c^U V \leftrightarrow a \lhd_c U \cup V \qquad \text{and} \qquad a \lhd_{c,S} V \leftrightarrow (\exists s \;\epsilon\; S)(a \cdot s \lhd_c U)$$

for all $a \in A$, $U, V \subseteq A$, $S \subseteq A$ monoid (or filter).

As before, the quotients and the localisations are represented in the category of rings by quotient rings and localized rings:

Proposition 2.7. *Let A be a commutative ring, $(A_{/U}, \lhd_{c,/(U)}, \ltimes_{c,/(U)}, \cdot)$ the formal Zariski topology on the quotient ring $A_{/U}$, and $(A, \lhd_c^U, \ltimes_c^U, \cdot)$ the quotient of the formal Zariski topology on A defined by $U \subseteq A$. The relations*

$$\pi_U^- : A_{/U} \to A \;\; \textit{and} \;\; id_A : A \to A_{/U}$$

form an isomorphism between those formal topologies.

Proposition 2.8. *Let A be a commutative ring, $(A_S, \lhd_c^S, \ltimes_c^S, \cdot)$ the formal Zariski topology on the localized ring A_S, and $(A, \lhd_{c,S}, \ltimes_{c,S}, \cdot)$ the localisation in S of the formal Zariski topology on A. The relations*

$$\varphi_S^- : A_S \to A \;\; \textit{and} \;\; \psi_S : A \to A_S$$

where $a\psi_S \frac{x}{s} \equiv a \lhd_{c,S} x$ form an isomorphism between the formal topologies under consideration.

In particular, we recall that properties such as *spatiality* and *reducibility* are preserved by continuous and convergent morphisms, as all the properties of topological nature.

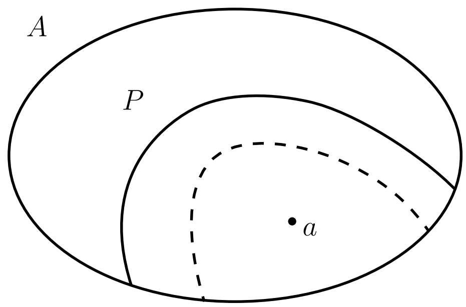
2.2.1 Spatiality and Reducibility of $\mathrm{Zar}_f(A)$

As previously shown, the formal points of the basic Zariski topology and of the formal Zariski topology coincide. In particular, the formal topology induced impredicatively by the points of A, namely by the basic pair $\Vdash : \mathscr{P}t(A) \longrightarrow A$, coincides with $\mathrm{Zar}_{\mathscr{P}t}(A)$. We therefore have the chain of implications

$$a \lhd U \to a \lhd_c U \to a \lhd_{\mathscr{P}t} U$$

for all $a \in A$ and $U \subseteq A$. In other words, $\mathrm{Zar}_f(A)$ is a better approximation of $Zar_{\mathscr{P}t}(A)$ with respect to $\mathrm{Zar}(A)$, and we are prompted to address again the issue of spatiality and reducibility for the formal topology $\mathrm{Zar}_f(A)$.

The reducibility of the formal Zariski topology asserts that for every radical coideal $P \subseteq A$ and every $a \;\epsilon\; P$, there exists a prime filter α containing a and contained in P.



By Corollary 1.6.1, with classical logic and open induction, the formal Zariski topology is spatial and reducible; the formal points coincide with the complements of the prime ideals and the statement of spatiality/reducibility has the familiar form

$$a \mathrel{\epsilon} \bigcap_{U \subseteq \mathfrak{p}} \mathfrak{p} \to a \mathrel{\epsilon} R(U) \tag{2.8}$$

where $a \in A$ and $U \subseteq A$. One has in fact

$$\begin{aligned} \forall\alpha(a \mathrel{\epsilon} \alpha \to U \between \alpha) \to a \lhd_c U \quad &\text{iff} \quad \forall\alpha(\neg(U \between \alpha) \to \neg(a \mathrel{\epsilon} \alpha)) \to a \lhd_c U \ \text{ iff} \\ &\text{iff} \quad \forall\alpha(U \subseteq -\alpha \to a \mathrel{\epsilon} -\alpha) \to a \lhd_c U \ \text{ iff} \\ &\text{iff} \quad \forall_{\mathfrak{Spec}(A)}\mathfrak{p}(U \subseteq \mathfrak{p} \to a \mathrel{\epsilon} \mathfrak{p}) \to a \mathrel{\epsilon} R(U). \end{aligned}$$

The implication (2.8), or its contrapositive, is usually called "Krull's Lemma": starting from the collection of points, it allows to deduce a concrete information on the ring side, that is, an equational witness for $a \mathrel{\epsilon} R(U)$. Nevertheless this existence is a consequence of open induction and classical logic used in the proof of Theorem 1.13, and a priori has no effective content [Sch13].

Vice versa, from the spatiality of the formal Zariski topology on every discrete ring A, one deduces the *restricted excluded middle* (REM). In this setting, REM amounts to saying that $U \cup -U \equiv S$ for all S set and $U \subseteq S$. The proof makes use of the following general lemma, already present in [Neg02] and [GS07]. The only difference lies in the reformulation by means of $\ltimes$:

Lemma 2.9. *Let $(S, \lhd, \ltimes)$ be a finitary and spatial formal topology. Then for all $a \in S$:*

$$a \lhd \emptyset \vee a \ltimes S.$$

Proof. We consider the subset U_a defined by

$$x \mathrel{\epsilon} U_a \equiv x = a \;\&\; a \ltimes S,$$

and show $\forall\alpha(a \mathrel{\epsilon} \alpha \to \alpha \between U_a)$; by spatiality, $a \lhd U_a$ follows. If $a \mathrel{\epsilon} \alpha$, then $a \ltimes \alpha$ because α is reduced, and therefore $a \ltimes S$; we then have a witness for $\alpha \between U_a$. Hence $a \lhd U_a$ and, since the cover is finitary, there exists a finite $U_0 \subseteq U_a$ such that $a \lhd U_0$. It is decidable whether U_0 is empty or inhabited. If U_0 is empty, then $a \lhd \emptyset$; if U_0 is inhabited, then so is U_a, and thus $a \ltimes S$. □

In particular, Lemma 2.9 applies to the formal Zariski topology.

Remark 12. Since the class of Zariski formal topologies is stable under forming quotients, from the hypothesis that the class of Zariski formal topologies is spatial follows, in the light of Lemma 2.9, that

$$a \lhd_c^U \emptyset \vee a \ltimes_c^U A,$$

that is,

$$a \lhd_c U \vee a \ltimes_c -U$$

for every ring A, every $a \in A$ and every $U \subseteq A$. Moreover, by compatibility, we have $a \ltimes_c -U \to \neg(a \lhd_c U)$ and we obtain

$$a \lhd_c U \vee \neg(a \lhd_c U)$$

or, in other words, every radical coideal is complemented for every ring A.

Proposition 2.10. *If spatiality holds for the class of Zariski formal topologies, then every subset U of a discrete set S is complemented.*

Proof. Take the ring $A = \mathbb{Z}[S]$ freely generated by S and consider the formal Zariski topology $(A, \lhd_c, \ltimes_c)$. We regard S and U as subsets of A. We are going to prove

$$a \lhd_c U \leftrightarrow a \,\epsilon\, U$$

for any $a \,\epsilon\, S$. Notice that A is the free $\mathbb{Z}$-module with a basis given by the monic monomials of A. Therefore every element of A can be written in a unique way as $\mathbb{Z}$-linear combination of such monomials. If $a \lhd_c U$ then there is $k \in \mathbb{N}$ such that

$$a^k = \sum_{i=0}^{n} b_i \cdot u_i \quad \text{where} \quad b_i \in A, u_i \,\epsilon\, U;$$

more explicitly, we have $b_i = \sum_{m \in M_i} \lambda_{i,m} \cdot m$ with M_i a finite set of monic monomials in A and $\lambda_{i,m} \in \mathbb{Z}$. One therefore has

$$a^k = \sum_{i=0}^{n} \sum_{m \,\epsilon\, M_i} \lambda_{i,m} \cdot (m \cdot u_i).$$

Since a^k and every $m \cdot u_i$ are monomials and therefore basis elements, this equation can be realized if and only if $a^k = m \cdot u_i$ for some $i = 1, \ldots, n$ and $m \in M_i$; this amounts to $m = a^{k-1}$ and $u_i = a$ and $a \,\epsilon\, U$. By remark 12 we get

$$a \,\epsilon\, U \vee \neg(a \,\epsilon\, U)$$

for all $a \in S$. □

Not even reducibility can be accepted constructively for every ring A. In fact, assuming it, we can prove a version of Russell's *Multiplicative Axiom* in the following form ACF*:

ACF*. *Let S be a discrete set and $\{U_i\}_{i \in I}$ a partition of S in finite and inhabited subsets, with I discrete. Then, there exists $\alpha \subseteq S$ such that*

$$\forall i (U_i \between \alpha) \qquad \textit{and} \qquad \forall t, t' \forall i (t, t' \,\epsilon\, U_i \cap \alpha \to t = t'). \tag{2.9}$$

We now fix a set S equipped with a partition $\{U_i\}_{i \in I}$ in finite and inhabited subsets, with I and S discrete. We will define a formal topology on S such that the formal points coincide precisely with the subsets α that satisfy (2.9). Consider on S the following generated basic topology:[3]

$$a \lhd_P U \equiv a \,\epsilon\, U \vee (\exists i \in I)(U_i \subseteq U)$$

$$a \ltimes_P F \equiv a \,\epsilon\, F \;\&\; (\forall i \in I)(U_i \between F).$$

It follows that a subset $U \subseteq S$ is saturated if and only if

$$\exists i (U_i \subseteq U) \to S = U$$

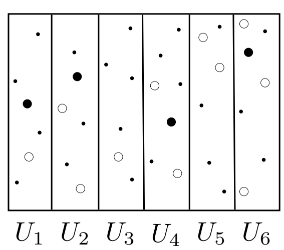
and is reduced if and only if

$$S \between F \to \forall i (U_i \between F)$$

for all $i \in I$. We define on the topology the following operation

$$a \,\epsilon\, t \circ t' \equiv \exists i, j \in I (i \neq j \;\&\; t \,\epsilon\, U_i \;\&\; t' \,\epsilon\, U_j) \vee t = t' = a$$

where $t, t' \in S$. With this operation, the filtering subsets (viz. the $U \subseteq S$ such that $t, t' \,\epsilon\, U \to t \circ t' \between U$), are the ones which have at most one element in each subset U_i of the partition.



In the picture above, we give as example $S = U_1 \dot\cup \cdots \dot\cup U_6$: the fat black dots form a filtering subset, the white ones a reduced subset.

The formal points for this topology are, by direct observation, exactly the inhabited subsets $\alpha \subseteq A$ which satisfy the conditions (2.9).

Remark 13. The existence of a formal point for this topology is equivalent to ACF*.

[3] Following [MLSng], the topology is generated by setting $\{U_i\}_{i \in I}$ as axiom-set for all $a \in S$.

Remark 14. Let A be a discrete ring. The subset $A \setminus \{0\}$ is a coideal, and inhabited by the unit if $1 \neq 0$. In fact

$$a + b \neq 0 \to a \neq 0 \vee b \neq 0$$

for all $a, b \in A$. Since the ring is discrete, $a \neq 0 \vee a = 0$; in the first case, we are done, in the second case, $a + b = 0 + b = b$ and by hypothesis $b \neq 0$. For the product one argues in the same way.

If moreover the ring A is reduced, that is, $a^n = 0 \to a = 0$, then $A \setminus \{0\}$ is a radical coideal, that is to say

$$\frac{a \neq 0}{a^n \neq 0}$$

for all $n \in \mathbb{N}$.

Lemma 2.11. *Let S be a discrete set and $\{U_i\}_{i \in I}$ a partition of S in finite and inhabited subsets, with I a discrete set; let $(S, \lhd_P, \ltimes_P, \circ)$ be the basic topology with operation assigned to S as above. Then there exists a non-trivial, reduced and discrete ring A, and a continuous and convergent morphism from* $\mathrm{Zar}_f(A)$ *to $(S, \lhd_P, \ltimes_P, \circ)$.*

Proof. Consider the ring $\mathbb{Z}[S]$ freely generated by the elements of S. We apply successively he following transformations: first, we quotient $\mathbb{Z}[S]$ by the ideal $I(H)$ generated by

$$H = \{t \cdot t' : (\exists i \in I)(t, t' \ \epsilon \ U_i) \ \& \ t \neq t'\};$$

secondly, we localize it in the monoid $M(K)$ generated by

$$K = \{\sigma_i : i \in I\} \quad (\sigma_i = \sum_{u \ \epsilon \ U_i} u)$$

for all[4] $i \in I$. Let A be the resulting ring. First of all, one can prove that A is non-trivial, that is, $M(K) \between I(H)$ leads to a contradiction. This follows from the structure of the elements of $M(K)$ and $I(H)$.

Secondly, the equality on A is decidable. To see this, let $x = \frac{a}{k} \in A$; we can suppose that a is a monomial, namely $a = s_1^{m_1} \cdots s_n^{m_n} \in \mathbb{Z}[S]$, and $k \in M(K)$. We want to show $x = 0 \vee x \neq 0$; if two variables s_i belong to the same element of the partition U_j for some $j \in I$, then $a \ \epsilon \ I(H)$ and therefore $\frac{a}{k} = 0$. Suppose instead that all the s_i lie in distinct subsets U_j of the partition; we show $k' \cdot a \notin I(H)$ for all $k' = \sigma_{i_1}^{l_1} \cdots \sigma_{i_h}^{l_h} \ \epsilon \ M(K)$. In details, we have:

$$k' \cdot a = \sigma_{i_1}^{l_1} \cdots \sigma_{i_h}^{l_h} \cdot s_1^{m_1} \cdots s_n^{m_n} = \sum_{(s'_1, \ldots, s'_h) \in U_{i_1} \times \ldots, \times U_{i_h}} s'_1, \ldots, s'_h \cdot s_1^{m_1} \cdots s_n^{m_n}.$$

At least one element of this sum does not lie in $I(H)$: it is enough to choose $s'_j = s_l$ if $s_l \in U_{i_j}$. This is sufficient to assert $k' \cdot a \notin I(H)$ and therefore $x \neq 0$.

[4]Since every U_i is finite, this sum is well-defined.

Finally, the ring A is reduced; a proof can be obtained similarly to the previous point, by making explicit $\frac{a^n}{k^n} = 0$, where $n \in \mathbb{N}$, $a \in \mathbb{Z}[S]$ is a monomial and $k \epsilon M(K)$.

We therefore have a chain of morphisms

$$S \overset{i}{\hookrightarrow} \mathbb{Z}[S] \xrightarrow{\pi_H} \mathbb{Z}[S]/I(H) \xrightarrow{\varphi_K} A, \qquad A \overset{r}{\longrightarrow} S$$

where i is the canonical inclusion, π_H is the projection on the quotients and φ_K is the localisation homomorphism in $M(K)$. Finally, let r be the composition of the three morphisms, considered as relation in the opposite direction; we will prove that r is a continuous and convergent morphism if S is endowed with the topology $(S, \lhd_P, \ltimes_P, \circ)$. For the sake of convenience, we identify S with the corresponding subset of A and the relation i with the identity.

To prove continuity, we have to check that r respects the axiom sets $\{U_i\}_{i\in I}$ for every $a \in A$. This is equivalent to showing $a \lhd_c U_i$ for all $i \in I$ and $s \in S$, which is obvious because $\sigma_i \in I(U_i)$ is invertible in the localized ring. As for the convergence of r, since $A \lhd_c S$ as a consequence of the proof of continuity, we only have to prove

$$t \cdot t' \lhd_c t \circ t'$$

for all $t, t' \epsilon S$. If $t \in U_i$ and $t' \in U_j$ with $i \neq j$, then $t \circ t' = S$. If instead $t, t' \epsilon U_i$ for the same $i \in I$, then either $t \neq t'$ and $t \cdot t' = \emptyset$, or $t = t'$ and $t \circ t' = \{t\} = \{t'\}$. In the first case, $t \cdot t' \epsilon I(H)$ and therefore $t \cdot t' \lhd_c \emptyset$; in the second case, the convergence condition becomes $t \cdot t \lhd_c t$, which is always true by $Left$. Hence r is a continuous and convergent morphism. □

Proposition 2.12. *If every formal Zariski topology is reducible, then ACF* holds.*

Proof. Proving ACF* is equivalent to proving the existence of a formal point for the formal topology $(S, \lhd_P, \ltimes_P, \circ)$. By Lemma 2.11, there exists a discrete, reduced and non-trivial ring A and a morphism r from $\mathrm{Zar}_f(A)$ to $(S, \lhd_P, \ltimes_P, \circ)$.

By Remark 14, the subset $A \setminus \{0\}$ is a radical coideal and $1 \epsilon A \setminus \{0\}$. In particular $1 \ltimes A \setminus \{0\}$, so, by reducibility, there exists a formal point α of the formal Zariski topology such that $\alpha \subseteq A \setminus \{0\}$. The direct image of α through r is a formal point for $(S, \lhd_P, \ltimes_P, \circ)$. □

To conclude this section, we list some principles constructively equivalent to spatiality for the class of Zariski formal topologies. Before, we make the following observations:

1. For all the monoids $S \subseteq A$ and for all ideals $I \subseteq A$ one has

$$S \between R(I) \leftrightarrow S \between I.$$

In particular, for $S = \{1\}$, one gets $A = R(I) \leftrightarrow A = I$.

2. For every radical ideal I and monoid S of A, if we set

$$a \epsilon L_S(I) \equiv (\exists s \epsilon S)(a \cdot s \epsilon I);$$

then $L_S(I)$ is the saturation of I with respect to the cover $\lhd_{c,S}$, and moreover every saturated subset is obtained in this way. Also, $S \between L_S(I)$ if and only if $S \between I$. By definition

$$\exists s(s \epsilon S \,\&\, s \epsilon L_S(I)),$$

that is, there exists $s' \epsilon S$ such that $s \epsilon S$ and $s \cdot s' \epsilon I$; then $s \cdot s'$ is a witness of $S \between I$.

3. Putting together the previous remarks, for every monoid S and every ideal I, we get

$$S \between L_S(R(I)) \leftrightarrow S \between R(I) \leftrightarrow S \between I.$$

We can now prove the following equivalences:

Proposition 2.13. *Asserting the spatiality of* $\mathrm{Zar}_f(A)$ *for every ring A is equivalent to each of the following:*

1. *For every ring A, for every monoid $S \subseteq A$ and every ideal $I \subseteq A$,*

$$(\forall \alpha \in \mathscr{P}t(A))(S \subseteq \alpha \to \alpha \between I) \to S \between I.$$

In particular, this holds for every filter S.

2. (Sufficiency) *For every ring A and for every $a \in A$,*

$$(\forall \alpha \in \mathscr{P}t(A))(\neg(a \epsilon \alpha)) \to a \lhd_c \emptyset.$$

In terms of prime ideals, this property can be rewritten with classical logic as

$$(\forall \alpha \in \mathscr{P}t(A))\forall \mathfrak{p}(a \epsilon \mathfrak{p}) \to a \epsilon \sqrt{(0)}.$$

3. *For every ring A and for every ideal $I \subseteq A$,*

$$(\forall \mathfrak{p} \in \mathfrak{Spec}(A))(\alpha \between I) \to A = I.$$

Assuming classical logic, this statement corresponds to

$$(\forall \mathfrak{p} \in \mathfrak{Spec}(A))(I \not\subseteq \mathfrak{p}) \to I = A.$$

Proof. (1) ($\leftarrow$) Let A be a commutative ring which satisfies 1. Given $a \in A$ and $U \subseteq A$, let $S(a)$ be the monoid generated by a and $I = R(U)$. Then

$$S(a) \subseteq \alpha \leftrightarrow a \epsilon \alpha, \qquad \alpha \between I \leftrightarrow \alpha \between U, \qquad S(a) \between I \leftrightarrow a \lhd_c U.$$

By substituting these equivalents we get

$$\forall \alpha(a \epsilon \alpha \to \alpha \between U) \to a \lhd_c U,$$

which is spatiality. (→) Suppose that the Zariski formal topologies are spatial, and fix a ring A, a monoid S and a radical ideal I in A. Let A_S be the localisation in S, equipped with the formal topology $\mathrm{Zar}_f(A_S)$. Thanks to the isomorphism of Proposition 2.8, this formal topology is spatial if and only if the formal topology $\mathrm{Zar}_{c,S}(A)$ is spatial. Remembering that the formal points in $\mathrm{Zar}_{c,S}(A)$ are precisely the formal points of $\mathrm{Zar}_c(A)$ containing S, the spatiality in $(1, L_S(R(I)))$ becomes

$$\forall \alpha \in \mathscr{P}t(A)(S \subseteq \alpha \to \alpha \between L_S(R(I))) \to 1 \;\epsilon\; L_S(R(I)).$$

Since $\alpha \between L_S(R(I)) \leftrightarrow \alpha \between I$ and $1 \;\epsilon\; L_S(R(I)) \to S \between L_S(R(I)) \leftrightarrow S \between I$, we obtain

$$\forall \alpha \in \mathscr{P}t(A)(S \subseteq \alpha \to \alpha \between I) \to S \between I,$$

for every ideal I.

(2) (→) Let A be a commutative ring satisfying spatiality. For every $a \;\epsilon\; A$, we have in particular

$$\forall \alpha (a \in \alpha \to \alpha \between \emptyset) \to a \lhd_c \emptyset,$$

that is, $\forall \alpha (\neg (a \in \alpha)) \to a \lhd_c \emptyset$, or *Sufficiency*.

(←) Let $a \in A$ and $U \subseteq A$. By hypothesis, $A_{/U}$ satisfies *Sufficiency*, so that

$$\forall \alpha \;\epsilon\; \mathscr{P}t(A_{/U})(\neg (a \;\epsilon\; \alpha)) \to a \lhd_c^U \emptyset.$$

Since $\alpha \;\epsilon\; \mathscr{P}t(A_{/U})$ if and only if $\alpha \;\epsilon\; \mathscr{P}t(A)$ and $\neg(\alpha \between U)$, and $a \lhd_c^U \emptyset$ if and only if $a \lhd_c U$, we have

$$\forall \alpha \;\epsilon\; \mathscr{P}t(A)(a \;\epsilon\; \alpha \to \alpha \between U) \to a \lhd_c U,$$

which is spatiality.

(3) (→) We give a sketch, following the previous points. Let A be a commutative ring satisfying spatiality. For every ideal $I \subseteq A$, spatiality in $(1, R(I))$ is expressed by

$$\forall \alpha (1 \;\epsilon\; \alpha \to \alpha \between R(I)) \to 1 \;\epsilon\; R(I)$$

which is to say that $\forall \alpha (\alpha \between I) \to A = I$.

(←) Let $a \in A$ and $U \subseteq A$. Applying the hypothesis to $A_{S(a)}$, the localisation of A in $S(a)$, and using the isomorphism of Proposition 2.8, we get

$$\forall \alpha \;\epsilon\; \mathscr{P}t(A)(S(a) \subseteq \alpha \to \alpha \between U) \to 1 \lhd_{c,S(a)} U$$

for all $U \subseteq A$. Since $S(a) \subseteq \alpha$ if and only if $a \;\epsilon\; \alpha$, and $1 \lhd_{c,S(a)} U$ if and only if $a \lhd_c U$, we finally have

$$\forall \alpha \;\epsilon\; \mathscr{P}t(A)(a \;\epsilon\; \alpha \to \alpha \between U) \to a \lhd_c U,$$

that is, spatiality for A. □

It is worthwhile to stress that the proof of (←) in point 1 and that of (→) in points 2 and 3 hold instance by instance of the ring A.

2.2.2 Connections with the Zariski Lattice

We present a different and well-known point-free interpretation of the Zariski spectrum (due to Joyal [Joy76]) and briefly recall the link to the formal Zariski topology. In particular, we give a alternative proof of the so-called *formal Nullstellensatz*.

We recall that the Zariski topology on a ring A has as basis the subsets of the form

$$D(a) = \{\mathfrak{p} \in \mathfrak{Spec}(A) : a \notin \mathfrak{p}\}$$

with $a \in A$. This $\mathfrak{Spec}(A)$ is a spectral space: that is, it is sober, i.e. every non-empty irreducible closed subset is the closure of a unique point, and the compact opens form a basis for the topology. It is possible to describe this basis in an formal way, as the collection of opens of the form $D(a_1) \cup \cdots \cup D(a_n)$ with $a_1 \ldots, a_n \in A$; these opens form a distributive lattice satisfying

$$D(0) = \emptyset, \quad D(1) = \mathfrak{Spec}(A),$$
$$D(ab) = D(a) \cap D(b), \quad D(a+b) \subseteq D(a) \cup D(b)$$

for all $a, b \in A$. At this point, one can avoid reference to the collection $\mathfrak{Spec}(A)$ of the prime ideals and formally describe the distributive lattice freely generated by the expressions $\{D(a)\}_{a \in A}$ and subject to the relations

$$D(0) = 0, \quad D(1) = 1, \quad D(ab) = D(a) \wedge D(b), \quad D(a+b) \leqslant D(a,b) \tag{2.10}$$

for all $a, b, b' \;\epsilon\; A$, where

$$D(a_1, \ldots, a_n) \equiv D(a_1) \vee \cdots \vee D(a_n).$$

This lattice is called the *Zariski lattice* of A [CL06, Lom06]. We notice that every element of the lattice can be written in the form $D(a_1, \ldots, a_n)$. Hence the Zariski lattice can be identified with the set $\mathcal{Z} = \mathcal{P}_\omega(A)$ of finite subsets of A, equipped with the minimal partial order relation $\leqslant$ satisfying

$$\{0\} \leqslant \emptyset \quad \{a \cdot b\} \leqslant \{a\} \quad \frac{\{c\} \leqslant \{a\} \quad \{c\} \leqslant \{b\}}{\{c\} \leqslant \{a \cdot b\}} \quad \{a+b\} \leqslant \{a, b\}. \tag{2.11}$$

and

$$U_0 \leqslant V_0 \leftrightarrow (\forall u \;\epsilon\; U_0)(\{u\} \leqslant V_0)$$

for all $a, b \in A$ and $U_0, V_0 \in \mathcal{Z}$. The condition $\{a\} \leqslant \{1\}$ is already entailed by $\{a \cdot b\} \leqslant \{a\}$.

In particular, the distributive lattice $(\mathcal{Z}, \leqslant)$ satisfies

$$\{0\} \leqslant \emptyset \quad \{a \cdot b\} \leqslant \{a\} \quad \{a+b\} \leqslant \{a, b\} \quad \{a\} \leqslant \{a^2\}.$$

for all $a, b \in A$, and therefore $U_0 \lhd_c V_0 \to U_0 \leqslant V_0$ by the induction rule associated to $\lhd_c$. Vice versa, $\lhd_c$ satisfies Convergence (Proposition 2.5) and thus the conditions (2.11) from which we derive $\lhd_c = \leqslant$. In other terms, $(\mathcal{Z}, \lhd_c)$ is exactly the lattice generated by the conditions (2.10). The characterization (2.7) of the formal cover can then be rewritten as follows:

Proposition 2.14. *For all* $a, b_1, \ldots, b_n \,\epsilon\, A$*,* $D(a) \leqslant D(b_1, \ldots b_n)$ *holds if and only if* $a \,\epsilon\, R(\{b_1, \ldots b_n\})$.

The statement above is also called *formal Nullstellensatz* and establishes the link between the lattice and the given structure of the ring, and is the core theorem for the application of formal methods [Coq09, CL06, CLS07, Lom06]. In our treatment, Proposition 2.14 corresponds to the explicit characterization (2.7) of the formal cover.

Chapter 3

A Constructive Notion of Codimension

Note. This chapter is based on the following publication [Rin13]: Davide Rinaldi. *A constructive notion of codimension.* Journal of Algebra, 383:178-196, 2013.

Introduction

In the previous chapter, we have described constructively the Zariski spectrum of a commutative ring A as a suitable inductively generated formal topology[1], the points of which correspond to its prime filters [Sam03, Sch06a, Sch08].

In this chapter, on the same path, we will define suitable formal topologies whose points correspond to the minimal primes, or primes of any fixed height n. By means of the corresponding principle of spatiality, and simple inductive arguments, we can find an elementary characterization of the classical notion of codimension (or height) of an ideal I, which with classical logic and open induction (see Corollary 1.6.1) can be shown equivalent to the customary one. This is based upon the characterization of Krull dimension that was provided earlier by Coquand, Lombardi, and Roy [CL02, CLR05] in the context of distributive lattices, and which we adapted in Section 1.6.1 for general formal topologies. In Section 3.2.1, also a constructively acceptable notion of equidimensionality is given.

We can thus state the Krull's Principal Ideal Theorem in a point-free way. The theorem asserts that in any Noetherian ring A

$$\text{Codim}\,(x_1, \ldots, x_n) > n \quad \to \quad 1 \,\epsilon\, (x_1, \ldots, x_n)$$

for any $x_1, \ldots, x_n \,\epsilon\, A$. As reported in [CL01], this theorem is considered by Kaplansky as "probably the most important single theorem in the theory of

[1] Corresponding to the inductive generation of radical ideals.

Noetherian rings" [Kap74]. The proof we give takes partially advantage of the work of Ducos [Duc09, DQ06] but from a fully constructive standpoint. In particular, it turns out necessary to analyse constructively the notion of Artinian ring. Compared to the constructive proof described in [CL01] by Coquand and Lombardi, this proof is more demanding but follows rather closely the classical theory.

3.1 Krull Dimension of the Zariski Formal Topology

Let A be a commutative ring with unit and $\mathrm{Zar}_f(A) \equiv (A, \lhd_c, \cdot)$ the corresponding Zariski formal topology. We briefly recall that the Zariski formal cover $\lhd_c$ is generated by the following rules

$$\frac{a \;\epsilon\; U}{a \lhd_c U} \qquad \frac{\top}{0 \lhd_c U} \qquad \frac{a \lhd_c U \quad b \lhd_c U}{a + b \lhd_c U} \qquad \frac{a \lhd_c U \quad \lambda \in A}{\lambda \cdot a \lhd_c U} \qquad \frac{a^2 \lhd_c U}{a \lhd_c U} \tag{3.1}$$

and can be characterized explicitly as follows:

$$a \lhd_c U \leftrightarrow a \;\epsilon\; \sqrt{(U)}.$$

for all $a \in A$ and $U \subseteq A$.

We will use in this chapter the simplified notation (U) (resp. $\sqrt{(U)}$) instead of $I(U)$ (resp. $R(U)$) to denote the ideal (resp. radical ideal) generated by $U \subseteq A$.

In particular, the formal opens of the formal Zariski topology are precisely the radical ideals of A. They form a locale, whose compact elements are the *radically finitely generated* ideals:

Definition 3.1. An ideal $I \subseteq A$ is said to be *radically finitely generated* if it is the radical of a finitely generated ideal. In other words, $I = \sqrt{(a_1, \ldots, a_n)}$ for some $a_1, \ldots, a_n \;\epsilon\; A$.

In other words, I is a radically finitely generated ideal if and only if $I =_{\lhd} \{a_1, \ldots, a_n\}$ for some $a_1, \ldots, a_n \;\epsilon\; A$. The set of radically finitely generated ideals inherits the structure of distributive lattice from the frame of formal opens, as shown in Section 2.2.2.

As already stressed, the intuition behind a formal topology is that of a set of basis indices for an ideal space of points. Vice versa, starting from a formal topology, a space of formal points $\mathscr{P}t(A)$ is always canonically, even though impredicatively, provided. In particular, in the case of the Zariski formal topology, formal points coincide with the complements of prime ideals.

This topological space induces a new formal topology structure on A, say $(A, \lhd_{\mathcal{P}t}, *)$. Explicitly, the cover $\lhd_{\mathcal{P}t}$ takes the following form

$$a \lhd_{\mathcal{P}t} U \equiv \forall \alpha (a \;\epsilon\; \alpha \to \alpha \between U). \tag{3.2}$$

One has directly $a \lhd U \to a \lhd_{\mathcal{P}t} U$, while the reverse implication

$$\forall \alpha (a \ \epsilon \ \alpha \to \alpha \between U) \to a \lhd U$$

is non-trivial and called *spatiality*. This corresponds to the *Krull's Lemma*, and is proved with classical logic and Zorn's Lemma. In our setting, this is consequence of Corollary 1.6.1, and works for all finitary formal topologies in presence of classical logic and Open Induction.

In the classical exposition, it is often the case that, in the presence of topological spaces arising in the algebraic context, ideal reasoning with points produces finally a concrete statement by means of the corresponding instance of spatiality. Nevertheless this is often hidden and implicit. In what follows, we will give a clear example of this procedure.

One of the most useful notion in commutative algebra is that of Krull's dimension, which gives a measure of the complexity of a ring A. The customary definition raises many issues of impredicativity.

Definition 3.2 (Classical,[Eis95, MR89]). Let A be a commutative ring. The *Krull dimension* of A is the supremum of the lengths of chains of distinct prime ideals:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \cdots \subsetneq \mathfrak{p}_n$$

In Section 1.6.1, we have shown how to make this definition constructively available for all finitary formal topologies with finitary operation. We make it explicit here for the Zariski formal topology and we give some equivalent versions.

For all $U, V \subseteq A$, we define the following subset (and, in fact, radical ideal):

$$c \ \epsilon \ (U \to_c V) \equiv c \cdot U \lhd_c V.$$

One defines the boundary ideal $N(a)$ for all $a \in A$ as the radical ideal generated by the subset

$$\{a\} \cup a \to_c \emptyset.$$

Definition 3.3 ([CLR05]). For all $n \in \mathbb{N} \cup \{-1\}$, we write Kdim $A \leqslant n$ and we say that the ring A has Krull dimension smaller or equal to n, if Kdim $\mathrm{Zar}_f(A) \leqslant n$.

Notice that, by Proposition 2.7, the quotient of $\mathrm{Zar}_f(A)$ in $N(a)$ is isomorphic to $\mathrm{Zar}_f(A/N(a))$, that is, the Zariski formal topology on the quotient ring $A/N(a)$. Then, Kdim $A \leqslant n$ holds if and only if:

$(n = -1)$ $1 = 0$ in A;

$(n \geqslant 0)$ For all $a \ \epsilon \ S$, Kdim $A/N(a) \leqslant n - 1$.

Notice that the Zariski cover $\lhd_c$ satisfies the following property

$$a \lhd_c U \cup V \leftrightarrow (u \ \epsilon \ U)(v \ \epsilon \ V)(a \lhd_c \{u\} \cup \{v\}) \tag{3.3}$$

for all $a \in A$ and $U, V \subseteq A$. This follows directly from the explicit characterization of $\lhd_c$. By using this property, and unfolding the previous definition, we can characterize the Krull dimension as follows:

Proposition 3.1. *Let A be a commutative ring. Then* Kdim $A \leqslant n$*, with $n \geqslant 0$, if and only if for all $a_0, \dots, a_n \in A$, there exist $b_0, \dots, b_n \in A$ such that*

$$1 \lhd_c \{a_0, b_0\}, \;\; a_0 \cdot b_0 \lhd_c \{a_1, b_1\}, \;\; \dots \;\; a_{n-1} \cdot b_{n-1} \lhd_c \{a_n, b_n\}, \;\; a_n \cdot b_n \lhd_c 0.$$

Proof. We show it for $n = 0, 1$. We have Kdim $A \leqslant 0$ if and only if $A/N(a)$ is the trivial ring, i.e. $1 \mathrel{\epsilon} N(a)$, for all $a \in A$. This amounts to say, because of the property (3.3), that for all a_0 there is $b \mathrel{\epsilon} a \to_c \emptyset$ (that is, $a \cdot b \lhd_c \emptyset$) such that $1 \lhd_c \{a_0, b_0\}$. We have Kdim $A \leqslant 1$ if and only if Kdim $A/N(a_1) \leqslant 0$ for all $a_1 \in A$, that is, because of the previous case, if and only if for all $a_0 \in A$ we can find $b_0' \in A$ such that

$$1 \lhd_c \{a_0, b_0'\} \cup N(a_1), \quad a_0 \cdot b_0' \lhd_c N(a_1).$$

Because of 3.3, we can find $b_0 \mathrel{\epsilon} \{b_0'\} \cup N(a_1)$ such that $1 \lhd_c \{a_0, b_0\}$, and, in particular, $a_0 \cdot b_0 \lhd_c a_0 \cdot b_0' \cup N(a_1) \lhd_c N(a_1)$. Similarly, we can find $b_1 \mathrel{\epsilon} a_1 \to \emptyset$ such that $a_0 \cdot a_1 \lhd_c \{a_1, b_1\}$. In all, we have Kdim $A \leqslant 1$ if and only if, for all $a_0, a_1 \in A$ there are $b_0, b_1 \in A$ such that

$$1 \lhd_c \{a_0, b_0\}, \quad a_0 \cdot b_0 \lhd_c \{a_1, b_1\}, \quad a_1 \cdot b_1 \lhd_c \emptyset.$$

The same argument can be carried on by induction on n. □

Example 3.2. A ring A is called zero-dimensional if Kdim $A \leqslant 0$. This means that for all $a \in A$ exists $k \in \mathbb{N}$, $\lambda \in A$ such that $a^k \cdot (1 - \lambda \cdot a) = 0$.

We define iteratively the so-called *k-th boundary ideals* ($k \geqslant 0$) as follows

$$x \mathrel{\epsilon} N_0(a_0) \equiv x \mathrel{\epsilon} N(a_0) \equiv x \mathrel{\epsilon} \sqrt{\{a_0\} \cup (a_0 \to_c \emptyset)}$$
$$x \mathrel{\epsilon} N_k(a_0, \dots, a_k) \equiv x \mathrel{\epsilon} \sqrt{\{a_k\} \cup (a_k \to_c N(a_0, \dots, a_{k-1}))}$$

for any $k \in \mathbb{N}$ and $a_0, \dots, a_k \in A$. Essentially, the $k-$th boundary ideal $N_k(a_0, \dots, a_k)$ corresponds to the boundary ideal of a_k in the quotient ring $A/N_{k-1}(a_0, \dots, a_{k-1})$. One then checks that Kdim $A \leqslant n$ (with $n \geqslant 0$) if and only if

$$1 \mathrel{\epsilon} N_n(a_0, \dots, a_n) \tag{3.4}$$

for all $a_0, \dots, a_n \in A$. More generally, we have:

Proposition 3.3. *Let A be a commutative ring, $k, n \in \mathbb{N}$ and $k \leqslant n$. Then*

$$\text{Kdim } A \leqslant n \leftrightarrow \forall a_0, \dots, a_k (\text{Kdim } A/N_k(a_0, \dots, a_k) \leqslant n - k - 1). \tag{3.5}$$

3.2 The $\leqslant k$-topologies and Ring Codimension

In this section we will obtain a predicative candidate for the notion of codimension for an ideal in a commutative ring [Eis95, MR89]. However, most of the procedures here sketched could be generalized to any spectral space without big effort. To this end, we first recall the classical impredicative notion of codimension:

Definition 3.4 (Classical)**.** Let A be a commutative ring and $I \subseteq A$ an ideal. One defines

$$\text{Codim } I = \inf_{\mathfrak{p} \supseteq I} \text{Kdim } A_{\mathfrak{p}}.$$

It is direct to see that Codim I = Codim $\sqrt{I}$, since $\mathfrak{p} \supseteq I \leftrightarrow \mathfrak{p} \supseteq \sqrt{I}$. Given $k \in \mathbb{N}$, one has Codim $I \leqslant k$ if and only if there exists a prime ideal $\mathfrak{p}$ in R such that Kdim $A_{\mathfrak{p}} \leqslant k$ and $I \subseteq \mathfrak{p}$. In a more compact way,

$$\text{Codim } I \leqslant k \quad \leftrightarrow \quad \exists \mathfrak{p}(I \subseteq \mathfrak{p} \;\&\; \text{Kdim } A_{\mathfrak{p}} \leqslant k).$$

The sentence is not yet stated in positive terms since the localization in $\mathfrak{p}$ requires a complementation. We are then interested in expressing it through prime filters. A few classical equivalences lead finally to

$$\text{Codim } I > k \quad \leftrightarrow \quad \forall \alpha(\text{Kdim } A_{\alpha} \leqslant k \to I \between \alpha).$$

By Proposition 2.8, we know that the Zariski formal topology on A_α is isomorphic to the localization $(A, \lhd_{c,\alpha}, \cdot)$ of $\text{Zar}_f(A)$ in α. The cover $\lhd_{c,\alpha}$ is defined explicitly as follows

$$a \lhd_c^{\alpha} U \equiv (\exists b \in \alpha)(a \cdot b \lhd_c U).$$

Therefore, given $\alpha \subseteq A$ prime filter, the condition Kdim $A_\alpha \leqslant k$ can be expressed explicitly, through the characterization (3.4), as

$$N(a_0, \ldots, a_k) \between \alpha, \tag{3.6}$$

for all $a_0, \ldots, a_k \in A$.

We will proceed now by defining a formal topology on A, coarser than the Zariski formal topology and whose points α are precisely the prime filters satisfying condition (3.6). In order to do this, we need to impose some condition on the ring A.

The classical notion of Noetherianity does not behave well constructively, since it is not provable even for discrete fields[2]. In the following, we will adopt instead the notion proposed by Richman [LQ12, MRR88, Ric74, Sei75] :

Definition 3.5 (Richman-Seidenberg)**.** A module M over a commutative ring R is said to be *Noetherian*, if for any ascending sequence $\{I_n\}_{n \in \mathbb{N}}$ of finitely generated submodules, there exists $\overline{n} \in \mathbb{N}$ such that $I_{\overline{n}} = I_{\overline{n}+1}$. A ring A is Noetherian if it is Noetherian as an A-module.

[2] See [Ric74] for a Brouwerian counterexample.

Here follows the definition of *coherent*, *discrete* and *strongly discrete* ring [LQ12]:

Definition 3.6. A commutative ring with unit A is said to be:

1. *coherent* if every finitely generated ideal is finitely presented;
2. *discrete* if the underlying set A is discrete;
3. *strongly discrete* if the membership relation is decidable on every finitely generated ideal $I \subseteq A$. Equivalently, if the quotient of A on a finitely generated ideal I is discrete.

One proves [LQ12] that, in a coherent commutative ring A, the *transporter ideal*

$$a \;\epsilon\; (V_0 : U_0) \equiv b \cdot U_0 \lhd V_0 \equiv b \cdot U_0 \;\epsilon\; V_0$$

is finitely generated, for all $U_0, V_0 \subseteq A$ finite.

Proposition 3.4 ([CL02])**.** *Let A be Noetherian and coherent. Then, for all $a \in A$ and $U_0 \subseteq A$ finite, $a \to_c U_0$ is radically finitely generated.*

Proof. Fix $a \in A$ and consider the following ideals, for all $k \in \mathbb{N}$:

$$b \;\epsilon\; I_k \equiv a^k b \lhd U_0 \equiv b \;\epsilon\; (U_0 : a^k).$$

The ideals $\{I_k\}_{k\in\mathbb{N}}$ form an ascending chain, and, since A is coherent, each I_k is finitely generated. Since A is Noetherian, there must exist $\overline{k}$ such that $I_{\overline{k}} = I_{\overline{k}+1}$. This implies that $\{I_k\}_{k\in\mathbb{N}}$ is stationary, and therefore $\bigcup_{k\in\mathbb{N}} I_k$ is finitely generated, say by $V_0 \subseteq_\omega A$. Then $V_0 =_\lhd a \to U_0$. □

In particular, if A is Noetherian and coherent, every k-th boundary ideal $N(a_0, \ldots, a_k)$ is radically finitely generated by a finite subset $N_{a_0,\ldots,a_k}$.

We need the following general lemma:

Proposition 3.5. *Let $(A, \lhd_c, \cdot)$ be the Zariski formal topology on A and $\mathcal{U} = \{U_i\}_{i\in I}$ a family of finite subsets of A. Let $\lhd_\mathcal{U}$ a new cover on A, defined by induction starting from the rules (3.1) of $\lhd_c$ and adding the induction rules ($i \in I$)*

$$\frac{x \cdot U_i \lhd_\mathcal{U} U}{x \lhd_\mathcal{U} U}\; R_i \quad . \tag{3.7}$$

The cover $\lhd_\mathcal{U}$ can be then characterized as follows:

$$a \lhd_\mathcal{U} U \leftrightarrow (\exists I_0 \subseteq_\omega I)(a \cdot \prod_{i \;\epsilon\; I_0} U_i \lhd_c U) \tag{3.8}$$

for all $a \in A$ and $U \subseteq A$. As a consequence[3]*, $(A, \lhd_\mathcal{U}, \cdot)$ is a formal topology on A.*

[3] A similar lemma could be proved for more general kinds of formal topology.

Proof. The implication ($\leftarrow$). We denote by $a \,\epsilon\, P$ the right member of (3.8) and we show the converse implication by induction on $\lhd_{\mathcal{U}}$: if $a \,\epsilon\, U$ then $a \lhd_c U$, and therefore $a \,\epsilon\, P$. If $a, a' \,\epsilon\, P$, this means

$$a \cdot \prod_{i \,\epsilon\, I_0} U_i \lhd_c U, \qquad a' \cdot \prod_{i \,\epsilon\, I_0'} U_i \lhd_c U$$

for some $I_0, I_0' \subseteq I$ finite. In particular, by $Left$, we have

$$a \cdot \prod_{i \,\epsilon\, I_0 \cup I_0'} U_i \lhd_c U, \qquad a' \cdot \prod_{i \,\epsilon\, I_0 \cup I_0'} U_i \lhd_c U$$

and therefore

$$(a + a') \cdot \prod_{i \,\epsilon\, I_0 \cup I_0'} U_i \lhd_c U$$

for all $a, a' \in A$ and $U \subseteq A$. With similar arguments, one shows that P is closed under the other rules for $\lhd_c$. Finally, if $a \cdot U_i \subseteq P$, and $U_i \equiv \{u_1^i, \ldots, u_m^i\}$, then for all $j = 1, \ldots, m$ there is $I_0^j \subseteq I$ finite such that

$$a \cdot u_j^i \cdot \prod_{i \,\epsilon\, I_0^j} U_i \lhd_c U.$$

By setting $I_0 \equiv \{i\} \cup \bigcup_{j=1}^m I_0^j$, we have $a \cdot \prod_{i \,\epsilon\, I_0} U_i \lhd_c U$, that is $a \,\epsilon\, P$. Through the characterization (3.8), it can be directly verified that $(A, \lhd_{\mathcal{U}}, \cdot)$ satisfies $Left$ and $Right$. □

Thanks to this lemma, if A is Noetherian and coherent, we can define, for each $k \,\epsilon\, \mathbb{N}$, a new finitary formal cover on A, generated by induction through the same rules of the Zariski formal topology, together with

$$\frac{x \cdot N_{a_0, \ldots, a_k} \lhd_{\leqslant k} U \quad a_0, \ldots, a_k \in A}{x \lhd_{\leqslant k} U} \tag{3.9}$$

For each $k \in \mathbb{N}$, the product $\cdot$ provides $(A, \lhd_{\leqslant k}, \cdot)$ with the structure of formal topology. The points of this topology consist of the prime filters α which split the new axioms, namely,

$$x \,\epsilon\, \alpha \to x \cdot N_{a_0, \ldots, a_k} \between \alpha \leftrightarrow N(a_0, \ldots, a_k) \between \alpha.$$

for all $a_0, \ldots, a_k \,\epsilon\, A$. In the light of (3.6), these are exactly the prime filters such that Kdim $A_\alpha \leqslant k$. As shown in the preceding section (3.2), these points define, impredicatively, a cover on A, which takes here the following form

$$a \lhd_{\leqslant k}^{\mathcal{P}t} U \equiv \forall \alpha (\text{Kdim } A_\alpha \leqslant k \;\&\; a \,\epsilon\, \alpha \to U \between \alpha).$$

Remembering that 1 lies in every prime filter α, we get

$$1 \lhd_{\leqslant k}^{\mathcal{P}t} I \equiv \forall \alpha (\text{Kdim } A_\alpha \leqslant k \to I \between \alpha).$$

The cover $\lhd_{\leqslant k}$ is finitary, and therefore, thanks to Corollary 1.6.1, with classical logic and open induction we have $a \lhd_{\leqslant k}^{\mathcal{P}t} U$ if and only if $a \lhd_{\leqslant k} U$. We can summarize this observations as follows:

Proposition 3.6. *Let A be a Noetherian and coherent ring. For every ideal I of A and $k \in \mathbb{N}$, the following equivalences hold*

$$1 \lhd_{\leqslant k} I \overset{(*)}{\leftrightarrow} 1 \lhd^{\mathcal{P}t}_{\leqslant k} I \leftrightarrow \text{Codim } I > k,$$

where (←), in (∗), requires classical logic and open induction.

The requirements on the ring A hold classically for a Noetherian ring. Our constructive assumptions are nevertheless wide enough, since there seems to be no basic classical application of the notion of codimension involving non-Noetherian rings [Eis95, MR89].

By using the characterization (3.8), we can explicit the cover $\lhd_{\leqslant k}$ as follows:

Proposition 3.7. *If A is a Noetherian and coherent ring, then*

$$a \lhd_{\leqslant k} U \leftrightarrow (\exists \{u_{i,j}\}_0^{l,k} \in A)(a \cdot \prod_{j=0}^{l} N(u_{0,j}, \ldots, u_{k,j}) \subseteq \sqrt{(U)}),$$

for all $k \in \mathbb{N}$, $a \in A$ and $U \subseteq A$.

As a direct corollary, we have an explicit form for the notion of codimension:

Corollary 3.8. *(OI+CL) If A is a Noetherian and coherent ring, then the following equivalences hold*

$$\text{Codim } I \ > k \ \overset{OI+CL}{\leftrightarrow} \ 1 \lhd_{\leqslant k} I \ \leftrightarrow \ \exists \{u_{i,j}\}_0^{l,k} (\prod_{j=0}^{l} N(u_{0,j}, \ldots, u_{k,j}) \subseteq \sqrt{I}).$$

In plain terms, Codim $I > k$ if and only if $\sqrt{I}$ contains a product of k^{th}-boundary ideals. Using the classical definitions, the following inequality is immediate:

$$\text{Kdim } A \geqslant \text{Kdim } A/I + \text{Codim } I.$$

A constructive version of this property can be carried out, by means of the following lemma [LQ12, Ch.XIII, §3]:

Definition 3.7 (Lombardi, Quitté)**.** Let $I_0, \ldots, I_m \subseteq A$ ideals. Then:

$$\text{Kdim } A/\prod_{s=0}^{m} I_s = \sup_s \text{Kdim } A/I_s.$$

Suppose from now on A to be a Noetherian and coherent ring.

Proposition 3.9. *Let $I \subseteq A$ be an ideal of a Noetherian and coherent ring A. Then for all $k, m \in \mathbb{N}$,*

$$\text{Codim } I > k \ \& \ \text{Kdim } A \leqslant m \ \longrightarrow \ \text{Kdim } A/I \leqslant m - k - 1.$$

Proof. By definition of codimension, we can find $u_{i,j} \in A$ such that

$$\prod_{j=0}^{l} N(a_{0,j}, \ldots, a_{k,j}) \subseteq \sqrt{I}.$$

By Remark 3.3, for all $j = 0, \ldots, l$, Kdim $A/N(a_{0,j}, \ldots, a_{k,j}) \leqslant m - k - 1$, and by the previous lemma we get Kdim $A/\prod_{j=0}^{l} N(a_{0,j}, \ldots, a_{k,j}) \leqslant m - k - 1$. Since

$$\prod_{j=0}^{l} N(a_{0,j}, \ldots, a_{k,j}) \subseteq \sqrt{I},$$

we have finally

$$\text{Kdim } A/I \leqslant \text{Kdim } A/\prod_{j=0}^{l} N(a_{0,j}, \ldots, a_{k,j}) \leqslant m - k - 1.$$

□

Given $I \subseteq A$ ideal, we denote in the following by $N_I(x_0, \ldots, x_k)$ the k-th boundary ideal of $x_0, \ldots, x_k$ in the ring A/I. For $k = 0$, this ideal is defined explicitely by

$$N_I(x) = \sqrt{(x) + (x \to_c I)}.$$

Notice that $N_I(x) \cdot N_J(x) \subseteq N_{I \cdot J}(x)$, for all ideals $I, J \subseteq A$ and $x \in A$.

3.2.1 Nilregular Elements and Equidimensionality

An element $a \in A$ is said to be *nilregular* [CLS05] if and only if $(a \to_c 0) \lhd_c 0$. In other words, a is nilregular if and only if for all $x \in A$, if ax is nilpotent then so is x. In the same fashion, a subset $U \subseteq A$ is called *nilregular* if and only if $(U \to_c 0) \lhd_c 0$. It is straightforward to show that a product of nilregular elements (resp. subsets) is nilregular, and that if $U \lhd_c V$ and U is nilregular, then also V is.

The following theorem introduce the so-called *nilregular element property.*

Theorem 3.10. *[CLS05] Let A be a Noetherian, coherent and discrete ring. Then every radically finitely generated nilregular ideal contains a nilregular element.*

Suppose A to be a Noetherian, coherent and discrete ring. By the previous theorem, every (0-th) boundary ideal must contain a nilregular element, since it is nilregular and radically finitely generated. Notice moreover that, if $a \in A$ is nilregular, then $N(a) \equiv \sqrt{(a)}$. Therefore, every nilregular radical ideal contains a boundary ideal. Since every boundary ideal is also nilregular, we have:

Corollary 3.11. *[CLS05] Let A be a Noetherian, coherent and discrete ring and S the monoid of its nilregular elements. Then* Kdim $A_S \leqslant 0$.

Let $I \subseteq A$ be with Codim $I > 0$; that means, there exist $u_0, \ldots, u_k \in A$ such that $\prod_{i=0}^{k} N(u_i) \subseteq \sqrt{I}$. Since $\prod_{i=0}^{k} N(u_i)$ is nilregular, there exists $u \,\epsilon\, A$, such that

$$N(u) \lhd_c \prod_{i=0}^{k} N(u_i) \subseteq \sqrt{I}.$$

We collect these observations in the following proposition:

Proposition 3.12. *Let A be a Noetherian, coherent and discrete ring and $I \subseteq A$ ideal. Then the following are equivalent:*

1. Codim $I > 0$,

2. $\sqrt{I}$ *contains a boundary ideal, i.e.,* $\exists u(N(u) \subseteq \sqrt{I})$,

3. I *is nilregular.*

If the ring is also strongly discrete, we can extend the argument as follows:

Proposition 3.13. *Let A be a Noetherian, coherent and strongly discrete ring and $I \subseteq A$ ideal. Then the following are equivalent:*

1. Codim $I > k$,

2. $\sqrt{I}$ *contains a k-th boundary ideal, i.e.,* $\exists u_0, \ldots u_k(N(u_0, \ldots, u_k) \subseteq \sqrt{I})$.

Proof. The implication (2⇒1) holds by definition of codimension. We prove (1⇒ 2) by induction on the length k. The case $k = 0$ is part of Proposition 3.12. If the statement is true for $k-1$, suppose Codim $I > k$, that is, $\sqrt{I}$ contains the product of the boundary ideals $\{N(y_i^0, \ldots, y_i^k)\}_{i=0}^{n}$. For each i, $N(y_i^0)$ is a nilregular ideal, so that we can find $u_i^0 \,\epsilon\, N(y_i^0)$ such that

$$N(u_i^0, y_i^1, \ldots, y_i^k) \subseteq N(y_i^0, \ldots, y_i^k).$$

The element $u_0 = \prod_{i=0}^{n} u_i^0$ is nilregular and satisfies

$$N(u_0, y_i^1, \ldots, y_i^k) \subseteq N(u_i^0, y_i^1, \ldots, y_i^k).$$

for each i. Since $N(u_0, y_i^1, \ldots, y_i^k)$ coincides with $N_{N(u_0)}(y_i^1, \ldots, y_i^k)$ in the quotient $A/N(u_0)$, by induction hypothesis we can find $u_1, \ldots, u_k$ such that

$$N(u_0, u_1 \ldots, u_k) \subseteq \prod_{i=0}^{n} N(u_i^0, y_i^1, \ldots, y_i^k) \subseteq \prod_{i=0}^{n} N(y_i^0, \ldots, y_i^k).$$

Hence, we have found $u_0, \ldots, u_k$ such that $N(u_0, u_1 \ldots, u_k) \subseteq \sqrt{I}$. □

This characterization gives a further simplification of the notion of codimension in the strongly discrete case. Moreover, the elements

$$u_0, u_1, \ldots, u_k \,\epsilon\, \sqrt{I}$$

found in the preceding proof satisfy the following property:

1. u_0 is nilregular,
2. u_i is nilregular over $A/(u_0, \ldots, u_{i-1})$, for all $1 \leqslant i \leqslant k$.

Since every regular element is, in particular, nilregular, we have as usual a clear link with the notion of depth of an ideal I [Eis95, MR89].

We conclude this section with some considerations about the notion of equidimensionality. Let us start from the classical definition:

Definition 3.8 (Classical)**.** Let A be a commutative ring. We say that A is equidimensional if for each $\mathfrak{p}_1, \mathfrak{p}_2$ minimal prime ideals we get

$$\text{Kdim } A/\mathfrak{p}_1 = \text{Kdim } A/\mathfrak{p}_2.$$

Since in that case Kdim $A/\mathfrak{p}_1 =$ Kdim A, this can be restated as:

Definition 3.9. Let A be a commutative ring. We say that A is equidimensional if for each $n \in \mathbb{N}$ and $\mathfrak{p}$ minimal prime ideal the following implication holds

$$\text{Kdim } A/\mathfrak{p} \leqslant n \to \text{Kdim } A \leqslant n.$$

The source of impredicativity is still given by the quantification on the minimal prime ideal $\mathfrak{p}$. To get rid of the minimal prime ideals, since we can deal only with the prime filters, we pose it as follows

$$\exists_{Max}\alpha(\text{Kdim } A/I \leqslant n \;\&\; \neg(\alpha \between I)) \to \text{Kdim } A \leqslant n$$

for each finitely generated ideal I, where α varies over the maximal filters of A. A few classical equivalences lead to

$$\text{Kdim } A > n \to (\text{Kdim } A/I \leqslant n \to \forall_{Max}\alpha(\alpha \between I)).$$

Finally, we can state it as follows

$$\text{Kdim } A > n \to (\text{Kdim } A/I \leqslant n \to \text{Codim } I > 0).$$

This is already a constructively acceptable characterization of the notion of equidimensionality. Proposition 3.12 allows a further simplification:

Corollary 3.14. *A Noetherian, coherent and discrete ring A is equidimensional if and only if*

$$\text{Kdim } A > n \to (\text{Kdim } A/I \leqslant n \to I \textit{ nilregular})$$

for any finitely generated ideal $I \subseteq A$.

3.3 Krull's Principal Ideal Theorem

In this section, we will give, under suitable hypotheses, a constructive proof of Krull's principal ideal theorem [Eis95, MR89]. This theorem links, in the Noetherian setting, the codimension of an ideal I to the number of its generators. Here follows the (contrapositive of) the classical statement:

Theorem 3.15 (PIT). *Let A be a Noetherian ring and $x_1, \ldots, x_n \ \epsilon \ A$. Then*

$$\text{Codim } (x_1, \ldots, x_n) > n \quad \to \quad 1 \ \epsilon \ (x_1, \ldots, x_n).$$

Along the lines of Richman's definition of Noetherianity, we introduce the dual constructive notion of Artinian module.

Definition 3.10. A module M over a commutative ring A is said to be *Artinian*, if it satisfies the descending chain condition, that is, for any descending sequence $\{I_n\}_{n\in\mathbb{N}}$ of finitely generated submodules, there exists $\overline{n} \in \mathbb{N}$ such that $I_{\overline{n}} = I_{\overline{n}+1}$. A ring A is Artinian if it is an Artinian A-module.

In the classical setting, one proves that any zero-dimensional Noetherian ring is Artinian[4]. In the constructive setting, this implication is more delicate. Here is our attempt to dress it constructively.

Notice that, if $\{I_n\}_{n\in\mathbb{N}}$ is an ascending (resp. descending) sequence of radically finitely generated ideals in a Noetherian (resp. Artinian) ring, still there exists $\overline{n} \in \mathbb{N}$ such that $I_{\overline{n}} = I_{\overline{n}+1}$; in fact, let K_n a set of radical generators for I_n, and we can suppose that $(K_n) \subseteq (K_{n+1})$ for every $n \in \mathbb{N}$ (resp. $(K_{n+1}) \subseteq (K_n)$). Then, there exists $\overline{n} \in \mathbb{N}$ such that $(K_{\overline{n}}) = (K_{\overline{n}+1})$ and therefore

$$I_{\overline{n}} = \sqrt{(K_{\overline{n}})} = \sqrt{(K_{\overline{n}+1})} = I_{\overline{n}+1}.$$

In other words, a Notherian (resp. Artinian) ring gives rise to a Noetherian (resp. Artinian) lattice of radically finitely generated ideals.

Remark 15. Spelling out Definition 3.3, a ring A is zero dimensional if the corresponding lattice of radically finitely generated ideals is a Boolean algebra. This can be rephrased by saying that, for each finitely generated ideal I,

$$1 \ \epsilon \ I + (I \to_c 0).$$

If the ring under consideration is Noetherian and coherent, one proves that $(I \to_c 0)$ is a radically finitely generated ideal. If the ring is reduced, then $(I \to_c 0)$ coincides with $(0 : I)$, the usual quotient ideal.

Proposition 3.16. *Let A be a coherent, Noetherian and zero-dimensional ring, and let $\{I_n\}_{n\in\mathbb{N}}$ be a descending sequence of ideals. Then there exists $n \in \mathbb{N}$ such that $I_{\overline{n}} \subseteq I_{\overline{n}+1} + \sqrt{(0)}$.*

[4] Classically, one proves also the converse [AM69].

Proof. Since the sequence $\{I_n\}_{n\in\mathbb{N}}$ is descending, the sequence $\{(I_n \to_c 0)\}_{n\in\mathbb{N}}$ of radically finitely generated ideals[5] is ascending, and by Noetherianity there exists $\overline{n}$ such that $I_{\overline{n}} \to_c 0 = (I_{\overline{n}+1} \to_c 0)$. Since the ring A is zero-dimensional

$$1 \;\epsilon\; I_{\overline{n}+1} + (I_{\overline{n}+1} \to_c 0) = I_{\overline{n}+1} + (I_{\overline{n}} \to_c 0).$$

Then, multiplying both sides by $I_{\overline{n}}$, we finally obtain

$$I_{\overline{n}} \subseteq I_{\overline{n}+1} + \sqrt{(0)},$$

since $I_{\overline{n}} \cdot (I_{\overline{n}} \to_c 0) \subseteq \sqrt{(0)}$. □

Remark 16. As a consequence, if A is a reduced, zero-dimensional, coherent and Noetherian, then it is Artinian, since $\sqrt{(0)} = (0)$.

The constructive notion of Artinianity is (almost) an additive property, like its classical counterpart:

Proposition 3.17. *If we have a short exact sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \longrightarrow 0$$

of A-modules, M, M'' are Artinian and coherent, and M is of finite type, then also M' is Artinian.

Proof. We can suppose $M < M'$ finitely generated submodule and $M'' = M'/M$. Let $\{I_n\}_{n\in\mathbb{N}}$ be a descending sequence of finitely generated submodules in M'. Since M'/M is Artinian, there exists m_0 such that $\beta I_{m_0} = \beta I_{m_0+1}$, that is, a finitely generated submodule $K_0 < M$ such that $I_{m_0} = I_{m_0+1} + K_0$. We can take $K_0 = I_{m_0} \cap M$, that is finitely generated by coherence. Iterating this construction, we construct a descending sequence $\{K_{m_i}\}_{i\in\mathbb{N}}$ of finitely generated submodules of M, such that

$$I_{m_i} = I_{m_i+1} + K_{m_i}.$$

By Artinianity of M, we find k such that $K_k = K_{k+1}$ and therefore $K_k \subseteq K_{k+1} \subseteq I_{m_{k+1}} \subseteq I_{m_k+1}$, so that finally $I_{m_k} = I_{m_k+1}$. □

Remark 17. Notice that, in the previous lemma, also the converse holds. More precisely, if M' is Artinian and coherent, and M is finitely generated, then both M and M'' are Artinian. To prove this, it is enough to reverse the inclusions in the corresponding proof of Theorem 2.1 [MRR88]. Therefore, if A is a coherent and Artinian ring and I a finitely generated ideal, then A/I is still Artinian (and coherent).

The following lemma emerged while looking for the constructive meaning of [Eis95, Theorem 2.14]:

[5] Coherence is needed to prove that they are all radically finitely generated.

Proposition 3.18. *Let A be a coherent and strongly discrete ring, and $c \in A$ such that $A/(c)$ is Artinian. Then $A/(c^2)$ is Artinian.*

Proof. Consider the short exact sequence of A-modules

$$0 \longrightarrow A/(c^2 : c) \xrightarrow{(-)\cdot c} A/(c^2) \xrightarrow{\pi} A/(c) \longrightarrow 0.$$

By the previous remark, since the finitely generated ideal $(c^2 : c)$ contains (c) and $A/(c)$ is Artinian, then also $A/(c^2 : c)$ is so; by the previous lemma, it follows that $A/(c^2)$ is Artinian. □

Remark 18. Iterating the Lemma we can prove more generally that, if $A/(c)$ is Artinian, then so is $A/(c^n)$ for any $n \in \mathbb{N}$.

Corollary 3.19. *Let A be a coherent and strongly discrete ring, and I finitely generated ideal such that also $\sqrt{I}$ is a finitely generated ideal. Then $A/\sqrt{I}$ Artinian implies A/I Artinian.*

Proof. Let $\sqrt{I} = (a_1, \ldots, a_n)$. By strong discreteness, we can decide whether $a_n \in I$ or not. If the second case holds, since $a_n^k \in I$ for k big enough, we can use Remark 18 to prove that $A/I + (a_1, \ldots, a_{n-1})$ is Artinian. Iterating this procedure, we eliminate all the generators and we prove the Artinianity of A/I. □

Theorem 3.20. *Let A be a zero-dimensional, Noetherian, coherent and strongly discrete ring such that the nilradical is finitely generated. Then A is Artinian.*

Proof. By Lemma 3.16, the ring $A/\sqrt{(0)}$ is Artinian. We are then under the hypothesis of Corollary 3.19, with $I = (0)$. □

A useful constructive and alternative notion of Noetherianity was pointed out by Martin-Löf and worked out in [JL91]. We sketch briefly the relationship between this notion and the former one in this specific case.

Definition 3.11. Let A be a ring. We say that a finitely generated ideal $I \subseteq A$ is *blocked* if every finitely generated ideal J strictly containing I is blocked. The ring A is said to be *(ML)-Noetherian* if it is coherent and strongly discrete, and (0) is a blocked ideal.

In an (ML)-Noetherian ring we therefore have at our disposal the following induction principle on finitely generated ideals

$$\forall I(\forall J(J > I \to \mathcal{U}(J)) \to \mathcal{U}(I)) \to \forall K(\mathcal{U}(K)),$$

the so-called Noetherian induction.

Proposition 3.21. *A ring A has a* radicality test *if for each finitely generated ideal $I \subseteq A$ we can decide whether I is radical or not, and, if the answer is negative, we can find $b \in A$ such that $b^2 \in I$ and $b \notin I$.*

Proposition 3.22. *Let A be an (ML)-Noetherian ring with a radicality test. Then every radically finitely generated ideal is finitely generated.*

Proof. We prove it by Noetherian induction[6]. Let $\mathcal{U}$ be the following predicate

$$\mathcal{U}(I) \equiv \sqrt{I} \text{ is finitely generated}$$

where I is a finitely generated ideal. Let's fix I and suppose $\mathcal{U}(J)$ for all $J > I$. Since we have a radicality test, we can decide whether I is radical or not, and, if the answer is negative, we can find $b \in A$ such that $b^2 \in I$ and $b \notin I$. In the first case, $I = \sqrt{I}$ is trivially finitely generated so $\mathcal{U}(I)$ holds; in the second case, for such b we have

$$(I : b) > I \quad \text{and} \quad I + (b) > I$$

and since by coherence $(I : b)$ is finitely generated, we have $\mathcal{U}(I : b)$ and $\mathcal{U}(I + (b))$. This means that both $\sqrt{(I : b)}$ and $\sqrt{I + (0)}$ are finitely generated and, by coherence, it follows that

$$\sqrt{I} = \sqrt{(I : b)} \cap \sqrt{I + (b)}$$

is finitely generated. So $\mathcal{U}(I)$ holds. By Noetherian induction, $\mathcal{U}(K)$ follows for every finitely generated ideal K. □

As a consequence, a zero-dimensional (ML)-Noetherian ring with a radicality test is Artinian. We are now about to prove a constructive version of Krull's PIT [Eis95]. The following lemma pops up while unravelling constructively the original proof of Krull's PIT [Eis95] and was originally pointed out and proved by Ducos [Duc09]. With our stronger hypotheses, the original proof of Ducos's result is constructively valid.

Proposition 3.23 (Ducos,[Duc09])**.** *Let A be Noetherian, coherent and strongly discrete, and $x \,\epsilon\, A$ in the Jacobson radical*[7] *$Rad(A)$ of A, such that $\sqrt{(x)}$ is finitely generated. Then*

$$\text{Kdim } A/(x) \leqslant 0 \quad \Rightarrow \quad \text{Kdim } A_x \leqslant 0.$$

Proof. We have to show that, for any $y \in A$, there exist $n, k \in \mathbb{N}$ such that

$$y^n x^k \,\epsilon\, (y^{n+1}).$$

Consider therefore the following ideals of A

$$a \,\epsilon\, I_n \equiv \exists k(a \cdot x^k \,\epsilon\, (y^n)).$$

[6] A similar form of proof pattern is treated in [Sch12] in more abstract terms.

[7] Constructively, we define the Jacobson radical as follows

$$a \,\epsilon\, Rad(A) \equiv \forall y(1 - ay \,\epsilon\, A^*).$$

It is enough to show that $I_{\overline{n}} = I_{\overline{n}+1}$ for a suitable $\overline{n}$. Notice that $\{I_n\}_{n\in\mathbb{N}}$ form a descending sequence of ideals, finitely generated by coherence and Noetherianity[8], and each I_n satisfies the property

$$(x) \cap I_n = (x) \cdot I_n. \tag{3.10}$$

Since the ring $A/(x)$ is zero-dimensional and satisfies the hypothesis of Proposition 3.20, the descending sequence of ideals $\{I_n + (x)\}_{n\in\mathbb{N}}$ must pause, let's say in $\overline{n}$. Namely

$$I_{\overline{n}} \subseteq I_{\overline{n}+1} + (x). \tag{3.11}$$

Notice that

$$(I_{\overline{n}+1} + (x)) \cap I_{\overline{n}} = I_{\overline{n}+1} \cap I_{\overline{n}} + (x) \cap I_{\overline{n}}.$$

In fact, $\supseteq$ holds in general. Vice versa, let $t = a + sx \,\epsilon\, I_{\overline{n}} \cap (I_{\overline{n}+1} + (x))$; since $a + sx \,\epsilon\, I_{\overline{n}}$, then $sx \,\epsilon\, I_{\overline{n}}$ because $a \,\epsilon\, I_{\overline{n}+1} \subseteq I_{\overline{n}}$. Hence, $a \,\epsilon\, I_{\overline{n}} \cap I_{\overline{n}+1}$ and $sx \,\epsilon\, I_{\overline{n}} \cap (x)$, so that $t \,\epsilon\, I_{\overline{n}} \cap I_{\overline{n}+1} + I_{\overline{n}} \cap (x)$. If we take in (3.11) the intersection with $I_{\overline{n}}$ on both sides, we get

$$I_{\overline{n}} = I_{\overline{n}+1} \cap I_{\overline{n}} + (x) \cap I_{\overline{n}} \overset{(3.10)}{=} I_{\overline{n}+1} + (x) \cdot I_{\overline{n}}.$$

Recalling that $x \in Jac(A)$, a straight application of Nakayama's Lemma[9] leads to $I_{\overline{n}} = I_{\overline{n}+1}$. □

Kaplansky deemed the preceding lemma one of the most fundamental results in the theory of Noetherian rings [Kap74]. For non-Noetherian rings, a clear counterexample can be found in every valuation ring A with Kdim $A > 1$ [Duc09].

We recall the notion of Lasker-Noether ring, widely studied in the literature [MRR88, Sei84, Per04]:

Definition 3.12. [MRR88] A ring A is called *Lasker-Noether* if it is Notherian, coherent, strongly discrete, and the radical of any finitely generated radical ideal is the intersection of a finite number of finitely generated prime ideals.

In the following, we will need a weaker notion, which does not mention prime ideals:

Definition 3.13. A ring A is called *weakly Lasker-Noether* if it is Noetherian, coherent, strongly discrete and every radically finitely generated ideal is finitely generated.

---

[8] In fact, $I_n = \bigcup_{i\in\mathbb{N}}(y^n : x^i)$ and $\{(y^n : x^i)\}_{i\in\mathbb{N}}$ is an ascending sequence of finitely generated ideals, so it must have a pause. It is direct to check that it stabilizes after this break. Therefore $I_n = (y^n : x^{k_n})$ for some k_n.

[9] By Nakayama's Lemma, we mean the following fact: if M is a finitely generated A-module, N is a submodule of M and $I \subseteq Rad(A)$ is an ideal, then $M = IM + N$ implies $M = N$. A constructive proof can be found in [LQ12, Chap. IX, §2].

If A is a Lasker-Noether ring, the radical of a finitely generated ideal is the intersection of finitely many finitely generated primes, and then, by coherence, it must be finitely generated. In particular this class of rings include finitely presented ring over $\mathbb{Z}$, or over finite fields.

The problem of finding generators of radical ideals in polynomial rings is addressed in [FGT02], from a computational point of view.

Proposition 3.24. *Let A be a weakly Lasker-Noether ring. Then for every $x, y \in A$ there exists $y' \in A$ such that*

$$N(x, y') \subseteq N(y, x).$$

Proof. Let S be the monoid formed by the elements z such that

$$(z \to_c N(x)) \subseteq N(x),$$

and let S' denote the monoid $S+(x)$. Then $x \in Rad(A_{S'})$ and Kdim $A_{S'}/N(x) \leqslant 0$ (Corollary 3.11). Lemma 3.23 implies Kdim $A_{x^{\mathbb{N}}.S'}/(x \to_c 0) \leqslant 0$, and therefore

$$\text{Kdim } A_{x^{\mathbb{N}}.S'} \leqslant 0,$$

since $A_{x^{\mathbb{N}}.S'}/\sqrt{(0)} \cong A_{x^{\mathbb{N}}.S'}/(x \to_c 0)$. Hence, $x^{\mathbb{N}} \cdot S' \between N(y)$, namely, there exists $y' \in S$, $n \in \mathbb{N}$ and $\lambda \in A$ such that $x^n(y' + \lambda x) \in N(y)$. We can rewrite this as $y' \in N(y, x)$, and thus $N(x, y') \subseteq N(y, x)$. ☐

As a consequence of Lemma 3.7, we have in particular

$$\text{Kdim } A/(x) \leqslant d \ \&\ \text{Kdim } A/(x \to_c 0) \leqslant d \ \Leftrightarrow\ \text{Kdim } A \leqslant d \tag{3.12}$$

for any $x \in A$ and $d \in \mathbb{N}$.

Proposition 3.25. *Let A be a weakly Lasker-Noether ring. Then the following hold, for every $x, x_1, \ldots, x_d \in Jac(A)$:*

1. Kdim $A/N(x) \leqslant n \quad \Rightarrow \quad$ Kdim $A/(x \to_c 0) \leqslant n + 1$;
2. Kdim $A/(x) \leqslant n \quad \Rightarrow \quad$ Kdim $A \leqslant n + 1$;
3. *More generally, we have*

$$\text{Kdim } A/(x_1, \ldots, x_d) \leqslant n \quad \Rightarrow \quad \text{Kdim } A \leqslant n + d.$$

Proof. (1) We prove it by induction on n. From Kdim $A/N(x) \leqslant -1$, we get $1 \in (x) + (x \to_c 0)$, and then $1 \in (x \to_c 0)$ because x is in the Jacobson radical. Hence

$$\text{Kdim } A/(x \to_c 0) \leqslant -1 \leqslant 0.$$

Suppose now that the statement holds for $n - 1$ and let uss prove

$$\text{Kdim } A/N_{(x \to_c 0)}(t) \leqslant n$$

for a generic $t \,\epsilon\, A$. By applying (3.12) to the ring Kdim $A/N_{(x\to_c 0)}(t)$, it is enough to prove

$$\text{Kdim } A/(x) + N_{(x\to_c 0)}(t) \leqslant n, \qquad \text{Kdim } A/(x \to N_{(x\to_c 0)}(t)) \leqslant n.$$

The left side follows immediately, since by hypothesis Kdim $A/N(x) \leqslant n$ and $N(x) \lhd (x) + N_{(x\to_c 0)}(t)$. To prove the rightmost condition, notice that by hypothesis we have Kdim $A/N(t,x) \leqslant n-1$; in fact, by Lemma 3.24, there exists $t' \,\epsilon\, A$ such that $N(x,t') \subseteq N(t,x)$ and this implies

$$\text{Kdim } A/N(t,x) \leqslant \text{Kdim } A/N(x,t') \leqslant n-1.$$

By inductive hypothesis, we have Kdim $A/(x \to_c N(t)) \leqslant n$ and therefore

$$\text{Kdim } A/(x \to_c N_{(x\to_c 0)}(t)) \leqslant 0,$$

because $(x \to_c N(t)) \subseteq (x \to_c N_{(x\to_c 0)}(t))$.

(2) By means of (3.12), Kdim $A \leqslant n+1$ follows from

$$\text{Kdim } A/(x) \leqslant n+1, \qquad \text{Kdim } A/(x \to_c 0) \leqslant n+1.$$

The left condition is true by hypothesis, the second one follows from (1), since

$$\text{Kdim } A/N(x) \leqslant \text{Kdim } A/(x) \leqslant n.$$

(3) It is enough to apply the statement (2) d-times. □

Here follows immediately the general case of Krull's Principal Ideal Theorem.

Theorem 3.26 (PIT). *Let A be a weakly Lasker-Noether ring, and $x_1, \ldots, x_n \in A$. Then*

$$\text{Codim } (x_1, \ldots, x_n) > n \quad \to \quad 1 \,\epsilon\, (x_1, \ldots, x_n).$$

Proof. Let S be the monoid formed by the elements $t \in A$ such that

$$(t \to_c (x_1, \ldots, x_n)) \lhd_c (x_1, \ldots, x_n)$$

and let S' denote the monoid $S + (x_1, \ldots, x_n)$. Then $x_1, \ldots, x_n \,\epsilon\, Rad(A_{S'})$ and

$$\text{Kdim } A_{S'}/(x_1, \ldots, x_n) \leqslant 0$$

because of Corollary 3.11. By Proposition 3.25 (3), Kdim $A_{S'} \leqslant n$. Since

$$\text{Codim } (x_1, \ldots, x_n) > n,$$

there exist $y_0^0, \ldots, y_k^0, \ldots, y_0^n \ldots, y_k^n$ such that

$$\prod_{i=0}^{k} N(y_i^0, \ldots, y_i^n) \subseteq \sqrt{(x_1, \ldots, x_n)}.$$

Since Kdim $A_{S'} \leqslant n$, we have $S' \between N(y_i^0, \ldots, y_i^n)$ for any i, so that

$$S' \between \prod_{i=0}^{k} N(y_i^0, \ldots, y_i^n) \subseteq \sqrt{(x_1, \ldots, x_n)},$$

which gives $S' \between (x_1, \ldots, x_n)$. The latter implies $S \between (x_1, \ldots, x_n)$, that is, there exists $t \in (x_1, \ldots, x_n)$ such that $(t \rightarrow_c (x_1, \ldots, x_n)) \lhd_c (x_1, \ldots, x_n)$. In particular

$$1 \; \epsilon \; (t \rightarrow_c (x_1, \ldots, x_n)) \lhd_c (x_1, \ldots, x_n),$$

from which $1 \; \epsilon \; (x_1, \ldots, x_n)$ follows. □

For the sake of completeness, we reformulate the last theorem as a statement about the formal topologies $\mathrm{Zar}_{\leqslant k}\ (A)$ defined in the preceding section. We recall in fact that

$$Codim(x_1, \ldots, x_n) > k \quad \Leftrightarrow \quad 1 \lhd_{\leqslant k} \{x_1, \ldots, x_n\},$$

so that, the Theorem 3.26 can be stated as follows

$$1 \lhd_{\leqslant k} \{x_1, \ldots, x_k\} \quad \Rightarrow \quad 1 \lhd_c \{x_1, \ldots, x_k\}.$$

If we consider it as a statement over the ring A_a, we get an easy generalization:

Corollary 3.27 (PIT for $\mathrm{Zar}_{\leqslant k}$**).** *Let A be a weakly Lasker-Noether ring. Then*

$$a \lhd_{\leqslant k} \{x_1, \ldots, x_k\} \quad \Leftrightarrow \quad a \lhd_c \{x_1, \ldots, x_k\}.$$

holds for any $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in A$.

Written in this form, the principal ideal theorem looks as a statement on the cover $\lhd_c$, that is, of completely topological nature. Many of the lemmas leading to it had the same character. It is then natural to ask if such a theorem could be stated more properly in the realm of finitary formal topologies satisfying suitable conditions.

However, all of the corresponding proofs strongly rely on the algebraic properties of the ring structures, and such generalization looks, for the moment, difficult to realise. Further studies will be undertaken in this direction.

Chapter 4

The Formal Projective Eisenbud-Evans-Storch Theorem

Note. This chapter is based on the following publication [Rin12]: Davide Rinaldi. *A formal proof of the projective Eisenbud-Evans-Storch theorem*, Archiv der Mathematik, 99:9–24, 2012.

Introduction

In the previous chapters, the Zariski spectrum of a commutative ring A was described in a formal way through a formal topology, whose formal points corresponds to the prime ideals of A. Krull's Lemma was substituted by the effective existence of a proof certificate, in form of an algebraic identity. Such description will enable us to state and prove constructively the projective Eisenbud-Evans-Storch Theorem.

In 1972, U. Storch [Sto72] proved that if A is a Noetherian commutative ring of finite Krull dimension d, then for every ideal $\mathfrak{b} \subseteq A[X]$ one can find $d+1$ polynomials $f_1, \ldots, f_{d+1}$ such that

$$V(\mathfrak{b}) = V(f_1, \ldots, f_{d+1}).$$

In other words, he showed that each radical ideal in $A[X]$ is the radical of an ideal generated by at most $d+1$ elements, provided that A has Krull dimension d.

Independently, in 1973, Eisenbud & Evans [EE73] proved the same result, and extended it also to the projective case. More precisely, they proved that if B is a graded Noetherian ring of the form $B = A[X]$ for some graded ring A of finite projective dimension d, and[1] $I \subseteq A^+B$ is a homogeneous ideal, then there

[1] A^+ is the subset of elements of positive degree in A.

exist $d+1$ homogeneous elements $g_1, \ldots, g_{d+1} \in I$ such that

$$\sqrt{I} = \sqrt{(g_1, \ldots, g_{d+1})}.$$

There is a rather interesting history behind these results, briefly sketched in [EE73, Kne60]. They both strengthen a classical result, announced by Kronecker in 1882 [Kro82], stating that each radical ideal in an n-dimensional polynomial ring is the radical of an ideal generated by $n+1$ elements[2].

Both proofs make extensive use of prime ideals [CSar], and the existence of concrete algebraic witnesses (for instance, corresponding to $a \in \sqrt{(f_1, \ldots, f_{d+1})}$) is finally achieved by the classical Nullstellensatz. In the statement itself, the classical definition of Krull dimension by means of chains of prime ideal is impredicative.

The constructive definition of Krull dimension for lattices and rings [CL02, CLR05], described in the last section, led to an elementary[3] constructive proof of Kronecker's result [Coq04] and to a constructive rebuilding [CLS05] of the Storch proof. We extend the latter proof to the projective case as stated by Eisenbud & Evans.

To this end, we use a description of the projective spectrum as a formal topology [CLS05] and get a topological proof which extends the affine case. It is in order to point out the algorithmic character of each proof, which can be regained on demand.

The central role played by techniques from formal topology prompts us to conjecture that these theorems, regarding the number of generators, could be stated more properly in the wider context of distributive lattices or formal topologies.

4.1 The Projective Spectrum as a Formal Topology

In this section, we will show how to associate to a graded ring a finitary formal topology with finitary operation, that represents its *projective spectrum*. A similar constructive approach has been carried out in [CLS07] by means of distributive lattices.

Definition 4.1. A commutative ring A is called *graded*, if it is a direct sum of abelian groups

$$A = \bigoplus_{d \geqslant 0} A_d.$$

and the product operation restricts to maps

$$A_d \times A_e \to A_{d+e}.$$

for all $d, e \in \mathbb{N}$. We say that $a \in A$ is *homogeneous* if $a \in A_d$ for some $d \geqslant 0$. In this case, we write $d(x) = d$ and say that x *has degree* d.

[2] It was originally proved for the polynomial ring of n variables over a field.

[3] Using direct properties of the logical theory considered, and explicit algebraic identities.

Notice that, in particular, $d(0) = d$ for each $d \geqslant 0$. We fix for simplicity the following notation

$$A^{\oplus} = \bigoplus_{d>0} A_d, \qquad A^{+} = \bigcup_{d>0} A_d,$$

In particular, to any element $a \in A$ is uniquely associate a finite subset

$$H(a) \equiv \{a_{i_1}, \dots, a_{i_n}\}$$

of homogeneous elements the so-called *homogeneous components* of a, such that $a = a_{i_1} + \cdots + a_{i_n}$.

We can generate a finitary basic topology $\lhd_h$ on A^+, the *projective basic topology*, by reflexivity and the following generating axioms:

$$\frac{\top}{0 \lhd_h U}\,0, \quad \frac{a \lhd_h U \quad b \lhd_h U \quad a, b \,\epsilon\, A_d}{a + b \lhd_h U}\,\Sigma, \quad \frac{a \lhd_h U}{a \cdot b \lhd_h U}\,\Pi, \quad \frac{a \cdot a \lhd_h U}{a \lhd_h U}\,Sq, \tag{4.1}$$

for all $a, b \,\epsilon\, A^+$ and $U \subseteq A^+$. Notice that these axioms almost coincide with those of the Zariski formal topology. However, we are restricting it to the elements of A^+, and, correspondingly, the rule Σ has to be restricted in a meaningful way. These analogies are made explicit in the following proposition.

Proposition 4.1. *Let A be a graded ring, and $\lhd$ and $\lhd_h$ be, respectively, the Zariski and the projective formal covers on A. Then we have*

$$a \lhd_h U \leftrightarrow a \lhd U, \tag{4.2}$$

for all $a \in A^+$ and $U \subseteq A^+$.

Proof. The implication $(\rightarrow)$ is direct, since the axioms of $\lhd_h$ are more restrictive than those of $\lhd$. Vice versa, we use induction on $a \lhd U$: if $a \,\epsilon\, U$, then $a \lhd_h U$; if $a \lhd_h U$, and $a \cdot b \in A^+$ then $b \in A^+$, so that $a \cdot b \lhd_h U$; if $a \lhd_h U$ and $b \lhd_h U$ and $a, b \,\epsilon\, A^+$, then $a + b \,\epsilon\, A^+$ is homogeneous if and only if $a, b \,\epsilon\, A_d$ for some $d > 0$. Hence, we have $a + b \lhd_h U$; finally, if $a^2 \,\epsilon\, A^+$ and $a^2 \lhd_h U$, then $a \,\epsilon\, A^+$, and then $a \lhd_h U$. □

Thanks to the equivalence (4.2), one can easily show that the product operation $\cdot$ on $(A^+, \lhd_h)$ gives structure of formal topology, as in the case of the Zariski formal topology. We will call $(A^+, \lhd_h, \cdot)$ the *projective formal topology* and we will denote it by $\mathrm{Proj}(A)$.

We make the assumption that A is generated as an A_0-algebra by finitely many $x_0, \dots, x_k \in A^+$ with $k \geqslant 1$, that is[4]

$$A = A_0[x_0, \dots, x_k].$$

with $d(x_i) = d_i$ for suitable $d_i \in \mathbb{N}$. Notice that, in particular, $A^+ \lhd_h \{x_0, \dots, x_k\}$, and, by *Loc*, $a \lhd_h a \cdot \{x_0, \dots, x_k\}$ for all $a \,\epsilon\, A^+$. In particular

$$a \lhd_h U \leftrightarrow \forall i (a \cdot x_i \lhd_h U) \tag{4.3}$$

[4]Some of the x_i are possibly 0.

for all $a \in A^+$ and $U \subseteq A^+$. A *homogeneous prime ideal* of A is a prime ideal which

1. is generated by homogeneous elements, or, equivalently, it contains an element if and only if it contains all of the homogeneous components;

2. does not contain the whole of $A^\oplus$, that is, under the assumptions, it doesn't contain all the x_i.

In the usual treatment, the homogeneous prime ideals are gathered together, as for the Zariski spectrum, into a topological space $\mathfrak{Proj}(A)$; a base of this space is given by the family

$$D(a) = \{\mathfrak{p} \in \mathfrak{Proj}(A) : a \in A \setminus \mathfrak{p}\} \quad (a \in A^+).$$

We can prove with classical logic that the space $\mathscr{P}t(A^+)$ of the formal points of $(A^+, \lhd_h, \cdot)$ corresponds homeomorphically to $\mathfrak{Proj}(A)$: each formal point α is mapped to the homogeneous ideal generated by its complement[5] $A^+ \setminus \alpha$.

Classically it is proved that $\mathfrak{Proj}(A)$ is homeomorphic to the result of gluing together the affine spectra[6] $\mathfrak{Spec}(A[\frac{1}{x_i}]_0)$. We will see now how this fact can be expressed in terms of the corresponding formal topologies:

Proposition 4.2. *The localization $(A^+, \lhd_{x_i,h}, *)$ of the projective formal topology $(A^+, \lhd_h, *)$ in x_i and the Zariski formal topology $(A[\frac{1}{x_i}]_0, \lhd^i, \cdot)$ on the ring $A[\frac{1}{x_i}]_0$, are isomorphic, for all $i = 0, \ldots, k$.*

Proof. (Sketch) Consider the function φ_i which sends $a \in A^+$ to $a^{d_i}/x_i^{d(a)}$. Then (φ_i, φ_i^-) is an isomorphism between the corresponding formal topologies. □

Remark 19. If we quotient the graded ring A by a homogeneous radically finitely generated ideal $I = \sqrt{(c_1, \ldots, c_n)}$, with $c_1, \ldots, c_n \in A^+$, then we get a graded ring A/I. The corresponding projective lattice is isomorphic to the quotient $(A^+, \lhd_h^I, \cdot)$ in $\{c_1, \ldots, c_n\}$ of the formal topology $(A^+, \lhd_h, \cdot)$ defined by

$$a \lhd_h^I U \equiv a \lhd_h U \cup \{c_1, \ldots, c_n\}.$$

for all $a \in A^+$ and $U \subseteq A^+$. We notice also that the ring A/I can be presented as $A_0[\overline{x}_0, \ldots, \overline{x}_k]$, where $\overline{x}_i$ is the image of x_i under the quotient map.

Remark 20. Analogously, let $S \subseteq A^+$ be a convergent subset, and consider the ring

$$A_S = A[\{\frac{x_j^{d(s)}}{s^{d_j}} : s \in S, j = 0, \ldots k\}].$$

[5] We could have defined, isomorphically, the projective formal topology just by describing the generation process of homogeneous ideals.

[6] The ring $A[\frac{1}{x_i}]$ is $\mathbb{Z}$-graded in a natural way, and $A[\frac{1}{x_i}]_0$ is the subring consisting of the elements of degree 0.

It is a graded ring, still of the form

$$A_S = B_0[x_1, \ldots, x_n] \quad \text{where} \quad B_0 = A_0[\{\frac{x_j^{d(s)}}{s^{d_j}} : s \in S, j = 0, \ldots k\}].$$

The corresponding projective formal topology is isomorphic to the localization in S of $(A^+, \lhd_h, \cdot)$, which is defined by

$$a \lhd_{h,S} U \equiv (\exists s \ \epsilon \ S)(a \cdot s \lhd_h U),$$

for all $a \in A^+$ and $U \subseteq A^+$.

4.2 The h-nilregular Element Property and Projective Dimension

Suppose now A to be strongly discrete[7], coherent and Noetherian. In particular, these assumptions are shared by the localized rings $\{A[\frac{1}{x_i}]_0\}_{i=0}^k$.

The nilregular element property says that if A is coherent, Noetherian and discrete, then every ideal boundary contains a nilregular element. Ideal boundaries and nilregular elements are objects of topological character and the nilregular element property can be suitably rephrased and proved for the projective formal topology.

We recall the following notation

$$b \ \epsilon \ a \to_h \emptyset \equiv a \cdot b \lhd_h \emptyset.$$

This notation can be extended to subsets as follows:

$$b \ \epsilon \ U \to_h \emptyset \equiv a \cdot U \lhd_h \emptyset \equiv (\forall u \ \epsilon \ U)(u \cdot b \lhd \emptyset).$$

Definition 4.2. An element $a \in A^+$ is called *h-nilregular* if $a \to_h \emptyset \lhd_h \emptyset$.

One has $a \lhd_h \emptyset$ if and only if a is nilpotent. We denote by $N_h \subseteq A^+$ the subset of the homogeneous nilpotent elements. In particular, $a \lhd_h \emptyset$ if and ony if $a \ \epsilon \ N^h$. An element $a \ \epsilon \ A^+$ is nilpotent if and only if, for all $b \ \epsilon \ A^+$, $a \cdot b \ \epsilon \ N^h$ implies $b \ \epsilon \ N^h$.

Remark 21. The h-nilregular elements of A form a convergent subset, that we denote with S^h. In fact, if a, a' are h-nilregular and $(a \cdot a') \cdot b \ \epsilon \ N^h$ then $a' \cdot b \ \epsilon \ N^h$, because a in nilregular, and then $b \ \epsilon \ N^h$, because a' is nilregular. In other words, if a and a' are nilregular, then $a \cdot a'$ is h-nilregular.

Proposition 4.3 (H-nilregular element property)**.** *Let A be a discrete, Noetherian and coherent graded ring; then every h-nilregular ideal contains an h-nilregular element.*

Before approaching the proof, we need some intermediate steps:

[7]A ring A is said to be *strongly discrete* if the quotient of A by a f.g. ideal is a discrete ring. In other terms, if the membership relation is decidable on every f.g. ideal.

Lemma 4.4. *Let A be a Noetherian and coherent graded ring. Then the formal open $\{b_1, \ldots, b_n\} \to_h \emptyset$ is finitely generated for all $b_1, \ldots, b_n \;\epsilon\; A^+$. In other words, there exist $c_1, \ldots, c_m \;\epsilon\; A^+$ such that*

$$\{c_1, \ldots, c_m\} =_{\lhd_h} \{b_1, \ldots, b_n\} \to \emptyset.$$

Proof. It is enough to prove it for $n = 1$. If A is Noetherian and coherent, then so are the localized rings $A[\frac{1}{x_i}]_0$ for all $i = 0, \ldots, k$. We have

$$a \;\epsilon\; b \to_h \emptyset \equiv ab \lhd_h \emptyset \leftrightarrow \forall i(abx_i \lhd \emptyset) \leftrightarrow \forall i(a \;\epsilon\; b \to_{x_i,h} \emptyset).$$

Thanks to the isomorphism of Proposition 4.2, and Proposition 3.4, $b \to_{x_i,h} \emptyset$ is finitely generated by some $c^i_1, \ldots, c^i_{m_i} \;\epsilon\; A^+$ for all $i = 0, \ldots, k$. Hence, we define

$$\{c_1, \ldots, c_m\} = \prod_{i=0}^{k} \{c^i_1, \ldots, c^i_{m_i}\}.$$

□

Lemma 4.5. *[CLS05] Let A be a discrete, Noetherian and coherent ring and $N \subseteq A$ the subset of nilpotent elements. Then, for any $a \in A$, one has $a \;\epsilon\; N$ or $a \notin N$.*

Corollary 4.6. *Let A be a discrete, Noetherian and coherent graded ring and $a \in A^+$; then one has $a \;\epsilon\; N^h$ or $a \notin N^h$.*

Proof. We have $a \;\epsilon\; N$ or $a \notin N$, by the previous lemma, that is, since $a \in A^+$, $a \;\epsilon\; N^h$ or $a \notin N^h$. □

Proposition 4.7. *Let A be a Noetherian, coherent and strongly discrete graded ring; then for all $b_0, \ldots, b_m \in A$ one can decide whether*

$$\{b_0, \ldots, b_m\} \to_h \emptyset \lhd_h \emptyset;$$

if instead $\neg(\{b_0, \ldots, b_m\} \to_h \emptyset \lhd_h \emptyset)$, then one can find $b_{m+1} \in A$ such that

$$\neg(b_{m+1} \lhd_h \emptyset) \quad \text{and} \quad b_{m+1} * \{b_0, \ldots, b_m\} \lhd_h \emptyset.$$

Proof. By Lemma 4.4, $\{b_0, \ldots, b_m\} \to_h \emptyset$ is finitely generated by some

$$c_1, \ldots, c_l \in A^+.$$

Using the lemma above we can check for each i if $c_i \lhd_h \emptyset$. If $\neg(c_j \lhd_h \emptyset)$ for some j, then take $b_{m+1} = c_j$ for any such j. □

Remark 22. Let $a_1 \in A_{d_1}$ and $a_2 \in A_{d_2}$ with $d_1, d_2 > 0$; then

$$\{a_1, a_2\} =_{\lhd_h} \{a_1^{d_2} + a_2^{d_1}, a_1 \cdot a_2\}.$$

The right hand side is obviously covered by the left hand side. Vice versa, since

$$a_1^{d_2+1} = (a_1^{d_2} + a_2^{d_1}) \cdot a_1 - (a_1 \cdot a_2) \cdot a_2^{d_1-1}$$

then

$$a_1 \lhd_h a_1^{d_2+1} \lhd_h \{(a_1^{d_2} + a_2^{d_1}) \cdot a_1, (a_1 \cdot a_2) \cdot a_2^{d_1-1}\} \lhd_h \{a_1^{d_2} + a_2^{d_1}, a_1 \cdot a_2\}. \tag{4.4}$$

We can work symmetrically for a_2. In particular, if $a_1 \cdot a_2 \lhd_h \emptyset$, then

$$a_1^{d_2} + a_2^{d_1} \lhd_h \{a_1, a_2\}.$$

The same observation can be extended to any finite sequence $a_1, \dots, a_l \in A^+$ with a_j lying in A_{d_j} for all j:

$$\{a_1, \dots, a_l\} =_{\lhd_h} \{a_1^{\alpha_1} + \cdots + a_l^{\alpha_l}\} \cup \{a_i b_j\}_{i,j=1}^n$$

where $\alpha_i = \sum_{j \neq i} d_j$. In particular, if $a_i b_j \lhd \emptyset$ for all $i, j = 1, \dots, n$, then

$$\{a_1, \dots, a_l\} =_{\lhd_h} a_1^{\alpha_1} + \cdots + a_l^{\alpha_l}.$$

We are now ready to prove Theorem 4.3:

Proof of the theorem. Let $I = \sqrt{(c_1, \dots, c_l)}$ an h-nilregular ideal, i.e.

$$\{c_1, \dots, c_l\} \to_h \emptyset \lhd_h \emptyset$$

By means of Lemma 4.7, we construct a sequence $b_0, \dots, b_n$ as follows: to start with, let b_0 be one of the c_i such that $\neg(c_i \lhd_h 0)$. If $b_0 \to_h \emptyset \lhd_h \emptyset$ we are done, otherwise, by 4.7, there must exist $b_1 \in A^+$ such that

$$\neg(b_1 \lhd_h \emptyset) \quad \text{and} \quad b_1 * b_0 \lhd_h \emptyset.$$

We can assume b_1 to be a multiple of one[8] c_i and then $b_1 \in I$. Having constructed $b_0, \dots, b_m$, if $\{b_0, \dots, b_m\} \to \emptyset \lhd_h \emptyset$ we're done, otherwise there must exist $b_{m+1} \in A^+$, multiple of one c_i, such that

$$\neg(b_{m+1} \lhd_h \emptyset) \quad \text{and} \quad b_{m+1} * \{b_0, \dots, b_m\} \lhd_h \emptyset.$$

By Noetherianity, this procedure must end in a finite number of steps, supplying a sequence $b_0, \dots, b_n$ such that

$$\{b_0, \dots, b_n\} \to \emptyset \lhd_h \emptyset \quad \text{and} \quad b_i * b_j \lhd_h \emptyset.$$

Hence, by Remark 22, there exist coefficients $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ such that

$$\{b_0, \dots, b_n\} \lhd_h b_0^{\alpha_1} + \cdots + b_n^{\alpha_n}.$$

In particular $b_0^{\alpha_1} + \cdots + b_n^{\alpha_n}$ is h-nilregular and belongs to I. □

[8] Otherwise, one can take $b_1 * c_i$. In fact, if $b_1 * c_i \lhd_h \emptyset$ for each i, then $b_1 \;\epsilon\; \{c_1, \dots, c_l\} \to \emptyset$, against the hypothesis of h-nilregularity.

We end this section with a brief analysis of the notion of Krull dimension for the case of the projective formal topology $(A^+, \lhd_h, \cdot)$.

If $a \in A^+$, the *homogeneous boundary* $N^h(a) \subseteq A^+$ is defined as $\{a\} \cup a \to_h \emptyset$. We denote by N_a^h the homogeneous radical ideal generated by $N^h(a)$.

If the ring A is Noetherian, coherent, then $N^h(a)$ is generated, as a formal open, by finitely many elements. In this case, after Remark 20, the quotient of the formal topology $(A^+, \lhd_h, \cdot)$ in $N^h(a)$ is isomorphic to the projective formal topology in the ring A/N_a^h.

Every homogeneous boundary is h-nilregular; in fact, $b \cdot N^h(a) \lhd_h \emptyset$ if and only if $a \cdot b \lhd_h \emptyset$ and $(a \to_h \emptyset) \cdot b \lhd \emptyset$. In particular, $b \lhd_h b \cdot b \lhd_h \emptyset$. As a consequence, thanks to Theorem 4.3, each $N^h(a)$ has a h-nilregular element $c \in N^h(a)$, or, in other words

$$S^h \between N_a^h \tag{4.5}$$

for all $a \in A^+$.

The projective dimension Projdim(A) of the graded ring A is defined as the Krull dimension of the associated projective formal topology $(A^+, \lhd_h, \cdot)$. More explicitly, we have, for all $n \in \mathbb{N} \cup \{-1\}$:

1. Projdim$(A) \leqslant -1$ if $x_0, \ldots, x_k$ are nilpotent,
2. Projdim$(A) \leqslant n$ if Projdim $(A/N_a^h) \leqslant n-1$ for all $a \in A$.

By spelling out the definitions, as done in Proposition 3.1, Projdim$(A) \leqslant n$ is equivalent [DQ06], for any $a_0, \ldots, a_n \in A^+$ and $i = 0, \ldots, k$, to the existence of $b_0^i, \ldots, b_n^i \in A^+$ such that

$$x_i \lhd_h \{a_0, b_0^i\}, \quad a_0 \cdot b_0^i \lhd_h \{a_1, b_1^i\}, \quad \ldots \quad a_n \cdot b_n^i \lhd_h 0.$$

This condition implies that each of the $A[\frac{1}{x_i}]_0$ has Krull dimension less or equal than n.

Example 4.8. As an example, we have Projdim $(A) \leqslant 0$ if for each $a \in A^+$, there exist $b^0, \ldots, b^n$ such that, for each $i = 0, \ldots, k$

$$x_i \lhd_h \{a, b^i\}, \qquad a \cdot b^i \lhd_h 0,$$

which means, respectively $\exists m_i \exists \lambda_i (x_i^{m_i} = \lambda_i \cdot a + b^i)$ and ab^i is nilpotent.

Recall that S^h is the multiplicative subset of h-nilregular elements and we have proved that $S^h \between N_a^h$ for all $a \in A^+$; then Projdim $(A_{S^h}) \leqslant 0$, since in A_{S^h}/N_a^h the elements $x_0, \ldots, x_k$ are sent to zero for each $a \in A^+$.

For each h-nilregular element $s \in A^+$,

$$\text{Projdim } (A) \leqslant n+1 \quad \Rightarrow \quad \text{Projdim } (A/(s)) \leqslant n$$

In fact, $N^h(s) = \{s\} \cup s \to_h \emptyset = \{s\}$ and therefore $\text{Proj}(A/(s)) \cong \text{Proj}(A/N_s^h)$.

4.3 The Projective Eisenbud-Evans-Storch Theorem

Let $B = A[X]$ the ring of polynomials over A and $A^+B \subseteq B$ the subset of polynomials with coefficients in A^+. We define a new formal topology $(A^+B, \lhd_H, \cdot)$, subject to reflexivity and the following generating axioms

$$\frac{\top}{0 \lhd_H U}\ 0_{,} \quad \frac{F \lhd_H U \quad G \lhd_H U \quad F+G\ \epsilon\ A^+B}{F+G \lhd_H U}\ \Sigma_{,}$$
$$\frac{F \lhd_H U}{F \cdot G \lhd_H U}\ \Pi_{,} \quad \frac{F \cdot F \lhd_H U}{F \lhd_H U}\ Sq_{,}$$

for all $F, G \in A^+B$. Since the relations are the same as in the projective case, most of the properties still hold and can be proved in the same way. For instance, we can prove, as in Proposition 4.2, that

$$F \lhd_H \{G_1, \ldots, G_m\} \leftrightarrow F \lhd \{G_1, \ldots, G_m\} \tag{4.6}$$

for all $F, G_1, \ldots, G_m \in A^+B$, where $\lhd$ is the Zariski formal cover. This should also convince that the axiom of formal topology are satisfied for $(A^+B, \lhd_H, \cdot)$. We can characterize this formal topology as follows:

Proposition 4.9. *Let A and B as above. For every $i \in \{0, \ldots, k\}$ the localization in x_i of $(A^+B, \lhd_H, \cdot)$ is isomorphic to the Zariski formal topology $(A[\frac{1}{x_i}]_0[X], \lhd, \cdot)$.*

Proof. Consider the function φ_i^X mapping $\sum_j \lambda_j X^j \in A^+B$ to $\sum_j \frac{\lambda_j^{d_i}}{x_i^{d(\lambda_j)}} X^j$, where $\lambda_j \in A_{d_j}$. Then the pair $(\varphi_i^X, (\varphi_i^X)^-)$ is an isomorphism between the corresponding formal topologies. □

Notice that, if $a, b_1 \ldots, b_n \in A^+ \subseteq A^+B$, then

$$a \lhd_h \{b_1, \ldots, b_n\} \quad \text{iff} \quad a \lhd_H \{b_1, \ldots, b_n\}.$$

The proof we are going to see now is based on induction over the dimension and follow closely the affine case. In the affine case, the base step consists of the following proposition:

Definition 4.3. [CLS05, LQ12] If R is a commutative ring such that Kdim$R \leqslant 0$, then every finitely generated ideal I of $R[X]$ is radically principal, i.e. $\sqrt{I} = \sqrt{(F)}$ for some $F \in R[X]$.

In terms of the Zariski lattice associated to $R[X]$, this means that for all $G, G' \in R$ there is $F \in R[X]$ such that $\{G, G'\} =_{\lhd_H} F$. We are now about to extend this observation to the projective case:

Definition 4.4. Let $A = A_0[x_0, \ldots, x_k]$ be a graded ring of projective dimension smaller or equal to 0. Then for all $F, G \in A^+B$, there exists $H \in A^+B$ such that $D_P^X(F, G) = D_P^X(H)$.

Proof. We consider the case $k = 1$, easier to understand. The generalization comes easily. Since Projdim $A \leqslant 0$, there exist $c_0, c_1 \in A^+$ such that

$$x_0 \lhd_h \{x_1, c_0\}, \qquad x_1 \cdot c_0 \lhd \emptyset,$$

$$x_1 \lhd_h \{x_0, c_1\}, \qquad x_0 \cdot c_1 \lhd_h \emptyset.$$

We can suppose $D_P(c_0 \cdot c_1) = 0$, taking $c_0 \cdot x_0$ instead of c_0. Under this hypothesis, we have

$$\{x_0c_0, x_1c_1, x_0x_1\} =_{\lhd_h} \{x_0, x_1\} =_{\lhd_h} A^+ \tag{4.7}$$

since

$$x_0 \lhd_h x_0^2 \lhd_h \{x_0x_1, x_0c_0\},$$
$$x_1 \lhd_h x_1^2 \lhd_h \{x_0x_1, x_1c_1\}.$$

Since Kdim $A[\frac{1}{x_i}]_0 \leqslant 0$, and by way of the isomorphism described in Theorem 4.9, there exist $H_0, H_1 \in A^+B$ such that

$$H_0 \cdot x_0 =_{\lhd_H} \{F \cdot x_0, G \cdot x_0\},$$
$$H_1 \cdot x_1 =_{\lhd_H} \{F \cdot x_1, G \cdot x_1\}.$$

Moreover, we can suppose that $H_0 + H_1 \in A^+B$, multiplying, respectively, by suitable powers of x_0 and x_1 the coefficients of H_0 and H_1. Consider now

$$H = H_0 \cdot x_0^{d_1} \cdot c_0^{d(c_1)d_1} + H_0 \cdot x_0^{d_1} \cdot x_1^{d(c_0)d(c_1)} + H_1 \cdot x_1^{d_0} \cdot c_1^{d_1 d(c_0)}.$$

We have $H \in A^+B$ and we immediately get

$$H \cdot x_0 \cdot c_0 = H_0 \cdot x_0 \cdot c_0 =_{\lhd_H} \{F, G\} \cdot x_0 \cdot c_0.$$

Similarly

$$H \cdot x_1 \cdot c_1 = H_1 \cdot x_1 \cdot c_1 =_{\lhd_H} \{F, G\} \cdot x_1 \cdot c_1,$$
$$H \cdot x_0 \cdot x_1 = H_0 \cdot x_0 \cdot x_1 =_{\lhd_H} \{F, G\} \cdot x_0 \cdot x_1.$$

Comparing left members and right members we arrive at

$$H \cdot \{x_0 \cdot c_0, x_1 \cdot c_1, x_0 \cdot x_1\} =_{\lhd_H} \{F, G\} \cdot \{x_0 \cdot c_0, x_1 \cdot c_1, x_0 \cdot x_1\}.$$

By (4.7), we finally obtain $H =_{\lhd_H} \{F, G\}$. □

We can now approach the Projective Eisenbud-Evans-Storch Theorem, following the affine case as treated in [CLS05].

Definition 4.5 (Projective Eisenbud-Evans-Storch)**.** Let $A = A_0[x_0, \ldots, x_n]$ be a coherent, Noetherian and strongly discrete graded ring. Suppose also Projdim $(A) \leqslant d$ and let $B = A[X]$. Then for all $G_1, \ldots G_m$ in A^+B there exist $F_0, \ldots, F_d \in A^+B$ such that

$$\{F_0, \ldots, F_d\} =_{\lhd_H} \{G_1, \ldots G_m\}.$$

In the light of (4.6), this means

$$\sqrt{(F_0, \ldots, F_d)} = \sqrt{(G_1, \ldots G_m)}.$$

Proof. The proof is done by induction on d. The basic case $d = 0$ is precisely Theorem 4.4. Let $S^h \subseteq A$ be the multiplicative subset of h-nilregular elements. Since for all $a \in A^+$ one has $S^h \between N_a^h$, it follows directly that A_{S^h} has projective dimension $\leqslant 0$. Therefore we can find F and $s \in S^h$ such that

$$F \cdot s \lhd_H \{G_1, \dots, G_m\}, \qquad sG_i \lhd_H F;$$

setting $F_0 = F \cdot s$, we get

$$\{s \cdot G_1, \dots s \cdot G_m\} \lhd_H F_0 \lhd_H \{G_1, \dots G_m\}.$$

By induction hypothesis, since Projdim $(A/sA) \leqslant d-1$, we can find $H_1, \dots,$ $H_d \in A^+B$ such that

$$\{H_1, \dots, H_d, s\} =_{\lhd_H} \{G_1, \dots, G_m, s\}$$

and in particular $H_i^{n_i} = \lambda s + F_i$ with $F_i \lhd_H \{G_1, \dots, G_m\}$, for which also $\{F_i, s\} =_{\lhd_H} \{H_i, s\}$. Now, on the other hand,

$$\{F_0, F_1, \dots, F_d\} \lhd_H \{G_1, \dots, G_m\}$$

and we are done. On the other hand, we have for each $i \leqslant m$,

$$G_i \lhd_H \{s, F_1, \dots, F_d\}$$

and thus

$$G_i \lhd_H \{sG_i, F_1, \dots, F_d\};$$

in addition, $sG_i \leqslant_H F_0$ and we finally arrive at

$$G_i \lhd_H \{F_0, F_1, \dots, F_d\}$$

as desired. □

The Projective Eisenbud-Evans-Storch theorem looks rather as a statement on the formal topologies underlying the algebraic structure. The studies undertaken in order to find a common generalization have shown, for the moment, little success. Still this algebraic intuition turns out to be very helpful for the formal topologist, as we will show in the next two chapters.

Chapter 5

A Universal Krull-Lindenbaum Theorem

Note. This chapter is based on the following submitted article [RS14]: Davide Rinaldi and Peter Schuster. *A Universal Krull-Lindenbaum Theorem*.

Introduction

Several indirect proofs with Zorn's Lemma have recently allowed for being turned upside down into direct proofs with Raoult's Open Induction [Rao88, Ber04, Coq92, CP99], which is transfinite induction limited to Scott-open predicates. Under sufficiently concrete circumstances one may further reduce to induction over finite partial orders, and thus achieve a constructive proof. Then mathematical induction suffices unless one fixes the size of the objects under consideration, in which case one even gets an entirely first-order proof. Case studies pertain to the ideal theory of commutative rings [Sch12] and more specifically to the Gelfand theory of Banach algebras [HS12].

Toward a systematic treatment we now classify, by representative proof patterns, the cases that can be found in mathematical practice, of which there are plenty. During this undertaking we have come across an extensive generalisation of Krull's theorem [Kru29] on prime ideals and of Lindenbaum's lemma [Tar30] on complete consistent theories. This generalisation subsumes various instances from diverse branches of algebra, such as the Artin-Schreier theorem, as well as the Henkin approach to Gödel's completeness theorem for first-order predicate logic.

Following Scott [Sco74] we put our theorem in universal rather than existential form, in which it is related to what is known as the formal Hilbert Nullstellensatz, and more generally to the concept of spatiality in locale theory and formal topology [Joh82, CL02, GS07, Sam03]. This move moreover allows us to prove the theorem in a relatively direct way, with the aforementioned Open Induction in place of Zorn's Lemma. By reduction to the corresponding theorem on irreducible ideals (due to Noether [Noe21], McCoy [McC38], Fuchs [Fuc49] and Schmidt [Sch52]) we further shed light on what prime ideals and related concepts have to do with transfinite methods.

Although (or just because) the universal Krull–Lindenbaum theorem can rightly be viewed as even more abstract than each of its instances, the availability of a relatively direct and inductive proof is likely to have impact on a partial realisation [CL06, LQ12, MS05, Sam12] of the revised Hilbert Programme à la Kreisel and Feferman. We expect that, as in the case studies mentioned above, one will eventually be able to do with finite methods and without ideal objects whenever it comes to prove any concrete instantiation of the theorem; and that just its universal character will suggest a general method.

5.1 A Variant of Open Induction

We first recollect some requisites from [Sch12], which includes standard material. Let $(X, \leqslant)$ be a partial order. Every quantification over the variables x, x', y, and z is to be understood as over the elements of X. We sometimes identify a predicate φ on X with $\{x \in X : \varphi(x)\}$. As usual, $x = y \wedge z$ means that x is the *greatest lower bound*, *infimum* or *meet* of y and z: that is,

$$\forall x' \,(x' \leqslant x \iff x' \leqslant y \wedge x' \leqslant z)\,.$$

Likewise, $x = \bigvee Y$ says that x is the *least upper bound*, *supremum* or *join* of $Y \subseteq X$: that is,

$$\forall x' \,(x' \geqslant x \iff (\forall y \in Y)(x' \geqslant y))\,.$$

Arbitrary meets and binary joins are dealt with accordingly. Note that it is not required that X always has the meets and joins in question, though this will often be the case.

Let O be a predicate on X. We say that O is *progressive* if

$$\forall x \,((\forall y > x)\, O(y) \Longrightarrow O(x)),$$

where $y > x$ is understood as the conjunction of $y \geqslant x$ and $y \neq x$. By *induction for O on X* we mean the following:

If O is progressive, then $\forall x\, O(x)$.

Note that this is induction from above rather than, as is more common, from below.

If X has a *least element* $\bot$, then $O(\bot)$ is equivalent to $\forall x\, O(x)$ whenever O is *monotone:* that is, if $x \leqslant y$, then $O(x)$ implies $O(y)$. If O is progressive, then O is satisfied by every maximal element of X, and thus by the *greatest element* $\top$ of X whenever this exists.

The predicate O is *meet-closed* if $x = y \wedge z$ implies that $O(x)$ follows from $O(y) \wedge O(z)$; and O is a *filter* of X if O is monotone and meet-closed such that if X has $\top$, then $O(\top)$. The prime example of a filter is the *principal filter*

$$\uparrow u = \{x \in X : u \leqslant x\}$$

generated by an element u of X.

Any $D \subseteq X$ is *directed* if every finite subset of D has an upper bound in D; in particular, D has at least one element: an upper bound of $\emptyset$. Also, X is a *directed-complete partial order* (for short, a *dcpo*) if every directed $D \subseteq X$ has a least upper bound $\bigvee D$ in X.

Now let X be a dcpo. A predicate C on X is *closed* [Zor35] or *admissible* [Str06] if

$$(\forall x \;\epsilon\; D)C(x) \Longrightarrow C(\bigvee D)$$

for every directed $D \subseteq X$.

Dually, a predicate O on X is *(Scott) open* [Mos06, Rao88] precisely when it (is monotone and) satisfies

$$O(\bigvee D) \Longrightarrow (\exists x \;\epsilon\; D)O(x)$$

for every directed $D \subseteq X$. In particular, an open filter is automatically Scott open.

Lemma 5.1. *Let X be a dcpo, and let any $Y \subseteq X$ have the induced partial order.*

1. *If $Y \subseteq X$ is monotone, then Y is closed.*
2. *Any $Y \subseteq X$ is closed if and only if it is a dcpo.*
3. *If $Y \subseteq X$ is closed and $O \subseteq X$ is open, then $O \cap Y$ is open in Y.*

In this chapter, the prime example of a dcpo will be a closed $X \subseteq \mathcal{P}(S)$ where S is any given set. Here $\mathcal{P}(S)$, ordered by $\subseteq$, actually is a complete lattice: that is, has arbitrary suprema and infima, which are the unions and the intersections, respectively. Hence that X is closed in $\mathcal{P}(S)$ means that it is closed under directed unions.

Example 5.2. Let S be a set. For every closed $X \subseteq \mathcal{P}(S)$, if $M \subseteq S$ and $a \in S$, then

$$\{F \in X : M \between F\} \quad \text{and} \quad \{F \in X : a \;\epsilon\; F\}$$

are Scott open and an open filter, respectively.

If O is an open predicate on a closed $X \subseteq \mathcal{P}(S)$, then O is closed under arbitrary unions, and O is closed under finite intersections, too, whenever it is monotone: that is, Scott open.

A principal filter $\uparrow u$ is open—and thus even Scott open—precisely when its generator u is *compact*: that is, for every directed $D \subseteq X$,

$$u \leqslant \bigvee D \Longrightarrow (\exists x \;\epsilon\; D)\;(u \leqslant x)\,.$$

A dcpo X is *algebraic* if every $x \in X$ is the supremum of the compact $u \in X$ with $u \leqslant x$.

Raoult's [Rao88] has coined the following principle, which is induction for an open predicate O on a dcpo X:

Open Induction (OI) *If X is a dcpo, and O is open and progressive, then $\forall x\, O(x)$.*

While Raoult [Rao88] has deduced OI from *Zorn's Lemma (ZL)*, this OI is actually equivalent, in a natural way [HS13], to an appropriate form [Fel67] of ZL. This form reads as

If X is a dcpo, and C is closed and inhabited, then C has a maximal element

where C is a predicate on X. In fact, the latter principle can be rephrased as

If X is a dcpo, and C is closed and unbounded, then C is empty

where C is *unbounded* if

$$\forall x\, (C(x) \Longrightarrow (\exists y > x) C(y)) \,,$$

which is to say that C has no maximal element. Now if O and C are complements of each other, i.e. $O \cup C = X$ and $O \cap C = \emptyset$, then

1. $O = X$ if and only if $C = \emptyset$;
2. O is open if and only if C is closed;
3. O is progressive if and only if C is unbounded.

In all, OI and ZL are equivalent by complementation, even instance by instance.

An element x of X is *reducible* if there are $y, z \in X$ such that $x = y \wedge z$ but $y > x$ and $z > x$; whence $x \in X$ is *irreducible* if

$$x = y \wedge z \Longrightarrow x = y \vee x = z \,.$$

If x is a maximal element of X, then x is irreducible—in fact, if x is maximal, then both $x = y$ and $x = z$ hold already if $x \leqslant y \wedge z$. We write

$$\mathrm{Irr}(X) = \{x \in X : x \text{ irreducible}\} \,.$$

The following consequence of OI, in slightly varied terms, has proved useful [HS12, Sch12]:

Theorem 5.3 (OI). *If X is a dcpo, and $O \subseteq X$ an open filter, then*

$$\mathrm{Irr}(X) \subseteq O \Longrightarrow X = O \,.$$

Proof. To apply OI it remains to show, from the given hypotheses, that O is progressive. To this end let $x \in X$ such that $O\,(y)$ for every $y > x$. If x is irreducible, then $O\,(x)$ by hypothesis. If x is reducible, say $x = y \wedge z$ with $x < y$ and $x < z$, then $O\,(y)$ and $O\,(z)$ and thus $O\,(x)$. □

To relativise this to any suitable $Y \subseteq X$ with the induced partial order, the given partial order X has to have *binary meets*:[1] that is, for any $y, z \in X$ there is $y \wedge z \in X$.

[1] Binary meets need not distribute over (directed) joins whenever they exist.

Corollary 5.4 (OI). *Let X be a dcpo that has binary meets. If $Y \subseteq X$ is a filter and $O \subseteq X$ an open filter, then*

$$\mathrm{Irr}(Y) \subseteq O \Longrightarrow Y \subseteq O\,.$$

Proof. We first use Lemma 5.1. Since Y is monotone, it is closed; whence Y is a dcpo and $O \cap Y$ is open in Y. Next, as X has binary meets, $O \cap Y$ is a filter. Finally, Theorem 5.3 with $O \cap Y$ and Y in place of O and X, respectively, yields $Y = O \cap Y$ as required. □

The case of principal filters can be put in a particularly slick way whenever the dcpo X is algebraic and has *arbitrary meets*: that is, every $Z \subseteq X$ has a greatest lower bound $\bigwedge Z$.

Corollary 5.5 (OI). *If X is an algebraic dcpo that has arbitrary meets, then, for every $v \in X$,*

$$\bigwedge \mathrm{Irr}(\uparrow v) \leqslant v\,.$$

Proof. Set $Y = \uparrow v$. For an arbitrary compact $u \in X$ we apply Corollary 5.4 to $O = \uparrow u$ and get that if u is a lower bound of $\mathrm{Irr}(Y)$, then u is a lower bound of Y, or, equivalently,

$$u \leqslant \bigwedge \mathrm{Irr}(\uparrow v) \Longrightarrow u \leqslant v\,.$$

As X is algebraic, this means that $\bigwedge \mathrm{Irr}(\uparrow v) \leqslant v$ as desired. □

With the subsequent proposition we finally observe that there is no need to distinguish, for the elements of a monotone $Y \subseteq X$, between irreducibility in Y and irreducibility in X.

Proposition 5.6. *Let X be a partial order that has binary meets. If $Y \subseteq X$ is monotone, then*

$$\mathrm{Irr}(Y) = \mathrm{Irr}(X) \cap Y\,.$$

Proof. To see $\mathrm{Irr}(Y) \subseteq \mathrm{Irr}(X)$, let $x = y \wedge z$ in X. If $x \in Y$, then $y, z \in Y$ because Y is monotone; whence if $x \in \mathrm{Irr}(Y)$, then $x = y$ or $x = z$. □

5.2 A Universal Krull–Lindenbaum Theorem

In this section, we will prove a theorem which generalises both Krull's Lemma in commutative algebra and Lindenbaum's completeness theorem for first-order logic. This will be expressed in terms of basic finitary covers, and is, in some sense, a revisitation of the completeness theorem of Section 1.4.2.

We introduce some minimal changes in the terminology used so far, inspired by universal algebra [Bir48, Coh81]. Let S be a set and $\lhd$ a finitary basic cover on it. To give a cover $\lhd$ amounts to give the closure operator

$$\mathscr{A}(U) = \{a \in S : a \lhd U\}$$

of every subset U of S, for which clearly $\mathscr{A}(U) \lhd U$. In this chapter, we will call *ideals*, or $\lhd$–*ideals*, the formal opens of the basic cover $(S, \lhd)$, namely, the fixed points of the operator[2] $\mathscr{A}$. A subset F of S is an ideal precisely when

$$U \subseteq F \Longrightarrow \mathscr{A}(U) \subseteq F \tag{5.1}$$

for every subset U of S, which implication actually is an equivalence.

In particular, an ideal is said to be *finitely generated* if it is finitely generated as a formal open, that is, it is of the form $\mathscr{A}(\{a_1, \ldots, a_n\})$ with $n \geqslant 0$. While $\mathscr{A}(\emptyset)$ is the smallest ideal, the largest ideal is S itself. An ideal I is *proper* if $I \neq S$.

As seen in Section 1.2, the $\lhd$-ideals form a complete lattice that we denote here by $\mathrm{Idl}(\lhd)$ of $\mathcal{P}(S)$. In particular, $\mathrm{Idl}(\lhd)$ is a bounded complete lattice with $\top = S$, $\bot = \mathscr{A}(\emptyset)$ and

$$\bigwedge D = \bigcap D\,, \quad \bigvee D = \mathscr{A}(\bigcup D)$$

for arbitrary $D \subseteq \mathrm{Idl}(\lhd)$. Reflexivity and transitivity alone do not in general suffice for the lattice $\mathrm{Idl}(\lhd)$ to be distributive, let alone a frame.

If $\sqsubseteq$ is a *preorder* on S (that is, a reflexive and transitive binary relation), then

$$a \lhd U \equiv \exists b \,\epsilon\, U \ (a \sqsubseteq b)$$

defines a cover $\lhd$ on S for which the saturation of $U \subseteq S$ is its *downset*

$$\downarrow U = \{x \in S : \exists u \,\epsilon\, U \ (x \sqsubseteq u)\}\,.$$

Any cover $\lhd$ induced by a preorder is *unitary*: that is, $a \lhd U$ amounts to $a \lhd b$ for some $b \,\epsilon\, U$. Conversely, every unitary cover $\lhd$ determines a preorder $\sqsubseteq$ by

$$a \sqsubseteq b \equiv a \lhd b\,.$$

Since these constructions are inverse to each other, the preorders on S are exactly the unitary covers on S. Note that if $\sqsubseteq$ is a preorder on S, then so is the reverse relation $\sqsupseteq$. We will treat this kind of topologies in a different context in the next chapter.

More generally, a cover $\lhd$ is *finitary* if $a \lhd U$ implies that $a \lhd U_0$ already for some finite subset U_0 of U. A finitary cover corresponds to a closure operator that is *algebraic* in the sense that $\mathscr{A}(U)$ is the directed union of the $\mathscr{A}(U_0)$ where U_0 ranges over the finite subsets of U.

A cover $\lhd$ is finitary if and only if $\mathrm{Idl}(\lhd)$ is closed under directed unions [Sch52]. If $\lhd$ is a finitary cover, then

1. $\bigvee D = \bigcup D$ for every directed $D \subseteq \mathrm{Idl}(\lhd)$;
2. $\mathrm{Idl}(\lhd)$ is closed in $\mathcal{P}(S)$ with directed union;

[2] This terminology is clearly justified in the case of the Zariski basic topology, where the formal opens consist of the ideals of the correspondent commutative ring.

3. every ideal is the directed union of its finitely generated subideals;

4. the finitely generated ideals are the compact elements of $\mathrm{Idl}(\lhd)$.

In all, if $\lhd$ is a finitary cover, then $\mathrm{Idl}(\lhd)$ is an algebraic dcpo, of which the finitely generated ideals form a basis of compact elements, and in which binary meets distribute over directed joins.

In addition to a cover $\lhd$, we now assume that we are given an *operation* $*$ on S: that is, a map $* : S \times S \to S$. Unlike the general case treated in Section 1.3, we restrict here the operation $*$ to have values in singletons, or, more directly, in S. As usual we lift $*$ from elements to subsets of S, by setting

$$U * V = \{a * b : a \;\epsilon\; U, b \;\epsilon\; V\}\,.$$

Apart from a certain compatibility with the cover (as specified below), no further properties will now be required from the operation, not even that it be associative. For the purposes of this Section 5.2, $(S, *)$ only needs to be a magma or groupoid, and need not even be a semigroup; in Section 5.3, however, $(S, *)$ will normally be a monoid.

In many but not all of the instantiations we will discuss in Section 5.3, S comes with a distinguished element e that is *convincing* in the sense that $\mathscr{A}(\{e\}) = S$ or, equivalently,

$$e \;\epsilon\; I \Longrightarrow I = S \tag{5.2}$$

for every ideal I. Since this implication actually is an equivalence, an ideal I is proper precisely when $e \not\epsilon\; I$. We assume the basic cover with operation $(S, \lhd, *)$ to satisfy Weak-Right (WR), that is,

$$\mathscr{A}(U \cup \{a\}) \cap \mathscr{A}(U \cup \{b\}) \subseteq \mathscr{A}(U \cup \{a * b\})$$

for all $a, b \in S$ and $U \subseteq S$.
Inasmuch as WR holds in all the instantiations given in Section 5.3, it is quite a natural property.

If $\sqsubseteq$ is a preorder on S, then WR for the corresponding unitary cover boils down to

$$x \sqsubseteq a \wedge x \sqsubseteq b \Longrightarrow x \sqsubseteq a * b\,. \tag{5.3}$$

If S even has binary meets $a \sqcap b$ with respect to $\sqsubseteq$, which are determined by requiring

$$x \sqsubseteq a \wedge x \sqsubseteq b \Longleftrightarrow x \sqsubseteq a \sqcap b$$

from all $x \in S$, then (5.3) is equivalent to $a \sqcap b \sqsubseteq a * b$. Analogous observations can be made when $\sqsubseteq$ and $\sqcap$ are replaced by[3] $\sqsupseteq$ and $\sqcup$.

We recall from Section 1.4.2, that if the finitary cover $(S, \lhd, *)$ satisfies WR, then it satisfies contraction and *Right*. In particular, if $(S, \lhd, *)$ satisfies *Left*

[3] This gives moreover a formal topology, that will be studied in details in the next chapter.

and WR, then it is a formal topology with operation. In this case, for all ideals F and G, we have

$$\mathscr{A}(F * G) = F \cap G\,.$$

$Left$ would bring the $\lhd$–ideals closer to the ideals of rings and lattices, and contraction would make them into a sort of radical ideals. $Left$, however, holds only in some of the instantiations given in Section 5.3, whereas WR is common to all of them.

A Theorem by Noether, McCoy, Fuchs and Schmidt

Let $\lhd$ be a finitary cover on a set S. An ideal I is *irreducible* if it is is irreducible as an element of $\mathrm{Idl}\,(\lhd)$, which means that

$$I = F \cap G \Longrightarrow I = F \vee I = G$$

for all ideals F and G. For example, if $I = S$, then I is irreducible. Correspondingly, I is *reducible* if $I = F \cap G$ for some ideals F and G with $F \supsetneqq I$ and $G \supsetneqq I$.

Theorem 5.7. *If S is a set with a finitary cover $\lhd$, then*

$$\bigcap \{I : I \supseteq F\} \subseteq F \tag{5.4}$$

for every ideal F where I ranges over the (proper) irreducible ideals.

Proof. Apply Corollary 5.5 to $X = \mathrm{Idl}\,(\lhd)$ and $v = F$. □

Corollary 5.8. *Let S be a set with a finitary cover $\lhd$. For every ideal F and $a \in S$, if $a \notin F$, then there is an irreducible ideal $I \supseteq F$ such that $a \notin I$.*

Now let $*$ be an operation on S. The following variants of Theorem 5.7 an Corollary 5.8 require $Left$. We recall that a subset M of S is *convergent* if

$$a \,\epsilon\, M \;\&\; b \,\epsilon\, M \rightarrow a * b \,\epsilon\, M \tag{5.5}$$

for all $a, b \in S$.

Proposition 5.9. *Let S be a set with a finitary cover $\lhd$ and an operation $*$. Assume that $Left$ is satisfied. Let $F, M \subseteq S$ such that F is an ideal and $M \neq \emptyset$ is a convergent subset. If $I \between M$ for every (proper) irreducible ideal I with $I \supseteq F$, then $F \between M$.*

Proof. We want to apply Corollary 5.4 to $X = \mathrm{Idl}\,(\lhd)$, $Y = \uparrow F$ and

$$O = \{G \in X : G \between M\}\,.$$

Clearly, O is Scott open (Example 5.2)—but why is O is meet-closed and thus an open filter? If $G, H \,\epsilon\, O$, then $G * H \,\epsilon\, O$ for M is convergent; whence $G \cap H \,\epsilon\, O$ by $Left$. □

Corollary 5.10. *Let S be a set with a finitary cover $\lhd$ and an operation $*$. Assume that $Left$ is satisfied. Let $F, M \subseteq S$ such that F is an ideal and $M \neq \emptyset$ is a convergent subset. If $F \cap M = \emptyset$, then there is an irreducible ideal $I \supseteq F$ such that $I \cap M = \emptyset$.*

The Universal Krull–Lindenbaum Theorem

Let $\lhd$ be a cover and $*$ an operation on a set S. We call an ideal I a *prime ideal* if

$$a * b \;\epsilon\; I \Longrightarrow a \;\epsilon\; I \vee b \;\epsilon\; I \tag{5.6}$$

for all $a, b \in S$. For example, if $I = S$, then I is a prime ideal. Notice that the prime ideals are, with classical logic, precisely the complements of the formal points. In this chapter, classical topological considerations play a secondary role.

Remark 23. If $Left$ is satisfied, then every prime ideal is irreducible.

Proof. Let an ideal I be reducible, say $I = F \cap G$ for ideals F and G with $F \supsetneqq I$ and $G \supsetneqq I$. Pick $a \;\epsilon\; F \setminus I$ and $b \;\epsilon\; G \setminus I$. Now

$$a * b \;\epsilon\; F * G \subseteq F \cap G = I$$

by $Left$. Hence I is not a prime ideal. □

While Remark 23 will not be used but for heuristics, Lemma 5.11 is crucial.

Lemma 5.11. *If WR holds, then every irreducible ideal is prime.*

Proof. If an ideal I is not prime, i.e. there are $a, b \in S \setminus I$ with $a * b \;\epsilon\; I$, then

$$\mathscr{A}(I \cup \{a\}) \cap \mathscr{A}(I \cup \{b\}) = I$$

by WR, and thus I is reducible. □

Unlike WR, $Left$ is not required for the *Universal Krull–Lindenbaum Theorem*:

Theorem 5.12. *Let S be a set with a finitary cover $\lhd$ and an operation $*$. If WR holds, then*

$$\bigcap\{P : P \supseteq F\} \subseteq F \tag{5.7}$$

for every ideal F where P ranges over the (proper) prime ideals.

Proof. In the presence of WR the prime ideals P include, by Lemma 5.11, the irreducible ideals I; whence

$$\bigcap\{P : P \supseteq F\} \subseteq \bigcap\{I : I \supseteq F\}\,. \tag{5.8}$$

Since F is trivially contained in the left-hand side, this equals F whenever so does the right-hand side, and the latter is the case by Theorem 5.7. □

Corollary 5.13. *Let S be a set with a finitary cover $\lhd$ and an operation $*$. Assume that WR is satisfied. For every ideal F and $a \in S$, if $a \notin F$, then there is a prime ideal $P \supseteq F$ such that $a \notin P$.*

Negation Operations

In addition to the *convergence* operation $*$, there sometimes also is a *negation* operation $\sim$ on S: that is, a map $\sim: S \to S$. We then understand by a *complete ideal* an ideal I for which

$$\sim c \in I \vee c \in I$$

for every $c \in S$.

Lemma 5.14. *Let S be a set with a covering $\lhd$, a convergence operation $\circ$ and a nagation operation $\sim$. Let $I \subseteq S$.*

1. *If I is a prime ideal, and $\sim c * c \in I$ for every $c \in S$, then I is a complete ideal.*
2. *If I is a complete ideal satisfying either (a) or (b) below, then I is a prime ideal.*
   (a) *$*$ is associative, $b = (\sim a * a) * b$ for all $a, b \in S$ and I is multiplicative.*
   (b) *Weak-Right holds, and $S \lhd \{c, \sim c\}$ for every $c \in S$.*

Proof. Part 1 is clear. As for part 2, assume that I is complete, and let $a, b \in S$ with $a * b \in I$. In situation 2a, either $a \in I$ and we are done, or $\sim a \in I$ and thus $b \in I$, for

$$b = (\sim a * a) * b = \sim a * (a * b)\,.$$

In situation 2b, we proceed in two steps. First, note that, by Weak-Right,

$$S \lhd \{a * b, \sim a, \sim b\}\,,$$

because $x \lhd \{a, \sim a\}$ and $x \lhd \{b, \sim b\}$ for every $x \in S$. Secondly, we can distinguish two cases: (i) either $a \in I$ or $b \in I$; (ii) both $\sim a \in I$ and $\sim b \in I$. In case (i) we are done. In case (ii), $I = S$ by the first step. This is impossible if I is proper, but it ensures that I is prime. □

Corollary 5.15. *Let S be a set with a covering $\lhd$, a convergence operation $*$ and a negation operation $\sim$. Assume that Weak-Right holds, and that there are $d, e \in S$ such that $e \in S$ is convincing, and for which $\sim c * c = d$ and $e \lhd \{c, \sim c\}$ for every $c \in S$. Let $I \subseteq S$ be an ideal with $d \in I$. Then I is a prime ideal if and only if I is a complete ideal.*

Corollary 5.16. *Let S be a set with a covering $\lhd$. Assume in addition that S is a group with operation $*$, identity d and inverse $\sim$. Let $I \subseteq S$ be an ideal that also is a multiplicative subset of S and satisfies $d \in I$. Then I is a prime ideal if and only if I is a complete ideal.*

Corollary 5.15 clearly has instantiations for Boolean algebras and classical logic (Sections 5.3 and 5.3). It can, however, also be applied to ordered fields (Section 5.3), though in this context Corollary 5.16 will do as well. In fact, Corollary 5.16 will be applied both to the multiplicative group and to the additive group of a field (Sections 5.3 and 5.3). As one can read off the proof of Lemma 5.14, for Corollary 5.16 one does not really need $(S, *, d, \sim)$ to be group proper: it would suffice to require that $*$ is associative, d is left-neutral and $\sim$ yields left-inverses.

Discussion

Corollary 5.13 is perhaps more widely known than Theorem 5.12: as Krull's Lemma, the Prime Ideal Theorem, the Prime Filter Theorem, the Separation Lemma, or the like. It was allegedly first proved by Krull [Kru29] for prime ideals of a commutative ring (Section 5.3), using the Well-Ordering Theorem. Tarski [Tar30, p. 394] ascribed to Lindenbaum the instance of Corollary 5.13 in first-order logic (Section 5.3).[4] To our knowledge only Scott [Sco74] has put Lindenbaum's lemma in a way similar to Theorem 5.12.

Theorem 5.7 for commutative rings is due to McCoy [McC38], who deduced it from the Well-Ordering Theorem; Fuchs [Fuc49] later gave a proof with Zorn's Lemma. Long before, Noether [Noe21] had given the specific case for Noetherian rings, for which no transfinite proof method is required, and every ideal is a finite intersection of irreducible ideals. Schmidt [Sch52] has eventually transferred Fuchs's proof to the context of closure operators.

Schmidt [Sch52] has further observed that Fuchs's proof [Fuc49] equally works when "I is irreducible" is understood à la Grell [Gre51]: that is, as I being *completely irreducible*. This means that I is not the intersection of any family of proper superideals—or, equivalently, I is a proper subset of the intersection of all its proper superideals. With this notion of irreducibility, Theorem 5.7 is provable from OI in the same way as above, and implies Theorem 5.7 as it stands just as this implies Theorem 5.12: every completely irreducible ideal is irreducible.

As usual, by a *maximal ideal* we understand an ideal I that is maximal among the proper ideals. Since every maximal ideal is completely irreducible, in all the contexts in which Weak-Right holds—such as the one of commutative rings—every maximal ideal is prime. In general, however, there are fewer maximal ideals than irreducible or prime ideals, and the intersection of all maximal ideals over a given ideal is bigger than this ideal. See also [HS12, HS13].

A straightforward attempt to prove Theorem 5.7 makes it clear that Zorn's Lemma, say, is indispensable, but also hints at a proof by Open Induction as the one we have given. To see this, let $a \in I$ for every *irreducible* ideal $I \supseteq F$, and note that proving $a \in F$ amounts to proving $a \in I$ for *all* ideals $I \supseteq F$. To prove the latter, let I be an arbitrary ideal with $I \supseteq F$. If I is irreducible, then $a \in I$ by hypothesis. If I is reducible, say $I = F \cap G$ for ideals F and G with $F \supsetneq I$ and $G \supsetneq I$, then one can only proceed by showing that both $a \in F$ and $a \in G$. To ensure that this process terminates in general one needs transfinite methods.

This attempt further shows that transfinite methods come somewhat naturally with irreducible ideals, whereas it is not clear a priori why prime ideals would have anything to do with transfinite methods. Now a satisfying answer to this question can be obtained: by repeating the attempt above with Theorem 5.12 in place of Theorem 5.7. In fact, if I is not prime, then I is reducible in

[4] The axioms of an algebraic closure operator are Tarski's axioms I.2–I.4 for the operator assigning to a set of assertions the set of its consequences [Tar30]; his axiom I.1 says that S is countable, which we do not require.

the presence of WR; see the proof of Lemma 23.

The relation between Theorem 5.7 and Corollary 5.8 is just as the one between Theorem 5.12 and Corollary 5.13: in both cases, the corollary is the contrapositive of the theorem. Unlike Theorem 5.12 and Corollary 5.13, Theorem 5.7 and Corollary 5.8 do not require a convergence operation $*$: they are results about sheer closure operators. In view of this, the following remarks on Theorem 5.12 and Corollary 5.13 almost literally hold for Theorem 5.7 and Corollary 5.8: just replace "prime" by "irreducible" and drop WR together with the convergence operation $*$.

First, why have we given priority to Theorem 5.12 over Corollary 5.13? It is natural and direct to prove Theorem 5.12 and Corollary 5.13 by Open Induction and Zorn's Lemma, respectively; and, as we have outlined in the introduction, we want to give precedence to Open Induction over Zorn's Lemma. Moreover, in concrete instantiations (see Section 5.3) it often is more direct and more natural to use Theorem 5.12 rather than Corollary 5.13.

Secondly, the reverse inclusion in (5.7) and the reverse implication in Corollary 5.13 hold anyway; whence the former may be put as an equality and the latter as an equivalence. Next, in (5.7) the intersection may be taken only over the *minimal* (proper) prime ideals containing F, by simply dropping the ones which are not minimal and thus are redundant for the intersection. Further, Theorem 5.12 and Corollary 5.13 can be formulated equivalently by replacing every occurrence of the ideal F by the saturation $\mathscr{A}(U)$ of an arbitrary $U \subseteq S$, for which moreover $P \supseteq \mathscr{A}(U)$ can be simplified to $P \supseteq U$; whence one can rephrase (5.7) as

$$\bigcap\{P : P \supseteq U\} \subseteq \mathscr{A}(U)\,.$$

As we have indicated in Theorem 5.12, the intersection in (5.7) can be restricted to the *proper* prime ideals, simply because $P = S$ is redundant for this intersection. To do so, however, requires to decide, for any given prime ideal P, whether $P = S$. In the presence of a convincing element e this can be reduced to deciding whether $e \in P$. Apart from this, a convincing element e has not been needed yet, though in many of the instantiations in Section 5.3 there is a natural candidate for such e .

Corollary 5.13 says, in particular, that if $\lhd$ is finitary and WR is satisfied, then for every proper ideal F there is a proper prime ideal P with $I \supseteq F$. Since $P = S$ is a prime ideal, the left-hand side of (5.7) can only be the intersection of the empty set—and thus equal to S—if it is restricted to the *proper* prime ideals. If, however, there is no proper prime ideal P with $P \supseteq F$, then $F = S$ anyway, by what we have just observed as a consequence of Corollary 5.13.

If $a \in S$ is such that $a * a = a$, then Proposition 5.7 and Corollary 5.8 are the special case $M = \{a\}$ of Proposition 5.9 and Corollary 5.10, respectively.

Why have we refrained from carrying over Proposition 5.9 and Corollary 5.10 from irreducible to prime ideals? This move would have required to suppose that both Weak-Right and *Left* be valid, under the conjunction of which, however, the prime ideals are exactly the irreducible ideals anyway (Remark 23, Lemma 5.11). Since, moreover, neither Proposition 5.9 nor Corollary 5.10 will be used in this chapter, we have also refrained from presenting even more variants of Theorem 5.7 and Corollary 5.8 or of Theorem 5.12 and Corollary 5.13, such as the ones that would correspond to [Per99, Corollary 3].

By complementation, the formal points are just the proper prime ideals. The terminology of (prime) ideals, however, is a natural choice for this chapter, especially in view of the intended instantiations (Section 5.3).

With a finitary cover $\lhd$ and an operation $*$ satisfying WR we believe to have singled out the bare minimum of data and properties that is needed to represent the proof pattern standing behind the universal Krull–Lindenbaum theorem.

For the sake of this theorem and its proof, the cover $\lhd$ and the operation $*$ need only be linked by WR and can otherwise be independent from each other. In all instantiations in Section 5.3, however, both $\lhd$ and $*$ emerge from the same structure.

5.3 Instantiations in Algebra and Logic

Although we will recall some of the crucial concepts,[5] we assume that the reader has some familiarity with the basics of the theories of commutative rings [AM69] and distributive lattices [BD74]. Every commutative ring or subring is supposed to have a unit 1. We write R^* for the multiplicative group of a commutative ring R. This consists of the $r \in R$ which are *invertible*: that is, there is $s \in R$ with $rs = 1$.

Commutative Rings

This is about Krull's theorem in its original form, for ideals, and its dual for filters. Let R be a commutative ring. We do not assume from the outset that R be *non–trivial*: that is, $1 \neq 0$ in R or, equivalently, $0 \notin R^*$.

Radical and Prime Ideals[6]. An *ideal* of R is a subset I which contains 0, is closed under addition, and satisfies

$$a \in I \Longrightarrow ab \in I \tag{5.9}$$

for all $a, b \in R$. We write (U) for the ideal generated by $U \subseteq R$: that is, (U) consists of all the $r_1u_1 + \ldots + r_nu_n$ with $u_1, \ldots, u_n \in U$ and $r_1, \ldots, r_n \in R$. As usual, (a) stands for $(\{a\})$ and the like. The sum $I + J$ of two ideals consists of the $x + y$ with $x \in I$ and $y \in J$.

A *radical ideal* is an ideal I such that

$$a^2 \in I \Longrightarrow a \in I$$

for every $a \in R$. If I is an ideal, then its *radical*

$$\sqrt{I} = \{r \in R : \exists \ell \geqslant 1\, (r^\ell \in I)\}$$

[5] For the sake of brevity we do not always indicate this by "recall that" or the like.

[6] This example has already been treated in Chapter 2, but we repeat it here in a different context.

is a radical ideal with $I \subseteq \sqrt{I}$; and I is a radical ideal if and only if $I = \sqrt{I}$. The smallest radical ideal is the *nilradical* $\sqrt{0}$: that is, the radical of the zero ideal 0.

An ideal I is *proper* if $1 \notin I$. For any ideal I, the following are equivalent: $I = R$, $I \between R^*$, $1 \in I$ and $1 \in \sqrt{I}$. The ring R is non–trivial if and only if 0 is a proper ideal.

A proper ideal I is a *prime ideal* if

$$ab \in I \Longrightarrow a \in I \vee b \in I$$

for all $a, b \in R$. (As is common for commutative rings with 1, prime ideals are understood to be proper.) Every prime ideal is a radical ideal.

Set $S = R$. As for the formal Zariski topology [Joh82, Sig95, Sch06a], we define $\lhd$ by setting $\mathscr{A}(U) = \sqrt{(U)}$. This $\lhd$ is a finitary covering, and the ideals of $\lhd$ are the radical ideals of R. Let further $*$ stand for multiplication in R. Hence the proper prime ideals of $\lhd$ are the prime ideals of R. Last but not least, $e = 1$ is convincing.

In this context, $Left$ holds in view of (5.9), and Weak-Right has the following form (see e.g. the proof of [AM69, Proposition 1.8], and [Sch12, Lemma 1]):

Lemma 5.17. *If I is an ideal of R, and $a, b \in R$, then*

$$\sqrt{I + (a)} \cap \sqrt{I + (b)} \subseteq \sqrt{I + (ab)}\,.$$

Proof. If $x^k = u + sa$ and $x^\ell = v + tb$ where $u, v \in I$, $s, t \in R$ and $k, \ell \geqslant 1$, then

$$x^k x^\ell = \underbrace{uv + utb + sav}_{\in I} + st\,ab$$

and thus $x^{k+\ell} \in I + (ab)$ as required. □

Theorem 5.12 has the following instance, that we have already encountered in chapter 2 and that can also be found as a definition of $\sqrt{I}$:

> *For every ideal I of R, the radical $\sqrt{I}$ is the intersection of the prime ideals of R that contain I.*

Here Corollary 5.13 is sometimes called the Prime Ideal Theorem for commutative rings:

> *If an ideal I of R is proper, then there is a prime ideal of R that contains I.*

In the particular case $I = 0$ these instances read as follows:

> *The nilradical $\sqrt{0}$ is the intersection of all the prime ideals of R.*
>
> *If R is non–trivial, then R has a prime ideal.*

Filters and Prime Filters. A subset F of R is a *filter* if it contains 1, is closed under multiplication, and satisfies

$$ab \in F \Longrightarrow a \in F$$

for all $a, b \in R$. In other words, any $F \subseteq R$ with $1 \in F$ is a filter if $ab \in F$ precisely when both $a \in F$ and $b \in F$.

Let $\langle U \rangle$ denote the filter generated by a subset U of R: that is, $\langle U \rangle$ consists of the $a \in R$ for which there are $b \in R$ and $u_1, \dots, u_n \in U$ with $n \geqslant 0$ such that $ab = u_1 \dots u_n$. These $u_1 \dots u_n$ need not be distinct. As usual we write $\langle c \rangle$ for $\langle \{c\} \rangle$ and the like. Since $\langle c \rangle$ consists of the divisors of the nonnegative powers of c, we have

$$a \in \langle c \rangle \Longleftrightarrow c \in \sqrt{a} \tag{5.10}$$

for all $a, c \in R$.

The smallest filter of R is the $R^* = \langle 1 \rangle$. A filter F is *proper* if $0 \notin F$; note that a filter F equals R if and only if $0 \in F$ (look at $a0 = 0$ for any $a \in R$). The ring R is non–trivial precisely when R^* is a proper filter.

A *prime filter* is a filter F satisfying

$$a + b \in F \Longrightarrow a \in F \vee b \in F$$

for all $a, b \in R$. The prime filters of R are nothing but the complements of the prime ideals, in the ring–theoretic sense, of R.

Set $S = R$, and define $\lhd$ by setting $\mathscr{A}(U) = \langle U \rangle$. This $\lhd$ is a finitary covering $\lhd$. Let $\circ$ stand for addition. Hence the (prime) $\lhd$–ideals are exactly the (prime) filters of R. Moreover, $e = 0$ is convincing.

With (5.10) at hand it is easy to see that $Left$ fails, actually that not even $a \circ a \lhd a$ holds: if $R = \mathbb{Z}$, then $1 \circ 1 \not\lhd 1$ because $1 \circ 1 = 2 \notin R^* = \langle 1 \rangle$. But Weak-Right looks as follows:

Lemma 5.18. *If U is a subset of R, and $c, d \in R$, then*

$$\langle U, c \rangle \cap \langle U, d \rangle \subseteq \langle U, c + d \rangle \, .$$

Proof. We first consider the special case $U = \emptyset$ and use (5.10). If $a \in \langle c \rangle \cap \langle d \rangle$, which is to say that $c, d \in \sqrt{a}$, then $c + d \in \sqrt{a}$ or equivalently $a \in \langle c + d \rangle$ as required.

In general, let $a \in \langle U, c \rangle \cap \langle U, d \rangle$, which means that

$$ag = uc^k \, , \quad ah = ud^\ell$$

where $u = u_1 \dots u_n$ for suitable common $u_1, \dots, u_n \in U$ with $n \geqslant 0$. This implies

$$ag' = (uc)^k \, , \quad ah' = (ud)^\ell$$

and thus $a \in \langle uc \rangle \cap \langle ud \rangle$, from which $a \in \langle u(c + d) \rangle$ and thus $a \in \langle U, c + d \rangle$ follow by the special case considered first. □

Theorem 5.12 now has the following instance:

Every filter F of R is the intersection of the (proper) prime filters of R that contain F.

This instance is sometimes [Laf77, p. 6, Corollaire] put as that F equals the intersection of the complements of the prime ideals of R disjunct from F.

Corollary 5.13 is sometimes called the Prime Filter Theorem for commutative rings R:

If F is a proper filter of R, then there is a proper prime filter of R that contains F.

In the particular case $F = R^*$ these instances read as follows:

The multiplicative group R^* is the intersection of all the (proper) prime filters of R.

If R is non–trivial, then R has a proper prime filter.

Here is a concrete application. An element a of R is a *zero divisor* if $ab = 0$ for some $b \in R$ with $b \neq 0$; and a is *regular* if it is not a zero divisor: that is, if $ab = 0$ implies $b = 0$ for every $b \in R$. The regular elements of R form a filter F, which therefore is the intersection of the prime filters of R containing F. By complementation, the set $R \setminus F$ of zero divisors of R is the union of the prime ideals of R which are disjoint from F [BIV89, Folgerung 1.19].

Distributive Lattices

With $\wedge$ and $\vee$ in place of multiplication and addition, respectively, the definitions of the following concepts for bounded lattices are the same as for commutative rings: ideal, proper ideal, prime ideal, filter, proper filter, prime filter. Since in any bounded lattice the notions of a (prime) ideal and of a (prime) filter are dual to each other, we can and will deal with them simultaneously.

Let L be a bounded lattice, with operations $\wedge$ and $\vee$, partial order $\leqslant$, least element 0, and greatest element 1. The lattice L need not be *non–trivial*, which would mean that $1 \neq 0$ in L. The ideal (respectively, the filter) of L that is generated by $U \subseteq L$ consists of the $a \in L$ for which there are $u_1, \ldots, u_n \in U$ with $n \geqslant 0$ such that $a \leqslant u_1 \vee \ldots \vee u_n$ (respectively, $a \geqslant u_1 \wedge \ldots \wedge u_n$). The smallest ideal (respectively, the smallest filter) of L is the zero ideal $0 = \{0\}$ (respectively, the unit filter $1 = \{1\}$).

Set $S = L$, and let $\mathscr{A}(U)$ denote the ideal (respectively, the filter) of L that is generated by $U \subseteq L$. This defines a finitary covering $\lhd$ such that the $\lhd$–ideals are the ideals (respectively, the filters) of L. Let further $\circ$ stand for $\wedge$ (respectively, for $\vee$). The (proper) prime $\lhd$–ideals are the (proper) prime ideals (respectively, prime filters) of L. Moreover, $e = 1$ (respectively, $e = 0$) is convincing.

In contrast to the situation for commutative rings, $Left$ holds in both cases, for any lattice L whatsoever. To discuss WR, we understand by *WR for ideals* (respectively, *WR for filters*) the case in which $\mathscr{A}(U)$ is the ideal (respectively, the filter) generated by U and $\circ$ stands for $\wedge$ (respectively, for $\vee$). Now

Weak-Right for ideals (and, dually, WR for filters) can be proved just as for commutative rings (Lemma 5.17) provided that the lattice L is distributive.[7]

In fact, to have WR both for ideals and filters is tantamount to the lattice being distributive. To see this, note first [Coh81, Chapter II, Proposition 4.4] that a lattice L is distributive precisely when

$$a \wedge (x \vee b) \leqslant x \vee (a \wedge b) \tag{5.11}$$

hold for all $a, b, x \in L$. Note that (5.11) is self-dual, and that for L to be modular [Coh81, p. 65] the condition (5.11) is required only when $x \leqslant a$, in which case (5.11) actually is an equality.

Proposition 5.19. *A lattice L is distributive if and only if both*

$$(u \vee c) \wedge (v \vee d) \leqslant u \vee v \vee (c \wedge d) \tag{5.12}$$

and the dual condition

$$(u \wedge c) \vee (v \wedge d) \geqslant u \wedge v \wedge (c \vee d) \tag{5.13}$$

are satisfied for all $c, d, u, v \in L$.

Proof. First, (5.11) can clearly be deduced from the case $u = v$ of either (5.12) or (5.13). Conversely, (5.12) is obtained by applying (5.11) twice:

$$(u \vee c) \wedge (v \vee d) \leqslant v \vee ((u \vee c) \wedge d) \leqslant v \vee u \vee (c \wedge d)$$

The deduction of (5.13) from (5.11) is dual to the one we have just carried out. □

Corollary 5.20. *A lattice L is distributive precisely when WR holds both for ideals and for filters.*

Proof. Weak-Right for ideals and filters are equivalent to (5.12) and (5.13), respectively. □

The instances of Theorem 5.12 for a bounded distributive lattice L are as follows:

> *Every ideal I of L equals the intersection of all the (proper) prime ideals of L that contain I.*
>
> *Every filter F of L equals the intersection of all the (proper) prime filters of L that contain F.*

Corollary 5.13 entails the following variants of the Prime Ideal Theorem and Prime Filter Theorem for bounded distributive lattices L:

[7] Actually the case of distributive lattices is considerably simpler than the one of commutative rings: unlike the latter, in the former there is no need to talk about powers and radicals, or about invertible elements.

> *If an ideal I of L is proper, then there is a proper prime ideal of L that contains I.*
>
> *If a filter F of L is proper, then there is a proper prime filter of L that contains F.*

The particular cases $I = 0$ and $F = 1$ of the foregoing may be of interest:

> *The zero ideal 0 is the intersection of all the (proper) prime ideals of L.*
>
> *The unit filter 1 is the intersection of all the (proper) prime filters of L.*
>
> *If L is non–trivial, then L has a proper prime ideal and a proper prime filter.*

Boolean algebras In addition to the structure of a distributive lattice, every Boolean algebra L has complementation $\neg$ as a negation operation. With $\neg$ as $\sim$, and with $d = 0$ and $e = 1$ if $\circ$ is $\wedge$ (respectively, with $d = 1$ and $e = 0$ if $\circ$ is $\vee$), the hypotheses of Corollary 5.15 are satisfied. For every $c \in L$ we indeed have $e \lhd \{\neg c, c\}$, because $1 \leqslant \neg c \vee c$ (respectively, $0 \geqslant \neg c \wedge c$) means that 1 belongs to the ideal (respectively, the filter) generated by c and $\neg c$. Hence Corollary 5.15 is a generalisation of the well-known fact that in every Boolean algebra the prime filters are precisely the ultrafilters: that is, in our terminology, the complete filters. Note that to prove this, or more generally to prove Corollary 5.15, does not require that the filters under consideration be proper, although this condition can be added at will.

Valuation Rings and Integral Closure

Let A be a subring of a field B. Let as usual $A[U]$ denote the subring of B containing A that is generated by $U \subseteq B$: i.e., the set of all the $f(u_1, \ldots, u_k)$ with $k \geqslant 0$ where $u_1, \ldots, u_k \in U$ and f is a polynomial in k variables with coefficients from A. Any $a \in B$ is *integral* over a subring R of B precisely when

$$a^n = r_{n-1}a^{n-1} + \ldots + r_1 a + r_0$$

for some $n \geqslant 1$ and certain $r_0, \ldots, r_{n-1} \in R$. The *integral closure* $\overline{R}$ of a subring R of B consists of the $a \in B$ which are integral over R; this $\overline{R}$ is the smallest integrally closed subring of B that contains R. A subring R of B is *integrally closed* in B if $R = \overline{R}$.

Let $S = B^*$, and set $\mathscr{A}(U) = \overline{A[U]}$ for every subset U of S. This defines a finitary covering $\lhd$ on S, and the ideals of $\lhd$ are the integrally closed subrings R of B which contain A. Let further $\circ$ stand for multiplication in B, whereas in this example we do not specify any (convincing) element e of S.

By Corollary 5.16 applied to the multiplicative group B^*, the prime ideals are precisely the complete ideals, which in turn are nothing but the *valuation rings* of $A \subseteq B$: that is, the subrings R of B with $R \supseteq A$ such that

$$c \in R \vee c^{-1} \in R \tag{5.14}$$

for every $c \in B^*$. Any such valuation ring R is an integral domain, as a subring of the field B; and R automatically is integrally closed in B; see, e.g., [AM69, Proposition 5.18.iii].[8]

Left fails in this context: for example, if $A = \mathbb{Z}$ and $B = \mathbb{Q}$, then $1 \circ \frac{1}{2} \not\lhd 1$ because $1 \circ \frac{1}{2} = \frac{1}{2}$ does not belong to $A[1] = \mathbb{Z}$. The required instance of WR is the case $R = A[U]$ of Lemma 5.21 below, which is somewhat hidden in the literature.

Lemma 5.21. *Let R be a subring of a field B. For all $x, y \in B$ we have*

$$\overline{R[x]} \cap \overline{R[y]} \subseteq \overline{R[xy]}\,.$$

Proof. In case $x = 0$ there is nothing to prove; whence we may suppose that $x \in B^*$. We first consider the specific case in which $y = x^{-1}$. Let $a \in \overline{R[x]} \cap \overline{R[x^{-1}]}$. Rewriting yields

$$\begin{array}{rcl} a^m + p_{m-1}a^{m-1} + \cdots + p_0 & = & 0 \\ a^n x^k + q_{n-1}a^{n-1} + \cdots + q_0 & = & 0 \end{array}$$

where $p_i, q_j \in R[x]$ and $\deg_x(q_j) < k$ for $i < m$ and $j < n$. Ordering these two equations with respect to $\deg_x$, and taking the resultant (see, for example, [Lom02, Theorem 2]), we get a monic polynomial equation in a with coefficients from R, as required. To deal with the general case, we consider $R' = R[xy]$, for which $R[x] \subseteq R'[x]$ and $R[y] \subseteq R'[x^{-1}]$. Now if $a \in \overline{R[x]} \cap \overline{R[y]}$, then $a \in \overline{R'[x]} \cap \overline{R'[x^{-1}]}$; whence $a \in \overline{R'}$ by the specific case. □

The following instance of Theorem 5.12 can be found in the literature (see, e.g., [Kap74, Theorem 57], [Laf77, p. 151, Corollaire 2] and [Lan93, Chapter VII, Proposition 3.6]):

> *The integral closure of a subring A of a field B is the intersection of all the valuation rings of $A \subseteq B$.*

This can be used to prove, among other things, Dedekind's Prague Theorem via the underlying theorem of Kronecker which says that, for any two polynomials, the products of the coefficients are integral over the coefficients of the product [CP01, Coq09, Edw90, Lom02].

Ordered Fields

Let K be a field. A *preorder* of K—also known as a *cone* in K—is a subset of K that contains all squares in K, and is closed under addition and multiplication:

$$P + P \subseteq P\,, \qquad P \cdot P \subseteq P\,.$$

[8] Unlike [AM69, Kap74] but following [Laf77, Lan93], we do not require B to be the quotient field of R, which in fact is an immediate consequence of (5.14): for every $c \in B^*$, either $c \in R$ or $c = \frac{1}{d}$ with $d = \frac{1}{c} \in R$.

In particular, the set Q consisting of the sums of squares in K is the smallest preorder of K.

A subset P of K is a preorder precisely when the relation $\leqslant$ defined by

$$a \leqslant b \equiv b - a \in P$$

is a preorder on K—that is, reflexive and transitive—and satisfies

$$a \leqslant b \Longrightarrow a + x \leqslant b + x\,, \qquad a \leqslant b \wedge x \geqslant 0 \Longrightarrow ax \leqslant bx\,.$$

Clearly, P can be recovered from $\leqslant$, because $P = \{x \in K : x \geqslant 0\}$.

For any preorder P, the preorder $P\langle U\rangle$ generated by $U \subseteq K$ consists of the $a \in K$ for which there are $n \geqslant 0$ elements $u_1, \dots, u_n$ of U such that

$$a = \sum_{v\, \epsilon\, \{0,1\}^n} \lambda_v u^v$$

where $u^v = u_1^{v_1} \dots u_n^{v_2}$ and $\lambda_v \in P$ for $v \,\epsilon\, \{0,1\}^n$. One can achieve that the $u_1, \dots, u_n \in U$ are mutually distinct. This $P\langle U\rangle$ is the smallest preorder of K that contains $P \cup U$.

An *order* of K is a preorder P for which $\leqslant$ is a *linear order*. Writing

$$-P = \{-x : x \in P\}\,,$$

this means that P satisfies the following conditions:

Antisymmetry $P \cap -P \subseteq \{0\}$

Dichotomy $P \cup -P \supseteq K$

In both conditions actually equality holds.

Let $S = K$, and set $\mathscr{A}(U) = Q\langle U\rangle$ for every subset U of S. The covering thus defined is finitary, and the ideals are nothing but the preorders of K. Let furthermore $\circ$ stand for $+$, and set $e = -1$, which is convincing at least if $\mathrm{char}(K) \neq 2$ (see below).

Let $a \in K^*$. Since

$$a^{-1} = \left(a^{-1}\right)^2 a\,, \qquad -1 = a^{-1}(-a)\,,$$

we have $a^{-1} \lhd a$ and $-1 \lhd \{a, -a\}$. In particular, if P is a preorder of K, and $a \in P$, then $a^{-1} \in P$; and if $a, -a \in P$, then $-1 \in P$.

The proper prime ideals are precisely the orders of K. On the one hand, Antisymmetry is equivalent to $-1 \notin P$; in fact, $P \cap -P$ contains an element of K^* precisely when $-1 \in P$, as we have just seen. Dichotomy, on the other hand, amounts to requiring

$$z \in P \vee -z \in P \tag{5.15}$$

from every $z \in K$. This means that P is a complete ideal or, equivalently, that P is a prime ideal: apply Corollary 5.16 to the additive group of the field K. (If

$\mathrm{char}(K) \neq 2$, then one can alternatively apply Lemma 5.14, part 1 with $d = 0$, and part 2b with $e = -1$: see below.)

If $\mathrm{char}(K) > 0$, then $-1 = 1 + \ldots + 1$ belongs to every preorder of K. If $\mathrm{char}(K) \neq 2$, then K is the only preorder containing -1, simply because

$$4x = (x+1)^2 - (x-1)^2$$

for every $x \in K$. Unlike an order, a preorder P may thus contain -1, in which case however $P = K$ whenever $\mathrm{char}(K) \neq 2$. In other words, if $\mathrm{char}(K) \neq 2$, then -1 is convincing. More precisely, for any given integer $n \geqslant 0$ we have $n \neq 2$ precisely when -1 is convincing in every field K of characteristic n. In fact, the indeterminate T is not a (sum of) square(s) in the field of rational functions $K = \mathbb{F}_2(T)$ with coefficients in the two-element field $\mathbb{F}_2$.

$Left$ fails, for example if the field K is *formal real*: that is, -1 is not a sum of squares in K. In any such K we indeed have $(-1) \circ 0 \not\triangleleft 0$ simply because $-1 + 0 = -1$.

The required instance of WR is the case $P = Q\langle U\rangle$ of the following lemma. For the time being we require that $\mathrm{char}(K) \neq 2$, but we will discuss this issue in a moment.

Lemma 5.22. *Let P be a preorder of a field K with $\mathrm{char}(K) \neq 2$. For all $a, b \in K$ we have*

$$P\langle a\rangle \cap P\langle b\rangle \subseteq P\langle a+b\rangle\,. \tag{5.16}$$

Proof. Let $x \in P\langle a\rangle \cap P\langle b\rangle$. This is to say that

$$x = ya + y' \quad \text{and} \quad x = zb + z'$$

with $y, y', z, z' \in P$, from which we get

$$(y+z)x = yz(a+b) + y'z + yz' \in P\langle a+b\rangle\,.$$

If $y + z \in K^*$, then

$$x = (y+z)^{-1}yz(a+b) + (y+z)^{-1}(y'z + yz') \in P\langle a+b\rangle\,.$$

If $y = 0$, then $x = y' \in P$ anyway. So we may suppose that $y + z = 0$ and $y \in K^*$. In this case, since $-y = z \in P$, we have that $y \in P \cap -P$; whence $-1 \in P$, and thus $P = K$ because $\mathrm{char}(K) \neq 2$. □

For any field K with $\mathrm{char}(K) \neq 2$, Theorem 5.12 has the following instance:

> *Every preorder P of K is the intersection of all the orders of K that contain P.*

This clearly is of particular interest in the specific case $P = Q$:

> *The sums of squares in K are exactly the elements of K that are nonnegative with respect to every order of K.*

Corollary 5.13 includes the *Artin–Schreier Theorem*:

> *A field K has an order if (and only if) K is formal real.*

As is well known, this has brought Artin to solve Hilbert's 17th Problem. Note that the Artin–Schreier Theorem trivially holds if $\mathrm{char}(K) > 0$: if K is formal real, then $\mathrm{char}\,(K) = 0$.

Apart from being required for -1 to be convincing (see above), the precondition that $\mathrm{char}\,(K) \neq 2$ was further used in our proof of WR (Lemma 5.22). What is the situation for a field K of arbitrary characteristic? While (5.16) trivially holds whenever $P = K$, following the proof of [Lor90, §20, Lemma 1] one can prove (5.16) for every preorder P with $-1 \notin P$, for which P the above instance of Theorem 5.12 can be inferred [Lor90, §20, Satz 1].

But what about adding the hypothesis $-1 \notin P$? It would make trivially valid the above instance of Theorem 5.12 whenever $\mathrm{char}(K) > 0$, for then every preorder P contains -1, so that there is no preorder P with $-1 \notin P$. If $\mathrm{char}\,(K) \neq 2$, then this instance of Theorem 5.12 equally holds if $-1 \in P$, for then $P = K$, and there is no order of K that would contain P, so that the intersection of all these orders equals K.

Complete Theories

This section is based on [CS12]. Let $\mathcal{L}$ be an arbitrary language—which need not be countable—of first-order predicate logic; we write $\mathcal{F}$ for the set of formulas of $\mathcal{L}$. Let $\vdash$ stand for deducibility with classical logic.[9] By a *theory* in $\mathcal{L}$ we understand a subset Γ of $\mathcal{F}$ that equals its *deductive closure* in $\mathcal{F}$ which in turn consists of the $\varphi \in \mathcal{F}$ for which $\Gamma \vdash \varphi$. The theories correspond to the filters of the Lindenbaum algebra.

A theory Θ is *consistent* if $\bot \notin \Theta$. A *prime theory* [RS63] is a theory Θ that satisfies

$$\varphi \vee \psi \in \Theta \Longrightarrow \varphi \in \Theta \vee \psi \in \Theta \tag{5.17}$$

for all $\varphi, \psi \in \mathcal{F}$. A theory Θ is *complete* if

$$\varphi \in \Theta \vee \neg\varphi \in \Theta \tag{5.18}$$

for every $\varphi \in \mathcal{F}$.[10]

Now set $S = \mathcal{F}$, and define a covering $\lhd$ on S by setting $\mathscr{A}(\Gamma)$ as the deductive closure of $\Gamma \subseteq \mathcal{F}$: that is, for every $\varphi \in \mathcal{F}$,

$$\varphi \lhd \Gamma \equiv \Gamma \vdash \varphi\,.$$

Hence $\lhd$ is a finitary covering, and the ideals are precisely the theories.[11] Let also e be $\bot$, which is convincing in view of *ex falso sequitur quodlibet.*

[9] We use classical logic also as our metalogic.

[10] Disjunction plays two different roles: on the left-hand side of (5.17) it belongs to the language, and can be seen as an algebraic operator on the formulas, whereas on the right-hand side of (5.17) and in (5.18) it is a connective in the metalogic.

[11] This covering on the set of formulas differs from the one studied before [CSSS00, Sam95], for which the saturated sets rather are the *complements* of the theories.

Let further $\circ$ stand for disjunction $\vee$, for which both Encoding and Weakening hold. In fact, Encoding is nothing but *disjunction elimination*:

> *If both* $\Gamma, \psi_1 \vdash \varphi$ *and* $\Gamma, \psi_2 \vdash \varphi$*, then* $\Gamma, \psi_1 \vee \psi_2 \vdash \varphi$ *.*

Moreover, Weakening corresponds to *disjunction introduction*:

> *If either* $\Gamma \vdash \psi_1$ *or* $\Gamma \vdash \psi_2$*, then* $\Gamma \vdash \psi_1 \vee \psi_2$ *.*

The proper ideals, prime ideals and complete ideals with respect to $\lhd$ correspond to the consistent theories, prime theories and complete theories, which in turn correspond to the proper filters, prime filters and ultrafilters, respectively, of the Lindenbaum algebra. As this is a Boolean algebra, one can use what we have said at the end of Section 5.3 to see that a theory is prime if and only if it is complete. Alternatively, one can verify this directly by applying Corollary 5.15, for which $d = \top$ and $e = \bot$ are indeed as required: while $\top$ clearly belongs to every theory, for every $\varphi \in \mathcal{F}$ we have $\neg\varphi \vee \varphi = \top$ and $\neg\varphi, \varphi \vdash \bot$.

In all, the present instance of Theorem 5.12 reads as follows [Sco74], for every $\Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$:

> *If* $\varphi \in \Theta$ *for every complete (consistent) theory* $\Theta \supseteq \Gamma$*, then* $\Gamma \vdash \varphi$*.*

The related variant of Corollary 5.13 is *Lindenbaum's Lemma* [Tar30]:

> *If a theory* Γ *is consistent, then there is a complete consistent theory* $\Theta \supseteq \Gamma$*.*

As all this is essentially an affair of propositional logic, we could simply have worked in the Lindenbaum algebra, applying our earlier treatment of Boolean algebras (Section 5.3). Dealing with predicate logic proper would instead have required—as in [CS12]—to speak of *completely* prime theories and filters, which are the ones that split *arbitrary* disjunctions. By the Henkin method, however, the case of predicate logic can be reduced to the one of propositional logic, as follows [Sho01].

The *Henkin expansion* $\mathcal{L}^*$ of $\mathcal{L}$ is such that if $\mathcal{F}^*$ denotes the set of formulas of $\mathcal{L}^*$, then for every $\psi \in \mathcal{F}^*$ for which $\exists\psi x$ is a closed formula there is a dedicated constant $c_{\psi,x} \in \mathcal{L}^*$. A *Henkin system* is a consistent prime theory Θ in $\mathcal{L}^*$ satisfying the following *Henkin condition*:

> *If* $\exists x\psi$ *is a closed formula and* $\exists x\psi \in \Theta$*, then* $\psi[x/t] \in \Theta$ *for some term* t*.*

Let H consist of the *Henkin axioms* $\exists x\psi \to \psi[x/c_{\psi,x}]$. If Θ is a theory in $\mathcal{L}^*$ with $\Theta \supseteq H$, then the Henkin condition is automatic by *modus ponens*, with $c_{\psi,x}$ for t. Among the theories in $\mathcal{L}^*$ that contain H, the Henkin systems thus are simply the consistent prime theories. In other words, while the Henkin systems *a priori* are the proper and completely prime filters of the Lindenbaum algebra, the ones containing H are the proper prime filters above H.

Now set $S = \mathcal{F}^*$, and redefine $\lhd$ by setting

$$\varphi \lhd \Gamma \equiv \Gamma, H \vdash \varphi$$

for every $\Gamma \subseteq \mathcal{F}^*$ and $\varphi \in \mathcal{F}^*$. Hence $\mathscr{A}(\Gamma)$ is the deductive closure of $\Gamma \cup H$ in $\mathcal{F}^*$, and the ideals are the theories in $\mathcal{L}^*$ that contain H. Moreover, $\lhd$ is a finitary covering.

Let again $\circ$ stand for $\vee$, and e for $\bot$. The proper prime ideals are just the Henkin systems that contain H. Encoding and Weakening hold as above but with $\Gamma \cup H$ in place of Γ; and the corresponding instance of Theorem 5.12 reads as follows, for any $\Gamma \subseteq \mathcal{F}^*$ and $\varphi \in \mathcal{F}^*$:

If $\varphi \in \Theta$ *for every Henkin system* $\Theta \supseteq \Gamma \cup H$, *then* $\Gamma, H \vdash \varphi$.

This is the essence of many a proof [Sho01] along the lines of Henkin's approach to Gödel's completeness theorem for not necessarily countable languages of first-order predicate logic. To see this more clearly, recall [Sho01] that for each $\Gamma \subseteq \mathcal{F}$ and every $\varphi \in \mathcal{F}$, if $\Gamma, H \vdash \varphi$, then $\Gamma \vdash \varphi$; and that the following are equivalent for each $\Theta \subseteq \mathcal{F}^*$ with $\Theta \supseteq H$: Θ is a Henkin system; Θ is a complete, consistent theory in $\mathcal{L}^*$; Θ is a maximal consistent subset of $\mathcal{F}^*$.

Chapter 6

Unary Formal Topologies and Finite Density

Scott introduced Information Systems [Sco82] to make available a simpler presentation of algebraic domains. Among the conceptual advantages of this axiomatic description, the definition of information system is of constructive nature. This aspect deserves a particular emphasis, as the main application of these structures is denotational semantics for programming languages.

Historically, the introduction of information systems prompted the question of a constructive axiomatisation of the notion of topological space, which finally resulted in Sambin's definition of a formal topology [Sam89, Sam03, Samng] in 1987. In its first stages, as Sambin himself reports [Sam97], information systems had a natural influence on the development of Formal Topology. This is witnessed, for instance, by the introduction of the *Pos* predicate, inspired by Scott's *Con*, and by the definition of continuous relation, modelled on Scott's approximable relations.

Since then, a few papers have described the connections between Formal Topology and Information Systems [Sam97, SVV96, Neg02]. The latter, and more generally constructive Domain Theory, can be naturally regarded as a branch of the former. In this vein, it is natural to ask how much of the well-developed theory of domains can be generalized to formal topology [Sam97] and, viceversa, whether formal topology is a convenient framework to deal constructively with denotational semantics.

In the present work, we will move along these lines and we propose unary formal topologies with finitary operation as a generalization of the notion of information system, for which the usual categorical constructions can be easily performed. To this end, the presence of a primitive generalized operation, appearing in the most recent formulation of formal topology [CMS13], is crucial. These structures benefit, unlike information systems, from the practical combinatorial notation of basic pairs. This feature represented in fact the initial motivation behind this work, since it makes dealing with a syntactic notion as confortable as playing with the point-theoretic counterpart. Nevertheless, the price to pay is a certain adjustment time, mainly because of the uncommon definitions and notations involved.

Schwichtenberg & al. described in [HKS10] a constructive formal theory TCF^+ of computable functionals, based on the partial continuous functionals as their intendend domain. Such a task was initially started by Dana Scott [Sco93], under the well-known abbreviation LCF (logic for computable functionals). TCF^+ differs from LCF, among other things, because the intended domains for base types are *non-flat* free algebras. In [HKS10], in particular, a proof of Kleene-Kreisel-Berger's Density Theorem is presented. We will show here how unary formal topologies with finitary operation furnish a natural substitute and we will give a version of Density Theorem for this setting. The choice of non-flat information systems, makes also possible a finite version of Berger's result. We then analyse the notion of finite totality from a constructive perspective.

We hope that this work can prompt further discussion on the interplay between constructive domain theory and formal topology, as it is a modest evidence that this can lead to new results. The proof of the Finite Density Theorem, in particular, could be of help for the ongoing formalization of TCF^+ on the proof assistant MINLOG.

6.1 The Scott Topology as a Basic Pair

In this section, we rephrase the topological space of ideals of an information system in terms of a natural basic pair. The basic idea behind this approach was already present in [Sam97]. We try here to push it forward and to set this as a substitute for information systems. A study of basic pairs generated by preorders can be already found in [SVV96, Samng].

Information systems provide an axiomatic setting to describe approximations of abstract objects (like functions or functionals) by concrete, finite ones. An arbitrary countable set of "bits of data" or "tokens" in the basic notion. In order to use such data to build approximations of abstract objects, one need a notion of "consistency", which determines when the elements of a finite set of tokens form a compatible information, and an "entailment relation", between consistent sets of data and single tokens. The axioms below are a minor modification of Scott's [Sco82], due to Larsen and Winskel [LW91].

Definition 6.1. A *Scott Information System* [Sco82, SW12] is a structure $\mathrm{A} \equiv (A, Con_A, \vdash_A)$, where A is a countable set (the *tokens*), Con_A is a non-empty set of finite subsets of A (the *consistent sets*), and $\vdash_A \subseteq Con_A \times A$ is a relation satisfying:

$$\frac{U_0 \subseteq V \quad V \in Con_A}{U \in Con_A}, \qquad \frac{a \ \epsilon \ A}{\{a\} \in Con_A}, \qquad \frac{a \ \epsilon \ U \quad U \in Con_A}{U \vdash_A a},$$

$$\frac{U \vdash_A a}{U \cup \{a\} \in Con_A}, \qquad \frac{U, V \in Con_A \quad U \vdash V \quad V \vdash_A a}{U \vdash_A a},$$

for all $a \in A$, where $U \vdash_A V \equiv (\forall a \ \epsilon \ V)(U \vdash a)$.

Example 6.1. As an example, given any countable set A, one defines an information system structure by taking as consistent subsets the empty subset and the singletons $\{a\}$, for all $a \ \epsilon \ A$, and as entailment relation the membership relation. This is the so-called *flat* domain associated to A.

Definition 6.2. A subset $x \subseteq A$ of an information system $\mathrm{A} \equiv (A, Con_A, \vdash_A)$ is called *ideal* [Sco82, SW12] if it is *consistent* and *deductively closed*, that means:

$$\frac{U \subseteq_\omega x}{U \in Con_A}\ \text{Cons.} \qquad \frac{U \subseteq x \quad U \vdash_A a}{a \ \epsilon \ x}\ \text{Ded. Closed}$$

One denotes by $|\mathrm{A}|$ the collection of the ideals of A.

One can show easily that $|\mathrm{A}|$ is a Scott domain with respect to the inclusion relation. Conversely, every Scott domain with countable basis can be represented as the set of all ideals of an appropriate information system.

Let us fix from now on an information system $\mathrm{A} \equiv (A, Con_A, \vdash_A)$. For all $U_0 \ \epsilon \ Con_A$, we can define an ideal

$$a \ \epsilon \ \overline{U}_0 \equiv U_o \vdash_A a.$$

This is called the *deductive closure* of U_0.

One introduces the following notation: for all $U, V \ \epsilon \ Con_A$, one writes $U \uparrow V$, and one says that U and V are *consistent with each other*, if $U \cup V \ \epsilon \ Con_A$.

Let $\mathrm{A} \equiv (A, Con_A, \vdash_A)$ be an information system. The collection $|\mathrm{A}|$ has structure of topological space with basic neighborhoods $B_{U_0} \subseteq |\mathrm{A}|$ defined by

$$x \ \epsilon \ B_{U_0} \equiv U_0 \subseteq x$$

for all $U_0 \ \epsilon \ Con_A$. This defines the well-known *Scott topology* on $|\mathrm{A}|$. In our setting, this topological space is represented by the concrete space $(|\mathrm{A}|, \Vdash, Con_A)$, where

$$x \Vdash U_0 \equiv U_0 \subseteq x.$$

Remark 24. A subset $D \subseteq |\mathrm{A}|$ is dense in the Scott topology if and only if, for all $U_0 \in Con_A$, there is $x \in D$, such that $U_0 \subseteq x$.

Proposition 6.2. *The concrete space* $(|\mathrm{A}|, \Vdash, Con_A)$ *is isomorphic to the sobrification of the concrete space* $(Con_A, \vdash_A, Con_A)$*, where* $\vdash_A$ *is the usual entailment relation between consistent subsets.*

Proof. First, we prove that $(Con_A, \vdash_A, Con_A)$ is a concrete space. The symbol $\downarrow$ is here defined by

$$W_0 \ \epsilon \ U_0 \downarrow V_0 \equiv \forall Z_0 (Z_0 \vdash_A W_0 \to Z_0 \vdash_A U_0 \ \& \ Z_0 \vdash_A V_0)$$

for all $U_0, V_0 \ \epsilon \ Con_A$, so that,

$$W_0 \ \epsilon \ U_0 \downarrow V_0 \quad \leftrightarrow \quad W_0 \vdash_A U_0 \ \& \ W_0 \vdash_A V_0 \quad \leftrightarrow \quad U_0 \uparrow V_0 \ \& \ W_0 \vdash_A U_0 \cup V_0.$$

The concrete space conditions are then easily verified: for the first condition, if $W_0 \vdash_A U_0$ and $W_0 \vdash_A V_0$ then $W_0 \vdash_A U_0 \cup V_0$, so that $W_0 \vdash_A U_0 \downarrow V_0$. Moreover, if $W_0 \,\epsilon\, Con_A$, then $W_0 \vdash_A W_0$, so that also the second condition trivially holds.

A point of this basic pair is given by an inhabited subset $\alpha \subseteq Con_A$, such that:

1. It is formal closed, that is, whenever $U_0 \,\epsilon\, \alpha$ we can find W_0 such that

$$W_0 \vdash_A U_0 \;\&\; \forall V_0 (W_0 \vdash_A V_0 \to V_0 \,\epsilon\, \alpha)).$$

   It is clear that if W_0 satisfies this condition, then also U_0 does. So that α is formal closed iff

$$U_0 \,\epsilon\, \alpha \quad \to \quad \forall V_0 (U_0 \vdash_A V_0 \to V_0 \,\epsilon\, \alpha)$$

   or, in other terms, if it is *deductively closed*.

2. It is convergent, that is, whenever $U_0, V_0 \,\epsilon\, \alpha$, then $U_0 \downarrow V_0 \between \alpha$. This explicitly means, that whenever $U_0, V_0 \,\epsilon\, \alpha$, then $U_0 \uparrow V_0$ and $U_0 \cup V_0 \,\epsilon\, \alpha$.

We have then two straight correspondences, inverse of each other:

$$\mathscr{P}t_{\vdash_A}(Con_A) \underset{G}{\overset{F}{\rightleftarrows}} |\mathrm{A}|$$

defined by $F(\alpha) = \bigcup_{U_0 \,\epsilon\, \alpha} U_0$ and $G(x) = \{U_0 : U_0 \subseteq x\}$. These correspondences lift to an isomorphism of concrete spaces

$$\begin{array}{ccc}
\mathscr{P}t_{\vdash_A}(Con_A) & \xrightarrow{\ \Vdash_{\ni}\ } & Con_A \\
{\scriptstyle G}\uparrow\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle id} \\
|\mathrm{A}| & \xrightarrow{\ \supseteq\ } & Con_A .
\end{array}$$

□

Motivated by the proof above, we introduce the following natural operation: for all $U_0, V_0 \,\epsilon\, Con_A$,

$$W_0 \,\epsilon\, U_0 * V_0 \equiv U_0 \uparrow V_0 \;\&\; W_0 = U_0 \cup V_0. \tag{6.1}$$

We have then $W_0 \vdash_A U_0 \downarrow V_0 \leftrightarrow W_0 \vdash_A U_0 * V_0$ or, equivalently, $\mathscr{A}_{\vdash_A}(U_0 \downarrow V_0) = \mathscr{A}_{\vdash_A}(U_0 * V_0)$.

Let us fix now two information systems

$$\mathrm{A} \equiv (A, Con_A, \vdash_A), \qquad \mathrm{B} \equiv (B, Con_B, \vdash_B),$$

determining, respectively, the concrete spaces

$$(Con_A, \vdash_A, Con_A), \qquad (Con_B, \vdash_B, Con_B).$$

A continuous morphism between them consists of a pair (r, s) making the following diagram commutative:

$$\begin{array}{ccc} Con_A & \xrightarrow{\vdash_A} & Con_A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ Con_B & \xrightarrow{\vdash_B} & Con_B \end{array}$$

that is, $s \circ \vdash_A = \vdash_B \circ r$. Notice that, since $\vdash_A$ and $\vdash_B$ are transitive, we have

$$\vdash_A \circ \vdash_A = \vdash_A, \qquad \vdash_B \circ \vdash_B = \vdash_B .$$

As a consequence, the relation $\overline{r} = \vdash_B \circ r \circ \vdash_A = \vdash_B \circ s \circ \vdash_A$ makes the diagram

$$\begin{array}{ccc} Con_A & \xrightarrow{\vdash_A} & Con_A \\ {\scriptstyle \overline{r}}\downarrow & \searrow^{\overline{r}} & \downarrow{\scriptstyle \overline{r}} \\ Con_B & \xrightarrow{\vdash_B} & Con_B \end{array}$$

commute. Moreover, the couple $(\overline{r}, \overline{r})$ is the saturation of (r, s). In fact:

$$V_0 \ \epsilon \ \mathsf{cl}_B(rU_0) \equiv \exists V_0'(V_0' \vdash_B V_0 \ \& \ U_0 r V_0') \equiv U_0(\vdash_B \circ r)V_0,$$

$$U_0 \ \epsilon \ \mathscr{A}_A(s^- V_0) \equiv \exists U_0'(U_0 \vdash_A U_0' \ \& \ U_0' r V_0) \equiv U_0(s \circ \vdash_A)V_0.$$

So that, we get

$$\vdash_B \circ r = \vdash_B \circ(\vdash_B \circ r) \overset{(cont.)}{=} \vdash_B \circ(s \circ \vdash_A) = \overline{r}$$

and symmetrically $s \circ \vdash_A = \overline{r}$. We will then identify continuous morphisms with the $\overline{r}$ making the diagram above commutative.

The convergence conditions for a continuous morphism $\overline{r}$ specialize here as follows:

(C1) $\overline{r}^- V_0 * \overline{r}^- V_0' \subseteq \mathscr{A}_A(\overline{r}^-(V_0 * V_0'))$, for all $V_0, V_0' \ \epsilon \ Con_B$. In explicit terms, this means that if $U_0 \overline{r} V_0$, $U_0' \overline{r} V_0'$ and $U_0 \uparrow U_0'$, then also $V_0 \uparrow V_0'$, and $(U_0 \cup U_0') \ \overline{r} \ (V_0 \cup V_0')$.

(C2) $Con_A = \mathscr{A}_A(\overline{r}^- Con_B)$. Explicitly, this means

$$(\forall U_0 \ \epsilon \ Con_A)(\exists V_0 \ \epsilon \ Con_B)(U_0 \overline{r} V_0),$$

and therefore, in presence of continuity, this is equivalent to $\emptyset_A \overline{r} \emptyset_B$.

In the general theory, to every continuous and convergent morphism $\overline{r} : \mathrm{A} \to \mathrm{B}$, is canonically associated a continuous and convergent function

$$\mathscr{P}t(\overline{r}) : |\mathrm{A}| \to |\mathrm{B}|.$$

More precisely, this correspondence gives rise to a functor $\mathscr{P}t$ from CSpa to Scott, the full subcategory of the category Top of topological spaces consisting of Scott Domains.

We claim therefore that Scott continuous functions between domains (induced by information systems) correspond to the continuous and convergent morphisms $\overline{r}$. This correspondence will be made precise in terms af approximable relations, the usual notion of morphism between information systems:

Definition 6.3. Let A $\equiv (A, Con_A, \vdash_A)$ and B $\equiv (B, Con_B, \vdash_B)$ be two information systems. An *approximable relation* (or map) between A and B is a relation $r \subseteq Con_A \times B$ which satisfies

(A1) If $U_0\ r\ b_1 \ldots, U_0\ r\ b_n$ then $\{b_1, \ldots, b_n\} \in Con_B$;

(A2) If $U_0\ r\ b_1 \ldots, U_0\ r\ b_n$ and $\{b_1, \ldots, b_n\} \vdash_B b$ then $U_0\ r\ b$;

(A3) If $U_0'\ r\ b$ and $U_0 \vdash_A U_0'$ then $U_0\ r\ b$,

for all $U_0, U_0' \ \epsilon\ Con_A$ and $b_1, \ldots, b_n \ \epsilon\ B$.

The condition (A2), for $n = 0$, tells that $U_0\ r\ b$ holds for all $b \ \epsilon\ B$ such that $\emptyset_B \vdash b$.

Let A $\equiv (A, Con_A, \vdash_A)$, B $\equiv (B, Con_B, \vdash_B)$, C $\equiv (C, Con_C, \vdash_C)$ be information systems, and $r \subseteq Con_A \times B$, $s \subseteq Con_B \times C$ be approximable maps. Their composition $s \circ r \subseteq Con_A \times C$ is defined as follows:

$$U_0\ (s \circ r)\ c \equiv (\exists V_0 \in Con_B)((\forall b \ \epsilon\ V_0)(U_0\ r\ b)\ \&\ V_0\ s\ c).$$

Let InfSys be the category of information systems and approximable maps, with identity of A being $\vdash_A \subseteq Con_A \times A$, and CSpaInf the full subcategory of CSpa whose objects are concrete spaces associated to information systems.

Proposition 6.3. *The correspondence which associates to an information system a concrete space can be extended to an isomorphism of categories*

$$\text{InfSys} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \text{CSpaInf}.$$

defined on morphisms by

$$U_0 \mathcal{F}(r) V_0 \equiv (\forall b \ \epsilon\ V_0)(U_0\ r\ b) \qquad U_0\ \mathcal{G}(\overline{r})\ b \equiv U_0 \overline{r} \{b\}$$

for all $U_0 \in Con_A$, $b \in B$ and $V_0 \in Con_B$.

Proof. Let r be an approximable map. To show that $\mathcal{F}(r)$ is a continuous morphism, we have to show that $\vdash_A \circ \mathcal{F}(r) = \mathcal{F}(r)$ and $\mathcal{F}(r) = \mathcal{F}(r) \circ \vdash_B$. The first equality is precisely (A3), while the second follows straight from (A2). To prove that $\mathcal{F}(r)$ satisfies (C1), suppose $U_0 \uparrow U_0'$, and $U_0 \mathcal{F}(r) V_0$, $U_0' \mathcal{F}(r) V_0'$. By (A3), for all $b \ \epsilon\ V_0 \cup V_0'$ we have $U_0 \cup U_0'\ r\ b$. By (A1), $V_0 \cup V_0' \ \epsilon\ Con_B$ (i.e. $V_0 \uparrow V_0'$) and we have by definition $U_0 \cup U_0' \mathcal{F}(r) V_0 \cup V_0'$. Finally, if $U_0 \in Con_A$,

$$U_0 \mathcal{F}(r) \emptyset_B \equiv (\forall b \in \emptyset_B)(U_0\ r\ b) \equiv \top.$$

Moreover,

$$U_0\mathcal{F}(id_A)U_0' \equiv (\forall a \;\epsilon\; U_0')(U_0 \;\vdash_A\; a) \equiv U_0 \vdash_A U_0' \equiv U_0 id_A U_0'$$

and

$$\begin{aligned} U_0\mathcal{F}(s\circ r)W_0 &\equiv (\forall c \;\epsilon\; W_0)(U_0\; (s\circ r)\; c) \equiv \\ &\equiv (\forall c \;\epsilon\; W_0)((\exists V_0 \in Con_B)((\forall b \;\epsilon\; V_0)(U_0\; r\; b) \;\&\; V_0\; s\; c)) \equiv \\ &\equiv (\forall c \;\epsilon\; W_0)((\exists V_0 \in Con_B)(U_0\mathcal{F}(r)V_0 \;\&\; V_0\; s\; c)) \equiv \\ &\equiv (\exists V_0 \in Con_B)(U_0\mathcal{F}(r)V_0 \;\&\; (\forall c \;\epsilon\; W_0)(V_0\; s\; c)) \equiv \\ &\equiv U_0\mathcal{F}(s)\circ\mathcal{F}(r)W_0. \end{aligned}$$

We omit here the proof that $\mathcal{G}$ is a well-defined functor[1]. It can be performed similarly. The two correspondences are inverse of each other: for every $U_0 \in Con_A$ and $b \;\epsilon\; B$

$$\begin{aligned} U_0\; \mathcal{G}(\mathcal{F}(r))\; b &\equiv U_0\mathcal{F}(r)\{b\} \leftrightarrow \\ &\leftrightarrow (\forall b' \;\epsilon\; \{b\})(U_0\; r\; b') \leftrightarrow \\ &\leftrightarrow U_0\; r\; b. \end{aligned}$$

Vice versa, for all $U_0, V_0 \;\epsilon\; Con_A$,

$$\begin{aligned} U_0\mathcal{F}(\mathcal{G}(\overline{r}))V_0 &\equiv (\forall b \;\epsilon\; V_0)(U_0\; \mathcal{G}(\overline{r})\; b) \leftrightarrow \\ &\leftrightarrow U_0 \between U_0 \;\&\; (\forall b \;\epsilon\; V_0)(U_0\; \overline{r}\; b) \overset{(U_0 \in Con_A)}{\leftrightarrow} \\ &\leftrightarrow (\forall b \;\epsilon\; V_0)(U_0\; \overline{r}\; \{b\}) \overset{(1)}{\leftrightarrow} U_0\overline{r}V_0, \end{aligned}$$

where (1) follows from $\vdash_B \circ\overline{r} = \overline{r}$, and the fact that the singletons are consistent. □

If r is an approximable mapping from A to B, we get a (Scott) continuous function

$$\begin{aligned} \Gamma(\overline{r}) : |\mathrm{A}| &\rightarrow |\mathrm{B}| \\ x &\mapsto \{b : \exists U_0(U_0\; r\; b \;\&\; U_0 \subseteq x)\} \end{aligned}$$

and every Scott continuous function is presented uniquely in this way. The correspondence Γ extends moreover to a functor between InfSys and Scott, the full subcategory of Top whose objects are Scott domains.

6.2 Unary Formal Topologies

In the last section, we have described the space of ideals of an information system $(A, Con_A, \vdash_A)$ as a suitable concrete space $(Con_A, \vdash_A, Con_A)$. Many properties of this concrete space depend on the fact it consists of a preorder $\vdash_A$ (i.e. a transitive and reflexive relation). We devote this section to the study of

[1] This fact is somehow contained in Proposition 6.4.

this class of formal spaces and we will show how they relates with information systems.

Let us fix a basic pair of the form $(A, \geqslant, A)$ where $(A, \geqslant)$ is a preorder. One can easily prove, similarly to Prop. 6.2, that this is a concrete space. It is therefore defined a formal topology structure on $(A, \lhd_{\geqslant}, \downarrow)$:

$$a \lhd_{\geqslant} U \equiv a \;\epsilon\; \mathscr{A}_{\geqslant}(U) \equiv \forall b(b \geqslant a \to \exists a'(b \geqslant a' \;\&\; a' \,\epsilon\, U)), \tag{6.2}$$

$$c \;\epsilon\; a \downarrow b \equiv c \lhd_{\geqslant} a \;\&\; c \lhd_{\geqslant} b \equiv c \geqslant a \;\&\; c \geqslant b \tag{6.3}$$

for all $a \;\epsilon\; A$ and $U \subseteq A$. We can simplify the rightmost member in (6.2), so that,

$$a \lhd_{\geqslant} U \equiv a \;\epsilon\; \mathscr{A}_{\geqslant}(U) \equiv \exists a'(a \geqslant a' \;\&\; a' \,\epsilon\, U). \tag{6.4}$$

In particular, $\lhd_{\geqslant}$ is finitary and, in addition, a *unary* cover [Samng, SVV96, Sam97, Neg02], i.e.

$$a \lhd_{\geqslant} U \equiv \exists u \;\epsilon\; U(a \lhd_{\geqslant} u).$$

Viceversa, any unary cover $\lhd$ is determined by a preorder $\geqslant_{\lhd}$ in the way above, if we define

$$a \geqslant_{\lhd} b \equiv a \lhd b \tag{6.5}$$

for all $a, b \in A$. Because of this, by abuse of terminology, we are going to call unary formal topology a concrete space of the form $(A, \geqslant_A, A)$.

The reason for keeping the notation of basic pairs, instead of switching completely to formal topologies, is because they enjoy a nice combinatorial notation for morphisms. Here follows a characterization of these morphisms, similar to that of the past section. Let us fix now two unary formal topologies $(A, \geqslant_A, A)$ and $(B, \geqslant_B, B)$. A continuous morphism between them consists of a pair (r, s) making the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{\geqslant_A} & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ B & \xrightarrow{\geqslant_B} & B \end{array} \tag{6.6}$$

that is, $s\circ \geqslant_A = \geqslant_B \circ r$. Notice that, since $\geqslant_A$ and $\geqslant_B$ are transitive, we have

$$\geqslant_A \circ \geqslant_A = \geqslant_A, \qquad \geqslant_B \circ \geqslant_B = \geqslant_B .$$

As a consequence, the relation $\overline{r} = \geqslant_B \circ r \circ \geqslant_A = \geqslant_B \circ s \circ \geqslant_A$ makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\geqslant_A} & A \\ {\scriptstyle \overline{r}}\downarrow & \overset{\overline{r}}{\searrow} & \downarrow{\scriptstyle \overline{r}} \\ B & \xrightarrow{\geqslant_B} & B \end{array}$$

commute. Moreover, the couple $(\overline{r}, \overline{r})$ is the saturation of (r, s). In fact:

$$b \;\epsilon\; \mathsf{cl}_B(ra) \equiv \exists b'(b' \geqslant_B b \;\&\; arb') \equiv a(\geqslant_B \circ r)b,$$
$$a \lhd_A s^- b \equiv a \;\epsilon\; \mathscr{A}_A(s^- b) \equiv \exists a'(a \geqslant_A a' \;\&\; a'rb) \equiv a(s\circ \geqslant_A)b$$

for all $a \in A$ and $b \in B$. In other terms, we get

$$\geqslant_B \circ r = \geqslant_B \circ(\geqslant_B \circ r) \overset{(cont.)}{=} \geqslant_B \circ(s\circ \geqslant_A) = \overline{r}$$

and symmetrically $s\circ \geqslant_A = \overline{r}$. We will then identify continuous morphisms with the relations $\overline{r}$ making commutative the diagram (6.6).

As seen in the introductory chapter of this thesis, finitary (and in particular, unary) formal topologies enjoy special properties and constructions once they are endowed with a finitary operation. The canonical operation $\downarrow$ on $(A, \geqslant_A, A)$ is not finitary unless the set A is finite.

However, under suitable hypotheses, the preorder relation $\geqslant_A$ can carry a finitary and topologically equivalent operation[2] $*$. Notice in fact, that such an operation $*$ exists if and only if $a \downarrow b$ is finitely generated as a formal open for all $a, b \in A$, namely, there exists $\{c_1^{ab}, \ldots, c_n^{ab}\} \subseteq A$ finite such that

$$a \downarrow b = \mathscr{A}(\{c_1^{ab}, \ldots, c_n^{ab}\}).$$

In this case, assuming (AC), we can define

$$a * b \equiv \{c_1^{ab}, \ldots, c_n^{ab}\}$$

for all $a, b \in A$, and the operation $*$ so defined is finitary and equivalent to $\downarrow$. In the concrete instances that we are going to analyse in the following, the finitary operation $*$ will be given explicitly and we will never have to perform a choice of generators. The present discussion has therefore just an heuristic character.

Vice versa, if we have a finitary operation $*$ equivalent to $\downarrow$, then $a \downarrow b$ is, by hypothesis, generated as a formal open by the finite subset $a * b$.

These considerations hold equally for all formal topologies. In the specific case of a unary formal topology $(A, \geqslant_A, A)$, $a \downarrow b$ is finitely generated as a formal open if and only if, for all $a, b \in A$,

1. $\{a, b\}$ is unbounded, in which case, $a \downarrow b$ is the empty subset, or
2. $\{a, b\}$ is bounded, and there exist $c_1^{ab}, \ldots, c_n^{ab}$ such that

$$c \geqslant_A a \;\&\; c \geqslant_A b \leftrightarrow c \geqslant_A c_1^{ab} \vee \cdots \vee c \geqslant_A c_n^{ab}. \tag{6.7}$$

This is an essentially order-theoretic condition. In particular, if we want $a \downarrow b$ to be finitely generated for all $a, b \in A$, we must be able to decide whether $\{a, b\}$ is bounded or unbounded for all $a, b \in A$.

[2] We recall that an operation $*$ on A is topologically equivalent to $\downarrow$ if $a * b =_{\lhd} a \downarrow b$ for all $a, b \in A$.

In analogy with the information systems, we introduce the following terminology: we write $a \uparrow b$, and we say that a is *consistent* with b, if $a \downarrow b$ is inhabited, that is to say, if $\{a, b\}$ is bounded. We generalize this notion to finite subsets: we will write $\uparrow \{a_1, \dots, a_n\}$, and we say that $\{a_1, \dots, a_n\}$ is consistent, if $\{a_1, \dots, a_n\}$ is bounded. Similarly, we write $a \not\uparrow a'$ if $\neg(a \uparrow a')$, and $\not\uparrow \{a_1, \dots, a_n\}$ if $\{a_1, \dots, a_n\}$ is unbounded. We denote by $\mathrm{Bou}(S) \subseteq \mathcal{P}_\omega(S)$ the family of bounded finite subsets of $S \subseteq A$.

In this chapter, we are mainly interested in unary formal topologies $(A, \geqslant_A, A)$ which carry a unary operation $*$, i.e. such that $a * b$ has formal cardinality smaller or equal to[3] 1.

This is motivated by the fact that the unary formal topology $(Con_A, \vdash_A, Con_A)$ induced by an information system $(A, Con_A, \vdash_A)$ has a unary operation $*$, defined as in (6.1). More precisely, in order to know that $U_0 * V_0$ is a finite subset for all $U_0, V_0 \in Con_A$, we need the consistency relation $\uparrow$ to be decidable.

Reasoning as above, one proves that $(A, \geqslant_A, A)$ carries a unary operation if and only if the following condition on the preorder $\geqslant_A$ holds: for all $a, b \in A$, $a \uparrow b$ or $a \not\uparrow b$ holds, and in the former case there exists c^{ab} such that

$$c \geqslant_A a \;\&\; c \geqslant_A b \leftrightarrow c \geqslant_A c^{ab}. \tag{6.8}$$

This is a very natural condition on the order: we can decide whether $a, b \in A$ have an upper bound, and if it is the case, they have a supremum $\mathsf{sup}(a, b) \equiv c^{ab}$. The preorder structures $(A, \geqslant_A)$ satisfying this condition and having a bottom element $\bot_A$ are called *bounded complete preorders*. The existence of a bottom element is not essential in general. However, it can always be added canonically and its presence turns out useful in the following.

In all, the concrete spaces induced by bounded complete preorders with decidable consistency relation $\uparrow$ generate unary formal topologies with unary operation. Moreover, since a bottom element $\bot_A$ is present, then the corresponding formal topology possess an element $\bot_A$, such that $A \lhd \bot_A$. This element is called a *convincing element* for the formal topology $(A, \lhd, *)$.

Vice versa, every unary formal topology $(A, \lhd, *, \bot_A)$, with unary operation and convincing element, is induced by a bounded complete preorder $\geqslant_\lhd$ defined as in (6.5), where the supremum of a bounded pair $a, b \in A$ is $c^{ab} \in A$ such that $a * b \equiv \{c^{ab}\}$. Moreover, since $a * b$ is finite, we can decide whether it is inhabited, and then $a \uparrow b$, or it is empty, in which case $a \not\uparrow b$.

For all $U_0 \equiv \{a_1, \dots, a_n\}$ subset of $(A, \lhd, *, \bot_A)$, we recall the following notation:

$$\prod U_0 \equiv a_1 * \dots * a_n.$$

In particular, if U_0 is bounded, we denote by $\mathsf{sup}(U_0)$ its supremum, for which we have $\prod U_0 = \{\mathsf{sup}(U_0)\}$.

We focus now on the study of unary formal topologies with unary operation and convincing element, and we analyse their relation with the information sys-

[3] Explicitly, $a * b$ is either a singleton or the empty set for all $a, b \in A$.

tems. As consequence of the discussion above, there are three ways of presenting these topologies:

1. as a proper formal topology structure $(A, \lhd, *, \bot_A)$;
2. as the corresponding concrete space $(A, \geqslant_A, A)$;
3. as the underlying bounded complete preorder $(A, \geqslant_A, \mathsf{sup}_A, \uparrow, \bot_A)$.

We have already noticed that if $\mathrm{B} \equiv (B, Con_B, \vdash_B)$ is an information system with decidable equality, then $\mathcal{H}(\mathrm{B}) \equiv (Con_B, \vdash_B, Con_B)$ is a unary formal topology with a natural operation $*$.

Vice versa, given a unary formal topology $(A, \lhd, *)$ with unary operation, we can associate an information system $\mathcal{I}(\mathrm{A})$ with decidable consistency: we pick A as set of tokens, $\mathrm{Bou}(A)$ as Con_A, and finally, for each $U_0 \;\epsilon\; \mathrm{Bou}(A)$, we define

$$U_0 \vdash_A a \equiv \prod U_0 \lhd a \equiv a \leqslant_\lhd \mathsf{sup}(U_0).$$

It is easy to verify that this provides indeed an information system structure with decidable consistency.

The convergence conditions for a continuous morphism $\overline{r}$ between two unary formal topologies with unary operation $(A, \lhd_A, *_A, \bot_A)$ and $(B, \lhd_B, *_B, \bot_B)$ specialize here as follows:

(C1) $\overline{r}^- b *_A \overline{r}^- b' \lhd_A (\overline{r}^- (b *_B b'))$, for all $b, b' \;\epsilon\; B$. In explicit terms, this means that if $a\overline{r}b$, $a'\overline{r}b'$ and $a \uparrow a'$, then also $b \uparrow b'$, and $a *_A a' \;\overline{r}\; b *_B b'$.

(C2) $A =_{\lhd_A} \overline{r}^- B$. Explicitly, in presence of continuity, this condition becomes $\bot_A \overline{r} \bot_B$.

We denote by UFTop_* the category of unary formal topology with unary operation. The identity morphism of UFTop_* associated to a unary formal topology $(A, \geqslant_A, A)$ is simply $\geqslant_A$.

The correspondence $\mathcal{H}$ extend to a functor from InfSys to UFTop_*, being just the functor $\mathcal{F}$ defined in the last section.

Proposition 6.4. *The correspondence $\mathcal{I}$ can be extended to a functor from* UFTop_* *to* InfSys*. More precisely, if $(A, \geqslant_A, A)$ and $(B, \geqslant_B, B)$ are two unary formal topologies with unary operation, and r is a continuous and convergent morphism between them, we define $\mathcal{I}(r) \subseteq Bou(A) \times B$ as*

$$U_0 \;\mathcal{I}(r)\; b \equiv (\mathsf{sup}(U_0))rb.$$

Proof. We prove that $\mathcal{I}$ is an approximable mapping; condition (A1) can be derived by applying (C1) a finite number of times: if $U_0 \; r \; b_1 \ldots, U_0 \; r \; b_n$ then $(\mathsf{sup}(U_0))rb_1$ and $(\mathsf{sup}(U_0))rb_2$, so that, by (C1), $\{b_1, b_2\}$ is bounded with supremum $b_1 * b_2$, and $U_0 \;\mathcal{I}(r)\; b_1 * b_2$. We repeat the argument with the next element on the list, till n, so to get $\{b_1, \ldots, b_n\}$ bounded and $U_0 \;\mathcal{I}(r)\; b_1 * \ldots * b_n$.

The conditions $(A2)$ and $(A3)$ are proved together: if $\mathsf{sup}(U_0) \geqslant_A \mathsf{sup}(U_0')$ and $\mathsf{sup}(U_0') \; r \; b_1 \ldots, \mathsf{sup}(U_0') \; r \; b_n$ and $b_1 * \ldots * b_n \geqslant_B b$ then we have just checked that $\mathsf{sup}(U_0')\ r\ b_1 * \ldots * b_n$; by continuity of r, we get $(\mathsf{sup}(U_0))rb$, i.e. $U_0\ \mathcal{I}(r)\ b$.

We have moreover:

$$U_0 \mathcal{I}(id_A)a \equiv \mathsf{sup}(U_0) \geqslant_A a \equiv U_0 \vdash_A a,$$

and, for $\mathrm{A} \stackrel{r}{\to} \mathrm{B}$ and $\mathrm{B} \stackrel{s}{\to} \mathrm{C}$ suitable unary formal topologies,

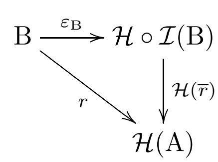
$$\begin{array}{rcl} U_0\ \mathcal{I}(s \circ r)\ c & \equiv & \mathsf{sup}(U_0)(s \circ r)c \equiv \\ & \equiv & (\exists b \in B)(\mathsf{sup}(U_0)rb\ \&\ bsc)) \leftrightarrow \\ & \leftrightarrow & (\exists b \in B)((\forall b'\ \epsilon\ \{b\})(\mathsf{sup}(U_0)rb')\ \&\ \mathsf{sup}(\{b\})sc)) \leftrightarrow \\ & \stackrel{(*)}{\leftrightarrow} & (\exists V_0 \in Con_B)((\forall b'\ \epsilon\ V_0)(V_0\mathcal{I}(s)b')\ \&\ \ V_0\mathcal{I}(s)c) \equiv \\ & \equiv & U_0\mathcal{I}(s) \circ \mathcal{I}(r)c. \end{array}$$

where to prove $\leftarrow$ in (*), one pick a V_0 satisfying the antecedent and use $\mathsf{sup}(V_0)$ to satisfy the consequent. □

We can now state the relation between the functors $\mathcal{H}$ and $\mathcal{I}$, and therefore between unary formal topologies with unary operation and information systems.

Theorem 6.5. *The category* InfSys *is a reflective subcategory of the category* UFTop_* *of unary formal topologies with unary operation, with* $\mathcal{H}$ *as inclusion functor and* $\mathcal{I}$ *as left-adjoint.*

Proof. We have to show that for each object $\mathrm{B} \equiv (B, \geqslant_B, B)$ in UFTop_* and $\mathrm{A} \equiv (A, \vdash_A, Con_A)$ in InfSys, there exists a continuous and convergent morphism $\varepsilon_{\mathrm{B}} : \mathrm{B} \to \mathcal{H} \circ \mathcal{I}(\mathrm{B})$ such that, for any continuous and convergent morphism $r : \mathrm{B} \to \mathcal{H}(\mathrm{A})$, there is a unique approximable map $\overline{r} : \mathcal{I}(\mathrm{B}) \to \mathrm{A}$ making the diagram



commutative. Notice that $\mathcal{H} \circ \mathcal{I}(\mathrm{B}) \equiv (Bou(B), \geqslant_B', Bou(B))$, where

$$V_0 \geqslant_B' V_0' \equiv \mathsf{sup}(V_0) \geqslant_B \mathsf{sup}(V_0').$$

We define then the canonical morphism ε_B as

$$b \varepsilon_B V_0 \equiv b \geqslant_B \mathsf{sup}(V_0).$$

We omit the proof that this is a continuous and convergent morphism. We define then $\overline{r} \subseteq \mathrm{Bou}(B) \times A$ as

$$V_0\ \overline{r}\ a \equiv (\mathsf{sup}(V_0))r\{a\}.$$

and we prove that the diagram commutes:

$$\begin{aligned} b(\mathcal{H}(\overline{r}) \circ \varepsilon_B)U_0 &\equiv \exists V_0(b\varepsilon_B V_0 \ \&\ V_0\mathcal{H}(\overline{r})U_0) \equiv \\ &\equiv \exists V_0(b \geqslant_B \mathsf{sup}(V_0) \ \&\ (\forall a \ \epsilon \ U_0)(V_0 \ \overline{r} \ a)) \overset{(b'=\mathsf{sup}(V_0))}{\leftrightarrow} \\ &\leftrightarrow \exists b'(b \geqslant_B b' \ \&\ b' r U_0) \overset{(cont.)}{\leftrightarrow} brU_0. \end{aligned}$$

If also $\overline{s}$ makes the diagram commute, then, for all $a \in A$

$$\begin{aligned} V_0 \ \overline{r} \ a \overset{(cont.)}{\leftrightarrow} (\mathsf{sup}(V_0))r\{a\} &\leftrightarrow (\mathsf{sup}(V_0))\mathcal{H}(\overline{s}) \circ \varepsilon_B\{a\} \leftrightarrow \\ &\leftrightarrow \exists V_0'(\mathsf{sup}(V_0) \geqslant_B \mathsf{sup}(V_0') \ \&\ V_0' \ \overline{s} \ a) \overset{(cont.)}{\leftrightarrow} \\ &\overset{(cont.)}{\leftrightarrow} \{\mathsf{sup}(V_0)\} \ \overline{s} \ a \leftrightarrow V_0 \ \overline{s} \ a. \end{aligned}$$

Hence $\overline{r} = \overline{s}$, and this completes the proof. □

As a first consequence, for any $\mathrm{B} = (B, \geqslant_B, B)$ in UFTop$_*$ and $\mathrm{A} = (A, \vdash_A, Con_A)$ in InfSys, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{UFTop}_*}(\mathrm{B}, \mathcal{H}(\mathrm{A})) \cong \mathrm{Hom}_{\mathrm{InfSys}}(\mathcal{I}(\mathrm{B}), \mathrm{A}). \tag{6.9}$$

We have then in particular, for B, B' in UFTop$_*$,

$$\mathrm{Hom}_{\mathrm{InfSys}}(\mathcal{I}(\mathrm{B}), \mathcal{I}(\mathrm{B}')) \cong \mathrm{Hom}_{\mathrm{UFTop}_*}(\mathcal{H} \circ \mathcal{I}(\mathrm{B}), \mathcal{H} \circ \mathcal{I}(\mathrm{B}')).$$

We can characterize the information systems of the form $\mathcal{I}(\mathrm{B})$ as follows:

Definition 6.4. An information system $\mathrm{A} = (A, \vdash_A, Con_A)$ is called *unary*, if for every $U_0 \ \epsilon \ Con_A$, there is $a_{U_0} \ \epsilon \ A$ such that $U_0 \vdash_A a_{U_0}$ and $\{a_{U_0}\} \vdash_A U_0$.

It is enough to verify the definitory property on the subsets with at most two elements[4]. In particular, there must be a token $\bot_A$ such that $\emptyset \vdash_A \bot_A$.

If A is of the form $\mathcal{I}(\mathrm{B})$, then for every $U_0 \ \epsilon \ Con_A$, we have $\mathsf{sup}(U_0) \vdash_A U_0$ and $U_0 \vdash_A \mathsf{sup}(U_0)$, and hence is unary. Viceversa, if $\mathrm{A} = (A, \vdash_A, Con_A)$ is a unary information system, we define canonically a unary formal topology with unary operation $\mathcal{H}_u(\mathrm{A}) \equiv (A, \geqslant_A, A)$, where

$$a \geqslant_A a' \equiv a \vdash_A a'$$

for all $a, a' \ \epsilon \ A$, and $a_{\emptyset_A}$ as convincing element. It is then direct to verify that A is exactly $\mathcal{I}(\mathcal{H}_u(\mathrm{A}))$.

Let UnInfSys be the full subcategory of InfSys whose objects are the unary information systems. The correspondence above defines a functor

$$\mathcal{H}_u : \mathrm{UnInfSys} \to \mathrm{UFTop}_*,$$

[4] The notion of unary information systems appears to be related, even if not equivalent, to that of *atomic information systems* [Kar13, Sch06b]. Further studies will be undertaken in this direction.

defined on an approximable mapping $r : \mathrm{A} \to \mathrm{B}$ as

$$a\mathcal{H}_u(r)b \equiv \{a\}\ r\ b.$$

The functor $\mathcal{H}_u$ is an inverse of the functor $\mathcal{I}$, in fact

$$U_0\ \mathcal{I}(\mathcal{H}_u(r))\ b \equiv \{\mathsf{sup}(U_0)\}\ r\ b \leftrightarrow U_0\ r\ b,$$

and vice versa, for $s : \mathrm{A} \to \mathrm{B}$ morphism of unary formal topologies,

$$a\ \mathcal{H}_u(\mathcal{I}(s))\ b \equiv \{a\}\ \mathcal{I}(s)\ b \leftrightarrow a\ s\ b.$$

Collecting these observations, we get:

Proposition 6.6. *The category* UFTop$_*$ *of unary formal topologies (with unary operation) with continuous and convergent morphism and the category* UnInfSys *of unary information systems are isomorphic.*

Example 6.7. Let A be a set. The flat information system associated to A is not unary, because it does not have a convincing element $\bot_A$ such that $\emptyset_A \vdash_A \bot_A$. We can canonically add an extra element $\bot_A$ to the set A, and we obtain an isomorphic unary information system.

Ideals as Formal Points. As usual in the general theory, formal topologies are provided with a notion of point. A formal point of a unary formal topology $(A, \geqslant_A, A)$ can be characterized as an inhabited, downward-closed subset $\alpha \subseteq A$ closed by $*_A$. More explicitly, $\alpha \subseteq A$ is a point if $\bot_A\ \epsilon\ \alpha$ and

$$\frac{a\ \epsilon\ \alpha \quad a \geqslant_A a'}{a'\ \epsilon\ \alpha}\ , \qquad \frac{a, a'\ \epsilon\ \alpha}{a *_A a'\ \epsilon\ \alpha}$$

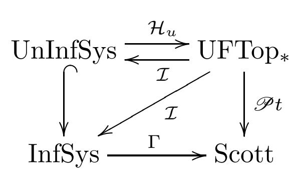
for all $a, a'\ \epsilon\ \alpha$. The collection of points of A is denoted by $\mathscr{P}t(A)$ and has structure of topological space.

Every continuous and convergent morphism $r : A \to B$ between two unary formal topologies $(A, \geqslant_A, A)$ and $(B, \geqslant_B, B)$ induces a continuous function

$$\mathscr{P}t(\overline{r}) : \mathscr{P}t(A) \to \mathscr{P}t(B).$$

and this correspondence is functorial between UFTop$_*$ and Scott.

Proposition 6.8. *The following diagram commutes*



Proof. We prove $\Gamma \circ \mathcal{I} = \mathscr{P}t$. First, notice that for any unary formal topology A , we have

$$\Gamma \circ \mathcal{I}(\mathrm{A}) = |\mathcal{I}(\mathrm{A})| = \mathscr{P}t(\mathrm{A}).$$

An ideal α in $|\mathcal{I}(\mathrm{A})|$ is a deductively closed subset such that all its finite subsets are consistent (i.e. bounded). In particular, $\bot_A$ is in α, because $\emptyset_A \vdash_A \bot_A$. Moreover, it must be closed downwards for the entailment relation on tokens, and any two-element set $\{a, a'\}$ in α is bounded. Hence $a * a' \ \epsilon\ \alpha$, being entailed by $\{a, a'\}$. Viceversa, if $\alpha \ \epsilon\ \mathscr{P}t(\mathrm{A})$, any finite subset $U_0 \subseteq \alpha$ is bounded by $\mathsf{sup}(U_0)\ \epsilon\ \alpha$ and, since α is downward closed, if $U_0 \subseteq \alpha$ (in particular, $\mathsf{sup}(U_0)\ \epsilon\ \alpha$) and $U_0 \vdash_A a \equiv \mathsf{sup}(U_0) \geqslant a$, then $a\ \epsilon\ \alpha$. If $r : \mathrm{A} \to \mathrm{B}$ is a continuous and convergent morphism, we have

$$\begin{aligned} b\ \epsilon\ \ \Gamma(\mathcal{I}(r))\ (\alpha) &\equiv \exists U_0 (U_0 \subseteq_\omega \alpha \ \&\ U_0\ \mathcal{I}(r)\ b) \equiv \\ &\equiv \exists U_0 (U_0 \subseteq_\omega \alpha \ \&\ (\mathsf{sup}(U_0)) r b) \leftrightarrow \\ &\leftrightarrow (\exists a\ \epsilon\ \alpha)(a r b) \equiv \\ &\equiv b\ \epsilon\ \mathscr{P}t(r)(\alpha). \end{aligned}$$

for all $\alpha\ \epsilon\ |\mathcal{I}(\mathrm{A})| (= \mathscr{P}t(\mathrm{A}))$, and $b\ \epsilon\ B$. □

In other terms, the points of a unary formal topology A with unary operation coincide canonically with the ideals of the corresponding information system $\mathcal{I}(\mathrm{A})$. By reason of this, we will call ideals[5] the formal points of a unary formal topology with unary operation.

6.3 Categorical Structure of UFTop$_*$

In this section, we show that the category UFTop$_*$ is cartesian closed category, that is, it has a terminal object and it is closed under binary products and exponential objects. This a necessary property for our tasks and it is well-known to be possessed by the categories InfSys and Scott. We will then show that the functor $\mathcal{I}$ described in the past section respects this structure.

A terminal object for the category is given by the unary formal topology

$$\mathbb{I} \equiv (\{*\}, =, \{*\}).$$

There is clearly only one continuous and convergent morphism from any unary formal topology to $\mathbb{I}$. We define the product as the usual product of preorders:

Definition 6.5. Let $\mathrm{A} = (A, \geqslant_A, A)$ and $\mathrm{B} = (B, \geqslant_B, B)$ be unary formal topologies. We define their product unary formal topology $\mathrm{A} \times \mathrm{B} = (A \times B, \geqslant_{A\times B}, A \times B)$, where

$$(a, b) \geqslant_{A\times B} (a', b') \equiv a \geqslant_A a' \ \&\ b \geqslant_B b',$$

for all $a, a'\ \epsilon\ A$ and $b, b'\ \epsilon\ B$.

[5] The term "ideal" was used in Chapter 5 with a different meaning. This clash is partially justified, since we are trying to follow the terminology of the classical traditions in algebra and domain theory.

It is direct to check that the usual projection morphisms determines two natural continuous and convergent morphisms $\overline{\pi_1} : \mathrm{A}\times\mathrm{B} \to \mathrm{A}$ and $\overline{\pi_2} : \mathrm{A}\times\mathrm{B} \to \mathrm{B}$. We show that this is actually a product in the category UFTop_*:

Proposition 6.9. *For all continuous and convergent morphisms* $r : \mathrm{C} \to \mathrm{A}$ *and* $s : \mathrm{C} \to \mathrm{B}$*, there is a unique morphism* $r \times s : \mathrm{C} \to \mathrm{A}\times\mathrm{B}$ *such that* $\overline{\pi_1} \circ (r \times s) = r$ *and* $\overline{\pi_2} \circ (r \times s) = s$*.*

Proof. We define $c\ (r \times s)\ (a, b) \equiv c\ r\ a\ \&\ c\ s\ b$, so that the $\overline{\pi_1} \circ (r \times s) = \pi_1 \circ (r \times s) = r$ and $\overline{\pi_2} \circ (r \times s) = \pi_2 \circ (r \times s) = s$ are clearly satisfied. Moreover, if $t : \mathrm{C} \to \mathrm{A}\times\mathrm{B}$ satisfies $\overline{\pi_1} \circ t = \pi_1 \circ t = r$ and $\overline{\pi_2} \circ t = \pi_2 \circ t = s$, then $t \subseteq r \times s$. Vice versa, if cra and csb, then there are $b' \in B$ and $a' \in A$ such that $ct(a, b')$ and $ct(a', b)$, so that $ct(a * a', b * b')$ because t is convergent. Since $(a * a', b * b') \geqslant_{A\times B} (a, b)$, then by continuity $ct(a, b)$. ☐

We recall that the category of information systems with approximable mappings, has products and exponential objects:

Definition 6.6. Let $\mathrm{A} = (A, Con_A, \vdash_A)$ and $\mathrm{B} = (B, Con_B, \vdash_B)$ be information systems. One defines $\mathrm{A}\times\mathrm{B} = (A \times B, Con_{A\times B}, \vdash_{A\times B})$ as follows

- $Con_{A\times B} \equiv Con_A \times Con_B$
- $(U_0, V_0) \vdash (a, b) \equiv U_0 \vdash_A a\ \&\ V_0 \vdash_A b.$

One defines $\mathrm{A} \to \mathrm{B} = (C, Con_C, \vdash_C)$ as:

- $C = Con_A \times B,$
- $\{(U_i, b_i) \mid i \in I\} \in Con_C := \forall_{J\subseteq I}(\bigcup_{j\in J} U_j \in Con_A \to \{b_j \mid j \in J\}) \in Con_B),$
- $W \vdash_C (U, b) := WU \vdash_B b$, where

$$\{(U_i, b_i) \mid i \in I\}U := \{b_i \mid U \vdash U_i\}.$$

The evaluation map $\mathrm{ev}^{\mathrm{B}}_{\mathrm{A}} : (\mathrm{A} \to \mathrm{B}) \times \mathrm{A} \to \mathrm{B}$ is defined as follows:

$$(W_0, U_0)\mathrm{ev}^{\mathrm{B}}_{\mathrm{A}} b \equiv W_0U_0 \vdash_B b \equiv W_0 \vdash_{A\times B} (U_0, b)$$

for all $W_0 \in Con_{(\mathrm{A}\to\mathrm{B})}$, $U_0 \in Con_{\mathrm{A}}$ and $b \in B$.

One proves that the approximable mappings between A and B are in canonical bijection with the ideals of $\mathrm{A} \to \mathrm{B}$. We have then, through the functor Γ, a canonical homeomorphism

$$|\mathrm{A} \to \mathrm{B}| \cong \mathcal{C}(|\mathrm{A}|, |\mathrm{B}|),$$

between the space of ideals of $\mathrm{A} \to \mathrm{B}$ and the space of continuous functions from $|\mathrm{A}|$ to $|\mathrm{B}|$. If α is an ideal of the left side, its corresponding image $T^B_A(\alpha)$ on the right side is defined on $\tau \in |\mathrm{A}|$ as:

$$b \in (T^B_A(\alpha))(\tau) \equiv (\exists U_0 \subseteq \tau)(\exists W_0 \subseteq \alpha)((W_0, U_0)\mathrm{ev}^{\mathrm{B}}_{\mathrm{A}} b).$$

We will present now a similar construction in the realm of unary formal topologies with unary operation. Let $\varphi \subseteq A \times B$ be a relation. We define the *source* of φ as

$$\mathrm{Sc}(\varphi) \equiv \pi_1(\varphi).$$

The relation φ will be called *consistent* if for all $U_0 \subseteq \mathrm{Sc}(\varphi)$ bounded and $V_0 \subseteq_\omega \varphi U_0$ finite, V_0 is bounded[6]. In particular, if φ is finite, φ is consistent if and only if $\varphi U_0 \in \mathrm{Bou}(B)$ for all $U_0 \in \mathrm{Bou}(\mathrm{Sc}(\varphi))$.

We define the *convergent closure* $\widetilde{\varphi}$ of a consistent relation φ as follows

$$\frac{a\varphi b}{a\widetilde{\varphi}b,} \qquad \frac{\top}{\bot_A\widetilde{\varphi}\bot_B} \qquad \frac{a\widetilde{\varphi}b \quad a'\widetilde{\varphi}b' \quad a \uparrow a'}{a *_A a'\widetilde{\varphi}b *_B b'}\ .$$

In the light of what done in the past section, we can define the space of morphisms between two unary formal topologies as a new unary formal topology:

Definition 6.7. Let $\mathrm{A} = (A, \geqslant_A, A)$ and $\mathrm{B} = (B, \geqslant_B, B)$ be unary formal topologies. We define the function space as the following unary formal topology $\mathrm{A} \to \mathrm{B} = (C, \geqslant_C, C)$:

- $C \subseteq \mathcal{P}_\omega(A \times B)$, $c \ \epsilon\ C$ if c is consistent (as a relation).
- $c \geqslant_C c' \equiv c' \subseteq \overline{c}$, where $\overline{c} \equiv \geqslant_B \circ \widetilde{c} \circ \geqslant_A$.

For $\mathrm{A} = (A, \geqslant_A, A)$ and $\mathrm{B} = (B, \geqslant_B, B)$ unary formal topologies,

$$\mathrm{A} \to \mathrm{B} = (C, \geqslant_C, C)$$

is easily seen to be a unary formal topology, since, if $c \geqslant_C c'$ ($\equiv c' \subseteq \overline{c}$) and $c \geqslant_C c''$ ($\equiv c'' \subseteq \overline{c}$), then $c' \cup c''$ must be also consistent, and is a supremum for $\{c', c''\}$. A convincing element is given by the relation $\bot_C \equiv \{(\bot_A, \bot_B)\}$.

We define an evaluation map $\mathrm{ev}_\mathrm{A}^\mathrm{B} : (\mathrm{A} \to \mathrm{B}) \times \mathrm{A} \to \mathrm{B}$ as follows:

$$(c, a)\mathrm{ev}_\mathrm{A}^\mathrm{B} b \equiv (a, b) \ \epsilon\ \overline{c}$$

for all $c \in C$, $a \in A$ and $b \in B$.

We show that this is indeed an exponential object in the category UFTop$_*$:

Proposition 6.10. *For any continuous and convergent morphisms* $r : \mathrm{D} \times \mathrm{A} \to \mathrm{B}$*, there is a unique morphism* $\lambda_r : \mathrm{D} \to (\mathrm{A} \to \mathrm{B})$ *such that* $r = \mathrm{ev}_\mathrm{A}^\mathrm{B} \circ (\lambda_r \times id_\mathrm{A})$.

Proof. We define $d\lambda_r c \equiv (\forall (a, b) \ \epsilon\ c)((d, a)rb)$. Notice then that

$$d\lambda_r c \equiv (\forall (a, b) \ \epsilon\ \overline{c})((d, a)rb).$$

We have then

$$\begin{array}{rcl} (d, a)(\mathrm{ev}_\mathrm{A}^\mathrm{B} \circ (\lambda_r \times id_\mathrm{A}))b & \leftrightarrow & \exists c(d\lambda_r c \ \&\ (c, a)\mathrm{ev}_\mathrm{A}^\mathrm{B} b) \equiv \\ & \equiv & \exists c((\forall (a', b') \ \epsilon\ c)((d, a')rb') \ \&\ (a, b) \ \epsilon\ \overline{c}) \leftrightarrow \\ & \equiv & \exists c((\forall (a', b') \ \epsilon\ \overline{c})((d, a')rb') \ \&\ (a, b) \ \epsilon\ \overline{c}) \leftrightarrow \\ & \overset{(*)}{\leftrightarrow} & (d, a)rb, \end{array}$$

[6] Even in this section, the notation with lower case letters indexed by 0 will be reserved to finite subsets.

where in ($*$) we pick $c \equiv \{(a,b)\}$. Moreover, if t satisfies $r = \mathrm{ev}^{\mathrm{B}}_{\mathrm{A}} \circ (t \times id_{\mathrm{A}})$, by continuity we have $dtc \equiv (\forall (a,b) \;\epsilon\; c)(dt\{(a,b)\})$ and

$$dt\{(a,b)\} \leftrightarrow \exists c(dtc \;\&\; (a,b) \;\epsilon\; \overline{c}) \leftrightarrow (d,a)(\mathrm{ev}^{\mathrm{B}}_{\mathrm{A}} \circ (t \times id_{\mathrm{A}}))b \leftrightarrow (d,a)rb$$

so that $t = \lambda_r$. □

We can conclude that UFTop$_*$ is a cartesian closed category. Notice that given any consistent relation $\varphi \subseteq A \times B$, and we can always define a continuous and convergent morphism $\overline{\varphi}$ in two steps:

1. We take its *convergent closure* $\widetilde{\varphi}$, defined as above;
2. We make it continuous, by defining $\overline{\varphi} \equiv \vdash_B \circ \widetilde{\varphi} \circ \vdash_A$.

The convergent closure is the smallest relation containing φ and satisfying the convergence condition $(C1)$. It is direct to verify that the second step does not affect $(C1)$, so that the result is actually a continuous and convergent continuous morphism. These observations show in particular:

Proposition 6.11. *If* $\mathrm{A} = (A, \geqslant_A, A)$ *and* $\mathrm{B} = (B, \geqslant_B, B)$ *are unary formal topologies, a point of* $\mathrm{A} \to \mathrm{B}$ *is precisely a continuous and convergent morphism from* A *to* B*.*

The functor $\mathcal{I}$ respects the structure of cartesian closed category. In particular:

Proposition 6.12. *Let* $\mathrm{A} \equiv (A, \geqslant_A, A)$ *and* $\mathrm{B} \equiv (B, \geqslant_B, B)$ *be two unary formal topologies. Then:*

1. $\mathcal{I}(\mathrm{A} \times \mathrm{B}) \cong \mathcal{I}(\mathrm{A}) \times \mathcal{I}(\mathrm{B})$,
2. $\mathcal{I}(\mathrm{A} \to \mathrm{B}) \cong \mathcal{I}(\mathrm{A}) \to \mathcal{I}(\mathrm{B})$,

naturally in A *and* B*.*

Proof. (Sketch) (1) We prove that $\mathcal{I}(\mathrm{A} \times \mathrm{B})$ is a product for $\mathcal{I}(\mathrm{A})$ and $\mathcal{I}(\mathrm{B})$. First of all we have two canonical approximable maps π'_1 and π'_2, defined as

$$\begin{array}{ll} \pi'_1 : \mathcal{I}(\mathrm{A} \times \mathrm{B}) \to \mathcal{I}(\mathrm{A}) & \qquad \pi'_2 : \mathcal{I}(\mathrm{A} \times \mathrm{B}) \to \mathcal{I}(\mathrm{B}) \\ W_0 \pi'_1 a \equiv \pi_1(W_0) \vdash_A a & \qquad W_0 \pi'_2 b \equiv \pi_2(W_0) \vdash_B b \end{array}$$

Let $r : \mathrm{C} \to \mathcal{I}(\mathrm{A})$ and $s : \mathrm{C} \to \mathcal{I}(\mathrm{B})$, and we define an approximable map

$$\begin{array}{c} r \times' s : \mathrm{C} \to \mathcal{I}(\mathrm{A} \times \mathrm{B}) \\ W_0(r \times' s)(a,b) \equiv W_0 ra \;\&\; W_0 sb. \end{array}$$

Then we have:

$$\begin{array}{c} W_0(\pi'_1 \circ (r \times' s))a \equiv \\ \equiv \exists_{\mathrm{Bou}(A \times B)} W_0((\forall (a,b) \;\epsilon\; W_0)(W_0 ra \;\&\; W_0 sb) \;\&\; \pi_1(W_0) \vdash_A a) \leftrightarrow \\ \leftrightarrow \exists_{\mathrm{Bou}(A) \times \mathrm{Bou}(B)} (U_0, V_0)((\forall a \;\epsilon\; U_0)(W_0 ra) \;\&\; (\forall b \;\epsilon\; V_0)(W_0 sb) \;\&\; U_0 \vdash_A a) \\ \leftrightarrow \exists_{\mathrm{Bou}(A)} U_0((\forall a \;\epsilon\; U_0)(W_0 ra) \;\&\; U_0 \vdash_A a) \leftrightarrow W_0(r \circ \vdash_A)a \leftrightarrow W_0 ra. \end{array}$$

Similarly $(\pi_2' \circ (r \times' s)) = s$. We omit the check of the uniqueness and naturality of $r \times' s$.

(2) We prove that $\mathcal{I}(\mathrm{A} \to \mathrm{B})$ is an exponential for $\mathcal{I}(\mathrm{A})$ and $\mathcal{I}(\mathrm{B})$. Let the evaluation map $\mathrm{ev}_{\mathrm{A}}^{\mathrm{B}} : \mathcal{I}(\mathrm{A} \to \mathrm{B}) \times \mathcal{I}(\mathrm{A}) \to \mathcal{I}(\mathrm{B})$ be defined as

$$(W_0, U_0)\mathrm{ev}_{\mathrm{A}}^{\mathrm{B}} b \equiv (\mathsf{sup}(U_0), b) \;\epsilon\; \overline{\cup W_0}$$

for all $W_0 \;\epsilon\; \mathrm{Bou}(A \to B)$, $U_0 \;\epsilon\; \mathrm{Bou}(A)$ and $b \;\epsilon\; B$.

Let now $r : \mathrm{D} \times \mathcal{I}(\mathrm{A}) \to \mathcal{I}(\mathrm{B})$ be an approximable map, with D an information system. We define the morphism $\lambda_r' : \mathrm{D} \to \mathcal{I}(\mathrm{A} \to \mathrm{B})$ as

$$Z_0 \lambda_r' c \equiv (\forall (a, b) \;\epsilon\; c)((Z_0, a) r b).$$

Notice then that

$$Z_0 \lambda_r' c \equiv (\forall (a, b) \;\epsilon\; \overline{c})((Z_0, a) r b).$$

We have then

$$\begin{aligned}(Z_0, U_0)(\mathrm{ev}_{\mathrm{A}}^{\mathrm{B}} \circ (\lambda_r' \times id_{\mathcal{I}(\mathrm{A})}))b \quad &\leftrightarrow \quad \exists W_0 (Z_0 \lambda_r' (\cup W_0) \;\&\; (W_0, U_0)\mathrm{ev}_{\mathrm{A}}^{\mathrm{B}} b) \leftrightarrow \\ &\leftrightarrow \quad \exists W_0 (Z_0 \lambda_r' \overline{\cup W_0} \;\&\; (\mathsf{sup}(U_0), b) \;\epsilon\; \overline{\cup W_0}) \leftrightarrow \\ &\leftrightarrow \quad Z_0 \lambda_r' \{(\mathsf{sup}(U_0), b)\} \equiv (Z_0, \{\mathsf{sup}(U_0)\}) r b \leftrightarrow \\ &\leftrightarrow \quad (Z_0, U_0) r b.\end{aligned}$$

We omit the check that the map λ_r' is unique and natural. □

Remark 25. Notice that the property of being unary is not respected by isomorphism of information systems, since, even if $\mathcal{I}(\mathrm{A} \to \mathrm{B})$ is unary, $\mathcal{I}(\mathrm{A}) \to \mathcal{I}(\mathrm{B})$ usually is not.

If $(A, \geqslant_A, A)$ and $(B, \geqslant_B, B)$ are two unary formal topologies, in the light of the Proposition 6.8 and 6.12, we get

$$\begin{aligned}\mathscr{P}t(\mathrm{A} \to \mathrm{B}) = |\mathcal{I}(\mathrm{A} \to \mathrm{B})| \cong |\mathcal{I}(\mathrm{A}) \to \mathcal{I}(\mathrm{B})| = \\ = \mathcal{C}(|\mathcal{I}(\mathrm{A})|, |\mathcal{I}(\mathrm{B})|) = \mathcal{C}(\mathscr{P}t(\mathrm{A}), \mathscr{P}t\mathrm{B})).\end{aligned}$$

If $\alpha \in_c \mathscr{P}t(\mathrm{A} \to \mathrm{B})$ is a formal point, then it is closed downwards and by union, so that it can be identified with the subset $\cup\alpha \subseteq A \times B$. Hence, its image $T_A'^B(\alpha)$ on the rightmost member can be characterized on $\tau \;\epsilon\; \mathscr{P}t(\mathrm{A})$ as its direct image through $\cup\alpha$, i.e.:

$$b \;\epsilon\; (T_A'^B(\alpha))(\tau) \equiv b \;\epsilon\; (\cup\alpha)\tau \equiv (\exists a \;\epsilon\; \tau)((a, b) \;\epsilon\; \alpha).$$

The functor of points behaves well with respect to product of unary formal topologies, since, unlike the general information systems case, points have to be inhabited.

Proposition 6.13. *Let* A $\equiv (A, \geqslant_A, A)$ *and* B $\equiv (B, \geqslant_B, B)$ *be two unary formal topologies. Then* $\mathscr{P}t(\mathrm{A} \times \mathrm{B}) \cong \mathscr{P}t(\mathrm{A}) \times \mathscr{P}t(\mathrm{B})$, *naturally in* A *and* B.

Proof. If α is in $\mathscr{P}t(\mathrm{A}\times\mathrm{B})$ then $\pi_A(\alpha)$ is in $\mathscr{P}t(\mathrm{A})$, in fact, it contains $\bot_A$, and it is obviously downward closed and closed by $*_A$, by definition of product order. The same holds for $\pi_B(\alpha) \in \mathscr{P}t(\mathrm{B})$. Then $\alpha = \pi_A(\alpha) \times \pi_B(\alpha)$: if $a \ \epsilon \ \pi_A(\alpha)$ and $b \ \epsilon \ \pi_B(\alpha)$, then there exist $a' \ \epsilon \ A$ and $b' \ \epsilon \ B$, such that $(a, b'), (a', b) \ \epsilon \ \alpha$, and therefore $(a, b) \ \epsilon \ \alpha$ because $(a, b) \leqslant (a *_A a', b *_B b') \ \epsilon \ \alpha$. We omit the easy proof that such correspondence is natural in A and B. □

6.4 The Unary Formal Topology Associated to an Algebra

We are going to settle in this section concrete examples of unary formal topologies with unary operation arising from *free algebras*. We first introduce a general system of types, as in [SW12, Sch10]: we start from a set of type variables ι, $\vec{\alpha}$ and we define inductively the *type forms* $Ty(\vec{\alpha})$, *constructor type forms* $KT_\iota(\vec{\alpha})$ and the *algebra forms* $Alg(\vec{\alpha})$:

$$\alpha_l \in Ty(\vec{\alpha}), \qquad \frac{\iota \in Alg(\vec{\alpha})}{\iota \in Ty(\vec{\alpha})}, \qquad \frac{\rho \in Ty \quad \sigma \in Ty(\vec{\alpha})}{\rho \to \sigma \in Ty(\vec{\alpha})}$$

$$\frac{\kappa_0, \dots, \kappa_{k-1} \in KT_\iota(\vec{\alpha})}{\mu_\iota(\kappa_0, \dots, \kappa_{k-1}) \in Alg(\vec{\alpha})} \ (k \geqslant 1)$$

$$\frac{\vec{\rho} \in Ty(\vec{\alpha}) \quad \vec{\sigma}_0, \dots, \vec{\sigma}_{n-1} \in Ty}{\vec{\rho} \to (\vec{\sigma}_\nu \to \iota)_{\nu<n} \to \iota \in KT_\iota(\vec{\alpha})} \ (n \geqslant 0)$$

For every constructor type $\kappa_i(\iota)$ of an algebra ι we provide a constructor symbol C_i of type $\kappa_i(\iota)$.

An algebra (type) ι is said to be *finitary* if all its constructor types only have finitary algebras as parameter argument types, and the $\vec{\sigma}_\nu$ in its explicit definition are all empty. To every finitary algebra type ι we can naturally associate a *flat* or *non-flat* unary formal topology with unary operation.

The flat case. For every finitary algebra type ι, we define a unary formal topology

$$(F_\iota, \geqslant_\iota, F_\iota).$$

The elements of $a \in F_\iota$, called *tokens*, are the bottom token $*$ or a type correct expressions of the form $Ca_1 \dots a_n$ where a_i is a token. The preorder is given by

$$\geqslant_\iota \equiv =_\iota \cup \{(a, *)\}_{a \ \epsilon \ F_\iota},$$

so that $a * a' \equiv a$, if $a =_\iota a'$. This is clearly a unary formal topology. For $\rho \to \sigma$, we define recursively

$$F_{\rho\to\sigma} \equiv F_\rho \to F_\sigma.$$

The unary formal topologies of this form are the one generated starting from the family of simple base spaces $\{F_\iota\}_{\iota \in Alg(\vec{\alpha})}$.

The non-flat case. For every finitary algebra type ι, we define a unary formal topology

$$(C_\iota, \geqslant_\iota, C_\iota).$$

The tokens $a \in C_\iota$ are $*$ or a type correct expressions $Ca_1^* \dots a_n^*$ where a_i^* is a token, and C is a constructor. In order to define the preorder relation, we have to introduce the notion of *one-step reduction* $\to_1$ of a token, by recursion on the token structure:

$$* \to_1 C * \dots *$$
$$Ca_1^* \dots a_n^* \to_1 Ca_1'^* \dots a_n'^* \equiv a_1^* \to_1 a_1'^* \ \& \ \dots \ \& \ a_n^* \to_1 a_n'^*.$$

The preorder $\leqslant_\iota$ is then the so-called *reduction relation*, that is, the transitive and reflexive closure of $\to_1$. We will say that a is a *reduction* of, a' if $a \geqslant_\iota a'$.

Lemma 6.14. *For all finitary algebra types ι, C_ι is a unary formal topology.*

Proof. A convincing element is clearly given by the token $*$. We have to check that if $a, a' \in C_\iota$ and a'' is a reduction both of a and a', then a, a' have a supremum $\mathsf{sup}(a, a')$. We prove this by induction on the structure of a and a'. If $a = * = a'$, then $*$ is also the supremum. Suppose now $a = Ca_1^* \dots a_n^*$, $a' = Ca_1'^* \dots a_n'^*$ and that the statement is true for all $a_i^*, a_j'^*$. Let $a'' = Ca_1''^* \dots a_n''^*$, a common reduction. Since for each i, $a_i''^*$ is a reduction both of a_i^*, and $a_i'^*$, we find by inductive hypothesis a supremum $c_i^* = \mathsf{sup}(a_i^*, a_i'^*)$, and we define

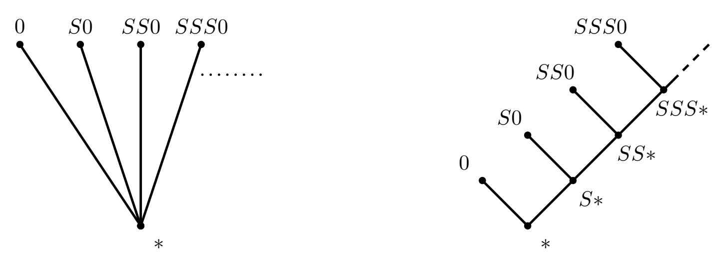
$$\mathsf{sup}(a, a') \equiv Cc_1^* \dots c_n^*.$$

The proof that this is indeed a supremum is straightforward. □

For $\rho \to \sigma$, we define recursively

$$C_{\rho \to \sigma} \equiv C_\rho \to C_\sigma.$$

Example 6.15. As basic example, we will consider the *flat* and *non-flat* bounded complete preorder structures associated to the algebra $\mu_{\mathbb{N}}(0^{\mathbb{N}}, S^{\mathbb{N} \to \mathbb{N}})$ of natural numbers, generated by a nullary constructor $0^{\mathbb{N}}$ and a unary constructor $S^{\mathbb{N} \to \mathbb{N}}$. We present them in tree-form:



The *flat* natural numbers $F_{\mathbb{N}}$ The *non-flat* natural numbers $C_{\mathbb{N}}$

The *token* $*$ is added as bottom element in both preorder structures. In the flat case, we have $SS0 \uparrow *$ (that is, the subset $\{SS0, *\}$ is bounded), while instead $SS0 \not\uparrow S0$. In the non-flat case, we have for example $S0 \uparrow S*$, but $S0 \not\uparrow SS*$.

Notice that the non-flat natural numbers are represented by a finitely branching tree, in contrast to the flat ones. This observation can be generalized to every finitary algebra and will be crucial in the following.

Example 6.16. Consider the following elements of $C_{\mathbb{N}} \to C_{\mathbb{N}}$:

$$c \equiv \{(*, S*), (SS*, SS0), (SS0, S*)\},$$
$$c' \equiv \{(SS0, SS0), (SSSS0, SS0), (SS0, *)\}.$$

Then $c' \leqslant c$, that is, $c' \subseteq \overline{c}$; in fact, $(SS0, SS0) \mathrel{\epsilon} \widetilde{c} \subseteq \overline{c}$; $SSSS0 \geqslant SS*$, $(SS*, SS0) \mathrel{\epsilon} c$ and $SS0 \geqslant SS0$, so that $(SSSS0, SS0) \mathrel{\epsilon} \overline{c}$; similarly $SS0 \geqslant *$, $(*, S*) \mathrel{\epsilon} c$ and $S* \geqslant *$ imply $(SS0, *) \mathrel{\epsilon} \overline{c}$.

Although not needed in the following, we prove the following standard facts to show the benefits of the basic pairs notation.

Proposition 6.17. *Let $\iota \mathrel{\epsilon} Alg(\vec{\alpha})$. Then every constructor C of arity $\vec{\rho} \to (\vec{\sigma}_\nu \to \iota)_{\nu<n} \to \iota$ induces an injective continuous function $\overline{C}$ from $\mathscr{P}t(\prod \vec{\rho} \times (\vec{\sigma}_\nu \to \iota)_{\nu<n})$ to $\mathscr{P}t(\iota)$.*

Proof. Let for convenience $\xi \equiv \prod \vec{\rho} \times (\vec{\sigma}_\nu \to \iota)_{\nu<n}$. A constructor C has the following property:

$$\frac{a_1^* \uparrow a_1'^* \quad \ldots \quad a_n^* \uparrow a_n'^*.}{Ca_1^* \ldots a_n^* \uparrow Ca_1'^* \ldots a_n'^*}$$

In fact, if for each i, $a_i''^*$ is a reduction both of a_i^*, and $a_i'^*$, then $a'' = Ca_1''^* \ldots a_n''^*$ is a common reduction of $Ca_1^* \ldots a_n^*$ and $Ca_1'^* \ldots a_n'^*$. Therefore C is consistent as a relation from C_ξ to C_ι, and moreover $C = \widetilde{C}$, since

$$C(\mathsf{sup}(a_1^*, a_1'^*)) \ldots (\mathsf{sup}(a_n^*, a_n'^*)) \equiv \mathsf{sup}(Ca_1^* \ldots a_n^*, Ca_1'^* \ldots a_n'^*). \tag{6.10}$$

The required partial morphism is then $\overline{C} \equiv \geqslant_\iota \circ C \circ \geqslant_\xi$. Let now be C^- be the inverse relation of C from C_ι to C_ξ. Since C is injective on tokens, (6.10) shows that C^- is consistent and $C^- = \widetilde{C^-}$. Hence, $\overline{C^-} \equiv \geqslant_\xi \circ C^- \circ \geqslant_\iota$ is a well defined partial morphism from C_ι to C_ξ. We have then

$$\begin{array}{rcl} \vec{a}\overline{C^-} \circ \overline{C}\vec{a}' & \equiv & \vec{a} \geqslant_\xi \circ C^- \circ \geqslant_\iota \circ \geqslant_\iota \circ C \circ \geqslant_\xi \vec{a}' \leftrightarrow \\ & \leftrightarrow & \vec{a} \geqslant_\xi \circ C^- \circ (\geqslant_\iota \circ C) \circ \geqslant_\xi \vec{a}' \leftrightarrow \\ & \overset{cont.}{\leftrightarrow} & \vec{a} \geqslant_\xi \circ C^- \circ C \circ \geqslant_\xi \circ \geqslant_\xi \vec{a}' \leftrightarrow \\ & \leftrightarrow & \vec{a} \geqslant_\xi \vec{a}' \equiv \vec{a}\ id_\xi\ \vec{a}'. \end{array}$$

The continuous map associated to C must then be injective, since it has a left inverse. □

In the light of Proposition 6.13, we get:

Corollary 6.18. *Let $\iota \,\epsilon\, Alg(\vec{\alpha})$. Then every constructor C of arity $\vec{\rho} \to (\vec{\sigma}_\nu \to \iota)_{\nu<n} \to \iota$ induces an injective continuous function from $\prod_{\rho\in\vec{\rho}} \mathscr{P}t(\rho) \times \mathscr{P}t((\vec{\sigma}_\nu \to \iota)_{\nu<n})$ to $\mathscr{P}t(\iota)$.*

Corollary 6.19 (Ideals of base type)**.** *Every ideal z of C_ι has the form $\overline{C}(\vec{x})$ for some constructor C and ideals $\vec{x}$.*

Proof. With the notation above, if $Ca_1^* \dots a_n^* \,\epsilon\, z$, then, by consistency, every element a of z is of the form $*$ or $Ca_1^* \dots a_n^*$. Consider $\overline{C^-}(z) \,\epsilon\, \prod_{\rho\in\vec{\rho}} \mathscr{P}t(\rho) \times \mathscr{P}t((\vec{\sigma}_\nu \to \iota)_{\nu<n})$, and we prove that $\overline{C} \circ \overline{C^-}(z) = z$. We have $*_\iota \overline{C} \circ \overline{C^-} *_\iota$ and, if $a' = Ca_1^* \dots a_n^*$:

$$\begin{aligned} a\overline{C} \circ \overline{C^-} a' \quad &\equiv \quad a \geqslant_\iota \circ C \circ \geqslant_\xi \circ \geqslant_\xi \circ C^- \circ \geqslant_\iota Ca_1^* \dots a_n^* \leftrightarrow \\ &\leftrightarrow \quad a \geqslant_\iota \circ C \circ \geqslant_\xi \circ \geqslant_\xi \circ C^- \circ (\geqslant_\iota \circ C)(a_1^*, \dots, a_n^*) \leftrightarrow \\ &\leftrightarrow \quad a \geqslant_\iota \circ C \circ \geqslant_\xi \circ \geqslant_\xi \circ C^- \circ C \circ \geqslant_\xi (a_1^*, \dots, a_n^*) \leftrightarrow \\ &\leftrightarrow \quad a \geqslant_\iota \circ C \circ \geqslant_\xi (a_1^*, \dots, a_n^*) \leftrightarrow \\ &\leftrightarrow \quad a\overline{C}(a_1^*, \dots, a_n^*) \equiv a = a'. \end{aligned}$$

Hence $a \,\epsilon\, \overline{C} \circ \overline{C^-}(z)$ if and only if $a \,\epsilon\, z$. □

6.5 A Revisitation of the Classical Density Theorem

We will prove in this section a version of the Kleene-Kreisel density theorem for unary formal topologies with unary operation. The proof will adapt Berger's argument [Ber93b] but it will let us state afterwards a finite version of it.

The unary formal topologies we consider from now on are always endowed with a unary operation and a convincing (i.e. bottom) element, but we omit to say this to avoid repetitions.

For the sake of clarity, we restrict here to a particular kind of unary formal topologies:

Definition 6.8. A unary formal topology $(A, \lhd_A, *, \bot_A)$ is called *coherent*, if

$$\uparrow \{a_1, \dots, a_n\} \;\leftrightarrow\; (\forall i, j)(a_i \uparrow a_j)$$

for all $a_1, \dots, a_n \,\epsilon\, A$.

In other words, a unary formal topology is coherent if and only if, for deciding the consistency of a finite subset, it is enough to check the consistency of its two-elements subsets.

Let $(A, \lhd_A, *, \bot_A)$ be a unary formal topology. We define

$$\prod_{i=1}^{n} U_i \equiv \{u_1 * \dots * u_n : u_i \,\epsilon\, U_i \text{ for } i = 1, \dots, n\}$$

for all $U_1, \dots, U_n \subseteq A$. Notice that, being just the extensional generalization of an operation on singletons, the $*$ operation distributes with union of subsets, i.e.

$$U *_A (V \cup V') \equiv U *_A V \cup U *_A V',$$

for all $U, V, V' \subseteq A$. We introduce also a generalization of $\uparrow$ and $\not\uparrow$ between subsets:

$$U \uparrow V \equiv (\exists u \,\epsilon\, U)(\exists v \,\epsilon\, V)(u \uparrow v), \qquad U \not\uparrow V \equiv (\forall u \,\epsilon\, U)(\forall v \,\epsilon\, V)(u \not\uparrow v),$$

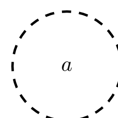
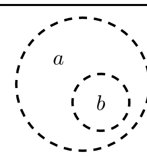
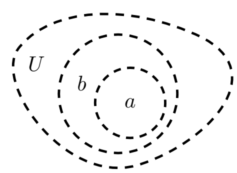
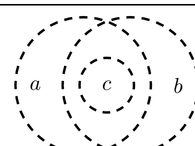
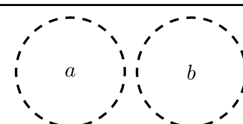
for all $U, V \subseteq A$. As done for general formal topologies and rings in the previous chapters, we can define a formal open $a \to \emptyset$ for every element $a \,\epsilon\, A$, as follows

$$b \,\epsilon\, a \to \emptyset \equiv a * b \lhd \emptyset \ \ (\leftrightarrow a * b = \emptyset \equiv a \not\uparrow b).$$

We recall that, in terms of the corresponding open subset in $\mathscr{P}t(A)$, $\mathsf{Ext}(a \to \emptyset)$ is the pseudocomplement of the base open subset $\mathsf{ext}(a)$. More generally, for every subset $U \subseteq A$, one defines $U \to \emptyset$ as follows

$$b \,\epsilon\, U \to \emptyset \equiv U * b \lhd \emptyset \ \ (\leftrightarrow U * b = \emptyset \equiv U \not\uparrow b).$$

For the sake of a better intuition, we draw in the table a topological intuition of some formal expressions:

Formal space $(A, \lhd, *, \perp)$	**Space of points** $\mathscr{P}t(A)$
$a \,\epsilon\, A$	
$b \lhd a$ (or $b \geqslant_{\lhd} a$)	
$a \lhd U \equiv$ $\equiv \exists b(a \geqslant_{\lhd} b \;\&\; b \,\epsilon\, U)$	
$c \lhd a * b$	
$b \,\epsilon\, a \to \emptyset$ (or $b \,\epsilon\, S_a$)	

In particular, we can define the boundary formal open

$$N(a) \equiv \mathscr{A}_{\lhd}(\{a\} \cup a \to \emptyset)$$

for all $a \epsilon A$. This corresponds, in the point-theoretic counterpart, to $\mathscr{P}t(A) \setminus \delta B_a$, where δB_a is the boundary of the open base subset B_a. More generally, we can define the boundary formal open of a subset $U \subseteq A$ as

$$N(U) \equiv \mathscr{A}_{\lhd}(U \cup U \to \emptyset).$$

We recall that a formal open $U \subseteq A$ is called *finitely generated* (for short, f.g.) if we can find a finite list $a_1, \dots, a_n \in A$ such that $U =_{\lhd} \{a_1, \dots, a_n\}$.

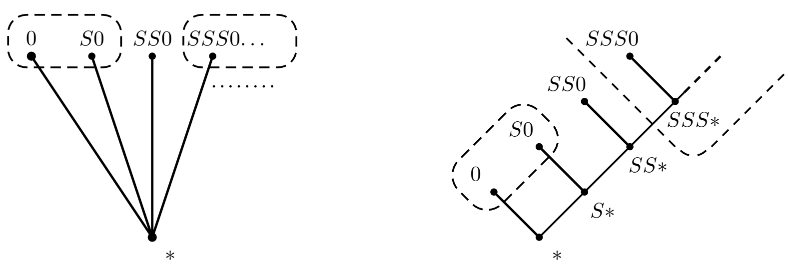
Example 6.20. In the unary formal topology $F_{\mathbb{N}}$ of flat natural numbers, the formal opens are the whole set (finitely generated by the element $*$) and the subsets not containing $*$ (and f.g. if and only if they are finite).

In the unary formal topology $C_{\mathbb{N}}$ of non-flat natural numbers, the subsets $\{S0, SS0\}$ and $\{S*, SS*, SSS*, \dots\}$ are formal opens, and are finitely generated, respectively, by $\{S0, SS0\}$ and $\{S*\}$.

The formal open $SS0 \to \emptyset$ of the token $SS0$ is the subset

$$\{0, S0, SSS0, SSSS0, \dots\}$$

in the flat case, while it is the subset $\{0, S0\} \cup \{a : a \geqslant SSS*\}$ in the non-flat case.



$SS0 \to \emptyset$ in the flat and non-flat algebra of naturals

Notice that, in the non-flat case, $SS0 \to \emptyset$ is f.g. by $\{0, S0, SSS*\}$, while it is not f.g. in the flat case. This is a consequence of the fact that non-flat tree is finitely branching, while the flat one is not. In particular, the boundary formal open $N(SS0)$ is f.g. in the non-flat case, while it is not so in the flat case.

The maximal ideals (i.e. formal points) can be characterized classically in terms of the boundary formal opens. The following proposition can be seen as a consequence of Proposition 1.19 and, even though classical, has to be considered simply as a justification for the definition of total ideal we are about to introduce.

Proposition 6.21 (CL)**.** *Let* $(A, \lhd_A, *, \perp_A)$ *be a unary formal topology*[7]. *A point* $\alpha \subseteq A$ *is maximal if and only if* $N(a) \between \alpha$ *for all* $a \in A$.

[7] The same result holds in any finitary formal topology with finitary operation.

Proof. The ($\to$) implication is proved classically; suppose α maximal and $a \,\epsilon\, A$. If $\neg(a \to \emptyset \between \alpha)$, that is, $a \uparrow \alpha$, then the downward (viz. deductive) closure of $\{a\} \cup \alpha$ is a point containing α. By maximality, we get $a \,\epsilon\, \alpha$.

($\leftarrow$) Let $\alpha' \subseteq \alpha$ and $a \,\epsilon\, \alpha$. We have $a \,\epsilon\, \alpha'$ or $(a \to \emptyset) \between \alpha'$. If the latter holds, that is, there is $a' \not\uparrow a$ such that $a' \,\epsilon\, \alpha'$, then $a, a' \,\epsilon\, \alpha$ in contradiction with the consistency of α. Hence $\alpha \subseteq \alpha'$. □

We fix a family $\mathscr{A} \equiv \{A_i\}_{i\in I} \equiv \{(A_i, \lhd_i, *, \bot_i)\}_{i\in I}$ of unary formal topologies. The elements of $\mathscr{A}$ will be called *base spaces*. Starting from $\mathscr{A}$, we generate the family $\overline{\mathscr{A}}$ of unary formal topologies generated from $\mathscr{A}$ by means of function spaces:

$$\frac{A \in \mathscr{A}}{A \in \overline{\mathscr{A}}} \qquad \frac{A, B \in \overline{\mathscr{A}}}{A \to B \in \overline{\mathscr{A}}.}$$

On each element A of the family $\overline{\mathscr{A}}$, we distinguish the *total points* (or *total ideals*)[8] $\mathbb{T}_A \subseteq \mathscr{P}t(A)$, defined as follows:

$$\begin{array}{rcl} A \in \mathscr{A} & \Rightarrow & \alpha \in \mathbb{T}_A \equiv (\forall a \in A)(N(a) \between \alpha)), \\ A \to B \in \overline{\mathscr{A}} & \Rightarrow & \alpha \in \mathbb{T}_{A\to B} \equiv \forall \tau(\tau \in \mathbb{T}_A \to \alpha\tau \in \mathbb{T}_B). \end{array} \tag{6.11}$$

The definition of total ideal for function spaces involves an unrestricted quantification over the ideals, which do not form a set in general, and therefore rises impredicativity issues. If we restrict the quantification to the subclass of *computable ideals*, that means, ideals whose underlying set of tokens is a recursively enumerable set, this is instead constructively acceptable.

Proposition 6.22. *Let $A \in \overline{\mathscr{A}}$ be a unary formal topology, $\alpha \in \mathbb{T}_A$, and $\alpha' \supseteq \alpha$ point. Then $\alpha' \in \mathbb{T}_A$.*

Proof. For base spaces, this is obvious, because α is maximal and $\alpha = \alpha'$. Suppose now that the statement is satisfied for $A, B \,\epsilon\, \overline{\mathscr{A}}$, and we prove it for $C \equiv A \to B$. Let $\gamma \,\epsilon\, \mathbb{T}_A$, $\gamma' \supseteq \gamma$ and $\alpha \in \mathbb{T}_A$. Since γ is total, we get $\gamma\alpha \in \mathbb{T}_B$ and therefore $\gamma'\alpha \in \mathbb{T}_B$ because $\gamma'\alpha \supseteq \gamma\alpha$. □

Definition 6.9. A unary formal topology $A \in \overline{\mathscr{A}}$ is called *dense* if $\mathbb{T}_A$ is *dense* in $Pt(A)$. In concrete terms, for each $a \,\epsilon\, A$, there is $\alpha \in \mathbb{T}_A$ such that $a \,\epsilon\, \alpha$.

The density theorem will consist in showing that each $A \in \overline{\mathscr{A}}$ is dense, provided natural hypotheses on the base spaces.

Example 6.23. Consider the algebra $F_{\mathbb{N}}$ of flat natural numbers. Its total ideals are those of the form $\{*, SS\ldots S0\}$, and are, in particular, maximal. The density theorem is in this case trivially satisfied.

The case of the function space $F_{\mathbb{N}} \to F_{\mathbb{N}}$ is quite similar: the total ideals must be defined on each $SS\ldots S0$ with total value. In particular the maximal ideals are total, and since every element of $F_{\mathbb{N}} \to F_{\mathbb{N}}$ can be extended to a maximal one, the density theorem hold easily also in this case.

[8] The notion of totality is here relative to the family $\mathscr{A}$ we started with.

The notions of maximality and that of totality start to diverge on higher types. Consider for instance the following example in $(F_{\mathbb{N}} \to F_{\mathbb{N}}) \to F_{\mathbb{N}}$: we define

$$cFk \equiv \exists k'(k \neq k' \ \& \ \{(0,0), \dots, (k-1, k-1), (k, k')\} \leqslant c),$$

for all $c \in F_{\mathbb{N}} \to F_{\mathbb{N}}$ and $k \in \mathbb{N}$. The functional F is maximal but not total, because it is not defined on the identity $id : \mathbb{N} \to \mathbb{N}$.

Remark 26. If A is dense, and $U_0 \subseteq A$ is bounded, then there is $\alpha \in \mathbb{T}_A$ such that $\prod U_0 \ \epsilon \ \alpha$ and therefore $U_0 \subseteq \tau$.

Definition 6.10. Let $A \in \overline{\mathscr{A}}$ be a unary formal topology. A subset $U \subseteq A$ is called *separating* if for all $\alpha \ \epsilon \ \mathbb{T}_A$, $\alpha \between N(U)$.

Remark 27. If $a \ \epsilon \ A$, with A base space, then every singleton $\{a\}$ is separating. The definition is satisfied since for total ideals of the base space we have $\forall a(\alpha \between N(a))$.

Remark 28. If $U, V \subseteq A$ are separating, then also $U * V$ is a separating pair. In fact, we have in general

$$N(U) * N(V) \lhd N(U * V).$$

Now, since a point α (and, in particular, a total one) is closed for $*_A$, if $\alpha \between N(U)$ and $\alpha \between N(V)$ then $\alpha \between N(U) * N(V)$, and, since α is formal closed, it follows $\alpha \between N(U * V)$.

Definition 6.11. A unary formal topology $A \in \overline{\mathscr{A}}$ is called *separating* if it has a set-indexed family of separating subsets $\{U_i\}_{i \in I}$ such that, for all $a, a' \in A$,

$$a \nsim a' \to \exists i, j(a \lhd U_i \ \& \ a' \lhd U_j \ \& \ U_i \nsim U_j). \tag{6.12}$$

Remark 29. If A is a base space, then it is separating. We pick the singletons $\{\{a\}\}_{a \ \epsilon \ A}$ as natural set-indexed family of separating subsets. Given $a, a' \ \epsilon \ A$, we satisfy (6.12) with $\{a\}, \{a'\}$.

Lemma 6.24. *If $A, B \ \epsilon \ \overline{\mathcal{A}}$, A is dense and B is separating then $C \equiv A \to B$ is separating.*

Proof. First we construct a set-indexed family of separating subsets for C, starting from the family of points $\{\alpha_a\}_{a \ \epsilon \ A}$ obtained on A by density, and the set-indexed family $\{U_i\}_{i \ \epsilon \ I_B}$ of separating subsets of B. This is indexed on $J \equiv A \times I_B$ and it is defined as follows:

$$V_{(a,i)} \equiv \{\{(a', u)\} : a' \ \epsilon \ \alpha_a, u \ \epsilon \ U_i\}.$$

for all $(a, i) \in J$. Each $V_{(a,i)}$ is separating, in fact, let $\gamma \in \mathbb{T}_C$. Since it is total, then $\gamma\alpha \in \mathbb{T}_C$ and then, by hypothesis, we have either $U_i \between \gamma\alpha$ or $(U_i \to \emptyset) \between \alpha$. In the first case, $V_{(a,i)} \between \alpha$, while in the latter $(V_{(a,i)} \to \emptyset) \between \alpha$.

Now we show that this family is making C into a separating space: let

$$c = \{(a_1, b_1), \ldots, (a_n, b_n)\}, c' = \{(a'_1, b'_1), \ldots, (a'_n, b'_n)\} \; \epsilon \; C,$$

with $c \not\uparrow c'$. Then there are some indices p, q with $a_p \uparrow a'_q$ but $b_p \not\uparrow b'_q$. Since B, is a separating space, we find $i_p, i_q \in I_B$ such that $b_p \lhd U_{i_p}$, $b_q \lhd U_{i_q}$ and $U_{i_p} \not\uparrow U_{i_q}$. It is now easy to see that $c \; \epsilon \; V_{(a_p * a'_q, i_p)}$, $c' \; \epsilon \; V_{(a_p * a'_q, i_q)}$ and $V_{(a_p * a'_q, i_p)} \not\uparrow V_{(a_p * a'_q, i_q)}$. □

We need the following slight generalization:

Lemma 6.25. *Let $A \; \epsilon \; \overline{\mathcal{A}}$ be a separating space. For all finite lists $a_1, \ldots, a_n \in A$, we can find separating subsets $U_1, \ldots, U_n \subseteq A$ such that:*

1. *for all $1 \leqslant i \leqslant n$, $a_i \lhd U_i$,*
2. *if $a_i \not\uparrow a_j$, then $U_i \not\uparrow U_j$.*

Proof. For all a_i, a_j with $a_i \not\uparrow a_j$ we have separating subsets $U_i^{ij}, U_j^{ij} \subseteq A$. We define therefore $U_i \equiv \prod_{a_i \not\uparrow a_j} U_i^{ij}$. By Remark 28, these subsets are separating and the required properties are direct to verify. □

Lemma 6.26. *If $A, B \; \epsilon \; \overline{\mathscr{A}}$, A is separating and B is dense then $C \equiv A \to B$ is dense.*

Proof. Given $c = \{(a_1, b_1), \ldots, (a_n, b_n)\} \in C$ and we find a total ideal γ extending it. Let $U_1, \ldots, U_n \subseteq A$ be separating subsets for $a_1, \ldots, a_n$ obtained by means of the previous lemma.

For each consistent (i.e. bounded) subset $V_0 \subseteq \{b_1, \ldots, b_n\}$ we find and fix, by density of B, a total ideal β_{V_0} containing it. We define an intermediate relation:

$$a \eta b \equiv a \; \epsilon \; \prod_{i=0}^{n} N(U_i) \; \& \; b \; \epsilon \; \beta_{\{b_i : a \lhd U_i\}}.$$

We make a few observations:

1. The subset $\{a_i : a \lhd U_i\}$ is consistent for all $a \; \epsilon \; \prod_{i=0}^{n} N(U_i)$, and so must be its image $\{b_i : a \lhd U_i\} \subseteq B$ through c. In fact, if $a \lhd U_i$ and $a \lhd U_j$, then $U_i \uparrow U_j$; then $a_i \not\uparrow a_j$ does not hold, because it would imply $U_i \not\uparrow U_j$. Since $\uparrow$ is decidable, we get $a_i \uparrow a_j$.

2. The relation η is consistent; in fact, let $(a, b), (a', b') \; \epsilon \; \eta$ and suppose $a \uparrow a'$. Since $a, a' \; \epsilon \; \prod_{i=0}^{n} N(U_i)$, we have[9] $a \lhd U_i$ or $a \lhd U_i \to \emptyset$, and $a' \lhd U_i$ or $a' \lhd U_i \to \emptyset$ for all $i = 1, \ldots, n$. If $a \lhd U_i$ and $a' \lhd U_i \to \emptyset$, then $a \not\uparrow a'$, a contradiction. Therefore $\{a_i : a \lhd U_i\} = \{a_i : a' \lhd U_i\}$, and $b, b' \; \epsilon \; \beta_{\{b_i : a \lhd U_i\}}$ must be consistent.

[9] For all j, we have $a \; \epsilon \; N(U_j) * S$, where $S \equiv \prod_{i \neq j} N(U_i)$, and hence $a \; \epsilon \; U_j * S$ or $a \; \epsilon \; V_j * S$.

3. The relation η is consistent with c. Let $(a,b) \,\epsilon\, \eta$, and $(a_i,b_i) \,\epsilon\, c$, for some $i = 0,\ldots,n$, with $a \uparrow a_i$. Since $a \,\epsilon\, \prod_{i=0}^n N(U_i)$, we have $a \lhd U_i$ or $a \lhd U_i \to \emptyset$. The latter case cannot occur: since $a_i \lhd U_i$, this would imply $a \not\uparrow a_i$. Thence $a_i \,\epsilon\, \{a_j : a \lhd U_j\}$ and $b, b_i \,\epsilon\, \beta_{\{b_j : a \lhd U_j\}}$ must be consistent.

4. The image of any element $a \,\epsilon\, \prod_{i=0}^n N(U_i)$ through η is a total ideal. Since each U_i is separating, if α is a total ideal, then $\alpha \between N(U_i)$ for all i, and therefore $\alpha \between \prod_{i=0}^n N(U_i)$. Therefore, the image $\eta\alpha$ of α through η contains a total ideal.

The continuous and convergent closure of $c \cup \eta$ is the total ideal we are looking for. □

Collecting the lemmas above, we can prove:

Theorem 6.27 (Density Theorem for unary formal topologies)**.** *If a family $\mathscr{A}$ consists of dense base spaces, so does $\overline{\mathscr{A}}$. In concrete terms, for each $a \,\epsilon\, A$, there is $\alpha \in \mathbb{T}_A$ such that $a \,\epsilon\, \alpha$.*

Proof. The base spaces are dense, by hypothesis, and separating. Using Lemmas 6.25 and 6.26, we can prove by simultaneous induction that each $A \,\epsilon\, \overline{\mathscr{A}}$ is both dense and separating. □

6.5.1 A Finite Version of the Density Theorem

In this section we will show that, under suitable hypotheses, as total ideal extending a given element $c \in C$, with $C \in \overline{\mathscr{A}}$, we can choose a *compact ideal*, that is, the downward closure of a special element $c' \geqslant c$.

A similar result, based on the same core idea, was obtained almost simultaneously by Basil Karádais in the setting of coherent information systems. Our interaction inspired the rest of this chapter.

The proof follows closely the pattern of the general case, and therefore some details will be omitted.

Definition 6.12. A unary formal topology $(A, \lhd_A, *, \bot_A)$ is called *good*, if for all $a \,\epsilon\, A$, $a \to \emptyset$ is a finitely generated formal open, that is, there exist $a_1,\ldots,a_n \,\epsilon\, A$ such that

$$a' \,\epsilon\, a \to \emptyset \quad \leftrightarrow \quad a' \lhd \{a_1,\ldots,a_n\}.$$

Definition 6.13. Let $A \in \overline{\mathcal{A}}$ be a unary formal topology. An element $a \,\epsilon\, A$ is called total if its downward closure $\overline{a}$ is a total ideal. We call A *finitely dense* if for each $a \,\epsilon\, A$, there is $a' \,\epsilon\, A$ total element such that $a \leqslant a'$.

In particular, every finitely dense $A \in \overline{\mathcal{A}}$ is dense.

Remark 30. We have equivalently that $A \in \overline{\mathcal{A}}$ is *finitely dense* if for each $a \,\epsilon\, A$, there is $a' \,\epsilon\, A$ such that $\overline{a'} \in \mathbb{T}_A$ and $a \uparrow a'$. In this case, $a * a'$ satisfies the previous definition.

Definition 6.14. Let $A \in \overline{\mathcal{A}}$ be a unary formal topology. A subset U is called *finitely separating* if it is finite, and there exists a finite subset $V_U \subseteq U \to \emptyset$ such that, for all $\alpha \,\epsilon\, \mathbb{T}_A$, $\alpha \between U \cup V_U$.

Remark 31. If A is a good base space and $a \,\epsilon\, A$, then every $\{a\}$ singleton is finitely separating. We can in fact define $V_{\{a\}} \equiv \{a_1, \dots, a_n\}$, where $\{a_1, \dots, a_n\}$ is a finite set of generators for $a \to \emptyset$.

Remark 32. If $U, U' \subseteq A$ are finitely separating, then also $U * U'$ is finitely separating. It is enough to define $V_{U*U'} \equiv V_U \cup V_{U'}$.

Definition 6.15. A unary formal topology $A \in \overline{\mathscr{A}}$ is called *finitely separating* if it has a set-indexed family of finitely separating subsets $\{U_i\}_{i \in I}$ such that, for all $a, a' \in A$,

$$a \nparallel a' \to \exists i, j (a \lhd U_i \;\&\; a' \lhd U_j \;\&\; U_i \nparallel U_j). \tag{6.13}$$

If A is a good base space, then it is finitely separating. The singletons $\{\{a\}\}_{a \,\epsilon\, A}$ are, as in the general case, a natural set-indexed family of separating subsets.

Lemma 6.28. *If $A, B \,\epsilon\, \overline{\mathcal{A}}$, A is finitely dense and B is finitely separating then $C \equiv A \to B$ is finitely separating.*

Proof. We construct a set-indexed family of finitely separating subsets for C, starting from the family of total elements $\{t_a\}_{a \,\epsilon\, A}$ obtained on A by density, and the set-indexed family $\{U_i\}_{i \,\epsilon\, I_B}$ of finitely separating subsets of B. This is indexed on $J \equiv A \times I_B$ and it is defined as follows:

$$U_{(a,i)} \equiv \{\{(t_a, u)\} : u \,\epsilon\, U_i\}.$$

for all $(a, i) \in J$. Each $U_{(a,i)}$ is separating, in fact, we can define the corresponding $V_{U_{(a,i)}}$ as follows

$$V_{U_{(a,i)}} \equiv \{\{(t_a, u)\} : u \,\epsilon\, V_{U_i}\}.$$

The rest of the proof follows now that of the general case. □

We can now specialize the main lemmas as follows:

Lemma 6.29. *Let $A \,\epsilon\, \overline{\mathcal{A}}$ be a finitely separating space. For all finite lists $a_1, \dots, a_n \in A$, we can find finitely separating subsets $U_1, \dots, U_n \subseteq A$ such that:*

1. *for all $1 \leqslant i \leqslant n$, $a_i \lhd U_i$,*
2. *if $a_i \nparallel a_j$, then $U_i \nparallel U_j$.*

Proof. As in the general case, for all a_i, a_j with $a_i \nparallel a_j$ we have separating subsets $U_i^{ij}, U_j^{ij} \subseteq A$. We define therefore $U_i \equiv \prod_{a_i \nparallel a_j} U_i^{ij}$. By Remark 32, these subsets are finitely separating. □

Lemma 6.30. *If $A, B \,\epsilon\, \overline{\mathcal{A}}$, A is finitely separating and B is finitely dense then $C \equiv A \to B$ is finitely dense.*

Proof. The proof is almost the same as in the general case. Given

$$c = \{(a_1, b_1), \ldots, (a_n, b_n)\} \in C,$$

we find a total relation extending it. Let $U_1, \ldots, U_n \subseteq A$ be finite separating subsets for $a_1, \ldots, a_n$ obtained by means of the previous lemma. For each consistent (i.e. bounded) subset $V_0 \subseteq \{b_1, \ldots, b_n\}$ we find and fix, by density of B, a total element b_{V_0} above it. We define the intermediate relation:

$$a\eta b \equiv a \,\epsilon\, \prod_{i=0}^{n} U_i \cup V_{U_i} \;\&\; b \,\epsilon\, b_{\{b_i : a \lhd U_i\}}.$$

Notice that the relation η defined in this way is finite. The same observations made in the proof of the general case hold for η, so that the finite relation $c \cup \eta$ is the total element required. □

We can then prove following by simultaneous induction:

Theorem 6.31 (Finite Density Theorem for good unary formal topologies)**.** *Let $\mathcal{A}$ be a family of finitely dense and good base spaces. For all $A \in \overline{\mathcal{A}}$, A is finitely dense.*

The concept of finite density is very concrete and algorithmic and no arbitrary ideals are employed in the proof, but still the concept of totality that we adopt is impredicative. In the next section, we will try to put forward a more abstract notion of totality which gives an answer to this issue.

We can apply this theorem to the case of non-flat free algebras. We have in fact

Lemma 6.32. *For all finitary algebra types ι, the unary formal topology C_ι is good.*

Proof. Let $P_1, \ldots, P_n$ be the constructors of ι, with arity m_j, for $j = 1, \ldots, n$. We prove it by induction on the structure of $a \,\epsilon\, C_\iota$. If $a = *$, then S_a is finitely generated by $N_a \equiv \{P_j \vec{*}\}_{j=1}^{n}$. If $a = P_j a_1^* \ldots a_{m_j}^*$, and we have finite subset of generators $N_{a_i^*}$ for $S_{a_i^*}$ we define

$$N_a \equiv \{P_j * \ldots b_i \ldots * : b_i \,\epsilon\, N_{a_i^*}\} \cup \{P_k \vec{*}\}_{k \neq j}.$$

It easy to see that S_a is finitely generated by N_a. □

Remark 33. In Example 6.20, we have noticed already that this does not hold in the flat case, since for instance $SS0 \to \emptyset$ in the flat algebra of natural numbers is not finitely generated.

We can now apply Theorem 6.31 to the family $\{C_\rho\}_\rho$:

Corollary 6.33 (Finite Density Theorem for non-flat free algebras)**.** *For all finitary algebra type ρ, C_ρ is finitely dense.*

In this thesis we are just focusing on the method, and therefore we just restrict to finitary algebras. The result can in fact be generalized, and will be subject to further study.

Example 6.34. We apply the density theorem in a specific case. The subset of total elements in $C_{\mathbb{N}}$ is given by

$$\{0, S0, SS0, SSS0, \dots\}$$

and is finitely dense. The topology $C_{\mathbb{N}}$ generates alone a family of unary formal topologies, called the family of *pure types*. In particular, the finite density theorem is satisfied on $C_{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$ and we can apply it to find a total element c' above

$$c \equiv \{(\underbrace{\{(S0, S*), (SS0, 0)\}}_{a1}, \underbrace{SS*}_{b1}), (\underbrace{\{(0, SSS*), (S*, S0)\}}_{a2}, \underbrace{S0}_{b2})\} \ \epsilon \ C_{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}.$$

The arguments a_1 and a_2 are incompatible, since $SS0 \uparrow S*$, but $0 \not\uparrow S0$. Then we can find two separating subsets for a_1 and a_2: on the left side, the total element $SS0$ is above both $SS0$ and $S*$. On the right side, the elements 0 and $S0$ are separated by $\{0\}$ and $\{S0\}$, where we can define $V_{\{0\}} \equiv \{S*\}$ and $V_{\{S0\}} \equiv \{0, SS*\}$. Therefore a_1 and a_2 are separated by

$$U_{a_1} \equiv \{\{(SS0, 0)\}\}, \quad U_{a_2} \equiv \{\{(SS0, S0)\}\},$$

for which we can in fact define

$$V_{U_{a_1}} \equiv \{\{(SS0, S*)\}\}, \quad V_{U_{a_2}} \equiv \{\{(SS0, 0)\}, \{(SS0, SS*)\}\},$$

The arguments of c' are in the set $(U_{a_1} \cup V_{U_{a_1}}) * (U_{a_2} \cup V_{U_{a_2}})$, that is, explicitly,

$$\{\underbrace{\{(SS0, 0)\}}_{a'_1}, \underbrace{\{(SS0, S0)\}}_{a'_2}, \underbrace{\{(SS0, SS*)\}}_{a'_3}\}.$$

We choose the values in M of a'_1, a'_2, a'_3 as follows:

$$B_{a'_1} \equiv \{b_i : a'_1 \lhd U_{a_i}\} \equiv \{SS*\}, \quad b'_1 = SS0,$$

$$B_{a'_2} \equiv \{b_i : a'_2 \lhd U_{a_i}\} \equiv \{S0\}, \quad b'_2 = S0,$$

$$B_{a'_3} \equiv \{b_i : a'_3 \lhd U_{a_i}\} \equiv \emptyset, \quad b'_3 = 0.$$

We can finally define the total element η as $\{(a'_1, b'_1), (a'_2, b'_2), (a'_3, b'_3)\}$, that is

$$\eta \equiv \{(\{(SS0, 0)\}, SS0), (\{(SS0, S0)\}, S0), (\{(SS0, SS*)\}, 0)\},$$

and then $c' = \eta \cup c$.

6.6 An Abstract Notion of Totality

Example 6.34 underlines the algorithmic and constructive nature of the finite density theorem. However, the notion of total element relies on the notion of total point defined in (6.11), that is not fully predicative, as already pointed out. More precisely, the definition of total element of a function space involves a quantification over all the total ideals of a given type, which are not proved to form a set.

The finite density theorem suggests us to restrict this quantification to the compact total ideals, since the quantification on these objects can be made constructively meaningful. In precise terms, once fixed a family $\mathscr{A} \equiv \{A_i\}_{i\in I} \equiv \{(A_i, \lhd_i, *, \bot_i)\}_{i\in I}$ of unary formal topologies, and $\overline{\mathscr{A}}$ the family it generates inductively by means of function spaces, we define inductively the collection $\mathbb{T}^{fin}_A \subseteq \mathscr{P}t(A)$ of *compactly total ideals*:

$$\begin{array}{rcl} A \in \mathscr{A} & \Rightarrow & \alpha \in \mathbb{T}^{fin}_A \equiv (\forall a \in A)(\alpha \between N(a)), \\ A \to B \in \overline{\mathscr{A}} & \Rightarrow & \gamma \in \mathbb{T}^{fin}_{A\to B} \equiv \forall a(a \,\epsilon\, M_A \to \gamma\overline{a} \in \mathbb{T}^{fin}_B), \end{array} \tag{6.14}$$

where we denote by M_A the subset of *compactly total elements* of $A \in \overline{\mathscr{A}}$, that is, the elements whose deductive closure is a compactly total ideal.

Unfortunately, but not surprisingly, this does not lead to the same notion of totality for general ideals: consider the collection of pairs $(a, 0)$ where a is a total element in $C_{\mathbb{N}\to\mathbb{N}}$. The deductive closure of this collection forms an ideal of $C_{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$ which clearly sends every compact total ideal to the total ideal $\{0, *\}$. However, it is not total, since it is undefined, for example, on the identity $id : \mathbb{N} \to \mathbb{N}$.

This notion of compact totality coincide instead on compact total ideals. In plain terms, an element is total (i.e. its deductive closure is a total ideal) if and only if it is compactly total. The proof of this fact is not direct and will engage us until the end of this chapter.

Remark 34. For the sake of clarity, we start by making a brief analysis of the structure of total and compactly total elements. Let

$$c \equiv \{(a_1, b_1), \ldots, (a_n, b_n)\} \,\epsilon\, A \to B$$

be a total element, with $A, B \in \overline{\mathscr{A}}$. First, since c is a finite list, the image of a total ideal α through c is a finite subset $c\alpha$, whose closure is by hypothesis a total ideal. In particular, this ideal is compact, generated by the (total) element $\mathsf{sup}(c\alpha)$. If we take the convergent closure $\tilde{c}$ of c, then all the total elements of the form $\mathsf{sup}(c\alpha)$ appear as second component of some pair in $\tilde{c}$. Then, we can get rid of all the pairs of $\tilde{c}$ whose second component is not a total element and still remain with a total element. The same argument and observation hold if we replace the word "total" with "compactly total".

Secondly, since $\overline{c}$ is a total ideal, then $c\alpha$ must be inhabited for all total ideals α, and in particular[10]

$$\forall \alpha \in \mathbb{T}_A(\alpha \between \{a_1, \dots, a_n\}).$$

Similarly, if $\overline{c}$ is compactly total ideal, we must have $\overline{a} \between \{a_1, \dots, a_n\}$ for all $a \in M_A$. This can be rephrased as

$$M_A \lhd \{a_1, \dots, a_n\}.$$

As a consequence, for every compactly total element c in a function space, there is a compactly total element c' such that $c \uparrow c'$ and c' is of the form $\{(a'_1, m_1), \dots, (a'_n, m_n)\}$ with $m_1, \dots, m_n$ total and $M_A \lhd c'^{-} M_B$. Viceversa, each c' such that $M_A \lhd c'^{-} M_B$ is compactly total. Following this remark, we are motivated to focus just on compactly total elements of this form.

We rephrase[11] finite totality in an abstract way, namely, independent from a generating family of objects. This is inspired by Berger's notion of abstract totality [Ber93b] and Normann's notion of domain with totality [Nor00a]. In this way, we are also able to stress the duality occuring between the concepts of dense space and separating space. In the next section, we will show the connections with the usual treatment.

In what follows, it is convenient to denote the unary formal topologies by referring to the corresponding order structure $(A, \geqslant, *, \uparrow, \perp_A)$. We assume also that the underlying set is discrete, that is, with decidable equality. We say that a subset $U \subseteq A$ is *fully inconsistent* if $u \not\uparrow u'$ for all $u, u' \in U$ such that $u \neq u'$.

We study special kinds of unary formal topology which carry two distinguished subset of, respectively, *(abstractly) total elements* $M_A \subseteq A$ (a *totality* on A) and of *special elements* $S_A \subseteq A$ (a *speciality* on A). These two subsets are related by the following relations:

(P_S) For all $s \in S_A$, we can find a fully inconsistent subset $U_s \subseteq S_A$ such that $s \in U_s$ and $M_A \lhd U_s$[12], i.e. $(\forall m \in M_A)(\exists s' \in U_s)(s' \leqslant m)$.

(P_M) For all $m \in M_A$, we can find a fully inconsistent subsets $V_m \subseteq M_A$ such that $m \in V_m$ and $S_A \lhd_\uparrow V_m$, that is, $(\forall s \in S_A)(\exists m' \in V_m)(s \uparrow m')$.

The property P_S says that the total elements must be, in a certain sense, maximal with respect to the S_A. More precisely, the following holds:

Proposition 6.35. *For all $s \in S_A$ and $m \in M_A$, if $s \uparrow m$ then $s \leqslant m$.*

Proof. Let $s \in S_A$ and $m \in M_A$ such that $s \uparrow m$. Thanks to (P_S), either we have $s \leqslant m$, or there is $u \in U_s$ finite such that $u \leqslant m$ and $s \not\uparrow u$; the latter leads to $s \not\uparrow m$, a contradiction. □

[10] Following an idea of Basil Karádais, we say in this case that the subset $\{a_1, \dots, a_n\}$ is *supportive*. We will introduce this notion properly in Section 6.7.

[11] And slightly restrict.

[12] The property (P_S) says exactly that the formal topology obtained by restricting the localization of the Scott topology on the total elements to the special elements has Krull dimension smaller or equal to zero.

In the light of the previous proposition, we can rewrite the property P_M as follows:

(P'_M) For all $m \,\epsilon\, M_A$, we can find a fully inconsistent subset $V_m \subseteq M_A$ such that $m \,\epsilon\, V_m$ and $(\forall s \,\epsilon\, S_A)(\exists m' \,\epsilon\, V_m)(s \leqslant m')$.

In particular, the total elements must be enough to have all the special elements below them.

In order to prove the density theorem in a finite way, we require a stronger and finite version of (P_S). We distinguish therefore

($P_{S,fin}$) For all $s \,\epsilon\, S_A$, we can find a fully inconsistent *finite* subset $U_s \subseteq S_A$ such that $s \,\epsilon\, U_s$ and $M_A \lhd U_s$, i.e. $(\forall m \,\epsilon\, M_A)(\exists s' \,\epsilon\, U_s)(s' \leqslant m)$.

We decided here to keep the treatment more general and symmetric, and to add this requirement when explicitly needed. As in the previous chapter, we assume the unary formal topologies $(A, \geqslant, *, \uparrow, \bot_A)$ of this section to be coherent. This hypothesis is satisfied by the examples of unary formal topology that we are going to discuss and they make the symmetry between dense and separating space more evident. They are nevertheless not essential to prove the main theorem of this section.

If we have "enough" special and total elements, the space will be called, respectively, (abstractly) *separating*, *dense* or *strong*:

Definition 6.16. Let $(A, \geqslant, *, \uparrow, \bot_A)$ be a unary formal topology with finitary operation, endowed with a totality M_A and a speciality S_A. We will say that A is:

1. *separating* if for all $a, a' \,\epsilon\, A$, we have
$$a \not\uparrow a' \rightarrow (\exists s_a, s_{a'} \,\epsilon\, S_A)((s_a \leqslant a \;\&\; s_{a'} \leqslant a') \;\&\; s_a \not\uparrow s_{a'}). \tag{6.15}$$
2. *dense* if for all $a, a' \,\epsilon\, A$, the following holds
$$a \uparrow a' \rightarrow (\exists m_a, m_{a'} \,\epsilon\, M_A)((a \uparrow m_a \;\&\; a' \uparrow m_{a'}) \;\&\; m_a \uparrow m_{a'}) \tag{6.16}$$
3. *strong* (or *with strong totality*) if for all $a, a' \,\epsilon\, A$, the following holds
$$a \uparrow a' \leftarrow (\exists m_a, m_{a'} \,\epsilon\, M_A)((a \uparrow m_a \;\&\; a' \uparrow m_{a'}) \;\&\; m_a \uparrow m_{a'}) \tag{6.17}$$

The converse implication of (6.15) is trivial, while in the case of density, one distinguishes the notion of strong space [Ber02, Ber93b, Nor00a]. Whenever the notion of totality we are using is closed by suprema, (6.16) rewrites as
$$a \uparrow a' \leftrightarrow (\exists m_a \,\epsilon\, M_A)(m_a \uparrow a \;\&\; m_a \uparrow a') \tag{6.18}$$
which, for $a' = a$, becomes
$$(\exists m_a \,\epsilon\, M_A)(a \uparrow m_a).$$

When the notion of totality is closed upwards for $\leqslant$, by taking $m'_a \equiv m_a * a \ \epsilon\ M_A$, this is equivalent to

$$(\exists m'_a \ \epsilon\ M_A)(a \leqslant m'_a).$$

for all $a \ \epsilon\ A$, precisely the usual way of stating density.

The alternative definition we gave has the benefit of showing the existing symmetry between this notion and that of separating space. In particular, one obtains the notion of dense space from that of separating space by replacing special elements with total elements, $\uparrow$ with $\not\uparrow$, and $\leqslant$ with $\uparrow$. Here follows the contrapositive of the formulas (6.15) and (6.16), which may let notice the symmetry even better:

$$a \uparrow a' \leftrightarrow (\forall s_a, s_{a'} \ \epsilon\ S_A)((s_a \leqslant a \ \&\ s_{a'} \leqslant a') \rightarrow s_a \uparrow s_{a'}),$$

$$a \not\uparrow a' \leftrightarrow (\forall m_a, m_{a'} \ \epsilon\ M_A)((a \uparrow m_a \ \&\ a' \uparrow m_{a'}) \rightarrow m_a \not\uparrow m_{a'}).$$

Notice that the properties (P_S) and (P_M) relates just the total elements with the special elements, while (6.16) (resp. (6.15)) relates the total (resp. special) elements with the general elements of the space. It looks as total elements and special elements could be carried on as independent notion, but they have to mix when we introduce function spaces, because of the contravariance. The *density theorem* will show how these properties together can be lifted to function spaces.

The following lemma will be needed in the following. It is not very important in our coherent setting, but if we extend our notion to a non-coherent setting, the following equivalences work only from right to left and we have to pick the right side as definition of the corresponding notions.

Lemma 6.36. *Let A be a unary formal topology with totality and speciality. Then:*

1. *A is separating if and only if, for all $a_1, \ldots, a_n$, we can find $s_{a_1}, \ldots, s_{a_n} \ \epsilon\ S_A$, with $s_{a_i} \leqslant a_i$ and if $a_i \not\uparrow a_j$ then $s_{a_i} \not\uparrow s_{a_j}$;*

2. *A is dense if and only if, for all $a_1, \ldots, a_n$, such that $\{a_1, \ldots, a_n\}$ is consistent, then there exists $m_A \ \epsilon\ M_A$ such that $m_A \uparrow a_i$ for all i;*

3. *A is strong if and only if, for all $a_1, \ldots, a_n$, and $m_A \ \epsilon\ M_A$ is such that $m_A \uparrow a_i$ for all i, then $\{a_1, \ldots, a_n\}$ is consistent.*

Proof. They are all direct and we show just $\rightarrow$ in 1. For all i, j, we can find $s_{ij}, s_{ji} \ \epsilon\ S_A$ such that $s_{ij} \leqslant a_i$, $s_{ji} \leqslant a_j$ and $s_{ij} \not\uparrow s_{ij}$ whenever $a_i \not\uparrow a_j$. Then we define $s_{a_i} \equiv \prod_j s_{ij}$ for all i and the requested properties can be easily verified. □

Given

$$(A, \geqslant, \uparrow, *, \perp, M_A, S_A), \qquad (B, \geqslant, \uparrow, *, \perp, M_B, S_B)$$

unary formal topologies with totality and speciality, we can endow the usual space of function $(C, \geqslant, \uparrow, *, \perp)$, where $C \equiv A \rightarrow B$ with a totality M_C and a speciality S_C defined as follows:

1. $c \,\epsilon\, M_C \equiv c \subseteq S_A \times M_B$, $M_A \lhd c^- M_B$ and $\pi_A(c)$ fully inconsistent; in other words, c is total[13]:
   (a) if it is made of pairs of the form (s_a, m_b) with s_A special and m_B total;
   (b) for all $m_A \,\epsilon\, M_A$ there exist $m_B \,\epsilon\, M_B$ and $s_a \leqslant m_A$ such that $(s_a, m_B) \,\epsilon\, c$;
   (c) if $(s_A, m_B), (s'_A, m'_B) \,\epsilon\, c$ and they are distinct, then $s_A \not\uparrow s'_A$.
2. $c \,\epsilon\, S_C \equiv c \subseteq M_A \times S_B$ and $\pi_A(c)$ fully inconsistent.

Remark 35. The hypothesis of "full inconsistence" both for $\pi_A(c)$ here and for the properties (P_S) and (P_M) is not really necessary, and all of the reasoning could be carried out without this assumption. It makes anyway the proofs shorter and cleaner, the whole discussion more symmetric and it is satisfied by the applications.

We show now that $(C, \geqslant, \uparrow, *, \perp, M_C, S_C)$ is a unary formal topology with totality and speciality. This amounts to show the properties P_M and P_S:

(P_S) We start by proving it for $c = \{(m_A, s_B)\}$. Since $s_B \,\epsilon\, S_B$, we can find $U_{s_B} \subseteq S_B$ finite and fully inconsistent such that $M_B \lhd \{s_B\} \cup U_{s_B}$ and $s_B \,\epsilon\, U_{s_B}$. We define then

$$U_c \equiv \{\{(m_A, b')\}\}_{b' \,\epsilon\, U_{s_B}}.$$

It is clearly finite, fully inconsistent, and $U_c \subseteq S_C$. If now $m_C \,\epsilon\, M_C$, we can find $(a, m_B) \,\epsilon\, m_C$, with $m_B \,\epsilon\, M_B$, such that $a \leqslant m_A$. By hypothesis we have $m_b \geqslant b'$ for some $b' \,\epsilon\, U_{s_B}$; for such b', we have $(m_A, b') \leqslant (a, m_B) \,\epsilon\, c$.

If c contains more pairs, then we can define

$$U_c \equiv \prod_{(m_i, s_i) \,\epsilon\, c} U_{\{(m_i, s_i)\}}.$$

(P_M) Let $m_C \,\epsilon\, M_C$; for any $(s_A, m_B) \,\epsilon\, m_C$, there is $V_{m_B} \subseteq M_B$ fully inconsistent such that $m_B \,\epsilon\, V_{m_B}$ and for all $s_B \,\epsilon\, S_B$, there is $m'_B \,\epsilon\, V_{m_B}$ such that $s_B \leqslant m'_B$. We define therefore

$$c \,\epsilon\, V_{m_C} \equiv c \,\epsilon\, \prod_{(s_A, m_B) \,\epsilon\, m_C} \{\{(s_A, m'_B)\}\}_{m'_B \,\epsilon\, V_{m_B}}.$$

It is clearly fully inconsistent, contains m_C and it is made of total elements, because each element has values in M_B and the same domain as m_C. Consider now

$$c = \{(m_1, s_1), \ldots, (m_n, s_n)\} \,\epsilon\, S_C.$$

[13] The notion of totality here presented is a bit more restrictive than the usual one, but the corresponding density theorem will in in fact show that for any m total in the usual sense, there is m_c total such that $m \sim m_c \equiv m \uparrow m_c$.

Since m_C is total, for each i, we can find $(s_{i,A}, m_{i,B}) \; \epsilon \; m_C$ with $s_{i,A} \leqslant m_i$ and $m_{i,B} \; \epsilon \; V_{m_{i,B}}$ such that $s_i \leqslant m_{i,B}$. We define then

$$c' \equiv \{(s_{1,A}, m_{1,B}), \ldots, (s_{n,A}, m_{n,B})\}.$$

This is clearly consistent, since the argument are fully inconsistent, and clearly $c' \; \epsilon \; V_{m_C}$. Since $(m_i, s_i) \leqslant (s_{i,A}, m_{i,B})$ for all i, we have also $c \leqslant c'$.

We fix

$$(A, \geqslant, \uparrow, *, \perp, M_A, S_A), \qquad (B, \geqslant, \uparrow, *, \perp, M_B, S_B)$$

unary formal topologies with totality and speciality and the corresponding space of functions $(C, \geqslant, \uparrow, *, \perp, M_C, S_C)$.

Lemma 6.37. *If A is dense unary formal topology and B is a strong unary formal topology, then $C \equiv A \to B$ is a strong unary formal topology.*

Proof. Let $m_C \; \epsilon \; M_C$ and $c, c' \; \epsilon \; C$ such that $m_C \uparrow c$ and $m_C \uparrow c'$. Let also $(a, b) \; \epsilon \; c$, $(a', b') \; \epsilon \; c'$ such that $a \uparrow a'$, and we show that $b \uparrow b'$. Since A is dense there is $m_A \; \epsilon \; M_A$ such that $a \uparrow m_A$ and $a' \uparrow m_A$, and since m_C is total, there is $s_A \; \epsilon \; S_A$ such that $s_A \leqslant m_A$ and $m_B \; \epsilon \; M_B$ such that $(s_A, m_B) \; \epsilon \; m_C$. Since $m_C \uparrow c$, and $s_A \uparrow a$, we have $m_B \uparrow b$ and similarly $m_B \uparrow b'$. Since B is strong, we get $b \uparrow b'$. □

We are now ready to prove the density theorem in this setting. This is a form of finite density, and we must suppose here that the property $P_{S,fin}$ holds in the unary formal topology A. The proof run as in the general case, by mutual induction.

Lemma 6.38. *If A is separating unary formal topology and B is a dense unary formal topology, then $C \equiv A \to B$ is a dense unary formal topology.*

Proof. Let $c \equiv \{(a_1, b_1), \ldots, (a_n, b_n)\} \; \epsilon \; C$. Thanks to separation on A, for each a_i, we can find $s_{a_i} \; \epsilon \; S_A$ such that $s_{a_i} \leqslant a_i$, and such that $s_{a_i} \not\uparrow s_{a_j}$ whenever $a_i \not\uparrow a_j$. For each s_{a_i}, we can find $U_{s_{a_i}} \subseteq S_A$ *finite* and fully inconsistent such that $s_{a_i} \; \epsilon \; U_{s_{a_i}}$. Since B is dense, for each consistent finite subset $V_0 \subseteq \{b_1, \ldots, b_n\}$ we choose an element $m_{V_0} \; \epsilon \; M_B$ with $m_{V_0} \uparrow b_1, \ldots, m_{V_0} \uparrow b_n$. We define therefore $m_C \; \epsilon \; M_C$ as follows:

$$a m_C b \equiv a \; \epsilon \; \prod_i U_{s_{a_i}} \;\&\; b = m_{\{b_i : s_{a_i} \leqslant a\}}.$$

Notice that $m_C \subseteq S_A \times M_B$, and $\pi_A(m_C)$ is fully inconsistent, so that m_C is consistent, and total, because $M_A \lhd \prod_i U_{s_{a_i}}$. It is well defined, in fact, $\{b_i : s_{a_i} \leqslant a\}$ is consistent: if $s_{a_i} \leqslant a$ and $s_{a_j} \leqslant a$, then $s_{a_i} \uparrow s_{a_j}$ and therefore $a_i \uparrow a_j$, so that $b_i \uparrow b_j$ because c is consistent. Moreover, $m_C \uparrow c$: if $(a, b) \; \epsilon \; m_C$ and $a \uparrow a_i$, this means $s_{a_i} \leqslant a$ and therefore $b \equiv m_{\{b_j : s_{a_j} \leqslant a\}} \uparrow b_i$. □

Lemma 6.39. *If A is a dense unary formal topology and B is a separating unary formal topology, then C is a separating unary formal topology.*

Proof. Let $c, c' \epsilon C$ such that $c \not\uparrow c'$. Then there are $(a, b) \epsilon c$, $(a', b') \epsilon c'$ such that $a \uparrow a'$ but $b \not\uparrow b'$. Since A is dense, we can find $m_A \epsilon M_A$ such that $m_A \uparrow a$ and $m_A \uparrow a'$. Since B is separating, we can find $s_B, s'_B \epsilon S_B$, such that $s_B \leqslant b$, $s'_B \leqslant b'$ and $s_B \not\uparrow s'_B$. Consider therefore

$$\{(m_A, s_B)\}, \{(m_A, s'_B)\} \epsilon S_C.$$

We have

$$(m_A, s_B) \leqslant (a, b) \epsilon c \quad \text{and} \quad (m_A, s_B) \leqslant (a', b') \epsilon c'$$

and $\{(m_A, s_B)\} \not\uparrow \{(m_A, s'_B)\}$. □

By composing the two lemmas above, we finally get:

Theorem 6.40 (Abstract Density)**.** *If A and B are both (abstractly) separating and (abstractly) dense unary formal topologies then so is the corresponding space of function $A \to B$.*

6.6.1 Application to Finitary Algebras

In Section 6.4, we have shown how to associate to every finitary algebra type ι a unary formal topology $(C_\iota, \leqslant_\iota, *)$. We can equip each of these topologies with a totality M_ι and a speciality S_ι:

(M_ι) $a = Ca_1^* \dots a_n^* \epsilon M_\iota$ if and only if every a_1^* is a total token.

(S_ι) All the tokens are special, i.e. $S_\iota \equiv A$.

This is a good choice, and we can prove that $(C_\iota, \leqslant_\iota, *, \uparrow, *_\iota)$ satisfies $P_{S,fin}$ and P_M:

($P_{S_\iota,fin}$) All the tokens in M_ι are maximal, so we just have to prove that for an arbitrary token $a = Ca_1^* \dots a_n^* \epsilon S_\iota \equiv A$ we can produce a finite and fully inconsistent subset U_a such that $a \epsilon U_a$ and $M_A \lhd U_a$. Let $P_1, \dots, P_n$ be the constructors of ι, with arity m_j, for $j = 1, \dots, n$. We prove it by induction on the structure of $a \epsilon C_\iota$. If $a = *$, then we take $U_a \equiv \{*_\iota\}$. If $a = P_j a_1^* \dots a_{m_j}^*$, and we have finite and fully inconsistent subset $U_{a_i^*}$ for all i, we define

$$U_a \equiv \bigcup_j \{P_j b_1 \dots b_n : b_i \epsilon U_{a_i^*}\}.$$

The subset U_a is clearly fully inconsistent and finite.

Let now $m \epsilon M_\iota$ be total and we reason by induction on the structure of m. If m is a nullary constructor, then $m \epsilon U_a$. Suppose then $m = P_i a'_1 \dots a'_{n_i}$ for some P_i and $a'_j \epsilon M_\iota$; by inductive hypothesis, we have $b'_j \leqslant a'_j$ for some $b'_j \epsilon U_{a'_j}$. Then $P_i b'_1 \dots b'_n \epsilon U_a$ and $P_i b'_1 \dots b'_n \leqslant P_i a'_1 \dots a'_{n_i} = m$.

(P_{M_ι}) Every token is smaller or equal to a total token, and the set of all the total tokens is fully inconsistent.

Lemma 6.41. *For all finitary algebra type ι, C_ι is dense (1), strong (2) and separating (3).*

Proof. (1) Every token can be extended to a total token, which is fact consistent to it. (2) Suppose $a, a' \in C_\iota$, $m \mathrel{\epsilon} M_\iota$, $a \uparrow m$ and $a' \uparrow m$. Then, since m is maximal, we have $a \leqslant m$ and $a' \mathrel{\epsilon} m$, so that $a \uparrow a'$. (3) Suppose $a \not\uparrow a'$ in C_ι. Since $a, a' \mathrel{\epsilon} S_\iota$, a, a' satisfy trivially the right side. □

For $\rho \to \sigma$, we define recursively $C_{\rho\to\sigma} \equiv C_\rho \to C_\sigma$ and we endow it with the totality and the speciality canonically associated to function spaces.

We are in the hypothesis of Lemma 6.37 and Theorem 6.40, with $\mathscr{A} \equiv \{C_\iota : \iota \in Alg(\vec{\alpha})\}$, and it follows:

Theorem 6.42 (Abstract Density Theorem, Non-Flat case)**.** *For all finitary algebra types ρ, C_ρ is (abstract) dense, strong and separating.*

The notion of totality we have put forward here is constructively acceptable, even if rather abstract. In the next section, we are going to show that this is in fact classically equivalent for the family of formal topologies $\{C_\rho\}_\rho$.

6.7 Comparing Classical and Abstract Totality

In this final section, we show the link between the notions of total element, discussed in Section 6.5.1, and that of abstract total element, introduced in Section 6.6.

Let $(A, \geqslant, \uparrow, *, \bot)$ be a unary formal topology. We recall that this is called *good* if for all $a \mathrel{\epsilon} A$, we can find a finite subset of generators $G_a \equiv \{a_1, \dots, a_n\}$ for $a \to \emptyset$. Starting from a family of good unary formal topologies, we were able to prove a finite version of the density theorem. We will consider a little strengthening of this notion:

Definition 6.17. A unary formal topology $(A, \geqslant, \uparrow, *, \bot)$ is called *very good* if

1. $(A, \geqslant)$ is a partial order.
2. For all $a \mathrel{\epsilon} A$, we can find a *fully inconsistent* finite subset of generators $G_a \equiv \{a_1, \dots, a_n\}$ for $a \to \emptyset$.
3. Every $a \mathrel{\epsilon} A$ is below a maximal element m_a.

In particular, every very good unary formal topology is good. If $(A, \geqslant, \uparrow, *, \bot)$ is very good, then we can associate to it naturally a totality M_A and a speciality S_A: we define $a \mathrel{\epsilon} M_A$ if a is maximal, and $S_A = A$. We can in fact prove the conditions $(P_{S_A,fin})$, (P_{M_A}):

$(P_{S_A,fin})$ If $a \in A$, we define $U_a \equiv \{a\} \cup G_a$, where G_a is a finite and fully inconsistent subset of generators of $a \to \emptyset$. Then U_a is fully inconsistent and generates $N(a)$. Since $M_A \lhd N(a)$, we have $M_A \lhd U_a$.

(P_{M_A}) If m_a is maximal, then we define

$$V_{m_a} \equiv \{m_a\} \cup \{m \ \epsilon \ M_A : m \not\uparrow m_a\}.$$

It is fully inconsistent, since two consistent maximal elements have to coincide in a partial order. Since A is very good, if $s \ \epsilon \ S_A \equiv A$, then we can find $m_s \ \epsilon \ M_A$ such that $s \leqslant m_s$. If $m_s \uparrow m_a$ then $m_s \leqslant m_a$, so that $s \leqslant m_a$. If $m_s \uparrow m_a$, then $m_s \ \epsilon \ V_{m_a}$. In both cases $s \lhd V_{m_a}$.

Example 6.43. The example to keep in mind is that of finitary algebras, discussed throughout this text. For all finitary algebra type ι, C_ι is very good. Because of these, each C_ι has a totality M_ι and a speciality S_ι, which coincide with those defined in the Section 6.6.1.

We fix a family $\mathscr{A} \equiv \{A_i\}_{i \in I} \equiv \{(A_i, \lhd_i, \uparrow, *, \perp_i)\}_{i \in I}$ of very good unary formal topologies, an let $\overline{\mathscr{A}}$ be the family generated from it by means of function spaces.

As discussed a few lines above, each of the base spaces A_i has a natural (abstract) totality M_{A_i} and speciality S_{A_i}. As a consequence, for all $A \in \overline{\mathscr{A}}$, we have a totality M_A and a speciality S_A defined on A.

For all $A \in \mathscr{A}$, the total and the abstractly total elements coincide with the maximal elements and therefore $M_A = T_{A,fin}$. We distinguish this as a property for each space $A \in \overline{\mathscr{A}}$:

Definition 6.18. We say that $A \in \overline{\mathscr{A}}$ is said to be *predicatively total* if its abstractly total elements are total.

As said above, every base space is predicatively total. We want to prove that each $A \in \overline{\mathscr{A}}$ is so. The proof will be done by mutual induction, as for the density theorem. We introduce therefore a dual notion.

Definition 6.19. Let $A \in \overline{\mathscr{A}}$. A subset $U \subseteq A$ is said to be *supportive* if

$$(\forall \alpha \in_c \mathbb{T}_A)(U \between \alpha)$$

and *abstractly supportive* if $M_A \lhd U$. We say that a unary formal topology $A \in \overline{\mathscr{A}}$ is *predicatively supportive*, if every abstractly supportive *finite* subset in S_A is supportive.

In other words, a subset U is supportive if $\mathsf{Ext}(U)$ contains all the total ideals. For instance, a subset $U \subseteq A$ is separating if and only if $N(U)$ is supportive.

Notice that the base spaces in $\mathscr{A}$ are predicatively total: in fact, every total ideal α must contain a total element, that is $M_A \between \alpha$. If $M_A \lhd U$, then also $U \between \alpha$. In all, base spaces are both predicatively total and predicatively supportive. We lift now these properties to function spaces:

Lemma 6.44. *If $A, B \ \epsilon \ \overline{\mathcal{A}}$, A is predicatively supportive and B is predicatively total then $C \equiv A \to B$ is predicatively total.*

Proof. Let $c \equiv \{(s_1, m_1), \dots, (s_n, m_n)\} \,\epsilon\, M_C$. Since B is predicatively total, $\overline{m}_i$ is a total ideal for all $i = 1, \dots, n$. Since A is predicatively supportive, and $M_A \lhd \{s_1, \dots, s_n\}$, then $\alpha \between \{s_1, \dots, s_n\}$ for all $\alpha \in_c \mathbb{T}_A$. Hence, if $\alpha \in_c \mathbb{T}_A$, $\overline{c}\alpha$ must contain a total ideal $\overline{m}_i$ for some i, and by Proposition 6.22 is itself total. □

Remark 36. Let $(A, \lhd, *)$ be a unary formal topology. The operation $*$ distribute "topologically" with respect to the union $\cup$. In other words, for all $U, U', V \subseteq A$, we have

$$(U \cup U') * V =_{\lhd} (U * V) \cup (U' * V),$$

In fact, since $\lhd$ is unary, we have $\mathscr{A}(U \cup U') = \mathscr{A}(U) \cup \mathscr{A}(U')$. Hence, we have

$$(U \cup U') * V =_{\lhd} \mathscr{A}((U \cup U') * V) = (\mathscr{A}(U) \cup \mathscr{A}(U')) \cap \mathscr{A}(V).$$

The right hand side is then equal to

$$(\mathscr{A}(U) \cap \mathscr{A}(V)) \cup (\mathscr{A}(U') \cap \mathscr{A}(V)),$$

and thus

$$(\mathscr{A}(U * V)) \cup (\mathscr{A}(U' * V)) = \mathscr{A}((U * V) \cup (U' * V)),$$

that is topologically equivalent to $(U * V) \cup (U' * V)$.

Lemma 6.45 (CL). *If $A, B \,\epsilon\, \overline{\mathcal{A}}$, A is predicatively total and B is predicatively supportive then $C \equiv A \to B$ is predicatively supportive.*

Proof. We have to show that if $U \subseteq S_C$ is finite and $M_A \lhd U$ then $U \between \gamma$ for all $\gamma \in_c \mathbb{T}_C$. We show it in three steps:

1. We suppose at first that U is of the form $\{\{(m, s_i)\}\}_{i=0}^n$. In this case, the subset $\{s_1, \dots, s_n\} \subseteq S_B$ must be abstractly supportive: let $m_B \,\epsilon\, M_B$, and $c \,\epsilon\, M_C$ extending $\{(m, m_B)\}$, obtained by means of the density theorem. In particular, we can suppose[14] $(s, m_B) \,\epsilon\, c$ for some $s \,\epsilon\, S_A$ such that $s \leqslant m$, and if $(s', m'_B) \,\epsilon\, c$, then $s \nUparrow s'$. Hence, $c \lhd U$ can happen if and only if there is i such that $\{(m, s_i)\} \leqslant (s, m_B)$. For such i, we have in particular $s_i \leqslant m_B$; since m_B is arbitrary, we have proved $m_B \lhd \{s_1, \dots, s_n\}$, that is, $\{s_1, \dots, s_n\}$ is abstractly supportive.

Since B is predicatively supportive, then $\{s_1, \dots, s_n\}$ is supportive, and, since A is predicatively total, $\overline{m}$ is a total ideal. Hence, if $\gamma \in_c \mathbb{T}_C$, then $\gamma\overline{m} \between \{s_1, \dots, s_n\}$. This implies $U \between \gamma$.

2. We suppose now that $U \subseteq S_C$ is of the form $\{\{(m_i, s_i)\}\}_{i=0}^n$. In other terms, U is a finite union of subsets of the previous kind

$$\{(m_1, s_{1j})\}\}_{j=1}^{n_1} \cup \dots, \cup \{(m_p, s_{pj})\}\}_{j=1}^{n_p},$$

where we recall that $m_i \nUparrow m_j$ if $i \neq j$. By the density theorem, for any p-tuple $m_1^B, \dots, m_p^B \,\epsilon\, M_B$ we can find:

[14] By exploiting fully the argument contained of the proof of the (abstract) density theorem.

1. $s_1^A, \ldots, s_n^A \in S_A$ such that $s_i^A \leqslant m_i$ for all $i = 1, \ldots, p$.
2. a total ideal $c \in M_C$ such that $(s_i^A, m_i^B) \in c$ for all $i = 1, \ldots, p$.

Since $c \lhd U$, there must be $i = 1, \ldots, p$ such that $m_i^B \lhd \{s_{i1}, \ldots, s_{in_i}\}$. Here we have to reason classically: since the total elements $m_1^B, \ldots, m_p^B \in M_B$ were chosen arbitrarily, for at least an $i = 1, \ldots, p$ the subset $\{s_{i1}, \ldots, s_{in_i}\}$ must be abstractly supportive, and therefore supportive. In other words, U contains a subset of the kind treated in the previous step and hence it is supportive.

3. Let's consider the general case. We pick $U \equiv \{c_1, \ldots, c_n\} \subseteq S_C$, where

$$\begin{aligned} c_1 &= \{(m_{11}, s_{11}), \ldots, (m_{1p_1}, s_{1p_1})\} \\ c_2 &= \{(m_{21}, s_{21}), \ldots, (m_{2p_2}, s_{2p_2})\} \\ &\vdots \\ c_n &= \{(m_{n1}, s_{n1}), \ldots, (m_{np_n}, s_{np_n})\} \end{aligned}$$

This step is of purely set-theoretic nature. Notice that we have

$$c_i = \{(m_{i1}, s_{i1})\} * \ldots * \{(m_{ip}, s_{ip})\}$$

for all i. In particular, Since $*$ and $\cup$ distribute topologically (see Remark 36), and we have

$$U \equiv \bigcup_i \{c_i\} = \bigcup_i \prod_{j=0}^{p_i} \{(m_{ij}, s_{ij})\} =_{\lhd} \prod_k U_k,$$

where U_k is a union of singletons of the form $\{(m, s)\}$, as in the step 2. In particular, if $M_C \lhd U$, then $M_C \lhd U_k$ for all k, that is, each U_k is abstractly supportive. By the previous step, each U_k is supportive. This means $\gamma \between U_k$ for all k and $\gamma \in_c \mathbb{T}_C$. This implies $\prod U_k \between \gamma$,and then $U \between \gamma$ for all $\gamma \in_c \mathbb{T}_C$. □

The use of classical logic can be avoided with proper decidability conditions on the base elements. By mutual induction, we can finally prove:

Theorem 6.46 (CL). *Every unary formal topology $A \in \overline{\mathcal{A}}$ is predicatively supportive and predicatively total.*

The notion of abstract totality we have described in the previous chapter is therefore stronger, at least classically, than the usual one, once we consider suitable formal topologies as base spaces. It is strictly stronger, because we asked a total element to have a particular stucture, that is, domain composed of fully inconsistent special elements. Both these two hypotheses could in fact be omitted, and the theorem above could have been proved in higher generality. This has not been done, essentially for three reasons:

1. The notion of total element is still wide enough to prove the density theorem. In particular, for all m compactly total, we can find by density an abstract total element m' such that $m \uparrow m'$.

2. The introduction of special elements make possible to shorten the proof considerably (informally speaking, a half), and let hidden symmetries be visible.

3. The example of non-flat free algebras satisfies all the additional hypotheses.

Special elements where introduced quite blindly, while looking for a proof of Lemma 6.45, where a canonical structure for a total element was needed; a better term-theoretic or domain-theoretic motivation has to be found.

Finally, the requirement of having a unary operation, instead of a general finitary one, is redundant for the proof of the finite density theorem[15]. In other words, we could have stated and proved a version of the density theorem for unary formal topologies with finitary operation. These objects are more general than information systems and thus can be related to non-deterministic computation. It would be worthwhile to undertake further studies on the subject.

[15] In this case, the function space has to be defined in a slightly different way.

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Eidesstattliche Versicherung

Hermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

München, 24.06.2014

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