Generalisations of Holographic Hydrodynamics

anomalous transport & fermionic universality

Dissertation by Stephan Steinfurt

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Zusammenfassung

In der vorliegenden Dissertation werden Eigenschaften stark gekoppelter hydrodynamischer Theorien untersucht, die mittels einer dualen Beschreibung als höherdimensionale gravitative Systeme aufgefasst werden können. Besonderes Augenmerk liegt hierbei auf der Berechnung physikalischer Größen wie Viskositäten oder Diffusionskonstanten. Diese werden hinsichtlich der Frage betrachtet, ob sie allgemeingültigen, universellen Gesetzmäßigkeiten folgen, die man aus der Beschreibung mittels einer Gravitationstheorie ableiten kann.

Die theoretische Grundlage bildet hierbei die Dualität konformer Quantenfeldtheorien im Minkowski Raum und höherdimensionaler Stringtheorien im Anti-de Sitter Raum, die AdS/CFT Korrespondenz. Einen besonders interessanten Grenzfall stellt der Limes starker Kopplung und hoher Anzahl von Freiheitsgraden der konformen Feldtheorie dar, in dem sich die duale Beschreibung zu klassischer Gravitationstheorie im AdS Raum vereinfacht. Mittels störungstheoretischer Betrachtung der Fluktuationen von Schwarzen Loch Lösungen der Gravitationstheorie lassen sich universelle hydrodynamische Eigenschaften der stark gekoppelten Feldtheorie beschreiben.

Eines der Hauptergebnisse dieses Forschungsgebietes ist der Nachweis, dass Fluide, die durch eine einfache duale Gravitationstheorie mit ungebrochener Rotationsinvarianz beschrieben werden können, ein universelles Verhältnis aus Scherviskosität und Entropiedichte besitzen. Erstaunlicherweise stimmt dieses Verhältnis parametrisch mit dem gemessenen Wert des stark gekoppelten Quark-Gluonen-Plasmas überein, ohne dass eine direkte Beschreibung dieser QCD Phase momentan möglich ist.

In der vorliegenden Arbeit wird die Konstruktion eines ähnlichen, universellen Zusammenhangs beschrieben. In der hydrodynamischen Beschreibung supersymmetrischen Feldtheorien existiert eine Diffusionskonstante, die, ähnlich der Scherviskosität, den spurfreien Teil der Konstitutivgleichung des Supersymmetriestroms beschreibt. Wir berechnen diese Konstante in supersymmetrischen Theorien allgemeiner Dimension mittels verschiedener unabhängiger Rechnungen. Dazu betrachten wir als duale Gravitationstheorie eine generische Supergravitationstheorie. Die Bewegungsgleichung des zum Supersymmetriestrom dualen Gravitinos in Schwarzen Loch Hintergründen wird gelöst und erlaubt die Berechnung der retardierten Greenschen Funktion des Supersymmetriestroms der Feldtheorie. Diese besitzt einen Pol, der die charakteristische Schalldispersionsrelation des Phoninos beschreibt, des Goldstonefermions spontan gebrochener Supersymmetrie aufgrund endlicher Temperatur. In dieser Dispersionsrelation findet sich die besagte Diffusionskonstante, die sich auch mittels einer neuartigen Kubo-Formel direkt aus der Greenschen Funktion berechnen lässt.

Das Hauptergebnis der Arbeit bildet hierbei die Etablierung eines Zusammenhangs

dieser Diffusionskonstante und eines universell gültigen Absorptionsquerschnitts auf der dualen Seite der Gravitationstheorie, der die Absorption von Spinoren von einem Schwarzen Loch Hintergrund beschreibt.

Eine weitere bedeutende Entwicklung besteht in der Entdeckung eines neuartigen Transportkoeffizienten, der einen beobachtbaren induzierten Strom aufgrund der Vortizität eines Fluids beschreibt. Dieser stellt die klassische Manifestation eines quantenmechanischen Effektes dar, der entsteht, wenn die zugrunde liegende mikroskopische Theorie eine quantenmechanische chirale Anomalie aufweist.

Wir untersuchen diesen Effekt mithilfe eines theoretischen Ansatzes, der verschiedene Zugänge zum Verhältnis von Hydrodynamik und Gravitation miteinander vereint. Dazu werden rotierende D3-Branen effektiv als asymptotisch flache Verallgemeinerungen von fünf-dimensionalen AdS Reissner-Nordström Schwarzen Löchern beschrieben. Die Fluktuationen dieses Hintergrundes beschreiben nun eine effektive hydrodynamische Theorie auf einer Fläche in festem Abstand zur Singularität des Schwarzen Lochs, auf der die Fluktuationen Dirichlet Randbedingungen annehmen.

Diese Herangehensweise erlaubt es uns den erwähnten Quanteneffekt nicht nur am Rand des AdS Raums zu betrachten, sondern auch am Horizont des Schwarzen Lochs, auf jeder Fläche mit konstantem Radius dazwischen oder sogar im asymptotisch flachen Raum.

Abstract

In the present thesis we study properties of strongly coupled hydrodynamic theories which may be described in terms of a dual higher dimensional gravitational system. Particular attention is given to the computation of physical quantities like the theories' viscosities and diffusion constants. These are analysed with regard to the question of whether they follow generally applicable, universal laws which may be derived from the description in terms of a gravitational theory.

The theoretical foundation for this is laid by the duality between conformal quantum field theories in Minkowski space and higher-dimensional string theories on Anti-de Sitter space, the AdS/CFT correspondence. A particularly interesting simplification is given by the limit of strong coupling and large number of degrees of freedom of the conformal field theory in which the dual description reduces to a classical theory of gravity on AdS space. By using a perturbative treatment of fluctuations of the gravitational theory's black hole solutions one may describe universal hydrodynamic properties of the strongly coupled field theory.

One of the main results within this area of research is the proof that fluids which may be described by a simple dual gravitational theory with unbroken rotational invariance possess a universal ratio of shear viscosity and entropy density. Astonishingly, this ratio parametrically agrees with the value measured for the strongly-coupled quark gluon plasma, although a direct treatment of this QCD phase is at present not available.

In the following work we describe the construction of a similar, universal relation. In the hydrodynamic description of supersymmetric field theories there exists a further diffusion constant which, similarly to the shear viscosity, appears in the traceless part of the constitutive relation of the supersymmetry current. We compute this constant in supersymmetric theories of arbitrary dimension via different independent calculations. For doing so we look at a generic supergravity theory as the gravitational dual. The equation of motion of a gravitino, which is the dual field to the supersymmetry current, is solved in a black hole background and allows for the computation of retarded Green's functions of the field theory's supersymmetry current. This has a pole which describes the characteristic sound dispersion relation of the phonino, the Goldstone fermion of spontaneously broken supersymmetry due to finite temperature. In this dispersion relation we find the aforementioned diffusion constant which we also obtain directly from the correlator via a new Kubo formula.

The main result of this project is the establishment of a relation of the supersound diffusion constant and a universally applicable absorption cross section on the dual gravitational side which describes the absorption of spinors by a black hole.

A further important development is the discovery of a new transport coefficient

which describes the observable current that is induced by the vorticity of a fluid. This illustrates the classical manifestation of a quantum mechanical effect which appears when the underlying microscopic theory possesses a quantum mechanical chiral anomaly.

We investigate this effect within a theoretical framework which unifies several different approaches at the interplay of hydrodynamics and gravitational physics. We effectively describe rotating D3-branes as asymptotically flat generalisations of five-dimensional AdS Reissner-Nordström black holes. The fluctuations of this background describe an effective hydrodynamical theory on a surface at a finite distance from the black hole's singularity, on which the fluctuations satisfy Dirichlet boundary conditions.

This approach allows us to study the mentioned quantum effect not only at the boundary of AdS space, but also at the black hole's horizon, at a surface in between at finite radius, or even in asymptotically flat space.

Publications

This dissertation is based on results the author obtained as a PhD student under the supervision of Prof. Dr. J. K. Erdmenger at the Max Planck Institute for Physics in Munich, Germany between July 2011 and June 2014.

Part of the covered material, in particular most of the content of chapter 5, has already been published in [1], although this chapter also contains some minor additional unpublished work. Chapter 4 is based on ongoing work [2] in collaboration with Johanna Erdmenger, Mukund Rangamani and Hansjörg Zeller.

[1] J. Erdmenger and S. Steinfurt, "A universal fermionic analogue of the shear viscosity," *JHEP* **1307** (2013) 018, `arXiv:1302.1869 [hep-th]`.

[2] J. Erdmenger, M. Rangamani, S. Steinfurt, and H. Zeller, "Hydrodynamic regimes of spinning black D3-branes." (work in progress), 2014.

Dedicated to my parents

Contents

Chapter 1

Introduction

What is reality?

The philosophical and scientific study of this question has a long history, which spans from the ancient Greeks through Newton and Einstein to quantum mechanics and contemporary high energy particle and gravitational physics. The Greek philosopher Plato already discussed aspects of this question in the seventh book of his famous *Politea*. We are going to see that aspects of this question have a somewhat surprising recurrence in recent developments in the research of a quantum theory of gravitation.

In his well-known *allegory of the cave*, Plato lets his mentor Socrates narrate about the perception of reality by prisoners in a cave, who are confined to only observe one of the cave's walls. These prisoners perceive the world outside the cave only through shadows on this very wall, which are projections of the outside reality; voices from outside are reflected by the wall and thus seem to originate from the shadows themselves as well. What is now real for these observers? If one of the prisoners were released to leave the cave and observed the outside world, he would be dazzled by the brightness of the sun, Plato argues. He would have a hard time convincing his fellow prisoners of the outside higher-dimensional reality which consists not only of two-dimensional shadows on the wall of the cave, but of three-dimensional coloured objects, whose shadows are projected onto this wall.

As argued by Polyakov [3], this is essentially what the holographic principle, proposed by 't Hooft [4] and Susskind [5], is about. In their seminal work, it is argued that within a quantum theory of gravitation the degrees of freedom of the theory are encoded in a lower-dimensional non-gravitational theory. This theory does not only describe the *projected* higher-dimensional reality, in which information on the perpendicular direction would be lost, but rather encodes the full (!) information of the higher-dimensional theory in a rather intricate way. Quantum gravity should be holographic in the sense that both descriptions contain the full information of the physical theory, very much like an optical hologram. Both descriptions can therefore be considered as a valid representations of reality although they look very different!

A concrete realisation of this holographic principle has been achieved within the context of string theory in terms of the celebrated AdS/CFT correspondence [6, 7, 8] (see also the classic review [9]), which will be the main focus of this thesis. Depending on the question that is pursued, we will see that either one or the other description is more convenient.

But before we introduce the AdS/CFT correspondence in some detail, let us back up for a while and dwell on the foundational principles[1] and insights that preceded this relatively new development. We want to embed the formulation and realisation of the holographic principle in the history of the description of fundamental interactions. The laws for the description of these have been formulated around few very powerful principles, which we will guide the reader through. As Abdus Salam once remarked [10, p. 149]

> "(...) it is not particles or forces with which nature is sparing, but principles."

The holographic principle and its realisation within AdS/CFT and more generally gauge/gravity duality appears to play a key role in the description of various mysterious phenomena, most importantly at the centre of a description of a quantum theory of gravitation itself.

1.1 Quantum field theory

For centuries, the principles of *symmetry* and *unification* have been of utmost importance either as guiding principles for new ideas that dealt with the laws of nature or at the centre of a deeper new understanding of these. Newton realised that the falling apple and the movement of the planets around the sun have the same origin in terms of the gravitational field that surrounds any massive body. One may argue that by this he unified the description of the celestial motion of planets and particular kinds of ordinary motion on the earth's surface.

It was Maxwell who, by summarising and extending the work of most prominently Gauss, Ampère and Faraday, formulated his famous equations that show the close interconnection of electric and magnetic forces. The underlying principle of the $U(1)$ *gauge invariance* of electromagnetism was formulated by Weyl[2] in 1929. Although nowadays regarded as a redundancy of the description rather than a symmetry, the invariance of the physical description and their connection to the symmetries of mathematical Lie groups has since been central in the understanding of the fundamental laws of nature.

In 1905 Einstein dealt with a slightly different question regarding electromagnetism. He realised that his theory of *special relativity* accounts for the appearance of electric and magnetic fields which depends on the inertial reference frame a moving observer performs his/her experiments in. On the one hand, this showed that the true physical object is the unified electromagnetic field that is just perceived as electric or magnetic field depending on the reference frame relative to the field's source. On the other hand, just from the basic principles underlying special relativity, the independence of the physical description on the particular reference frame and the constancy and equality of the (vacuum) speed of light in every such frame, it gave a unified description of space and time itself. Accordingly, there is a symmetry principle

[1] The views of the author have either directly or indirectly been influenced or sharpened by Weinberg's view of nature in terms of few fundamental principles – see e.g. [10, 11]. This should become apparent throughout this exposition. We however highly recommend the mentioned beautifully written literature!

[2] For many more details on the history of gauge invariance and references to the original literature see [12].

connected to this observation, the generalised rotational invariance in Minkowski space, *Lorentz invariance*, associated with the group $SO(1,3)$, or, also taking into account translational symmetries, *Poincaré invariance*.

Nowadays we understand electromagnetism in terms of a renormalisable quantum field theory, quantum electrodynamics, which on its own is already of unprecedented precision in terms of prediction and agreement with experiment (see e.g. [13]). But even more spectacularly, it is part of one of the most successful and best tested theories ever to be written down, the Standard Model of particle physics [14,15,16,17] (for an introduction see e.g. [13,18]). The Standard Model of particle physics describes not only electromagnetism in terms of a quantum field theory, but also two of the other fundamental interactions, the weak interaction and the strong force. Similarly to electromagnetism, these also rely on gauge invariance under a (non-Abelian) Lie group; in total the gauge redundancy of the Standard Model is described by the group

$$SU(3) \times SU(2) \times U(1)_Y \, , \tag{1.1}$$

in which $U(1)$ electromagnetism is unified with the weak interactions to the electroweak force, described by the $SU(2) \times U(1)_Y$ part of the Standard Model gauge group [15]; Y denotes the hypercharge.

So, after the description of the unity of motion due to gravitational force and the observation that electricity and magnetism stem from the underlying unified concept of electromagnetism we see that this electromagnetism is also unified with the weak interactions to the electroweak interactions. Matter particles like electrons or quarks transform as specific representations of the given symmetry groups and thus complete a theoretical structure, which is highly constrained due to its exact and/or approximate symmetries.

Elegant, symmetric and constrained as it is, the most important aspect of the Standard Model is of course that it works and indeed does describe nature within the realm of its validity and experimental accessibility. One spectacular highlight of the Standard Model success story was reached recently, when at the Large Hadron Collider at CERN in Geneva, the collaborations associated to the two general purpose detectors ATLAS and CMS independently confirmed the existence of a new particle [19,20], the long ago proposed Higgs boson [21,22,23,24]. This not only verified the existence of a new kind of fundamental elementary particle, a spinless scalar boson, but also provides a mechanism for *electroweak symmetry breaking*. This particular form of *spontaneous symmetry breaking* explains how the $W^{\pm}$ and Z bosons and other fundamental particles like the electron acquire masses via their interactions with the Higgs field and its condensate. Furthermore, it provides a promising window into new physics beyond the Standard Model should deviations from the Standard Model predictions of the self-couplings of the Higgs be found. Often, theories beyond the Standard Model predict a Higgs sector which resembles the Standard Model one to a large extent, but deviates from it on a more subtle level.

But the structure of quantum field theory in general and the Standard Model in specific is further constrained by some profound principles which the theories rely on. We have already mentioned *Lorentz invariance* and implicitly *quantum mechanics* as the fundamental framework. Together with the principle of *cluster decomposition*, that S-matrix elements factorise for far separated measurements, these directly lead to the description of fundamental interactions in terms of a quantum field theory, at least at the level of a low energy effective theory (see [25,26] and the concise summary in [11]). Together with the accompanying gauge principles, there is little room for what the theory of fundamental interactions could have been.

At present, the Standard Model is usually understood as such an *effective field theory* [11], which probably does receive non-renormalisable higher order corrections that are strongly suppressed at low energies by high energy scales, such as the GUT or Planck scale. These operators could e.g. induce baryon or lepton number violating processes [27, 28] which are needed for an explanation of the baryon asymmetry of the universe. For the principle of renormalisability, which was crucial for the original acceptance of the Standard Model following [17], this then however means the following: For the description of an *effective* theory at low energies renormalisability (in the power counting sense) should not be regarded as a principle on the same level as the aforementioned ones. However, for the description of a *fundamental* theory, renormalisability is still a very important ingredient.

Many models have been proposed which extend the known gauge interactions of the Standard Model (1.1) by unifying the electroweak interactions also with the theory of strong interactions, QCD, that is based on the $SU(3)$ gauge symmetry. In these, the whole Standard Model gauge group is embedded into a single grand unified gauge group like $SU(5)$ or $SO(10)$, following [29,30,31]. Often these models suffer exactly from the possibility of operators which in the effective theory well below the GUT scale induce the just mentioned higher-dimensional operators, like dimension six operators. These could induce proton decay at a rate incompatible with current observations. Nevertheless, grand unification with mechanisms to remedy this and other problems is still an attractive possibility, about which only experiment can make a conclusive statement.

Another proposal of physics beyond the Standard Model involves an extension of the usual symmetry principles we have encountered so far in a radical way, namely *supersymmetry*, see e.g. [32]. In its core, supersymmetry extends the current understanding of space and time to also include fermionic coordinates. In general this extension could address many theoretical and experimental challenges within one mathematically attractive framework. The accompanying newly predicted particles (every boson would have a new fermionic partner and vice versa) could via loop corrections contribute to the Higgs mass and by that be part of the solution to the hierarchy problem, which in its simplest form is the puzzle and search for an explanation of the Higgs mass' low value. Furthermore, it could lay out the path to the aforementioned unification of the three Standard Model forces via an improvement of gauge coupling unification at high energy scales and provide a stable candidate for the existing but little understood dark matter. Experimental signatures, however, have so far not really reached a positive result on this[3].

On the other hand, supersymmetry can be understood as just a particular symmetry of specific quantum field theories with a priori little relevance for beyond the Standard Model physics in a phenomenological sense. Many powerful exact results may be derived in such supersymmetric theories [33]. Most interestingly, one of the

[3]Occasionally, it is argued that the rather low Higgs mass may be interpreted as a quite favourable observation for supersymmetric extensions of the Standard Model since theories like the MSSM more or less generically predict a rather low Higgs mass. On the other hand the generic expectation for the Higgs mass in these models would have been for it to be even lighter, so also in such models some kind of tension arises.

very powerful concepts underlying certain quantum field theories can be made rather precise: the concept of *duality*.

As early as in the 1970s it was shown that in two dimensions perturbative excitations of certain fermionic theories, the massive Thirring model, and solitons of a seemingly different bosonic theory, the Sine-Gordon model, may be interchanged to form an equally valid description of physics in terms of apparently quite different fundamental building blocks [34,35]. It can be explained via an underlying *weak-strong duality* of the two descriptions. In four dimensions the principle of bosonisation, which was the key to understanding the aforementioned two dimensional case, does not apply anymore and identifying the duality of two specific quantum field theories is much more difficult. The idea of interchanging perturbative excitations like W bosons in gauge theories with solitonic excitations with topological charge (like magnetic monopoles) however remained [36,37]. Soon after, it was yet established in maximally supersymmetric (!) field theory in four dimensions [38], in which the understanding of the supersymmetry algebra and its central extensions [39] was imperative. In the case of minimally supersymmetric QCD electromagnetic duality was demonstrated many years later in [40], in which the low energy excitations like gauge bosons, meson and baryon operators can really be thought of as being composed out of similar but different excitations of the dual theory!

One particular aspect of a duality between two theories is that usually neither may be regarded as more fundamental. The seemingly fundamental building blocks of one theory are made out of the equally apparently fundamental looking building blocks of the other. There are also aspects which the two descriptions need to agree on, like they must respect the same global symmetries, independent of their various different perturbative descriptions. This usually poses good first checks of proposed dualities — but the question of what is emergent, fundamental or "real" is ambiguous.

The aspect of weak-strong duality of two perturbative descriptions also lies at the heart of the AdS/CFT or gauge/gravity correspondence, as we will see. The main difference as compared to the just mentioned examples is however that it comprises a duality between a quantum field theory and a specific quantum theory of gravitation.

1.2 Quantum gravity

In our discussion of the principles underlying quantum field theory and the Standard Model, we have so far left out the other cornerstone of fundamental physics, namely the theory of general relativity [41, 42]. After the successes of the theory of special relativity Einstein set out to find a theory of gravitation compatible with his special theory of relativity, which he finally successfully formulated in 1915 [43]. The deep underlying principle of general relativity, the *equivalence principle*, states that locally an accelerated observer cannot perform a measurement which distinguishes his situation from the one in a corresponding gravitational field. This principle and the accompanying general coordinate invariance of physical phenomena, lies at the centre of the current understanding of gravitation.

The relation of the theory of gravitation to quantum mechanics has however been subject to both practical and conceptual difficulties. From a field theoretic perspective, general relativity also has to be understood as an effective field theory [11]. Already on its own, general relativity is perturbatively non-renormalisable and there is no reason not to expect higher curvature terms in the action, which go beyond the Einstein-Hilbert term

$$\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R + O(R^2)\,. \tag{1.2}$$

At low energies, the theory is as reliable as any other non-renormalisable quantum field theory like Fermi's non-renormalisable theory of weak interactions. It describes the propagation and interaction of a spin two quantum field. The problem lies in the fact that at higher energy one needs infinitely many counterterms to properly renormalise and define the theory. More attractively, it might be superseded by a UV sensible theory of quantum gravity, which reduces to this particular one in an effective low-energy limit like string theory.

But before we elaborate on the achievements of the string theory description of a quantum theory of gravity, we have to explain one more essential aspect at the interface of quantum mechanics and the general theory of gravity, which we already alluded to at the very beginning of the introduction. The theory of black hole solutions provides an example in which the need for a quantum mechanical description of gravity becomes apparent.

Generically, theories of gravitation possess classical solutions to their field equations which describe the geometry of a black hole spacetime. Although these may classically be described only in terms of a few parameters like its mass, charge and angular momentum[4], they have a sizeable entropy S, which is given in terms of the area A of their horizon [44] and the fundamental constants of nature, Boltzmann's constant k_B, the vacuum speed of light c, Planck's constant $\hbar$ and Newton's gravitational constant G as

$$S = \left(\frac{k_B c^3}{\hbar}\right)\frac{A}{4G}\,. \tag{1.3}$$

Also, this horizon area classically satisfies a theorem, which allows it only to increase [45], stunningly similar to the second law of thermodynamics

$$\delta A \geq 0\,. \tag{1.4}$$

However, classically, the black hole does not radiate. For a thermodynamic analogy, the entropy interpretation of the horizon however requires the existence of a conjugate thermodynamical variable, a temperature. Matter can only fall into a black hole and eventually into the singularity, but never escape to asymptotic infinity once it crossed the horizon (this is basically the defining property of the horizon). Quantum mechanically however, Hawking showed that a black hole does radiate thermally [46], fixing the prefactor of (1.3) unambiguously. The appearance of Planck's constant shows the quantum mechanical origin of the radiation and shows that it is actually of enormous size (in SI units about 10^{69} times the area as measured in square meters). Also the appearance of the other fundamental constants of nature shows in which way many distinct areas of physics beautifully come together in this relation: Gravitational, relativistic physics (Newton's constant G and the speed of light c), but also statistical physics and thermodynamics through the notion of entropy itself and Boltzmann's constant k_B.

However the quantum mechanical origin of the *thermal* radiation already points in the direction of one of the classic clashes of quantum mechanics and black hole

[4] For details on the no-hair theorem see [42] and references therein.

gravitational physics, the information paradox [47]: If the black hole is formed via the collapse of a configuration that is in a pure state, how could the outgoing Hawking radiation be thermal given the supposed existence of a unitary S-matrix that describes the whole collapse and radiation process by taking pure ingoing states to pure outgoing asymptotic states? Black hole complementarity [48] has been proposed as a solution to this question, but this also runs into more recently found paradoxes [49] (more on these specific questions can be found in section 3.1). Clearly, this apparent paradox should be resolved in a satisfactory description of the black hole collapse and evaporation process within a consistent quantum theory of gravitation.

Such a theory is provided by string theory[5]. From the early days of a supposed description of the strong interaction via the string theory of flux tubes that has finally been superseded by QCD[6], this subject has undergone many different phases. Most importantly, it was realised that it does provide a consistent theory of quantum gravity: Left- and right-moving modes along a closed string may be quantised and at the lowest excited level describe a massless spin two particle, a graviton! Also, it can be directly shown that in an effective field theory way, governed by the mass scale corresponding to the length of the string, Einstein's theory of gravitation (or a very close cousin to it) may be recovered.

The various known string theories are interconnected by a web of dualities. These contain a strong-weak S-duality [57] very much related to the field theory duality of $\mathcal{N} = 4$ super Yang-Mills theory mentioned earlier, as we will see. A genuinely string theoretic duality is T-duality [58], which interchanges the winding of a string around a compactified dimension and the discretised momentum along it. Effectively, a string theory on a compactified dimension of radius R is therefore dual to one on a compactified dimension of radius α'/R, where $\sqrt{\alpha'}$ denotes the string length scale. This interchanges Dirichlet and Neumann boundary conditions for open strings [59], exchanges even and odd dimensional D-branes in type II string theory and more generally relates IIA to IIB, but also the two heterotic string theories with gauge groups $SO(32)$ and $E_8 \times E_8$ to one another. It is closely related to mirror symmetry of Calabi-Yau manifolds [60], which serve as compactified dimensions of string theory yielding a semi-realistic supersymmetric four-dimensional effective theory [61]. T- and S-duality close into a more general U-duality [62] and all the interconnected theories should be understood as limits of an eleven-dimensional theory, M-theory [63, 64]. Beyond the many dualities which relate string theories to string theories, gauge/gravity duality will provide a duality which relates a string theory to a gauge theory.

String theory also provides a counting of the black hole microstates [65] that are coarse grained to the description of a classical black hole compatible with Bekenstein's black hole entropy. It gave an impressive and deep result for the possible microscopic origin of the entropy formula (1.3) at the quantum mechanical level.

Even one of the most puzzling problems of gravitational and cosmological physics, the cosmological constant problem [66], may be addressed via the reference to the landscape of consistent string theory vacua (e.g. the rather recent [67, 68, 69], reviewed

[5]For an introductory standard see [50, 51], or more recently [52, 53]; for a lecture based introduction we recommend [54].

[6]Note that gauge/gravity duality again explains field theory flux tubes and Wilson lines by fundamental strings [55, 56]. So, for theories which exhibit gauge/gravity duality in the string theory context, the initial motivation and connection of these two kinds of strings is again much more strengthened.

in [70]). Although this may be unsatisfactory, it does indeed provide a plausible solution to this great puzzle of gravitational physics as well.

Despite the manifold successes on theoretical grounds, it is hard to test string theory experimentally due to its high scale relevance. Perhaps the best hope of direct relevance to current observations lies in descriptions of inflationary cosmology [71, 72, 73] within the context of string theory [74]. On theoretical grounds, models of inflation are generically sensitive to higher dimensional operators that require a theoretical understanding in a UV complete theory [75], unless one is satisfied with a significant fine-tuning. Furthermore, recent observations of the cosmic microwave radiation and its polarisation by the BICEP2 collaboration [76] point to an end of inflation at a rather high energy scale ($\sim$ GUT scale [77]), not too far from the Planck scale itself, where quantum gravitational effects become important. The paradigm of cosmic inflation itself provides a theoretical framework that explains many of the puzzles of previous big bang cosmology (the horizon, flatness and monopole problems as most conspicuous puzzles) in terms of a period of inflationary growth of the scale of the universe. The homogeneity and isotropy of the cosmic microwave background [78] are beautifully explained and the quantum mechanical fluctuations further provide a primordial source for its small anisotropies. Moreover, the seeds of the large-scale structure of the universe as encountered in the many galaxies, originates from these quantum mechanical fluctuations, or as Brian Greene put it [79],

> "According to inflation, the more than 100 billion galaxies, sparkling throughout space like heavenly diamonds, are nothing but quantum mechanics writ large across the sky. To me, this realisation is one of the greatest wonders of the modern scientific age."

The results of [76] are claimed to provide a signal for primordial gravitational waves, which within the theory of inflation originate from the *quantisation* of the gravitational field itself – its tensorial part [77]. An independent experimental verification or falsification of these claims has so far not yet been reported. Currently the results of [76] should therefore be considered as preliminary[7] but, if independently confirmed, certainly groundbreaking.

1.3 Holographic reality

The black hole entropy formula (1.3) is exceptional in another regard as well. It is holographic in the sense that the information encoded in the microstates of the black hole is proportional to the *area* of its horizon rather than the corresponding volume [80]. Thinking of an ordinary system like a gas of particles which could have collapsed to form a black hole (with the degrees of freedom of every individual particle being to move in all three dimensions) one would rather expect the entropy to scale like the volume of the containing space. But apparently this is not the case in a quantum mechanical treatment of gravity. Rather, quantum gravity behaves holographically [4, 5].

Further support for this behaviour of quantum gravitational theories comes from considerations about how much information may actually be stored in matter within

[7] Note in particular the cautious remarks with regard to polarised dust emission in the published version of [76] and the literature cited therein.

a finite volume [81], given that this matter might collapse to form a black hole when suitably compressed. It has long been conjectured[8] that the entropy to mass ratio of a general physical system is bounded by 2π times the radius of the smallest surrounding sphere [83], basically corresponding to the entropy being bounded by the corresponding black hole one [5]. Thus, a black hole has the maximal entropy for a given mass and area. This leads to the assertion that the information bound for storage in a given volume of space also behaves holographically.

An explicit realisation of the holographic principle has been achieved by the AdS/CFT correspondence [6, 7, 8] within string theory, which makes this general statement about quantum gravity concrete in particular examples:

The full symmetries and dynamics of a specific (quantum) string theory on the geometry of five-dimensional Anti-de Sitter space may be encoded holographically in the symmetries and dynamics of a corresponding specific four-dimensional (conformal) quantum field theory!

Rather than being defined on the horizon of a black hole, the theory should be thought of as living on the *boundary of AdS space*.

It is a *weak-strong duality* in the sense alluded to earlier. Whenever one of the two sides of the medal is accessible via weak coupling perturbation theory, the other side is very strongly coupled. This is also the main obstacle for a mathematical derivation of the duality, but countless checks, see e.g. [9], have convincingly been performed to test it in various limits, at least for specific dual pairs.

One of the main aspects of this particular new type of duality is that it indeed relates a more or less ordinary *quantum field theory* to a *quantum theory of gravitation*. On the one hand, it tightly binds string theory into the structure of at least particular kinds of quantum field theories – albeit often with a high degree of symmetry. String theory with all its complications and beauty then arises from these in a particular limit [84, 85]. Even more generally, it can actually be understood as a constructive definition of a quantum theory of gravitation itself [86].

Moreover, it makes the notion of reality ambiguous. The physical theories that feature such a duality can be seen either as a string theory in higher dimensional space or as a quantum field theory in lower dimensional space. A priori, neither of these two descriptions is preferred. It usually happens that in particular limits one of the two descriptions is more appropriate computationally or more natural for understanding certain kinds of phenomena. As for dualities between quantum field theories in the same space-time dimension, the fundamental building blocks of one description combine into the building blocks of the dual theory. For example, the graviton of the gravitational theory can to some degree be understood as a bound state of gauge bosons of the field theory [87, 86], the different dimensionality of space-time evading the famous Weinberg-Witten theorem [88] which seemed to forbid exactly such a relation.

So, in essence, gauge/gravity duality, which is the extension of AdS/CFT that departs from the conformality of the field theory or the exact AdS background of the gravity theory, possesses a somewhat striking similarity to the Platonian tale of the shadows on the wall. However, in the case of quantum gravity and the holographic principle, information is not lost via the projection onto the wall, but rather fully

[8] For a covariant generalisation which circumvents some of the problems of the earlier proposal see [82].

encoded on it.

1.4 Reductionism and holography

String theory has for a long time been a prime candidate for a unified theory of fundamental interactions. The history of these fundamental interactions as described in terms of quantum field theories has in the past already undergone significant unification, both in terms of different concepts and principles, but also in terms of their actual realisation in nature. As Steven Weinberg put it [10, p. 231f]:

> "In this century we have seen a convergence of the arrows of explanation, like the convergence of meridians toward the North Pole. Our deepest principles, although not yet final, have become steadily more simple and economical."

String theory provides a further unification with the theory of gravitation in terms of a consistent quantum theory of gauge interactions and gravitation, and it might indeed be the next step in this general unification process. Along this line of thinking, many semi-realistic models within string theory have been constructed which more or less resemble our Standard Model or grand unified extensions thereof (for a representative collection see e.g. [89]). However, conclusive statements of string theory's relation to nature cannot be made so far.

The understanding of string theory as a fundamental theory of interactions that unifies the descriptions of quantum field theories like the Standard Model and Einstein's theory of gravitation in terms of the vibration modes of open and closed strings is in a sense the endpoint of the reductionistic viewpoint in terms of deeper and deeper fundamental building blocks and principles that underlie the laws of nature: in the end everything is made of a string. In general however, it is not clear if such a "final theory" actually exists, although there are reasons in favour of such an assumption. From the few fundamental principles of such a theory other descriptions would emerge at lower energies, the way thermodynamic reasoning emerges from an effective treatment of statistical mechanics in the thermodynamic limit.

The holographic principle can have a two-fold interpretation in this context. On the one hand, it seems to be a truly fundamental, reductionistic additional principle of quantum gravity itself. As we have seen, reasoning about the entropy of black holes more or less directly leads to such a holographic understanding, and this seems to be true for general theories of gravitation. On the other hand, the duality aspect of the AdS/CFT correspondence points to it being an emergent phenomenon resulting from the treatment of particular quantum field theories in a special limit. In particular, (part of) space-time itself then is emergent [90]! One way or the other, it seems to be of tremendous importance in the formulation of a quantum theory of gravitation.

Via gauge/gravity duality, string theory might also be useful as a means for attacking many open questions in related areas of physics, in which often a much less fundamental and less reductionistic attitude towards nature's unexplained phenomena is taken. Clearly, this is also of tremendous interest. As Alvin Weinberg remarked [91] [emphasis in the original] (as cited by [10, p. 60]):

> "I would therefore sharpen the criterion of scientific merit by proposing that, other things being equal, *that field has the most scientific merit which*

contributes most heavily to and illuminates most brightly its neighboring scientific disciplines".

Certainly, one should not overestimate the scientific merit of string theory and to fully consent to the above statement in favour of string theory could appear derogatory towards other scientific disciplines; however, the productive interconnection with fields like e.g. mathematics cannot be disregarded either. In the following, we want to give two examples of very fruitful relationships of string theory to seemingly very different "neighboring disciplines" of physics, in which the gauge/gravity approach to understanding unresolved questions has been helpful: heavy-ion physics and the theory of hydrodynamics. These two examples are not only amongst the most prominent ones in this area, but also represent the main motivation for our original work in this thesis.

Due to its nature as a strong/weak duality, questions of strong coupling physics, e.g. on the field theory side, may be translated into questions within weak coupling physics, on the gravity side, via the gauge/gravity dictionary. The weakly coupled questions can then be addressed via the usual methods of perturbative analysis. The prospects of such an approach is twofold. Firstly, one can get qualitative insight into generic quantities of strongly coupled field theories. These might even possess universal properties in the sense that they are rather independent of the precise map of explicit dual theories. In the long term, one might even approach certain questions quantitatively in the sense that one might engineer a precise gravity analogue of the question under observation. Secondly, one may find new phenomena in rather well-established theories, which might be uncovered via the precise unambiguous holographic map.

1.4.1 Hydrodynamics applications

Shear viscosity

An example for the first prospect, the qualitative understanding of strong-coupling phenomena and their possible universality, is the universality of the ratio of shear viscosity η over entropy density s of strongly coupled field theories with a gravity dual [92]. Under the assumptions of a two derivative, Einstein-Hilbert gravity theory as the gravitational part of the duality and furthermore isotropy of space, the ratio is universally given by

$$\frac{\eta}{s} = \left(\frac{\hbar}{k_B}\right)\frac{1}{4\pi} \tag{1.5}$$

and as such surprisingly low. Usually, more familiar fluids have (in natural units) $\eta/s > 1$ or even bigger; the weak coupling result for a theory with coupling constant $g \ll 1$ also behaves completely differently (see e.g. [93])

$$\frac{\eta}{s} \sim \frac{1}{g^2} \gg 1\,. \tag{1.6}$$

So, one may conclude that a large class of strongly coupled theories has a rather low ratio of η and s, contrary to weak coupling intuition or measurements of very familiar fluids (like water, ...).

But even more interestingly, this value seems to be in parametric agreement with values measured at the relativistic heavy-ion collider RHIC at Brookhaven National

Laboratory near New York (see [93, 94] and references therein). At this experiment, also at the ALICE experiment at the LHC in Geneva, the quark-gluon plasma is studied, the phase of QCD matter at high temperatures and high densities which behaves like a strongly coupled almost perfect fluid. Although a precise holographic description of this phase of matter has not been obtained so far, it is encouraging to have a tool at hand, which can at least make qualitative predictions of properties of strongly coupled theories.

In [92] it was even speculated that (1.5) is a lowest bound for all relativistic quantum field theories by relating its value to the Heisenberg uncertainty principle. The viscosity of a plasma is proportional to the energy density and mean free time of the plasma's quasiparticles. Given that the entropy density is proportional to k_B times the number density of particles and that energy per quasiparticle and its mean free time are complementary in the sense of the Heisenberg uncertainty principle, it was argued that

$$\frac{\eta}{s} \gtrsim \left(\frac{\hbar}{k_B}\right) \tag{1.7}$$

with some prefactor. Correction terms to the universal result may be computed in specific setups[9]. String theory corrections $\sim (\alpha')^3 R^4$ are positive [95], but special curvature squared corrections $\sim R^2$ to the Einstein-Hilbert action, like in Gauss-Bonnet gravity, may violate the bound [96, 97, 98, 99]:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - O\left(N^{-1}\right) + O\left(\lambda^{-3/2}\right)\right) \tag{1.8}$$

Supersound diffusion constant

In our present work, in chapter 5, we will be engaged with the search for a similarly universal quantity, for which we will specifically look at the hydrodynamic limit of *supersymmetric* holographic theories.

The shear viscosity arises as the coefficient of the symmetric traceless part of the first derivative order correction to ideal hydrodynamics (in supersymmetric and non-supersymmetric theories)

$$T_{ij} \sim -2\eta\left(\partial_{(i}u_{j)} - \frac{1}{d-1}\delta_{ij}\partial^k u_k\right) . \tag{1.9}$$

Likewise, the supersymmetry current, which lies in the same supermultiplet as the energy-momentum tensor [100], has a similar constitutive relation

$$S^i \sim -D_{3/2}\left(\delta^i_j - \frac{1}{d-1}\gamma^i\gamma_j\right)\partial^j\rho\,, \tag{1.10}$$

where $\rho = S^0$ basically is the fermionic supercharge density[10]. The term written is the γ-traceless part of the supersymmetry current due to $\gamma^i\gamma_i = d-1$, but there is

[9] The parameters N and λ will be introduced later, see e.g. the discussion around eq. (2.51). Corrections in N^{-1} correspond to quantum gravity corrections while $\lambda^{-1/2}$ corrections are due to finite string length.

[10] This is *not* to be understood as a classical fermionic charge! We will give a precise interpretation of this later on.

also another one in complete analogy to the appearance of the bulk viscosity in the energy-momentum tensor, which vanishes for super-conformal theories.

So, one may wonder if $D_{3/2}$ possesses similarly universal properties as the shear viscosity. Can one maybe translate the shear viscosity's universality to the diffusion constant $D_{3/2}$?

The shear viscosity's universality may be shown from the universal properties of the off-diagonal part of the energy-momentum tensor's retarded Green's function $\sim \langle T_{xy}T_{xy}\rangle$. One could now hope that these universal properties directly translate into the supersymmetry current's retarded Green's function $\sim \langle S_x\bar{S}_x\rangle$, from which $D_{3/2}$ may be deduced. But in this straightforward sense it falls short due to a simple reason: To have a hydrodynamic description we need finite temperature; but this breaks supersymmetry spontaneously [101]. So the relation of the two correlators is non-trivial, because the supercharges do not annihilate the finite temperature ground state.

Given the spontaneous nature of supersymmetry breaking by finite temperature we may wonder about the corresponding Goldstone fermion, the so-called phonino [102, 103, 104], a massless excitation with sound dispersion relation

$$\omega = v_s k - iD_s k^2 \,. \tag{1.11}$$

The phonino contributes a pole in the retarded Green's function of the supersymmetry current and we will analytically derive its dispersion relation, for which the attenuation coefficient D_s will be closely related to $D_{3/2}$ the same way shear viscosity and sound attenuation coefficient are connected. But what about universality of these coefficients?

The universality of η may be derived [92] by relating it to a universal absorption cross section for a minimally coupled massless scalar [105]. A Kubo formula extracts η from the retarded Green's function of T_{xy}, but in the low energy-limit the same expression also describes the absorption cross section of a bulk graviton h_{xy} by the brane, on which the field theory is defined. The bulk graviton satisfies the equations of motion of a minimally coupled scalar and then the universality of the cross section applies.

To derive such a relation also for $D_{3/2}$ will be the basic goal of chapter 5. After computing it explicitly following and extending earlier work [106, 107], e.g. by extracting the phonino's dispersion relation from the retarded Green's function of the supersymmetry current, we indeed establish the relation of $D_{3/2}$'s newly derived Kubo formula to a universal absorption cross section [105]; this time it is the gravitational absorption cross section of a minimally coupled *spinor* by a black hole.

In the shear viscosity case the absorption cross section may be related to the entropy density of the field theory such that the universal relation (1.5) follows. In our case however, there does not seem to be a similar thermodynamic interpretation for the fermionic absorption cross section. Nevertheless, the establishment of universal properties for a further hydrodynamic transport coefficient is of intrinsic interest.

Anomalous transport phenomena

A further example of the usage of gauge/gravity duality lies within a very broad ansatz and reformulation of hydrodynamics [108]. In the *fluid/gravity correspondence* [109], for reviews see [110,111], one finds that the vacuum (!) Einstein equations in AdS space contain the full non-linear incompressible non-relativistic Navier-Stokes equations of fluid dynamics in an interesting way [112]! The basic idea comprises that the universal sector of stress tensor dynamics, which hydrodynamics describes, corresponds to the universal sector of graviton dynamics on the dual gravity side. One may consistently reduce to the effective dynamics of this sector.

Additionally, via the corresponding precise map for charged fluids it was found in [113, 114], that certain quantum field theories contain a term in their hydrodynamic constitutive relations, that has for decades been disregarded. Under certain circumstances, the charge current J^μ obtains a contribution proportional to the fluid's vorticity ω^μ as

$$J^\mu = nu^\mu + (\ldots) + \xi\omega^\mu \,, \qquad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}u_\nu\nabla_\lambda u_\rho \,, \tag{1.12}$$

where n is the charge density and u^μ the fluid velocity. It was believed that $\xi \neq 0$ violates the positive divergence of the entropy current [108], but this turned out not to be correct [115] after the original arguments were revisited given the explicit $\xi \neq 0$ result of [113,114]. It is actually imperative given certain quantum anomalies [116,117] of the underlying microscopic quantum field theory [115]. Moreover, this term has observable macroscopic effects in a vorticity induced current, the chiral vortical effect, similar to the separation of chiral matter due to an external magnetic field, the chiral magnetic effect. This is a fascinating consequence, given that its origin is genuinely quantum mechanical.

In chapter 4, we initiate the study of such phenomena in the context of the blackfold paradigm [118, 119, 120, 121]. We will give a more refined introduction to this interesting subject in section 3.4. But for the time being let us just record that it is a paradigm, which allows for the description of hydrodynamics via a gravitational system even more generally then the membrane paradigm [122] and the fluid/gravity correspondence [109]. Due to its generality, it allows for an explicit interpolation between the various well-established frameworks [123]. For implementing this one is motivated by ideas of the holographic Wilsonian renormalisation group [124, 125, 126], where part of the geometry is integrated out to recover a Wilsonian effective action on some intermediate radial scale. More precisely, and actually slightly different from the Wilsonian approach, a holographic screen at finite radial coordinate is implemented in [123], on which perturbations satisfy Dirichlet boundary conditions [127].

We aim for the extension of [123] to *charged* fluids, such that we may study the anomaly related transport coefficient ξ in (1.12) also in such setups. We will study non-extremal rotating D3-branes in ten dimensional type IIB supergravity which we effectively describe by a generalised five-dimensional Reissner-Nordström black hole after consistently truncating most of the dynamics of the other five dimensions [128]. These are generally given by a squashed Sasaki-Einstein manifold which may be seen as a $U(1)$ fibre bundle with a Kähler-Einstein base. This $U(1)$ will describe the rotation of the D3-branes and in the lower-dimensional theory a corresponding gauge field in the usual Kaluza-Klein sense.

We solve part of the equations of motion of fluctuations in this background which is further supplemented with space-time dependent terms according to the fluid/gravity and blackfold paradigms. From an effective Brown-York [129] like quasi-local stress-energy tensor and charge current, we extract the shear viscosity and anomaly related transport coefficient. These are then defined for any cut-off surface theory. This includes the membrane paradigm theory at the black hole horizon, the fluid/gravity analysis at the AdS boundary in the near-horizon limit, but also extends these to any cut-off theory in AdS space or even at asymptotically flat infinity.

The two examples presented show how much gauge/gravity duality might influence our understanding of physics even without direct application to, say, the Standard Model itself. A low viscosity over entropy ratio had not been anticipated, not to mention its universality. But given gauge/gravity duality, one was able to compute it in the strong coupling regime of the large N limit of $\mathcal{N} = 4$ super Yang-Mills theory [130] and show that this ratio is universally low for a wide class of theories [92].

The chiral vortical effect could have been found decades earlier, but the arguments seemed robust until proven wrong. Moreover, the reformulation of hydrodynamics completely within general relativity is already revolutionary on its own; additionally, its holographic map is so precise that it stimulates one to occasionally rethink the old lore.

1.4.2 Condensed matter applications

Beyond these two undoubtedly very significant developments which we partly aim to extend within this thesis, there are further attempts with regards to particular condensed matter systems, which we do not want to leave unmentioned (for reviews see [131, 132]).

One aspect deals with the search for a theoretical understanding of high temperature superconductors, possibly starting from a gravity dual description [133, 134, 135]. In this work it was shown that a (charged) black hole configuration in AdS space can be unstable to classical scalar perturbations below a critical temperature T_c. Below T_c, on the field theory side one obtains the condensation of a corresponding dual charged operator via a second order phase transition, an infinite DC conductivity and a gap for low frequency charged excitations. Although very promising, many questions are still open.

Also, many other aspects of materials like the high T_c superconducting cuprates remain mysterious. Potentially however, they may be described via a weakly coupled gravity dual of the presumed strongly interacting fermionic system. This is suggested by the strong coupling, but also because cuprate high T_c superconductors are related to the theory of quantum phase transitions, which are phase transitions at zero temperature [136]. Since critical phenomena are universally described by conformal field theories, one may hope for an AdS/CFT description since it is CFT's, which gauge/gravity duality describes most naturally.

The high temperature phase of these materials is hereby of particular interest and AdS/CFT quite generically allows for going to high temperature phases of described theories via putting a black hole in the bulk. In the case of the cuprates, this high temperature phase involves so-called strange metal behaviour. Although the system has a Fermi surface, the low energy excitations may not be described by conventional Landau Fermi liquid theory[11]. Its properties clearly deviate from these, most famously by a DC resistivity, which is linear in temperature opposed to T^2 behaviour for Fermi-liquids. Also for these system initial holographic studies have been pursued [141].

[11] Previous studies of general non-Fermi liquids via holography include [137, 138, 139, 140].

1.5 Recap

To summarise, there are many reasons why the study of the AdS/CFT correspondence is interesting and important. On the one hand it very generally ties string theory and by this a consistent theory of quantum gravity to the general structure of rather ordinary quantum field theories. This certainly extends our current understanding of each of these. One may even argue that, in principle, the rich structure of string theory could have actually been discovered via it.

AdS/CFT is furthermore a quite rigorous definition of what is actually meant by a theory of quantum gravity and might therefore apply even more generally to consistent theories of quantum gravity besides string theory. It is of holographic nature: one of the dimensions is emergent. Additionally, it explains the microscopic origin of black hole entropy, which was historically a precursor, and generically seems to answer the old information paradox question in favour of information preservation. However, it also does not show where Hawking's arguments actually fail [142].

Furthermore, in a non-reductionist view towards physical theories, it offers ways to describe specific strongly coupled field theories in a rather model-independent universal way. Therefore, it may help answer questions about strongly coupled theories which could not have been addressed earlier. These answers may yield qualitative insight for orders of magnitude of physical parameters like $\eta/s \ll 1$, rigorously help find new phenomena or raise the hope to attack long-standing mysteries like high temperature superconductivity or turbulence.

1.6 Results of the thesis

In the course of this thesis we mainly extend the current status of the hydrodynamic limit of gauge/gravity duality in two ways.

Firstly, we deal with a generalisation of previous studies on the interconnection of hydrodynamics and gravitational physics. In the blackfold paradigm, one may construct a hydrodynamic system, which comprises several previous such setups (membrane paradigm, AdS/CFT hydrodynamics, fluid/gravity, cut-off surface holography) as limiting cases. So far, the main results are

- We determined the appropriate consistent truncation of type IIB supergravity and applied it to a stack of rotating D3-branes, yielding a novel five-dimensional asymptotically flat doubly charged black brane spacetime, which has AdS Reissner-Nordström as its near-horizon limit.

- In the tensor sector of the corresponding intrinsic dynamics blackfold setup the equations of motion were completely integrated and allow for the determination of the shear viscosity which interpolates between all aforementioned hydrodynamic setups.

- In the vector sector, the very complicated coupled equations of motion were analytically integrated. Here, the extension of chiral anomaly related transport was for the first time extended to the blackfold paradigm.

Secondly, we apply the search for universal quantities in the hydrodynamic limit of generic field theories to the correlator of supersymmetry currents. The main results in this project, which were published in [1], are:

- Using and extending well-known holographic methods I explicitly computed a hydrodynamic transport coefficient, the supercharge diffusion constant D_s, for strongly coupled supersymmetric conformal field theories with supergravity dual in arbitrary dimensions via three independent methods.
- For one of these methods I derived a novel Kubo formula for the aforementioned transport coefficient, which was unknown in the previous literature on the subject. This went along with a redefinition of the more generic transport coefficients according to standard symmetry principles in the constitutive relation of the supersymmetry current.
- Most importantly, the relation to a known universal absorption cross section result and by this the establishment of a particular kind of universality for the diffusion constant was found.
- Also, the given results were partially extended to non-conformal Dp-brane world-volume field theories.

1.7 Outline of the thesis

The outline of the thesis is as follows:

In chapter 2, we are going to start with an introduction to the AdS/CFT correspondence, which will be the main underlying framework of this thesis. We are going to review the subject starting off from the Weinberg-Witten theorem and how a holographic description may circumvent it to yield a graviton as a bound-state made of gauge bosons. From 't Hooft's classical large N limit of gauge theories we will be led to the notion of AdS space (via a digression to Yang-Mills instanton moduli spaces) which appears suitable for a holographic description of degrees of freedom. Then the AdS/CFT correspondence is introduced starting from the two different descriptions of D-branes and the seemingly miraculous agreement of absorption cross section computations. Maldacena's original argument is followed by the exposition of the AdS/CFT dictionary (with examples for scalars and spinors). After a short discussion of the relation of chiral anomalies to Chern-Simons terms within AdS/CFT, we conclude by shortly outlining finite temperature and chemical potential configurations.

The following chapter 3 deals with the general relations of general relativity and the theory of hydrodynamics, which will set the stage for our original work in chapters 4 and 5. For doing so, we will revise the subject of black hole thermodynamics and its relation to the membrane paradigm. After a short description of black hole complementarity we will arrive at the AdS/CFT description of near boundary fluids via the analysis of linearised perturbations and in the more general framework of the fluid/gravity correspondence. For this, we are going to give sample computations which are of importance later on. Special emphasis is laid on relatively new developments for the description of anomalous transport phenomena. The ideas of defining a holographic screen at finite radial distance are then discussed both in the context of asymptotically AdS geometries, as well as within the so-called blackfold paradigm.

Chapter 4 then uses the aforementioned ideas to initiate the study of anomalous transport phenomena within the blackfold paradigm and the finite r Dirichlet problem. For pursuing this, we first have to identify a suitable supergravity theory, whose mode spectrum is truncated to a convenient consistent subset. This consistent truncation will then be applied to the background of a rotating stack of D3-branes on which hydrodynamics-like perturbations are defined. The tensor sector of fluctuations then shows the direct generalisations of similar results in charged hydrodynamics from the fluid/gravity correspondence and recent results for uncharged cut-off surface hydrodynamics within the blackfold paradigm. Then the vector sector is analysed in all its complications with fully integrated solutions of the complicated coupled equations of motion. From this one may compute the transport coefficients in the theory which interpolates between AdS/CFT, membrane paradigm and cut-off surface holography results. The complications of the computations in the scalar sector are briefly outlined.

In the last chapter 5, we analyse a certain diffusion constant which appears in the hydrodynamic limit of generic strongly coupled supersymmetric field theories with a gravity dual. We compute this diffusion constant in arbitrary dimension for various conformal theories and analyse its universality properties. The computations are performed via the so-called phonino pole in the two-point function of supersymmetry currents in a theory whose supersymmetry is broken spontaneously by temperature. Another computation comprises the transverse gravitino, which quite generally in the chargeless limit satisfies a minimally coupled spin 1/2 fermion equation. For this computation a novel Kubo formula is derived. The diffusion constant may then via the Kubo formula be related to a universal absorption cross section result such that a particular kind of universality may indeed be established. Further generalisations encompass non-conformal field theories with a gravity dual in which a particular computational trick is applied.

We then conclude with a short summary and the outline of further extensions of the discussed setups.

Chapter 2

Gauge/gravity duality

The AdS/CFT correspondence [6,7,8] provides an explicit realisation of the holographic principle [4, 5] by relating a four-dimensional quantum field theory, $\mathcal{N} = 4$ super Yang-Mills theory, to a quantum theory of gravity, namely type IIB superstring theory on the space $\mathrm{AdS}_5 \times \mathrm{S}^5$. Beyond the original literature there exists a large number of excellent reviews and lecture notes, e.g. [9,143,87,86]. We are going to give an overview of the basic construction and highlight some aspects which will be of importance later on.

2.1 Why and how?

Before we dive into the string theory construction of the original AdS/CFT duality [6], let us start with some general remarks about why and how a non-Abelian gauge theory can at all give rise to a quantum theory of gravity[1]. Many more precise statements like the concrete holographic dictionary and how one may test the duality will then follow afterwards.

We begin by an outline of a classic result [88], which seems to forbid that the quantum gravity's graviton could be made out of gauge bosons. This reasoning is then however evaded by AdS/CFT through general quantum gravitational observations relating directly to the holographic principle. To find a holographic description of some kind, we will argue that it is natural to look at field theories, as for instance Yang-Mills theories with gauge group $SU(N)$, in the limit where one takes the number of colours N to infinity. This $N \to \infty$ limit was long ago discussed by 't Hooft [144]. This takes us back directly to string theories defined on AdS space. The emerging radial dimension is related to the field theory's energy scale and arguments about the coupling constants suggest that many aspects of such a duality will be particularly apparent in supersymmetric theories.

2.1.1 Weinberg Witten theorem

Given that we would like to describe a quantum theory of gravity as emerging from an underlying gauge theory, one may ask the following: Is it possible to understand the graviton $h_{\mu\nu}$, seen as a perturbative spin 2 excitation of some vacuum metric

[1] This introductory approach is very much based on [87, 132, 86].

$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, as a bound state of gauge bosons?

$$h_{\mu\nu} \quad \longleftrightarrow \quad \mathrm{Tr}\,(A_\mu A_\nu) \tag{2.1}$$

Though a very appealing idea, such a construction immediately runs into problems and is therefore at first sight prohibited [88], unless the assumptions are circumvented:

> "A theory that allows the construction of a conserved Lorentz covariant energy-momentum tensor $\theta^{\mu\nu}$ for which $\int \theta^{0\nu} d^3x$ is the energy-momentum four-vector cannot contain massless particles of spin $j > 1$."

Now, why does this prohibit the graviton from being a bound state of gauge bosons and how is this circumvented by AdS/CFT?

Proof

The proof, which we will now quickly review following [88], is elegantly presented by considering the following matrix element for *massless* one-particle states, composite or not, with four-momenta p and p', and helicities j

$$\langle p', \pm j | \theta^{\mu\nu} | p, \pm j \rangle \,. \tag{2.2}$$

On the one hand the generic assumptions ensure it to *not vanish.* However for particles with $j > 1$, one may show that it *has to vanish*, which leads to a contradiction for these spins.

To show that it does not vanish, observe that *Lorentz invariance* ensures it be equal to a non-vanishing structure in the limit of no momentum transfer $p' \to p$:

$$\langle p', \pm j | \theta^{\mu\nu} | p, \pm j \rangle \to \frac{p^\mu p^\nu}{E(2\pi)^3} \neq 0\,, \tag{2.3}$$

so in particular $\langle p' | \theta^{00} | p \rangle \sim E$ for the physically measured energy of a one-particle state. Terms like $\eta^{\mu\nu}$ are prohibited because $\theta^{\mu\nu}$ is conserved. Given that $p' \to p$, only the structure $p^\mu p^\nu$ appears and part of the assumption is also that the *energy-momentum tensor does receive contributions* from the quantum field that corresponds to the one-particle state $|p, \pm j\rangle$.

On the other hand, $(p' - p)^2 \neq 0$ allows us to go to a rest-frame, in which $p' + p$ is time-like, so that the in-going and outgoing states move in *opposite* spatial directions. We may perform a rotation by an angle φ around this axis and the states or the energy momentum tensor transform as representations of this rotation such that rotational invariance leads to

$$e^{\pm 2i\varphi j} \langle p', \pm j | \theta^{\mu\nu} | p, \pm j \rangle = R(\theta)^\mu{}_\rho R(\theta)^\nu{}_\sigma \langle p', \pm j | \theta^{\rho\sigma} | p, \pm j \rangle \tag{2.4}$$

Now, the matrix $R(\varphi)^\mu{}_\nu$ can only have eigenvalues $e^{\pm i\varphi}$ or 1 and so for $j > 1$ rotational invariance implies $\langle p', \pm j | \theta^{\mu\nu} | p, \pm j \rangle = 0$. This is true in all frames due to Lorentz invariance/covariance and in contradiction to the aforementioned result that it must not vanish. Therefore massless particles with spin $j > 1$ are prohibited.

Consequences and ways out

General relativity, understood as an effective quantum field theory for massless spin 2 gravitons, should better circumvent this. But how? The answer lies in the general coordinate invariance of general relativity, which on a linearised level turns into a gauge symmetry for the graviton:

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu \Omega_\nu(x) + \partial_\nu \Omega_\mu(x) \tag{2.5}$$

This transformation property is of course important for reducing the unphysical degrees of freedom of the graviton to the remaining two physical polarization states. It is entirely analogous to the case of a massless spin 1 gauge boson A^μ. This transforms *not* like a Lorentz vector, but rather according to [25]

$$U(\Lambda,0)A^\mu(x)U^{-1}(\Lambda,0) = \Lambda^\mu{}_\nu A^\nu(\Lambda x) + \partial^\mu \Omega(x,\Lambda)\,. \tag{2.6}$$

Only, when assuming interactions which are invariant under both $A^\mu \to \Lambda^\mu{}_\nu A^\nu$ and $\delta A^\mu = \partial^\mu \Omega(x)$, we get consistent interactions for massless helicity ± 1 particles getting rid of the unphysical modes.

Analogously, the graviton does *not* transform as a Lorentz tensor either, but rather as

$$U(\Lambda,0)h^{\mu\nu}(x)U^{-1}(\Lambda,0) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma h^{\rho\sigma}(\Lambda x) + \partial^\mu \Omega^\nu(x,\Lambda) + \partial^\nu \Omega^\mu(x,\Lambda)\,. \tag{2.7}$$

We may build Lorentz-covariant energy-momentum pseudo-tensors $\theta^{\mu\nu}$ within general relativity, but these will not preserve general covariance [145]; there are no local gauge invariant observables in general relativity! So, when eliminating the unphysical degrees of freedom to fix the gauge, the energy-momentum tensor will not be Lorentz invariant anymore. Or said differently, $\theta^{\mu\nu}$ cannot be Lorentz covariant in a consistent quantum mechanical sense, in which it should also be invariant under 2.5, the gauge symmetry or general coordinate invariance. Therefore the Weinberg-Witten theorem does not prevent the understanding of gravity as a quantum theory of massless helicity ± 2 particles.

What it does prohibit however is the understanding of the graviton as a bound state of gauge bosons A^μ like gluons, in the ordinary sense of all fields living in four-dimensional Minkowski space. The bound state can just not account for the required general coordinate invariance (2.5) that is needed to circumvent the theorem.

Gauge/gravity duality circumvents the theorem in a very different way. The five-dimensional graviton may roughly be understood as a bound-state of four-dimensional gauge bosons, or more precisely being dual to it. The five-dimensional argument is then the same as above. We may construct energy-momentum tensors $\theta^{\mu\nu}$, but they will not be Lorentz-invariant. That was however an assumption of the Weinberg-Witten theorem and given that it is not satisfied, its conclusion need not apply. The graviton then however is non-dynamical in the four-dimensional field theory. This has a Lorentz-invariant conserved energy-momentum tensor, but the one-particle states of the field theory have spin ≤ 1.

2.1.2 Holographic principle

As we have just seen, the Weinberg-Witten theorem may be evaded by evoking a new dimension. Gravitons can be understood as a bound state of gauge bosons, in a dual sense as we will see later, however only if the assumptions of the Weinberg-Witten theorem are not valid. Different space-time dimensionality is one way to refrain from these.

Of course, this goes very well along with the holographic principle of section 1.3. The entropy of a coarse-grained quantum gravitational system like a black hole does not scale as its volume, but rather as its area. The effective degrees of freedom are therefore apparently captured by a lower-dimensional non-gravitational theory. Since we want a higher-dimensional graviton with all its emerging general coordinate reparameterisation invariance and we could in principle get it from lower-dimensional gauge bosons – at least the Weinberg-Witten theorem does not prohibit it – we may try to push this idea even further.

But there seems to be another obstacle related to the degrees of freedom counting. How can it be that the lower-dimensional gauge theory indeed describes the same degrees of freedom as a higher dimensional theory? Usually, one would assume the higher-dimensional theory to have many more degrees of freedom – just from the existence of an additional dimension. One-particle states are labelled by momenta in the space they propagate in [87], or said differently, a graviton may in a semi-classical treatment still move almost freely in all directions.

To also capture this rather classical effect, it is natural to ask for the lower-dimensional degrees of freedom to somehow be enlarged significantly.

How could we do this? As we have just asked for a lower-dimensional gauge theory of some kind, there is not much freedom. The number of spins for elementary fields is bounded, the space-time dimensionality is fixed; the most natural parameter to use for increasing the effective number of degrees of freedom seems to be to increase the number of colours N. We may thus take

$$N \to \infty \,. \tag{2.8}$$

In such a limit, we expect not to reveal the full quantum nature of quantum gravity, but rather its classical limit.

2.1.3 Large N limit of gauge theories

Such a large N limit of gauge theories had already been considered by 't Hooft in the 1970's [144]. Effectively, the Feynman diagrams in a perturbative treatment of gauge theories like $SU(N)$ Yang-Mills theory reorganise in a surprising way: In this $N \to \infty$ limit the original coupling constant g_{YM} is not really appropriate anymore for governing the perturbative order of individual Feynman diagram contributions, which it was for fixed N. Instead, Feynman diagrams are ordered according to a series in $1/N$ as

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \,. \tag{2.9}$$

The different orders in this sum describe different topologies of the Feynman diagrams, g being the genus of the surface. At every such order the diagrams obey a perturbative expansion

$$f_g(\lambda) = \sum_{i=0}^{\infty} c_{g,i} \lambda^i \,. \tag{2.10}$$

(ignoring logarithmic contributions etc.) in terms of the coupling constant $\lambda = g_{YM}^2 N$, which is held fixed in the large N limit.

The derivation of this is rather simple (we follow [9]). Writing the Yang-Mills action as

$$\int d^4x \left(-\frac{1}{2g_{YM}^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \right) , \tag{2.11}$$

we see that in an arbitrary Feynman diagram propagators scale like $g_{YM}^2 = \lambda/N$. For fields Φ^i_j in the adjoint of the gauge group – i and j are colour/anti-colour or fundamental/anti-fundamental indices respectively – the propagator has an index structure

$$\langle \Phi^i_j \Phi^k_l \rangle \propto \left(\delta^i_l \delta^j_k - \frac{1}{N} \delta^i_j \delta^k_l \right) \tag{2.12}$$

suggesting a double line notation with oriented lines (i.e. arrows) from a colour to the next contracted anti-colour index in every diagram, which of course has to be overall colour neutral. The generators of the adjoint representation of $SU(N)$ are trace free, thus we get the second correction term, which however is subdominant in the large N limit.

Every such closed loop in the double line notation contributes a factor of $\delta^i_i = N$ and the non-Abelian interaction vertices of (2.11) come along with $1/g_{YM}^2 = N/\lambda$. Thus a generic Feynman diagram with V vertices, E propagators and F loops comes at an order

$$N^{V-E+F} \lambda^{E-V} . \tag{2.13}$$

We may understand the diagram as a two-dimensional compact surface by adding a point at infinity – like the generalisation of the stereographic projection of a sphere to the plane, which is the simplest case of planar diagrams $\sim N^2$. Then topologically V, E and F may be understood as vertices, edges and faces of a simplex. The combination $V - E + F = \chi = 2 - 2g$ can be seen as the Euler characteristic of this surface and the expression in terms of the genus g holds since our surface is both closed and oriented, giving the promised perturbative series (2.9).

Relation to string theory

The perturbation series in terms of $1/N$, which is of topological nature, is very reminiscent of the way (oriented) closed string amplitudes without any vertex operator insertions on the string world-sheet are organised. In this analogy $1/N$ corresponds to the string coupling g_s. For example to lowest order the sphere amplitude, $g = 0$ in the string model, corresponds to the planar diagrams at order N^2 in the gauge theory. Since string theory does provide a consistent theory of quantum gravity, we see that we are on the right track for describing a theory of quantum gravity in terms of a gauge theory. But since critical superstring theory is defined in ten dimensions and we only considered gauge theories in four dimensions, we still need to understand how the different dimensions are interconnected in a holographic way.

Also, we see that at leading order in N, the string theory is dominated by tree-level closed string amplitudes, which is basically *classical* string theory. Every string theory contains a graviton in its spectrum; so by this reasoning we may justify our assertion on p. 22 that $N \to \infty$ should correspond to some classical gravitational theory. Sending also the string length scale $\sqrt{\alpha'} \to 0$ decouples the higher mass string modes. In string theory, Newton's constant G_N is proportional to the square of the string coupling $G_N \propto g_s^2$ as can be seen from the NS-NS part of the effective supergravity actions. So, we get

$$G_N \propto \frac{1}{N^2}\,. \tag{2.14}$$

But Newton's constant has length dimensions. In order for it to be parametrically small we need to compare it to a different length scale, which is going to be the AdS radius. The same argument holds for α' as well, which one also needs to compare to the AdS radius.

On the field theory side, the large N limit does not really correspond to a classical limit like $\hbar \to 0$, although one might naïvely think so, since $S \sim \frac{N}{\lambda} \int d^4x\, F_{\mu\nu}^2$. There are still loop contributions even in the planar limit. This makes the duality we are slowly approaching very interesting of course; that is we describe a quantum mechanical theory via a classical gravitational theory.

In the above elaborations, we have left out many interesting extensions, which may also be included: Fields in the fundamental representation of the gauge group correspond to boundaries of the surfaces ($\leftrightarrow$ open strings); other gauge groups may be related to non-orientable string theories and most importantly one may extend the reasoning from vacuum diagrams to insertions of gauge-invariant operators which then relate to string theory vertex operator insertions in a consistent description of $g_s \leftrightarrow 1/N$.

Instantons

Also non-perturbative objects like instantons may be mapped to one another, although we have so far only told the perturbative story of the large N limit. Also in this case we do not really need to go through the whole exact duality from string theory, but can argue relatively straightforwardly:

A detailed analysis of the moduli space of multi-instantons for $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ shows it to be exactly[2] $\mathrm{AdS}_5 \times \mathrm{S}^5$ [147]. But already a quite simple argument which we will present soon, when we explicitly deal with the radial dimension, directly yields the AdS part of $\mathrm{AdS}_5 \times \mathrm{S}^5$ by only considering $SU(2)$ one-instantons on S^4 [148]. Now of course these have finite action [149]

$$S = \frac{8\pi^2}{g_{YM}^2} \sim \frac{N}{\lambda}\,. \tag{2.15}$$

Since we argued that $1/N$ corresponds to the string coupling g_s, we see that the object on the gravity theory side, which seems to correspond to the gauge theory instanton, also contributes a non-perturbative effect to the quantum gravity path integral scaling like e^{-1/g_s}. So, the natural objects the Yang-Mills instantons will be mapped to are D-branes or more specifically D-instantons, i.e. $D(-1)$ branes [150]. Their moduli space is of course nothing else but $\mathrm{AdS}_5 \times \mathrm{S}^5$ itself, the space-time on which they are localised.

The heuristic relations, which we have just illuminated are made more precise by the AdS/CFT correspondence, in which we may study a specific gauge theory, $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$, being related to a specific quantum theory of gravitation, type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[2] A nice summary is presented in [146].

2.1.4 The radial scale and energy

But before we present the string construction, there is more we can argue about the additional dimension on the gravity side. As we have already mentioned, the moduli space of $SU(2)$ one-instantons on S^4 is AdS$_5$ [148].

First of all, let us recall that a solution to the self-dual instanton equations is provided by the classical Euclidean background [149]

$$A_\mu(x) = \frac{x^2}{x^2+z^2}\, g(x)^{-1}\partial_\mu g(x)\,, \qquad g(x) = \frac{x_4 + i x_i \sigma_i}{\sqrt{x^2}}\,. \tag{2.16}$$

It is important to note that the instanton background depends on five parameters[3], the Euclidean position and the size z: Translations lead to an instanton solution at a different place with x^μ shifted and space-time dilatations effectively rescale z.

What is now the moduli space for these parameters? For answering this (section 4.5 of [148]), it is useful to look at the conformal transformation of A_μ [151]: Inversions basically interchange instanton and anti-instanton up to a gauge transformation, rotations can be undone by an $SU(2) \times SU(2) \simeq SO(4)$ gauge transformation and special conformal transformations may be compensated by a combined translation and gauge transformation. In total, this invariance closes into $SO(5)$. The Euclidean conformal group $SO(1,5)$ takes a solution to a different (but equivalent) solution. Any of these solutions is however independent under a subgroup $SO(5) \subset SO(1,5)$. Therefore the moduli space is parametrised by the coset $SO(1,5)/SO(5)$, which however *is* Euclidean AdS$_5$ seen as a $SO(5)$-invariant hypersurface in six dimensional flat space with $SO(1,5)$ Lorentzian invariance.

Given the AdS radius L, the metric of AdS space in Poincaré coordinates is

$$ds^2 = \frac{L^2}{z^2}\left(dx^\mu dx_\mu + dz^2\right)\,, \tag{2.17}$$

or upon a coordinate change $r = L^2/z$ we can write it as

$$ds^2 = \frac{r^2}{L^2}\, dx^\mu dx_\mu + \frac{L^2}{r^2}\, dr^2\,. \tag{2.18}$$

Looking at the metric (2.17), we see that it is Lorentz invariant and in particular scale invariant under

$$x^\mu \to \lambda x^\mu\,, \qquad z \to \lambda z\,, \qquad r \to \lambda^{-1} r\,. \tag{2.19}$$

as is also apparent from the instanton solution (2.16). So given that the extra-dimension z corresponds to a generic length scale in the field theory, the size of the instanton, r represents an energy scale. We thus have collected our first hint, that our holographic description takes us in the simplest case to the AdS$_5$ gravitational description of a four-dimensional *scale-invariant* and actually *conformal* field theory!

[3]There are two important points to be considered for a slightly more rigorous treatment of the $SU(2)$ one-instanton (see sections 1.1.2 and 1.5.1 of [146]). Actually, the $SU(2)$ orientation should also be counted, because it corresponds to the *global* part of the gauge group, yielding the famous $4kN$ number of bosonic collective coordinates of a k-instanton in $SU(N)$ gauge theory. Also, the fermionic zero modes are of importance for yielding the AdS$_5$ measure. Only when counting these as well, $z = L^2/r$ can be seen as the radial AdS direction.

So, we might think of the radial scale as *geometrising the energy scale of the field theory.* This goes along with another very interesting observation, the fact that the renormalisation group equations of quantum field theory are *local* in energy scale. They are in general non-linear, which fits to the non-linearity of gravitational physics, but the RG equations describe the evolution of coupling constants *at* a particular energy scale[4] [87]. Also there are examples from QCD phenomenology, where one can argue that the energy scale behaves as an additional dimension, as remarked in [86].

AdS space has a boundary at $z = 0$ and we may quite literally think of the field theory being defined on that boundary. This is very much supported by the holographic counting of degrees of freedom [152], which we will present now (see [132]).

Holographic counting

How does the general holographic counting of degrees of freedom work? On the one hand, an $SU(N)$ gauge theory has of course an infinite number of degrees of freedom. But we may regulate the theory by introducing a UV cutoff ϵ_{QFT} and putting the theory into a box of scale R. Then the degrees of freedom will be finite and given by the number of fields in each cell, $\sim N^2$ for an $SU(N)$ gauge theory, times the number of cells, i.e.

$$N_{d.o.f.} \sim \left(\frac{R}{\epsilon_{QFT}}\right)^3 N^2 \tag{2.20}$$

Now the holographic principle asserts that this is captured by the regulated area of the boundary in Planck units. Does this work given AdS_5 space with metric (2.17)? Again the result is infinite, because the spatial directions x^i are infinitely extended, but also because areas diverge close to the boundary at $z \to 0$. We may regulate both again. The coordinates x^μ of AdS are identified with the ones of the field theory, thus the regulator R is identical to the one used before. For the $z \to 0$ divergence, we see that the divergence also has to be regulated by introducing a cutoff at $z = \epsilon_{AdS}$. However, it is not clear how this is mapped to the QFT regulator ϵ_{QFT}. The gravity side clearly allows for this preferred regulator, but there are many regularisation procedures in quantum field theory like dimensional regularisation, momentum cutoff or Pauli-Villars to name just a few standard ones. It is not quite clear how the regulators are mapped to one another [125].

But for our rough degree of freedom estimate, we need not bother with this subtle question and may just compute the area which encloses AdS space. It is given by

$$A = \int_0^R d^3x \sqrt{g}\Big|_{z=\epsilon_{AdS}} = \left(\frac{RL}{\epsilon_{AdS}}\right)^3 . \tag{2.21}$$

Then, according to the holographic principle, the maximal entropy which may be stored in this regulated space-time is given by this huge area in Planck units

$$\frac{A}{4G_N} = \left(\frac{R}{\epsilon_{AdS}}\right)^3 \left(\frac{L^3}{4G_N}\right) . \tag{2.22}$$

[4] We will have more to say about this and the relation of physics in AdS space to Wilsonian effective actions in section 3.3.

So, indeed the degree of freedom counting is of a holographic kind given that we assume a rough map of regulators. Large distance phenomena in AdS space[5] correspond to short distance, i.e. UV phenomena in the field theory – this is the IR/UV connection [152, 80]. The equivalence of (2.20) and (2.22) exactly needs the general relation of Newton's constant and the gauge theory's number of colours N mentioned earlier around (2.14), i.e.

$$\frac{L^3}{G_N} \propto N^2 . \tag{2.23}$$

This closes the narrative of how a gauge theory in the large N limit may holographically be related to a theory of gravity in AdS space. Most naturally, the Feynman diagrams of large N gauge theories are mapped to classical string theory amplitudes. Field theory instantons may be related to $D(-1)$ instantons living on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Thus, AdS space is the natural geometry to make the geometrisation of the field theory's energy scale concrete, when the field theory is conformal. We may now tie together these already very impressive observations and finally make them precise by providing the first explicit example [6].

2.2 The AdS/CFT correspondence

Historically, the key to uncovering the string theory realisation of the holographic principle via AdS/CFT lay in the physics of a stack of D3-branes in particular limits [6]. In particular, their low energy effective world-volume theory plays the role of the field theory side of the duality and seeing them as extremal black branes in IIB supergravity leads to $\mathrm{AdS}_5 \times \mathrm{S}^5$ as its near horizon geometry, the geometry on which the full string theory may be studied.

But more generally speaking, it is only important to study superstring theory on a particular AdS background without referring to it as the near-horizon geometry of a stack of branes. $\mathrm{AdS}_5 \times \mathrm{S}^5$ is just a particular maximally supersymmetric background, which solves the equations of motion of IIB supergravity and therefore is a convenient string theory background. One may justifiably look at other AdS backgrounds and then try to directly relate them to a dual conformal field theory, but usually it is very difficult to understand *which* conformal field theory one is actually talking about. So the main advantage in looking at intersecting brane constructions or the mentioned stack of D3-branes is that one may really explicitly deduce the dual field theory. It is however not a matter of principle and one expects many other dual field theories to the AdS flux vacua [153].

But let us start with the original D-brane "derivation" of the duality. As we already mentioned, one of the key insights which led to the discovery of the AdS/CFT correspondence was the different views on D-branes, which were first discovered along with orientifold planes in [59, 154]. These have two interpretations either as end-points of open strings or as black branes depending on parameter ranges.

There are plenty of detailed introductions to this, e.g. [9, 143] which will be among the main sources we draw from in the following. So we will restrict to the most important points and highlight some aspects, which will be important for later chapters. These include the relation of AdS/CFT to earlier absorption cross section

[5]The region $z \approx 0$ corresponds to a large distances $r \to \infty$, i.e. to IR phenomena on the gravity side.

calculations, the dimensions of operators dual to bulk fermions or the relation of chiral anomalies and Chern-Simons terms.

2.2.1 D-branes and open strings

On the one hand, D-branes may be understood as end-points of open strings which satisfy Dirichlet boundary conditions. T-duality of open strings on a compactified circular dimension interchanges Dirichlet and Neumann boundary conditions. Via this observation open strings whose endpoints were before freely moving in space may be tied to a higher-dimensional hypersurface, the D-brane, which is dynamical on its own.

The string end-points comprise dynamics along and perpendicular to the brane. If we quantise the open string and look at the first level of the open string spectrum

$$\alpha^{\mu}_{-1}\,|0\rangle \tag{2.24}$$

the string oscillators with μ perpendicular to the brane may be regarded as world-volume scalars. Those parallel to it carry a vectorial space-time index μ along the brane's world-volume and may thus be regarded as photons/gauge bosons. Among the D-branes with odd dimensionality, those within type IIB superstring theory, is the D3-brane with four world-volume space-time directions and six dimensions perpendicular; so the low energy effective world-volume theory on the brane includes a four-dimensional gauge boson and six scalars. Given that the insertion of a D3-brane in flat space halves the full supersymmetry of string theory and that a world-volume supersymmetry originates from this (basically as a kind of projection), we also get four Weyl fermions from the fermionic string modes. All these modes assemble into an $\mathcal{N}=4$ supersymmetry vector multiplet and the low energy theory on the world-volume of a D3-brane is already from the point of supersymmetry uniquely given by $\mathcal{N}=4$ super Yang-Mills theory with gauge group $U(1)$.

This may also be seen from considering the low-energy effective action of open string theory [155], the DBI action [156,157], which to lowest order in the α'-expansion yields

$$\mathcal{L}_{\mathrm{DBI}} = \mathcal{L}_{\mathrm{SYM}} + (\alpha')^2\,\mathrm{Tr}\,F^4 + \dots\,, \tag{2.25}$$

the non-Abelian Yang-Mills theory generalisation being discussed in [158]. From inspection of the quadratic term, we may however read off the field theory's coupling constant in terms of the string parameters. The DBI action[6] expands

$$S_{\mathrm{DBI}} = -\frac{2\pi}{(2\pi\sqrt{\alpha'})^4}\int d^4x\, e^{-\Phi}\sqrt{-\det\left(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} \tag{2.26}$$

$$\approx -\int d^4x \left(1 + \frac{1}{2g^2_{YM}}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}\right) \tag{2.27}$$

where the constant term is cancelled by the corresponding Wess-Zumino term $\sim \int d^4x\, C_4$ because of the aforementioned BPS property and higher order corrections are suppressed for small field strengths relative to α'. The Yang-Mills coupling thus relates to the string coupling as

$$g^2_{YM} = 4\pi g_s\,. \tag{2.28}$$

[6]The pull-back of the metric to the brane's world-volumne is left impilcit.

This already shows that $g_sN \sim \lambda$ for the 't Hooft coupling $\lambda = g_{YM}^2 N$, which is the natural coupling constant in the large N limit of a gauge theory.

A stack of N D3-branes yields an extended non-Abelian gauge group $U(N)$ with all fields in the adjoint representation [159]. The open strings then not only stretch from one brane to itself but carry matrix-valued labels, Chan-Paton factors, that assign which of the stacked D-branes they start and end on. If we separate k of the branes from the original stack of N branes, the gauge group breaks from $U(N)$ to $U(N-k) \times U(k)$. Strings that stretch from one stack to the other are in the bifundamental representation of $U(N-k) \times U(k)$ and their length r determines the mass of the corresponding W-bosons via

$$m_W^2 \sim r^2/(\alpha')^2 \,. \tag{2.29}$$

This energy scale, which corresponds to a Higgs effect in the $U(N)$ gauge theory, will be of importance at several points in subsequent sections. On the one hand we will keep it fixed while discussing D-branes as black p-branes, but it already is a more precise hint to understanding the radial coordinate r as an energy scale of the field theory compared to 2.1.4.

The picture just described is clearly from the point of view of string perturbation theory. But when is it valid? Now additionally to a factor of $g_s \ll 1$ at each further loop order, for every amplitude there is also another factor of N from the Chan-Paton labels. Intuitively, this may be understood because every loop may end on either of the N branes. For example the annulus open string one loop amplitude may be related via open-closed duality to a tree-level closed string amplitude, where a closed string is emitted from any one of the N branes. So, g_s and N always appear together at each loop order and string perturbation theory is therefore valid for

$$g_sN \ll 1 \qquad \text{or} \qquad \lambda \ll 1 \,. \tag{2.30}$$

2.2.2 Black p-branes and gravity

On the other hand D3-branes can be regarded as the long sought for sources of Ramond-Ramond flux [160], which curve space-time and therefore are described by closed string dynamics (which contains gravity). They *are* the black p-branes discovered a few years earlier [161]. The low energy effective action of the closed sector of type IIB superstring theory [162] is given by type IIB supergravity in ten dimensions, where higher curvature contributions[7] are suppressed by higher orders of α',

$$\mathcal{L} = \mathcal{L}_{\text{IIB SUGRA}} + \alpha' R^2 + (\alpha')^2 R^3 + (\alpha')^3 R^4 + \dots \,. \tag{2.31}$$

The equations of motion of the dominant contribution are solved by the following extremal black brane solution, which carries N units of R-R flux

$$\int \star F_5 = N \,. \tag{2.32}$$

Metric and R-R four-form are given by

$$ds^2 = H(r)^{-1/2}\, dx_{||}^2 + H(r)^{1/2} dx_\perp^2 \tag{2.33}$$

$$(C_4)_{0123} = 1 - H(r)^{-1} \,, \tag{2.34}$$

[7] We show them only schematically, see e.g. [163] and/or references in [164] for details.

in which $H(r) = 1 + \frac{L^4}{r^4}$ and $r^2 = x_\perp^2$ (the dilaton is constant). The length L may be related to the string coupling g_s, number of branes N and the string length scale $\sqrt{\alpha'}$ via

$$L^4 = 4\pi g_s N(\alpha')^2 \,. \tag{2.35}$$

D-branes are BPS objects, i.e. the state they are described by preserves part of the theory's supersymmetry. This is very much related to the fact that one may stack them on top of each other; gravitational attraction and RR charge repulsion exactly cancel. Looking at the metric ansatz (2.33), the supergravity equations of motion easily show, that $H(r)$ has to be a harmonic function of the transverse space $\triangle_\perp H(r) = 0$, whose linear structure directly shows that different D3-branes may be put at different points $\vec{x}_{\perp,i}$ in transverse space

$$H(r) = 1 + 4\pi g_s(\alpha')^2 \sum_i^N \frac{1}{(\vec{x}_\perp - \vec{x}_{\perp,i})^4} \tag{2.36}$$

or on top of each other. We may therefore understand the $1/r^4$ dependence just from the perspective of a Coulomb/Newton law in six transverse dimensions and the superposition naturally explains the factor N in (2.35).

Also the factor g_s is readily understood: The D-brane is a non-perturbative object, which has a tension that parametrically scales as $1/g_s$, very similar to a field theory soliton, for which however usually the exponent is -2. This may be equated with the p-brane's ADM mass $\sim L^4/\kappa^2$, in which Newton's constant appears as $G_N \sim \kappa^2$. Since we know from the NS-NS sector that $\kappa^2 \sim g_s^2$, we find $L^4 \propto g_s$.

A different way to argue this would be that tree-level open string interactions are at order $1/g_s$. The effect of the term L^4/r^4 may for $g_s N \ll 1$ be understood as a tree-level closed string effect, i.e. the exchange of gravitons etc. describes an effective Newtonian potential. By open-closed duality this effect may be described by an open string *one loop* diagram, the annulus amplitude, which results into an additional factor of g_s.

Since there is no other length scale in string theory apart from $\sqrt{\alpha'}$, we get the factor of $(\alpha')^2$ in (2.35) on dimensional grounds, where the remaining numerical factor may be obtained by a more detailed analysis of the mentioned arguments.

What about the validity of this black p-brane description of D3-branes? The solution (2.33) is clearly valid, when curvatures are small compared to the string scale; then higher curvature terms are suppressed. Since the components of the Riemann tensor of the solution (2.33) are on dimensional grounds set by L via $R \sim 1/L^2$, hence the classical supergravity solution is valid for

$$\frac{L^4}{(\alpha')^2} \gg 1 \qquad \text{or} \qquad g_s\, N = \lambda \gg 1 \,. \tag{2.37}$$

So, we see that (2.30) and (2.37) comprise totally inequivalent parameter regimes. From the point of view of string theory, the combination $g_s N$ appears either as the string perturbation theory expansion parameter or as the parameter which describes the validity of the supergravity approximation. From the point of view of pure field theory, the $g_s N$ does however also have the interpretation of the 't Hooft parameter in the large N limit of a generic gauge theory.

2.2.3 D-brane entropy and absorption cross sections

As we have seen there are maximally inequivalent parameter regimes for the two different descriptions of D-branes. But just from the fact that they do have these two interpretations, a lot of possibilities arise. In particular, one may count microscopic degeneracies of particular bound state configurations and by this provide a statistical derivation of the Bekenstein-Hawking entropy formula [65]. Generically, the Bekenstein-Hawking formula is of course valid only in the supergravity regime, in which string perturbation theory suffers from large quantum corrections. But for particular BPS configurations quantum corrections are under good control. In [65] a particular D-brane bound state was analysed, for which the string BPS state degeneracy is a protected quantity and it was thus possible to show agreement with the black hole result although the parameter regimes are inequivalent.

Given a quantum mechanical origin of the Bekenstein-Hawking formula, one may ask if analysing the aforementioned configuration or similar ones may also shed light on the black hole information paradox. Can one probe the microscopic degrees of freedom by absorption/reemission of low energy quanta by D-branes, in which the string perturbation picture would manifestly be unitary and the black hole picture would directly addresses Hawking's argument?

Most stunningly was the consideration of the absorption process in the case of D3-branes [165, 166, 167]. In the D-brane picture, one may consider how e.g. an external massless scalar like the bulk dilaton may be absorbed by the D-brane. Given the DBI effective action for the massless string modes (2.26), one knows the coupling to the world-volume gauge fields

$$S_{\rm int} \sim \int d^4x\, \Phi\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\,. \tag{2.38}$$

It is now a simple field theory computation (at weak coupling) to calculate the tree-level absorption cross section for such an external scalar of energy ω, which dissolves into the gauge fields. At leading order it is given by

$$\sigma_{\rm abs} = \frac{\kappa^2 N^2 \omega^3}{32\pi}\,. \tag{2.39}$$

Clearly, the factor N^2 comes from the degeneracies of gauge bosons the dilaton couples to. The dependence on κ stems from the non-canonical normalisation of Φ's kinetic term $\frac{1}{2\kappa^2}\int d^{10}x\,(\partial\Phi)^2$ and the factor of ω^3 then follows on dimensional grounds[8].

Likewise, we could have done the computation for a transverse graviton, which couples to the $\mathcal{N}=4$ super Yang-Mills' energy momentum tensor $T^{\mu\nu}$ via [166]

$$\int d^4x\, \frac{1}{2} h_{\mu\nu}T^{\mu\nu}\,. \tag{2.40}$$

Although the graviton may decay into the whole $\mathcal{N}=4$ vector multiplet, in the end the total sum of the absorption cross sections completely agrees with (2.39). Note that in both cases, the coupling of the external bulk fields is to *single trace gauge invariant*

[8] In ten dimensions the natural absorption cross section dimensionality is eight. Seeing it as "cross section per unit longitudinal volume of the brane" [165], we get down to five length dimensions. Noticing that Newton's constant in ten dimensions $G_N \sim \kappa^2$ already contributes eight, we arrive at the remaining factor $\sim \omega^3$.

operators. This observation will be of tremendous importance when we formulate the AdS/CFT dictionary [7, 8] in section 2.2.5.

Furthermore, since

$$T_{\mu\nu} \sim \mathrm{Tr}\left(F_\mu{}^\rho F_{\rho\nu} + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \dots\right) \tag{2.41}$$

we notice how the idea of having the graviton as a bound state of gauge bosons is implemented (see section 2.1.1): The graviton couples to the energy momentum tensor, which basically is such a strong coupling bound state $\mathrm{Tr}\,(\partial A\partial A)$; however also the other fields of the $\mathcal{N}=4$ multiplet contribute to it[9]. Momentarily, in section 2.2.5, the interpretation will be further refined and the graviton is then understood as being *dual* to exactly this operator, the energy-momentum tensor.

Now, we may ask what the absorption cross section of an s-wave minimally coupled scalar of energy ω by the gravity background (2.33) is. The precise computation [165] involves the matching method, in which the linearised gravitational wave equation is solved far away from the black brane and close to its horizon. Expansions in a particular overlapping region are then used to match the solutions and compute the absorption probability. This is basically given by the tunneling probability through a potential barrier, where for higher partial waves the bar is raised. For s-waves it is thus easiest to be absorbed by the black hole. The tunneling probability is then given in terms of the dimensionless quantity ωL since L sets the scale of the barrier; modes with higher frequency ω relative to L are more likely to be absorbed.

On dimensional grounds, one may argue for another factor $\sim 1/\omega^5$ for computing the cross section from the absorption probability to arrive at [165]

$$\sigma_{\mathrm{abs}} = \frac{\pi^4}{4\omega^5}(\omega L)^8\,. \tag{2.42}$$

which is identical to (2.39) given (2.35) and the precise relation of Newton's constant $\sim \kappa^2$ to g_s! Also the classical absorption computation for a transverse graviton completely matches (2.39). Its physical transverse components satisfy a minimally coupled scalar equation, such that the agreement of the gravity computation with the dilaton case is not surprising.

So, as for the black hole entropy counting [65], we find agreement of two related absorption cross section quantities, although they are defined in completely different parameter regimes. The gravity computation is hereby governed by the expansion parameter (2.35)

$$(\omega L)^4 \sim (g_s N)(\omega^2\alpha')^2 \ll 1\,. \tag{2.43}$$

Given that we also have $g_s N \gg 1$ to suppress finite string size corrections and $g_s \ll 1$ to suppress string loop corrections, we arrive at a double scaling limit [165]

$$g_s N \gg 1 \qquad \text{and} \qquad \omega^2\alpha' \ll 1\,. \tag{2.44}$$

in which naturally also the energy of infalling waves should be small compared to the string scale to trust the supergravity analysis.

The same way the entropy counting was protected by supersymmetry, there is also a non-renormalisation theorem [167] connecting the two absorption cross section

[9] The full tensor may be found in [167].

results from receiving corrections in λ. The Adler-Bardeen theorem [168] protects the one-loop exactness of anomalous contributions to the $\mathcal{N}=4$ super Yang-Mills $SU(4)$ R-symmetry currents, which by supersymmetry is related to the trace anomaly of the energy-momentum tensor $\langle T^\mu_\mu \rangle$ [100]. This then relates to the two-point function of the energy-momentum tensor [169] $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$, from which the absorption cross section may be deduced. The absorption cross section is then only determined by the c anomaly coefficient ($c = N^2/4$ for $\mathcal{N}=4$ SYM) via

$$\sigma_{\text{abs}} = \frac{c}{8\pi}\,\kappa^2 \omega^3 \tag{2.45}$$

This relation then extends to the dilaton case. So, again we see a very non-trivial agreement of world-volume field theory and supergravity analysis which was only possible due to particular non-renormalisation properties of the involved theories.

So, most naturally, it is suggestive that these are the same theory in different parameter regimes. But to make this statement more precise, we have to go back to the D-branes and the two interpretations, analyse the particular limits and then arrive at a revolutionary observation [6]. We are going to extensively draw mainly from [9] in the following sections.

2.2.4 The Maldacena limit

So, what is the low energy effective theory from full type IIB superstring theory with open and closed sectors, where a stack of N D-branes, seen as the hyperplane on which open strings end, is put in flat ten dimensional Minkowski space?

The low energy effective action at energies well below the string scale $1/\sqrt{\alpha'}$ contains the DBI action, describing the massless modes of the open sector, type IIB supergravity for the massless modes coming from the closed sector and interactions of both. But this is only half-way towards the desired low energy limit. We may go a step further and may also consider the low-energy limit of this combined system.

For excitations of fixed energy this means we can equivalently send the string mass scale to infinity, i.e. $\alpha' \to 0$, for which the DBI action reduces to the action of $\mathcal{N}=4$ super Yang-Mills theory with gauge group $U(N)$ as already explained in (2.26).

The supergravity sector becomes free in this limit since (g_s kept fixed)

$$\kappa^2 \sim g_s^2 (\alpha')^4 \to 0\,. \tag{2.46}$$

The supergravity action contains no mass parameters/coupling constants except for the dimensionful κ. So, if we canonically normalise the participating fields, it is clear that their relative interactions are suppressed by some power of this dimensionful parameter. In essence, this is the same argument as why Einstein gravity is non-renormalisable at the level of power-counting.

Likewise the interaction terms between the open and closed sectors vanish. Everything in the effective action is governed by the string length scale. If we send it to zero only the power-counting renormalisable interactions remain. For the open sector, this is $\mathcal{N}=4$ super Yang-Mills theory, for the closed sector it is free supergravity in flat space.

Next we want to compare the above reasoning with the situation, in which the D3-branes are understood as black p-brane solutions (2.33) of type IIB supergravity.

In this situation one may distinguish two kinds of perturbative excitations: those close to the brane and those far away. As we have already observed in the considerations on the absorption cross section of s-wave scalars by the black brane, the key to understanding and distinguishing these two excitations lies in the understanding of the Newtonian potential barrier. The tunneling probability for massless particles from the outer to the inner region is strongly suppressed for low energy, i.e. large wavelength excitations. The excitation's wavelength $\sim 1/\omega$ is so large, that it cannot resolve the black brane of size L anymore. These excitations are described by free supergravity in flat space.

On the other hand, low energy excitations close to the brane will not easily tunnel through the potential barrier either (nor climb it). As measured from infinity, the energy of particles close to the horizon is heavily redshifted the closer they are to the horizon. So we may actually consider all kinds of perturbations (even genuine string modes) close to the horizon $r \approx 0$ and regard them as low energy excitations. What we want to keep fixed in the $\alpha' \to 0$ low energy limit, which we also take on this side, is the mass scale (2.29) set by W-bosons if one of the branes is separated from the rest. So, while taking α' to zero, we also send $r \to 0$, keeping r/α' fixed. For the two terms in $H(r)$ of (2.33), we see that one is dominated by the other

$$\frac{L^4}{r^4} = 4\pi g_s N \left(\frac{(\alpha')^4}{r^4}\right) \frac{1}{(\alpha')^2} \gg 1\,. \tag{2.47}$$

Therefore, in the near horizon region $r \ll L$ we have $H(r) \to L^4/r^4$. Writing the transverse six-dimensional space in polar coordinates $dx_\perp^2 = dr^2 + r^2 d\Omega_5^2$, we thus arrive at

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \tag{2.48}$$

which is the metric of $\mathrm{AdS}_5\times \mathrm{S}^5$ with identical radii for AdS space and the sphere. So, we see, that the modes close to the horizon are supergravity/string theory modes on the background $\mathrm{AdS}_5\times \mathrm{S}^5$.

Hence, from the point of view as regarding D3-branes as black p-branes, we again get two sectors, which decouple: free supergravity in flat space and string theory on $\mathrm{AdS}_5\times \mathrm{S}^5$.

So, from both points of view we get two decoupled systems. In both cases, one is free supergravity in flat space. Assuming that both systems describe one and the same object, i.e. D3-branes are the black p-branes for $p = 3$, we also identify the corresponding other one and arrive at the identification:

$\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$

$\equiv$

type IIB superstring theory on $\mathrm{AdS}_5\times \mathrm{S}^5$ with N units of five-form flux

Before we make this more plausible by shortly considering some of the symmetries of the two theories and especially comment on the aforementioned coupling constants and possible corrections in these, let us quickly go into one detail, which we have skipped over hastily. In the field theory argument we have so far mentioned $U(N)$ gauge theories and in the above claim, only the $SU(N)$ part of it appears. This has the following reason(s) [8, 170]: On the one hand, the $U(1)$ part of $U(N) = SU(N) \times U(1)$ decouples and describes a free field theory. But AdS gravity cannot account for this, since in the gravity theory everything couples at least to gravity. The remaining $U(1)$ part can be thought of as describing the overall position of the stack of branes: There are six scalars in the vector multiplet clearly having this interpretation.

Another argument comes from the dual supergravity spectrum, which just does not contain a field with the corresponding mass [8, 171] (anticipating the AdS/CFT dictionary). The difference between $SU(N)$ and $U(N)$ single trace operators lies in the existence of so-called singleton operators like $\mathrm{Tr}\,\Phi^I$, where Φ^I for $I = 1, 2, 3$ describes one of the three chiral superfields of $\mathcal{N} = 4$ super Yang-Mills theory. For $SU(N)$ the trace over the generators of the algebra's adjoint representation automatically vanishes, for $U(N)$ it does not. The conformal dimension of $\mathrm{Tr}\,\Phi^I$ satisfies the unitarity bound $\Delta \geq (d-2)/2$ and thus describes a free field (as stated before) yielding a consistent picture of the difference between $U(N)$ and $SU(N)$ gauge groups.

Global symmetries

Coming back to our introduction, we see that the only parts that actually matter for Maldacena's duality are the low energy field theory and the string theory on the background $\mathrm{AdS}_5 \times \mathrm{S}^5$. There are a couple of immediate checks for this duality, like the match of global symmetries [6]: $\mathcal{N} = 4$ super Yang-Mills theory has an $SU(4) \simeq SO(6)$ R-symmetry which, in simplified terms, describes the $SO(6)$ rotation of the six scalars of the multiplet amongst each other or the corresponding $SU(4)$ rotation of the four Weyl fermions[10]. This corresponds to the $SO(6)$ isometry of the S^5. Furthermore, $\mathcal{N} = 4$ super Yang-Mills is conformal, the conformal group in four dimensions being $SO(2, 4)$. This is identical to the isometry of AdS space as we have already seen earlier, when we presented the foresighted argument dealing with the moduli space of field theory one- instantons (p. 25, although we had the Euclidean version $SO(1, 5)$ earlier).

Furthermore, the number of supersymmetries matches. On the supergravity side, one may show that $\mathrm{AdS}_5 \times \mathrm{S}^5$ is maximally supersymmetric, i.e. there are as many Killing spinors as for flat space[11] and this supersymmetry is connected with the aforementioned bosonic symmetries into the superconformal group $SU(2, 2|4)$ [172]. The field theory has 16 normal supercharges as $\mathcal{N} = 4$ indicates, but also 16 further supersymmtries are realised non-linearly closing the super-Poincaré and conformal group into the superconformal group.

Also S-duality of the two theories is immediately recognised and related to each other. Type IIB superstring theory possesses an $SL(2, \mathbb{Z})$ symmetry which mixes dilaton and RR axion, see e.g. [173] with its geometric realisation. Also $\mathcal{N} = 4$ super Yang-Mills has this symmetry, mixing coupling constant and θ parameter of the theory [38]. We already related the Yang-Mills coupling constant to the string coupling, the expectation value of the dilaton, in (2.28); also axion and θ parameter are related.

[10] More precisely, it rotates the supercharges Q^I_α with I an $SU(4)$ fundamental index.

[11] For a quick argument see [9].

Coupling constants

What are now the different parameters of both sides of the duality? On the field theory side, we have the coupling constant g_{YM} and the number of colours N or in 't Hooft's large N limit

$$\lambda = g_{YM}^2 N \qquad \text{and} \qquad N\,. \tag{2.49}$$

On the string theory side, we have g_s, α', the number of five-form flux quanta N, the radius of curvature L and Newton's constant $G_N \sim \kappa^2$. But these are not independent of each other. Clearly, we have

$$L^4 = 4\pi g_s N(\alpha')^2 \qquad \text{and} \qquad \kappa^2 = \frac{1}{2}(2\pi)^7 g_s^2(\alpha')^4\,. \tag{2.50}$$

The first relation has already been argued for after we first stated the D3-brane metric on p. 29. The relation for the gravitational coupling constant simply originates from the NSNS sector effective action, the expectation value of the dilaton and dimensional analysis. As usual, one may choose units in which e.g. the Planck mass is set to one or equivalently, $L = 1$, which is sometimes convenient. In essence, we are left with two parameters, which are the same as for the field theory (remember (2.28)) and govern finite string length and string perturbation theory corrections via

$$\alpha' \sim \frac{1}{\sqrt{g_s N}} \sim \frac{1}{\sqrt{\lambda}} \qquad \text{and} \qquad \kappa^2 \sim \frac{1}{N^2}\,. \tag{2.51}$$

As we have argued earlier, the classical supergravity approximation is valid for $N \to \infty$ and $\lambda \to \infty$ (p. 30), while the planar field theory's perturbative description is valid for $N \to \infty$ and $\lambda \to 0$ (p. 29).

Since the duality in not proven[12], although overwhelming support in favour of it exists, one may distinguish several limits of the conjectured duality:

- Full $\mathcal{N} = 4$ super Yang-Mills theory at all values of λ and N is dual to full quantum type IIB superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ with N units of five-form flux.
- *Planar* $\mathcal{N} = 4$ super Yang-Mills theory, i.e. the 't Hooft limit $N \to \infty$ at fixed λ is dual to *classical* type IIB superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$.
- The *large* λ limit of *planar* $\mathcal{N} = 4$ super Yang-Mills theory is dual to *classical* type IIB *supergravity* on $\mathrm{AdS}_5\times\mathrm{S}^5$.

As mentioned earlier, part of the problem at hand is the lack of a known quantisation of string theory on the curved background $\mathrm{AdS}_5\times\mathrm{S}^5$. However, the flat space perturbative string spectrum is of course known. In [84], it was shown how to reproduce this from the field theory. Even more so, it was derived how to reproduce the perturbative string spectrum for string theory on a pp-wave background, which arises as a Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$ and is a maximally supersymmetric background [174]. In that background the string modes may be quantised similarly to flat space [175]. In the world-sheet action in light-cone gauge, they acquire a mass, however they are

[12] One would need full non-perturbative quantisation of IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$, which is extremely difficult.

still free and quantisation is essentially as straightforward as in the massless case. In the dual field theory one considers states with parametrically large R charge $J \sim \sqrt{N}$ and finite $\Delta - J$. These results very much suggest that the field theory knows much more about the dual gravitational theory than just about the supergravity fields.

Wilson loops and Wilson lines are interpreted such that the field theory Wilson line extends into the bulk as a fundamental string sweeping out a minimal area surface [55, 56]. In a way this incorporates the old pre-QCD motivation for studying string theory as describing the flux tubes in the theory of strong interactions.

Also other genuine string theory objects like its various branes arise [148] as baryon vertices or instantons as already anticipated earlier around p. 24. We have seen that the field theory instantons are directly mapped to string D-instantons which have $\mathrm{AdS}_5 \times \mathrm{S}^5$ as their moduli space.

The baryon vertex is also straightforward to understand [148]: N external quarks on the boundary have fundamental strings stretching into the bulk with the same orientation. They may now end on a D5 brane, which wraps the S^5. The gauge field on the D5 brane now couples to N units of charge coming from the five-form flux $\int F_5 = N$, which have to be cancelled by N units of charge -1 coming from the N strings. So, indeed the wrapped D5 brane plays the role of a baryon vertex.

Tests at finite N exist too, related to the existence of a baryon vertex. In [170], the global symmetry arising from the $\mathbb{Z}_N$ center of the $SU(N)$ gauge group was matched when the field theory is put on a non-simply-connected manifold. On the one hand, this gives another reason, why the field theory's gauge group is actually $SU(N)$ and not $U(N)$, which would have a center $U(1)$, but furthermore it works at finite N.

Although not a precise test, one may also argue that IIB supergravity as a quantum theory is not complete although $\mathcal{N} = 4$ on the other hand is. So, it seems compulsory to also embed the supergravity part of the duality into a quantum theory of gravity, which in this case most naturally is quantum type IIB superstring theory.

We see that generically various tests and expectations for this specific example suggest that the duality is correct in its strongest form, although this is not conclusive of course.

What is however still unclear is how general this duality is for other more bottom-up motivated invocations of the duality. One may e.g. quite straightforwardly perform certain $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ computations in arbitrary space-time dimension d and we will actually do exactly this for a gravitino later in chapter 5. Everything in the computations looks fine and may indeed be carried out in arbitrary dimension, but it is very well known [176] that there is no supergravity theory for more than eleven dimensions, e.g. the supergravity algebra does not close anymore at the non-linear level. But computations for a gravitino assume a supergravity theory from the very beginning, since the gravitino is the corresponding gauge field of local supersymmetry. Also there are no superconformal algebras beyond six dimensions respectively no simple AdS supergroups beyond AdS_7 [172] – not to mention the critical dimension of string theory. In many cases of bottom-up holography one knows that there cannot be a further completed dual along the usually assumed lines of a supergravity theory although the computation on its own does not show this at all. In our case we will work at the linearised level, at which one may easily write down a supergravity theory in any dimension [176]. In a more intricate way [177], our computation in all d will also reveal features of non-conformal holographic duals [178] which do obey the above dimensionality constraints.

Therefore, it is always important to keep in mind that in many cases the duality will only work in simplified limits, in a more intricate way or just not work at all; in others it seems very likely that it is true in its strongest form.

2.2.5 The dictionary

We have now given plenty of reasons, why the AdS/CFT conjecture appears to be true. But what we have left out so far is maybe the most important aspect of the duality: How does one translate the AdS description into the CFT one? What is the dictionary?

The key to identifying this dictionary lies in the boundary conditions [7,8], which one has to impose at the boundary of AdS space[13]. Together with the intuition, which we gained by considering the two absorption cross section pictures [165, 166, 167] (section 2.2.3), the identification will be quite natural.

Since a conformal field theory does not have an S-matrix due to the lack of non-interacting asymptotic states[14], we have to map fields in AdS to gauge-invariant operators in the CFT which are the sensible observables.

But how exactly? As we have seen earlier from analysing the DBI action, the bulk dilaton field Φ couples to the CFT's single-trace gauge-invariant operator $\frac{1}{4g_{YM}^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}$ (2.38); the bulk graviton $h_{\mu\nu}$ couples to the field theory's energy-momentum tensor $T_{\mu\nu}$ (2.40), which is also single-trace and gauge-invariant. Also the generalisation to gauge fields/global currents via

$$\int d^4x\, J^\mu(x) A_\mu(x) \tag{2.52}$$

is straightforward. In general, this suggests a correspondence between the *single-trace gauge-invariant local operators* of the CFT and the *single particle states* in the AdS gravity theory, generalising to multi-trace operators and multi-particle states.

The duality thus comes into play, when we observe how the operators on one side couple to the fields on the other side. The coupling is via the gravity fields' boundary values. So, let us be more general and look at a generic bulk field $\phi(x,z)$ (for simplicity a scalar), which couples to a CFT operator $\mathcal{O}$ via

$$\int d^4x\, \phi_0(x)\mathcal{O}(x)\,, \tag{2.53}$$

where $\phi_0(x) = \phi(x,z)\big|_{z=0}$ is the restriction of $\phi(x,z)$ to the AdS boundary at $z=0$, a Dirichlet condition. Basically, we have to think of $\phi_0(x)$ as an *external source* for the operator $\mathcal{O}(x)$.

Both, field and operator have to have corresponding quantum numbers of the global symmetry $SU(2,2|4)$ for this coupling to be allowed[15]. From the supergravity

[13]Here we will consider the coordinate system (2.17), for which the AdS boundary is at $z=0$.

[14]A CFT has no length scale, therefore interactions are long-ranged.

[15]Clearly, the gauge degrees of freedom should not show up, because they are redundancies of the description. Thus we have *gauge-invariant* operators and gauge-fixed (physical) particle fields. The gauge redundancies are emergent. A nice example for such a phenomenon appears in condensed matter physics [87] and references therein, where sometimes an electron field $e(x)$ can split up in boson $b(x)$ and fermion $f(x)$ via $e(x) = b(x)f(x)^\dagger$. Then rotating boson and fermion by a local phase $e^{i\lambda(x)}$ is an emergent gauge symmetry.

point of view however, it is natural to expand the fields in spherical harmonics of $SO(6)$ [171], which then results into mass terms in the gravity theory on AdS; on the field theory, the most important quantum number of an operator $\mathcal{O}$ is its conformal dimension Δ, which we will momentarily relate to the aforementioned mass.

Then we would like to define a generating functional for correlation functions of this operator $\mathcal{O}$ in the CFT. This is then naturally identified with the partition function of the string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ supplemented with the boundary condition $\phi \to \phi_0$ or more concretely [7, 8]

$$\langle e^{\int d^4x\, \phi_0(x)\mathcal{O}(x)} \rangle_{\mathrm{CFT}} = Z_{\mathrm{string}} \left[\phi(x,z)\big|_{z=0} = \phi_0(x) \right] . \tag{2.54}$$

Given certain boundary conditions for the gravity theory on AdS, we may then solve the equations of motion subject to these boundary conditions and then on-shell understand the right hand side as being exactly what we wrote on the left hand side: a generating functional that depends on these boundary conditions.

In the large N, large λ limit, we may approximate the right hand-side by the exponential of the classical type IIB supergravity effective action.

$$Z_{\mathrm{string}} \left[\phi(x,z)\big|_{z=0} = \phi_0(x) \right] \approx e^{-I_{\mathrm{SUGRA}}} \tag{2.55}$$

and obtain the generating functional for connected Green's functions from evaluating at saddle points

$$W[\phi_0] = I_{\mathrm{SUGRA}}\big|_{\text{on shell}} , \tag{2.56}$$

with regularisation at $z = \epsilon$. The outlined prescription gives a procedure from which arbitrary n-point functions may be computed. The easiest ones are of course two-point functions, which are very much constrained by the conformal symmetry. Matching these is therefore not really a true check of the duality but rather enforced due to the underlying symmetries. However, from looking at these, one may unambiguously fix the operators' normalisations such that already three-point functions provide a very non-trivial test of the dynamics. As with all the tests we are going to mention, AdS/CFT also passed this one impressively [179, 180]. For comparing the three-point functions of both sides of the duality one had to work out the three field coupling on the string theory side, which was the most difficult part in the computation. Much more on three-point functions and their interesting renormalisation may be found in [143].

Scalars

Let us look at a specific standard example, that of a minimally coupled massive scalar in AdS_{d+1} (the description (2.54) easily generalises to higher dimensions). From its equation of motion

$$(\Box - m^2)\phi = 0 \quad \Leftrightarrow \quad \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) - m^2 \phi = 0 \,, \tag{2.57}$$

we easily observe that near the boundary the solutions, which are essentially modified Bessel functions, behave as z^{Δ} or $z^{d-\Delta}$ with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2} \,, \tag{2.58}$$

the exponents being the two solutions to $\Delta(\Delta - d) = m^2L^2$. For $m^2 = 0$, the term $z^{d-\Delta} = z^0$ describes the source for $\mathcal{O}$ as stated earlier, in which the notation $\phi(x,z)\big|_{z=0} = \phi_0(x)$ is just fine while $z^\Delta = z^d$ is subdominant. But for $m^2 \neq 0$, we see that our above notation (2.54) runs into trouble. We have to adjust for a more general description, which defines an external source that couples to an operator $\mathcal{O}$ in the field theory. Thus, we must renormalise[16]

$$\phi(x,\epsilon) = \epsilon^{d-\Delta}\phi_0(x)\,. \tag{2.59}$$

This then defines an analogous condition, in which we take

$$\lim_{\epsilon\to 0} \epsilon^{\Delta-d}\phi(x,\epsilon) = \phi_0(x)\,. \tag{2.60}$$

The computation so far was for a dimensionless field $\phi(x,z)$; we thus see that ϕ_0 has length dimension $\Delta - d$. Because the measure d^dx in (2.54) carries d length dimensions, the operator $\mathcal{O}$ has mass dimension Δ. This may be directly confirmed more concretely by computing the two-point function $\langle\mathcal{O}(x)\mathcal{O}(0)\rangle \sim 1/x^{2\Delta}$, which is obliged to be of this form due to conformal symmetry. As outlined above, one has to insert the full solution into the gravity action and by doing so arrive at a functional of ϕ_0. This is then interpreted as a generating functional for correlation functions, which one may compute via functional derivatives $\delta/\delta\phi_0$.

We see that one of the key aspects is that the conformal dimension Δ relates to the mass m^2L^2 in AdS units of the supergravity field (2.58). This relation generalises for fields of different spin (see [9] and references therein):

- scalars: $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}$
- spinors: $\Delta = \frac{d}{2} + |mL|$
- vectors: $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{(d-2)^2}{4} + m^2L^2}$
- spin 3/2: $\Delta = \frac{d}{2} + |mL|$
- massless spin 2: $\Delta = d$

The massless spin 2 field is easy to understand. Its dual operator is just the energy-momentum tensor, which clearly has dimension d; e.g. in its OPE with itself, the dominant short distance term is universally given by the conformal anomaly $\sim c/x^{2d}$. As we explained earlier, the correlators of the energy-momentum tensor are protected by supersymmetry and a non-renormalisation theorem (see p. 33 and [167]).

The vector case can also partly be understood without any computation. For $m^2 = 0$, it should be dual to a conserved current, which on dimensional grounds ($\leftrightarrow$ gauge invariance of the gauge field!) has to have dimension $d - 1$.

The generalisations to a Proca field or higher form fields require short computations similar to (2.57).

[16]It is very similar to the Z factors in usual renormalisation theory.

Spinors

Let us quickly give a rough overview on the spinorial cases, since those will be important for us later on. Compared to (2.57), it is clear, that we do not get a quadratic relation between conformal dimension and mass simply because the equations of motion are first order. However, the first order nature of the equations of motion also implies that when they are satisfied and the on-shell action is computed, the fermionic contributions seem to vanish. It was shown that supplementing the fermionic part with an additional boundary term solves the problem (for spin 1/2 fermions see [181, 182] and for spin 3/2 fermions see [183, 184]).

We first look at the near boundary asymptotics of a spinor field. The equation of motion for a Dirac spinor ψ in AdS$_{d+1}$ (2.17) is

$$(\nabla\!\!\!/ - m)\psi = 0\,, \tag{2.61}$$

where the radial dependence of the spin connection terms is not completely obvious. But one may perform a simple conformal rescaling of this (cf. [185]) into flat space, where the spin connection vanishes. This we may always do, say in $d+1$ space-time dimension, if we also rescale mass terms via $(\Omega^{d/2}\psi, \Omega^{-2}g_{\mu\nu}, \Omega m)$. Choosing $\Omega = L/z$, we recover

$$\left(\partial\!\!\!/ - \frac{mL}{z}\right)\left(\frac{L^{d/2}}{z^{d/2}}\psi\right) = 0 \tag{2.62}$$

in flat space with metric $\eta_{\mu\nu}$. Here, the near-boundary asymptotics of the spinor $\sim z^{-d/2}\psi$ is clear and one may already see $\Delta = \frac{d}{2} + |mL|$ for the dual operator to ψ appear[17]. For spin 3/2 fields the arguments are essentially the same (only with additional Christoffel symbols, which in the end also vanish after the conformal rescaling).

Reading off the conformal dimension requires the existence of a coupling of the boundary value of $\psi(x,z)$, which we denote by $\psi_0(x)$, to a spinorial operator $\bar{\mathcal{O}}$. This will be ensured by a proposed boundary term of the form

$$\sim \lim_{\epsilon\to 0}\int d^dx\,\sqrt{-G_\epsilon}\bar{\psi}\psi\,. \tag{2.63}$$

where G_ϵ is the induced metric originating from (2.17) and the normalisation has to be determined e.g. by supersymmetry, relating it known normalisations[18]. However, the boundary spinors are spinors in a dimension less than the bulk ones and we want to fix half of the degrees of freedom of the bulk spinors via a boundary condition as before in the scalar case. So, how does this work? Given a boundary spinor $\psi_0(x)$, we may use the equations of motion to integrate for the whole spinor $\psi(x,z)$. Now, one may show that to ensure normalisability of the solution[19] we have to set

$$\psi_{0,+}(x) = 0 \qquad \text{and} \qquad \bar{\psi}_{0,-}(x) = 0 \tag{2.64}$$

for the Γ^z-chiral boundary spinors $\Gamma^z\psi_{0,\pm} = \pm\psi_{0,\pm}$ with $\psi_0 = \psi_{0,+} + \psi_{0,-}$. When the boundary dimension d is even, then Γ^z may indeed be understood as the theory's

[17] We will assume $m > 0$ from now on; the arguments for $m < 0$ will be similar, but some of the boundary conditions/asymptotic behaviours of ψ and $\bar{\psi}$ have to be interchanged.

[18] We will perform exactly such a computation to fix a boundary term normalisation later on p. 127.

[19] This corresponds to choosing the boundary condition for the scalar such that the $z^{d-\Delta}$ term has a vanishing coefficient.

chirality matrix, the generalisation of γ_5 in usual four-dimensional field theories. In this case ψ_0 and $\bar{\psi}_0$ form two independent Weyl spinors since the relative other chirality part has been set to zero in the aforementioned normalisability condition. In odd dimensions there is no chirality matrix and Γ^z just projects on two independent Dirac spinors.

With these boundary conditions, the solutions to the equations of motion for given boundary data will furnish the aforementioned asymptotics

$$\lim_{\epsilon\to 0} \epsilon^{-d/2+m}\psi(x,z) \sim -\psi_{0,-}\,, \tag{2.65}$$

$$\lim_{\epsilon\to 0} \epsilon^{-d/2+m}\bar{\psi}(x,z) \sim \bar{\psi}_{0,+}\,. \tag{2.66}$$

We see, that we get a consistent picture for fermions as well. The two-point function computation immediately follows from the prescribed boundary term and the solutions.

We have identified the dominant term in the near boundary expansions as an external source for a dual operator both in the case of a scalar, as well as in a fermionic example. These modes are non-fluctuating and describe the background by defining the field theory's sources [186]. So when performing computations for some field in an existing asymptotically AdS background, one may set these modes to zero, unless one wants to introduce new sources.

What is the interpretation of the other solution, which is subdominant? First of all, one needs to point out that it is normalisable, in contrast to the aforementioned solution. In [187] it was shown that it determines the expectation value of the dual operator given a certain solution. Thus for a certain radial profile for e.g. a scalar, we may read off this very component if the operator dual to this scalar has condensed i.e. if it possesses a non-vanishing expectation value. To mention one example, this is of particular importance for the studies of holographic superconductors [133, 134, 135], where one has a charged operator which condenses similar to the Cooper-pair operator in BCS superconductors.

Furthermore, we have to mention that we only dealt with the *standard quantisation* involving $\Delta = \Delta_+$, which is the only valid quantisation for $m^2 > -\frac{d^2}{4}+1$. For particular lower masses, there is however a window [188]

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1\,, \tag{2.67}$$

in which both solutions are normalisable and either can be seen as source or expectation value depending on one's quantisation scheme [189] (the other one with Δ_- as the operator's conformal dimension is usually referred to as *alternative quantisation*). This realisation filled a gap in the understanding of allowed operator dimensions, in which only an operator with $\Delta = \Delta_+$ seemed to be allowed. Looking at the scalar case $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}$, we would have $\Delta_+ \geq d/2$, which would be significantly above the unitarity bound for the scalar field theory operators $(d-2)/2$. The conformal dimensions in between should however also be accessible by AdS/CFT, for which explicit examples are stated in [189]. The two quantisation schemes are related by a Legendre transform of their respective generating functional.

In addition, it is noteworthy that certain negative values of m^2 are indeed allowed as long as they are not too below the Breitenlohner-Freedman bound

$$m^2 \geq -\frac{d^2}{4}\,. \tag{2.68}$$

Lower masses would indeed trigger a classical instability, which may be very interesting from the point of view of the field theory.

2.2.6 AdS/CFT and anomalies

Of course, there are many further checks of the duality which it has impressively passed. Let us mention a few.

The structure of chiral anomalies is also exactly reproduced when $\mathcal{N} = 4$ super Yang-Mills is coupled to external gauge fields [8]. The theory has a global $SU(4)$ R-symmetry, which we already mentioned to be related to the $SO(6)$ rotation symmetry of the five-sphere. The global currents of the field theory couple to the gauge fields of the gravitational theory (2.52).

Then the conservation of the currents is broken in the usual way arising from a triangle diagram between R-symmetry currents with the chiral fermions running in the loop

$$(\mathcal{D}^\mu J_\mu)^a = \frac{N^2-1}{384\pi^2} d^{abc} \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma} \tag{2.69}$$

with $SU(4)$ structure constants d^{abc}. This effect is precisely matched (in the large N limit, where $N^2 - 1 \approx N^2$ by a Chern-Simons term of the gravitational theory's gauge fields

$$\frac{iN^2}{96\pi^2} \int_{\mathrm{AdS}_5} d^5x \left(d^{abc} \epsilon^{\mu\nu\lambda\rho\sigma} A^a_\mu \partial_\nu A^b_\lambda \partial_\rho A^c_\sigma + \ldots \right) . \tag{2.70}$$

Performing a gauge transformation of this, we exactly reproduce a boundary term, which describes the non-conservation of the above current.

This match of a parity violating Chern-Simons term on the one hand and a chiral anomaly will be of huge importance in our work in chapter 4, which builds on known results [113, 114, 115] reviewed in section 3.2.3.

Another beautiful check of anomalies, which we would like to at least mention, is the one of conformal anomalies in the theory [190], which started the precise holographic renormalisation program [191, 192, 193]. For our subsequent work, this will however not be of such a great importance.

2.3 Finite temperature and chemical potentials

There are many generalisations of the AdS/CFT correspondence which are interesting and worth studying. For our particular computations later on two ingredients will be of particular importance, the role of black holes/finite temperature configurations and the addition of chemical potentials for conserved charges.

The role of black holes is quite straightforward to understand. If we insert a large black hole into the bulk space-time then this will correspond to a deconfined finite temperature phase of the field theory.

Taking large N and large λ, from (2.54) we obtain the relation between supergravity action and generating functional of connected Green's functions

$$e^{-W} = e^{-I_{\mathrm{SUGRA}}} . \tag{2.71}$$

At finite temperature, one may relate the generating functional to the free energy $W = \beta F$, where β is the inverse temperature. Now there are usually several saddle-points which contribute to I_{SUGRA}; one of them may be a black hole in AdS space. For these configurations, the relation $\beta F = I_{\text{SUGRA}}$ has been the long-standing observation of Euclidean quantum gravity treating black holes in the canonical ensemble [194]. Basically one identifies the finite temperature state of the field theory with the one on the gravity side.

From the given relation one may directly obtain the black hole's entropy S, after the Hawking temperature T is computed by avoiding the conical singularity at the horizon in Euclideanised space; the periodicity then directly determines β. The entropy then indeed agrees with the Bekenstein-Hawking formula [194]. Also the first law of thermodynamics is reproduced, the pressure P being related to the free energy F via $P = -\partial_V F$. We once more see the power of the dual description: the field theory does describe the microstates of the black hole in the way outlined by Strominger and Vafa [65]. So, AdS/CFT makes the observations of [65] even more concrete: It correctly reproduces the black hole's entropy by a string theoretic microstate counting according to [65], but also identifies the whole field theory which effectively describes these microstates. Interestingly, the free field theory computation of the free energy only differs by a factor of 3/4 compared to the supergravity one [195]

$$F_{\text{SUGRA}} = \frac{3}{4} F_{\text{SYM}}\,, \tag{2.72}$$

which does not quite suggest a non-renormalisation theorem of it, but is still a striking fact. In [164] small corrections in $\lambda^{-3/2}$ were computed on the gravity side by considering higher curvature contributions and a smooth interpolating function was suggested which starts off at 1 for $\lambda \approx 0$ and reaches 3/4 at $\lambda \to \infty$, where supergravity is valid.

It is however clear, that the supergravity action will receive contributions from different saddle-points [196,8,197]. In the case of black holes, it makes a huge difference how big or small they are relative to the AdS size. In the case of small black holes the configuration of a gas of particles in AdS is energetically favoured in comparison to the black hole state. So, one may actually undergo a thermal phase transition, which on the dual field theory side is interpreted as a confinement/deconfinement transition [8, 197].

Likewise to putting the field theory at finite temperature, which corresponds to having a non-extremal black hole in the bulk, we may consider a black hole which is charged under one of the $U(1)$ gauge fields [198], i.e. an AdS Reissner-Nordström black brane [131]:

$$\begin{aligned}
ds^2 &= \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right) \\
f(z) &= 1 - \left(1 + \frac{z_+^2\mu^2}{\gamma^2}\right)\left(\frac{z}{z_+}\right)^d + \frac{z_+^2\mu^2}{\gamma^2}\left(\frac{z}{z_+}\right)^{2(d-1)} \\
\gamma^2 &= \frac{(d-1)g^2L^2}{(d-2)\kappa^2}
\end{aligned} \tag{2.73}$$

with gauge field profile

$$A_t = \mu\left[1 - \left(\frac{z}{z_+}\right)^{d-2}\right]. \tag{2.74}$$

The gauge field is massless, so this will correspond to a conserved current with $\Delta = d - 1$ in the field theory. We may now understand the non-normalisable and normalisable components $z^{d-\Delta-1} = z^0$ and $z^{\Delta-1} = z^{d-2}$ as source and expectation value[20] of a conserved current operator J^μ. In our Reissner-Nordström solution these are exactly the two terms for the gauge field A_t, μ is the chemical potential and the other term is up to a normalisation the expectation value for the charge operator J^0.

We may think of the current as being the number current for e.g. a $U(1)$ baryon symmetry of the field theory. This number operator generically couples to the corresponding chemical potential μ, which will therefore be identified with the non-normalisable term in the $z \to 0$ component of the time component of the gauge field, as we already indicated by our above notation.

Being a bit more explicit, we can have a coupling

$$\bar{\psi}(\nabla\!\!\!/ + A\!\!\!/)\psi \tag{2.75}$$

in the gravity action. Then at the boundary we have $A_t \sim \mu$ and the number operator $\hat{N} \sim \psi^\dagger\psi$. In thermodynamics of grand canonical ensembles, we thus get the familiar term $\sim \mu N$ in the Gibbs free energy. This will also automatically be present in Euclidean quantum gravity [194].

So, we see that finite temperature and finite chemical potential configurations are implemented in a straightforward manner. They correspond to putting a black hole in AdS space with a finite charge.

[20]Note that zA_t roughly behaves like a scalar [131]. That is why additional factors of $1/z$ appear compared to the previous scalar relations $z^{d-\Delta}$ and z^Δ.

Chapter 3

General relativity and hydrodynamics

Einstein's general theory of relativity is one of the beautiful pillars in the current understanding of our universe.

Firstly, it describes various phenomena in gravitational physics which can hardly be described otherwise within a self-consistent framework. The manifold successes with experimental verification range from the early description of the perihelion precession of Mercury and predictions like light deflection by the sun and gravitational redshift effects considered already by Einstein [43] up to a modern understanding of the various phases of cosmological evolution.

Moreover, Einstein's theory of general relativity is formulated in a way which many physicists perceive as beautiful [10, p.132ff]. This beauty is however mostly not of a subjective kind, the way this notion is often used in association with pieces of art or music. But there is a deeper sense of rigour and inevitability associated to it. As Einstein himself wrote in a letter to *The Times* on November 28, 1919 [199]

> "The chief attraction of the theory lies in its logical completeness. If a single one of the conclusions drawn from it proves wrong, it must be given up; to modify it without destroying the whole structure seems to be impossible."

A simple example for the inevitability of the precise predictions of general relativity, as compared to Newton's theory of gravitation, is the inverse square law for the gravitational force between two massive bodies. In Newton's theory this law could have been slightly different with the exponent to deviate from -2 e.g. at a subsequent digit. In Einstein's theory the exponent is exactly -2 within the Newtonian approximation. There are interesting and important correction terms to it, but the Newtonian result is rigorously derived. Newton's force is explained as descending from the curvature of space-time and thus only an approximate concept.

Ultimately, the rigour of the predictions of general relativity stems from the powerful concepts which led to its discovery: the principles of special relativity together with the equivalence principle. Furthermore, it may be described starting from an action principle. So, the wealth of experimental confirmations goes along with a structure with very few, very powerful underlying concepts making clear predictions.

The beauty also extends to solutions of general relativity's field equations. These have many solutions, but most notably and mysteriously among them are the black hole solutions. On the one hand they are the most simple solutions possible, as remarked by Chandrasekhar [200, p.1]:

> "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well."

In four dimensions they are only characterised by very few parameters (see [201] and references therein): mass, charge and angular momentum which characterise the Kerr-Newman black hole solution[1].

On the other hand quantities like the area of their horizon and their surface gravity can be given interpretations in terms of thermodynamic quantities, entropy and temperature [44, 46], suggesting that they are a coarse-grained macroscopic description of microscopic degrees of freedom. For an outside observer, these degrees of freedom may be described in terms of the ones of a hot membrane with properties known from the effective theory of hydrodynamics [122].

Departing from the familiar four dimensions of space-time, the wealth of gravitational solutions is increased. There can be black hole solutions with non-spherical horizons like black rings [202] or black Saturns [203] and within the supergravity/string theory context there are solutions with different charges and/or extended planar horizons [204, 161].

From these, one may make the statistical nature of the black hole entropy clear [65] and derive many further beautiful concepts like the AdS/CFT correspondence. For studying field theories at finite temperature the AdS/CFT correspondence leads back to general considerations of black holes in spaces with a negative cosmological constant.

The fluid/gravity correspondence [109] in asymptotically AdS spaces extends and clarifies a further beautiful direct connection of black hole physics to hydrodynamics: Via a systematic derivative expansion, one may construct black hole space-times with regular (!) event horizons which in a dual way describe hydrodynamical systems slightly out of local equilibrium.

All the mentioned aspects of the connection of black hole physics and hydrodynamics may explicitly be found within the general structure of the blackfold paradigm [118, 119]. Recently, this has been shown very explicitly in [123] and we are going to extend this explicit analysis in chapter 4 with regards to one of the conceptually most interesting results of the fluid/gravity correspondence: the appearance of transport coefficients [113, 114] which appear due to a quantum anomaly in the underlying microscopic field theory [115].

But before we get there, we give a detailed overview of the concepts and ideas which underlie this project. In section 3.1, we start off with a detailed review of black hole thermodynamics, the black hole membrane paradigm and black hole complementarity. In section 3.2 we present the fluid/gravity correspondence reviewing the basic aspects in terms of one of the basic but exemplary computations before

[1] In the prologue to his book Chandrasekhar actually refers only to mass and angular momentum – the Kerr solution – since astrophysical black holes are usually assumed to be chargeless. But of course the charged extension is also studied in detail in [200].

we account for the anomaly related computations. Section 3.3 is devoted to the physics on a cutoff surface at finite radial distance between black hole horizon and boundary. Interesting concepts that will be explained are the Wilsonian holographic renormalisation group flow and the general AdS Dirichlet problem. The latter will play an eminent role in the computation of [123] and our generalisation as shown in the subsequent section, which also features an introduction to the blackfold paradigm.

3.1 The membrane paradigm

In short, the black hole membrane paradigm makes concrete the notion that for an observer outside its horizon the black hole, or rather the horizon itself, can be regarded as a hot membrane exhibiting properties like resistivity or shear and bulk viscosities.

Historically, the membrane paradigm way of thinking about black holes arose from an evolutionary process of understanding some of their astonishing properties. In particular, the relation of these to usual concepts of thermodynamics proved to be a useful guiding principle (we partly follow [205] and [206]).

Originally, the known black hole solutions were just seen as gravitational backgrounds which e.g. induce redshift effects or explain corrections to the gravitational potential. After the discovery of the Schwarzschild and Reissner-Nordström solutions decades passed until further backgrounds like the Kerr [207] and Kerr-Newman [208, 209] metrics were found in the 1960's. But also these were for quite some time primarily seen as stationary backgrounds without any dynamics on their own.

The research about the global dynamics of these black hole solutions started, when it was realised that it is possible to extract part of the energy from a rotating black hole via the Penrose process [210]. Since classically nothing can leave the interior of a black hole, not all the energy may be removed. Only the rotational part of the energy may be extracted, effectively stopping the black hole from rotating; but there still remains the mass of a non-rotating black hole. This observation lead to the distinction of reversible and irreversible mass of a black hole [211, 212], similar to reversible and irreversible processes in thermodynamics. By also considering the collision of two black holes Hawking proved the area increase theorem [45], which is similar to the second law of thermodynamics that describes entropy increase. In the simple case of a Schwarzschild black hole, part of this area increase is very natural to understand. The Schwarzschild radius is proportional to the mass of the black hole and if matter cannot escape a black hole, the area of its horizon can therefore only grow. But the key to understanding this more rigorously, also for stationary black holes, laid in the analysis of the collision.

The similarities of black hole mechanics and thermodynamics were striking and culminated in the formulation of the four laws of black hole mechanics [213], although the true nature of the black hole's temperature and entropy at that time were still mysterious: Since a black hole does not radiate classically, it was difficult to understand how it could have a temperature at all, meaning: how could it be in thermal equilibrium with an outside system although it cannot radiate [213]? The interpretation of the black hole's area as its entropy S [44] and its surface gravity as its temperature T required an explanation of a radiation process of some kind, which was finally provided by Hawking in his famous paper [46]. The quantum mechanical nature of the radiation was revealed and not only fixed the previously unknown constant factor between S and T making the map to thermodynamic quantities unambiguous, but also introduced even more puzzling questions like the seeming loss of information during black hole evaporation [47].

It took another two decades until a microscopic origin of Bekenstein's black hole entropy formula was provided [65] within the context of string theory via an actual counting of black hole microstates within a certain supersymmetric D-brane configuration. But already in the 1970's it was possible to ask if the dynamics of the internal black hole microstates, whatever those may be, imprints itself on the dynamics of the horizon.

Indeed, it was found that an observer outside the black hole can regard it as a hot membrane with resistivity ρ [214, 215], bulk viscosity ζ and shear viscosity η [216, 217] (see also [218, 219]). Their respective values were computed to be

$$\rho = 4\pi\,, \qquad \zeta = -\frac{1}{16\pi G}\,, \qquad \eta = \frac{1}{16\pi G}\,. \tag{3.1}$$

where the negativity of ζ already indicates serious difficulties in the rigorous understanding as a true dynamical fluid. Note that for the Bekenstein entropy $S = \frac{A}{4G}$ and the corresponding density $s = \frac{S}{A}$, we have that the ratio of shear viscosity over entropy density is given by

$$\frac{\eta}{s} = \frac{1}{4\pi}\,, \tag{3.2}$$

which will soon be related to the universal AdS/CFT value by a renormalisation group flow.

Furthermore, the dynamics follows the appropriate equations of motion, Ohm's law, Joule's law and the non-relativistic incompressible Navier-Stokes equations after one has properly defined corresponding current densities. The equations of motion may be derived directly from the equations of motion of general relativity or electromagnetism in curved space, e.g. by projecting them along the horizon. Specific components of the extrinsic curvature of the horizon may be interpreted as the shear tensor for the energy momentum tensor of a fluid and other components as the surface gravity, which basically is the black hole's temperature. Since computations directly at the mathematical horizon suffer from the complication of it being a null surface, it is often useful to extend these concepts to the stretched horizon [220, 221] a Planckian distance away from the black hole, on which the same deductions hold. This was then coined the "membrane paradigm" [122].

At the stretched horizon, the surface is not light-like anymore, but rather time-like, which is more sensible for the description of dynamics. Also one may wonder about the nature of the black hole entropy in terms of an effective quantum field theory at or close to the horizon. Since the dynamics of the horizon shall incorporate the dynamics of the internal black hole microstates in some sense, one may try to relate it to the entropy of quantum fields within a quantum field theory defined on the curved space of the black hole's near-horizon region [80]. For non-extremal black holes this near-horizon geometry is universally Rindler space, the space seen by a uniformly accelerated observer in flat space, who accelerates to resist the gravitational pull inwards. One may consider e.g. free scalar fields in that geometry and estimate their entropy contribution[2]. Those computations are dominated by the Planckian region extremely close to the horizon, where quantum field theory breaks down and is replaced by a theory with a sensible UV completion. The effective degrees of freedom of that theory tame the divergence in the sense that not a lot is contributed anymore at UV distances $\lesssim l_P$ and that therefore the black hole entropy is mostly contained in the modes of the quantum field theory description at distances $\gtrsim l_P$. To account for the *finite* black hole entropy, one may therefore rather relate to the stretched horizon at $\sim l_P$ away from the mathematical horizon.

A further elegant refinement of the membrane paradigm was presented in [206], in which the membrane paradigm equations of motion were conveniently rederived from an action principle. The key observation is that an observer outside the black hole can (classically) not be influenced by the interior; thus, one should be able to recover the membrane's equations just from considerations outside the horizon, but with the amendment of suitable boundary conditions *at* the horizon. The authors stress the similarity of their derivation of Ohm's law, Joule's law and the Navier-Stokes equations with the way electrical potentials in electrostatics are derived via the method of image charges. The boundary conditions are implemented on the horizon ("boundary") by putting fictitious electromagnetic and gravitational sources on it. Let us quickly review their presentation for the case of a gravitational source:

For such a fictitious source, incorporated into S_{surface}, we look at the action outside (!) the black hole

$$S_{\text{outer}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,R + \frac{1}{\kappa^2}\int d^3x\sqrt{-h}\,K + S_{\text{matter}} + S_{\text{surface}}\,, \tag{3.3}$$

where we have introduced some matter fields in S_{matter} and put the Gibbons-Hawking boundary term. This involves the trace of the extrinsic curvature K, since we require a well-defined variational principle of the gravitational theory also in the presence of the boundary. Then the membrane equations of motion may be easily obtained from the action of the original system including these fictitious boundary terms [206].

An outside observer cannot tell the difference if the charges are real or not as long as he does not cross the horizon. The freely falling observer, who does cross the horizon, however finds that there were no actual sources whatsoever. Similarly to the observer who looks behind a conducting plate and finds that there is no object like an actual charge behind the plate (it is only an image incorporating the right boundary conditions), the observer who falls into a black hole does not find an electromagnetic potential as he would have expected from outside the black hole. In terms of our gravitational sources under consideration it is straightforward to see this from the Israel junction conditions for the inside/outside extrinsic curvatures $K^{\pm}_{ab}$ at the horizon: From the equations of motion of the just stated action, we find that on the (stretched) horizon, we get

$$\kappa^2 T^{\text{s}}_{ab} = (K_{ab} - h_{ab}K)\ . \tag{3.4}$$

In general, for a boundary surface for which $\pm$ refer to outside and inside of the boundary surface, the Israel junction conditions are given by $\kappa^2 T^s_{ab} = ([K]_{ab} - h_{ab}\,[K])$, in which $[K] = K^+ - K^-$. We thus find that, seen from the inside, the extrinsic

[2] Interactions are of course crucial for many aspects in this context, but not so much for the argument we follow here (for many more details, see [80]).

curvature vanishes $K^{-}_{ab} = 0$. So, the inside of the black hole is actually flat and looks like there is really no gravitational source at the horizon. The Gauss-Codazzi equation for the hypersurface

$$\hat{\nabla}^a T^{\rm s}_{ab} = -h_{bc} T^{cd} n_d \tag{3.5}$$

can then be manipulated to recover the Navier-Stokes equations, using that one may write the fictious horizon energy-momentum tensor $T^{\rm s}_{ab}$ at the stretched horizon in the usual hydrodynamic form as explained earlier[3].

The explanation in [206] makes the distinction between the different perceptions of black holes for infalling observers and those outside the horizon particularly clear. The observer outside the horizon (say, at asymptotic infinity) perceives the black hole as a hot membrane. Every object that falls into the black hole looks as if it completely dissolves in the hot membrane at the horizon. For a freely infalling observer the equivalence principle and the low curvature at the horizon strongly suggest that he does not notice anything special.

The difference in the different perceptions of horizon crossing as seen from the point of view of the infalling observer and the observer e.g. at spatial infinity, has been sharpened into the principle of black hole complementarity [48], following earlier work [222, 223], as a solution to the information loss paradox. In that description, both perceptions are equally valid and since no observer can verify both descriptions, this does not violate the principles of quantum mechanics, the no-cloning theorem. In particular, the observed bits which observers measure inside or outside the horizon are not two copies of the same original one, which falls into the black hole, but they are just two different perceptions of exactly the same bit. The proposal of black hole complementarity was put forward to resolve the black hole information paradox [47] by keeping an effective field theory description at the horizon, which should be valid since the curvatures at the horizon are small for a big black hole, and upholding the equivalence principle to explain the suggested unitarity of black hole collapse and evaporation, i.e. that information should *not* be lost. Only looking at the outside observer, infalling matter is diluted into the hot membrane at the horizon, which then just emits Hawking radiation again - in total, a completely unitary process for a thermal system. But at the same time the infalling observer sees the infalling matter inside the black hole, so locality is lost. There is no such thing as a *local* degree of freedom associated to the infalling bit of information and no observer can verify and compare both descriptions.

The advent of the AdS/CFT correspondence gave strong additional support for the loss of locality and the unitarity of black hole evaporation. The AdS/CFT correspondence is a holographic description of degrees of freedom in a boundary space-time, thus an explicit construction of the holographic principle, in which the radial dimension is emergent.

Recently, the viewpoint of black hole complementarity has come under strong pressure. The assumptions of [48], namely the no-information loss or unitarity of black hole evaporation, the existence of an effective field theory description outside the black hole and the “no-drama” hypothesis, i.e. that the infalling observer does not observe anything special at the horizon, have been shown to be inconsistent with each other [49], suggesting that one of the assumptions needs to be abolished. The solution proposed in [49] was that the infalling observer encounters a firewall at the horizon,

[3] We will exactly find such an equation in our later computations, see eq. (4.78).

i.e. highly excited modes which effectively burn the infalling observer, thus departing from the "no-drama" assumption. This suggestion would be a radical deviation from currently accepted principles. But abandoning the alternatives is maybe even more radical. In any case the paradox of [49] raises serious question for the understanding of the genuine quantum aspects of a theory of quantum gravity. Maybe ideas like the recently proposed ER = EPR proposal [224] will yield a satisfying solution to the paradox. In our work, these important recent developments will however not be of importance, because our description will entirely describe a gravitational system from outside its event horizon and also mainly in a classical limit.

We so far looked at the near-horizon region of a generic non-extremal black hole and found the interesting analogy with a fluid dynamical description. In a completely different limit, the near-boundary limit of black holes in asympotically AdS spaces, we also recover the description of a fluid of some kind. The description of this fluid will be reviewed in the following section. The connection to the membrane paradigm description of black holes is not immediately clear, but there are interesting similarities like the two fluids' identical value of η/s. The connection of these two fluid will then be elaborated on in subsequent sections starting from 3.3.

3.2 The fluid at the AdS boundary

Essentially, there are currently two complementary ways to think about the fluid at the AdS boundary.

On the one hand the AdS/CFT correspondence describes the duality of a quantum string theory on asymptotically AdS space to a conformal quantum field theory, which one may think of as being defined on the boundary of that AdS space. In the limit of large N and large λ the duality simplifies. The gravitational theory simplifies to classical (super-)gravity on asymptotically AdS space and then describes the large N limit of the dual conformal theory at infinitely strong coupling. We may then put a black hole into AdS space and by this describe the field theory at finite temperature, which softly breaks conformal invariance of the field theory. Long wavelength excitations of the field theory around local thermal equilibrium are then universally expected to be described by the theory of hydrodynamics [108, 225].

So, we may want to analyse this particular hydrodynamic limit of the assumed full AdS/CFT correspondence (in its strong or only weak form). This approach has been pioneered in the work of Policastro, Son and Starinets starting off from [130, 226, 227]. In the established framework the hydrodynamic limit of real-time AdS/CFT correlators is defined and various interesting results were derived from this. For $\mathcal{N} = 4$ super Yang-Mills theory one of the most interesting results of the hydrodynamic limit of strongly coupled field theories was found, the suprisingly low value of the ratio of shear viscosity and entropy density [130]

$$\frac{\eta}{s} = \frac{1}{4\pi} \,. \tag{3.6}$$

Its universality for isotropic strongly coupled field theories with an Einstein gravity dual was a subsequent landmark result [92]. There are various ways how to compute this and we will highlight some of these computations soon.

In general, however, the procedure following [226, 227] becomes increasingly difficult if one wants to apply it also to second order transport coefficients [228].

Yet, in chapter 5 we will extend these calculations in arbitrary dimensions to the similar case of a (first order) diffusion constant in the hydrodynamics of generic supersymmetric field theories with a gravity dual.

On the other hand the relation of two classical systems, weakly coupled gravitational physics and the hydrodynamic limit of a certain quantum field theory raises the suspicion that there might be a more direct map of gravitational degrees of freedom and the variables of hydrodynamics. In the work [109], the fluid/gravity correspondence was established and exactly provided such a direct map. For the description of particular theories both approaches will yield the identical answer, but for particular questions one or the other approach is more useful. In the first approach, the question of universality and corrections in $1/\sqrt{\lambda}$ and $1/N^2$ may be addressed more efficiently. In the second approach second order transport coefficients are easier to be computed and the directness of the approach helps to address more conceptual aspects like the hydrodynamic significance of underlying quantum anomalies [115].

We will give an introduction to both approaches in the following.

3.2.1 Holographic hydrodynamics

As reviewed in chapter 2, the AdS/CFT correspondence relates a gravity/string theory in AdS space to a dual field theory on the boundary of that space, which is a conformal field theory - the most prominent example of course being the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions (see chapter 2). The duality may be deformed in several ways.

One of the questions which immediately suggests itself is the interpretation of a black hole in the context of the duality; we have already partly covered this in section 2.3. Since the one side of the duality is a gravitational theory which generically has black hole solutions, there should also be a generic dual interpretation more or less independent of the particular realisation of the duality. Indeed, the basic aspects of this were already discussed in two of the foundational publications on the AdS/CFT correspondence [8, 197].

Two of the most basic properties of (non-extremal) black holes are their thermodynamic properties, in particular their temperature and entropy, which, when exponentiated, describes the number of coarse-grained internal microstates that make up the black hole. Now, given the duality of the gravitational system to a quantum field theory in flat space, what else could the dual interpretation of such a macroscopic state be but the thermodynamics of the corresponding field theory? In the limit of large number of colours N and large 't Hooft coupling λ, where we have a description in terms of classical (super-)gravity in AdS space, the state dual to the black hole can however be no gas of weakly interacting particles. It is rather a strongly interacting plasma with temperature and entropy identified with the black hole ones.

Unlike asymptotically flat black holes, which have negative specific heat, in AdS black holes can have either positive or negative specific heat, depending on their mass and its relation to the AdS scale [196, 8]. Only for large masses, the specific heat is positive and the interpretation in terms of deconfined $\mathcal{N} = 4$ super Yang-Mills theory makes sense. For small masses the specific heat is negative and the black hole evaporates. Already in [8] it was indicated that the existence of a dual, manifestly unitary description suggests the unitarity of the evaporation process. Furthermore, the properties of such black holes are in some respects very similar to those of black holes in flat space[4] and therefore also point towards a holographic resolution of the information paradox in the context of flat space in favour of the information conservation. The phase transition between these two configurations [196] can on the field theory be interpreted as a confinement-deconfinement transition [197].

In equilibrium, the stress-energy tensor of the field theory is completely determined by the "energy of the black hole" or, more precisely, its ADM mass. It may be computed via several complemetary procedures [229, 230] and [191, 192, 193], in which always the main aspect is the systematic cancellation of divergences close to the AdS boundary if the stress-energy tensor were defined by more traditional methods [129]. By this procedure, the time-time component

$$T_{00} = \epsilon \tag{3.7}$$

of the stress energy tensor is directly fixed; assuming rotational invariance of the plasma, we also get a pressure term

$$T_{ij} = P\,\delta_{ij} \tag{3.8}$$

isotropic in space.

Due to relativistic invariance of our boundary theory, we may perform a boost into a frame with velocity u_μ with $u_\mu u^\mu = -1$, where the energy momentum-tensor may then be written as

$$T_{\mu\nu} = \epsilon\, u_\mu u_\nu + P\,\left(g_{\mu\nu} + u_\mu u_\nu\right) . \tag{3.9}$$

In a conformal field theory like $\mathcal{N} = 4$ super Yang-Mills theory, we have a further constraint, which can be checked via the explicit construction of the field theory's energy momentum tensor or be determined from general principles. The conformality of the field theory requires a vanishing trace of the energy-momentum tensor

$$T^\mu_\mu = 0 \tag{3.10}$$

and therefore that $\epsilon = 3P$.

Also other thermodynamic properties of black holes may be identified with thermodynamic quantities of the field theory (cf. section 2.3). As a $U(1)$ charged black hole comes with a radial profile for the bulk gauge field, according to the AdS/CFT dictionary there will be a corresponding conserved current for a global $U(1)$ symmetry in the boundary field theory. This global symmetry group can basically be seen as the subgroup of bulk diffeomorphisms that describes the large gauge transformation that act non-trivially on the boundary. For example the conformal symmetry group itself can be seen as originating from the asymptotic symmetry algebra of bulk gauge transformations (diffeomorphisms) of AdS space, as already pointed out for the AdS_3 case in the 1980's [231]. Now, the chemical potential μ for a charge density of that global symmetry can be determined from the gauge field. It is just the asymptotic value of the time component of the bulk gauge field.

$$\mu = \lim_{r\to\infty} A_t(r)\,. \tag{3.11}$$

[4]Note however that the different boundary conditions at infinity modify this similarity quite substantially in many aspects.

Given such a setup, one may similarly to 3.9 determine the equilibrium configuration for the $U(1)$ current. Most precisely, it should be determined from holographic renormalisation again [232], but also the procedure of [129] usually suffices. The charge current is then given by the $U(1)$ charge density n via

$$J_\mu = n\, u_\mu\,, \tag{3.12}$$

where we have again performed the aforementioned boost to a fluid frame moving at the velocity u_μ. According to the AdS/CFT dictionary the CFT expectation value of this current can be computed from the sub-dominant component of the bulk gauge field profile as (see again section 2.3)

$$J_\mu \sim \lim_{r\to\infty} r^2 A_\mu(r)\,. \tag{3.13}$$

Diffusion constants from correlator poles

Beyond the convenient map of thermodynamic properties of the two sides of the duality, one may wonder about the situation mildly out of equilibrium. This can be analysed by looking at small perturbations of the black hole (e.g. scalar perturbations) [233]. These perturbations, the quasi-normal modes, can however not be stable because the black hole absorbs them, never to be seen outside again. Effectively, the black hole thus acts like a dissipative thermal medium, which again also fits the interpretation with a hot membrane at the horizon.

An important role is here played by the asymptotic boundary conditions [233]. AdS space has a boundary which basically acts like a box on the length scale of the AdS radius. Without a black hole there would therefore exist stable normal modes. But with a black hole modes going outside are reflected by the boundary and therefore eventually also fall into the black hole. The time scale of relaxation to the equilibrium situation after a small perturbations of a large black hole was found to generically be given by the time scale set by the black hole's temperature.

The study of linearised fluctuations may be taken further and be interpreted in the dual field theory [226,227,234], as reviewed in [93]. As we have seen in section 2.2.5, for linearised gravitational fluctuations of some asympotically AdS background there are two distinct near boundary modes. The non-normalisable term introduces a source for a dual single trace operator in the field theory, while the subdominant term sets this operator's expectation value. If we want to slightly perturb the equilibrated field theory, we should switch on a small source and see how the expectation value responds. So let us couple such a source ϕ_0 to the corresponding operator $\mathcal{O}$ via

$$\int d^4x\, \phi_0(x)\mathcal{O}(x)\,. \tag{3.14}$$

If the source is small, then we may use linear response theory, which connects source and the operator's expectation value via the operator's *retarded* Green's function

$$iG^R(x-y) = \theta(x^0 - y^0)\langle[\mathcal{O}(x), O(y)]\rangle \tag{3.15}$$

to obtain

$$\langle\mathcal{O}(x)\rangle = -\int d^4y\, G^R(x-y)\phi_0(y)\,, \tag{3.16}$$

where the use of the retarded Green's function clearly is required by causality. Now, it is this retarded Green's function $G^R(x-y)$, which we would like to study in the hydrodynamic limit [227] using a Minkowski space, i.e. real time description [226]. General field theory knowledge suggests that its low energy, low momentum poles should describe the fluid's excitation modes and via such linearised computations we may then derive the fluid's transport and diffusion coefficients [235].

Let us look at a simple example of a conserved current $\mathcal{O}(x) = J_\mu(x)$, for which we assume an initial charge density $\langle J^0 \rangle = 0$ in thermal equilibrium. We now slightly perturb the system and study its long wavelength, low energy response. The spatial part of the current will be determined via the time component due to a hydrodynamic constitutive relation

$$J_i = -D\partial_i J^0 \tag{3.17}$$

similar to (1.9) and (1.10), where D is the diffusion constant. Since hydrodynamics is, simply speaking, the effective theory of the dynamics of conserved currents $\partial_\mu J^\mu = 0$, we directly obtain Fick's law

$$\partial_0 J^0 - D\partial^i \partial_i J^0 = 0\,. \tag{3.18}$$

This suggests a dispersion relation $\omega = -iDk^2$ for the long wavelength perturbation and an exponential decay of $J^0(t)$.

Via gauge/gravity duality this analysis may be undertaken and the corresponding strong coupling value of D for $\mathcal{N} = 4$ super Yang-Mills theory at finite temperature T may be derived [227] starting off from a black brane space-time in AdS_5. The conserved current J^μ may most simply be one of the three gauge theory's R-symmetry currents in the $U(1)^3 \subset SU(4)$ Cartan subgroup, which couples to the Abelian supergravity gauge fields A_μ (2.52).

So, we look at Maxwell's equations in a black brane background in momentum space. We assume plane wave ansätze for the gauge field's perturbations with low frequency and low momentum compared to the field theory's temperature; this is the hydrodynamic limit of slight deviations from local equilibrium. The Maxwell equations may then in a fixed gauge be perturbatively solved in $\mathfrak{w} = \omega/2\pi T \ll 1$ and $\mathfrak{q} = q/2\pi T \ll 1$ using ingoing boundary conditions at the black hole's horizon and arbitrary boundary functions. The first boundary condition is suggested from the fact that we would like to describe a dissipative phenomenon in the field theory.

Now one would like to use these solutions to compute the on-shell gravity action, interpret the boundary functions as sources for the dual operator and take functional derivatives of it to compute the Green's functions. However, since we deal with a real time description, this is slightly subtle [226]. However on the level of Green's functions a prescription was suggested in [226] and later confirmed using the Schwinder-Keldysh approach [236] for finite temperature field theory. The upshot of the analysis is however that one may indeed also describe the Minkowski space correlator via gravitational methods and arrive at a long wavelength, low frequency retarded Green's function for J^0 [227]

$$G^R \sim \frac{1}{i\omega - Dq^2} \tag{3.19}$$

with diffusion constant

$$D = \frac{1}{2\pi T}\,. \tag{3.20}$$

So, indeed we see that the dimensionality of the diffusion constant is determined by the field theory's length scale, its temperature. The precise details of the computation will not be of much importance for us, so we were quite concise. However, a very similar computation will be performed later on for a diffusion constant which is defined in the hydrodynamic limit of a supersymmetric field theory.

Transport coefficients from Kubo formulae

Likewise, we may compute the shear viscosity η of $\mathcal{N}=4$ super Yang-Mills theory [130] from the hydrodynamic limit of the energy-momentum tensor's retarded Green's function [227]. However, also a related but technically slightly different approach may be used here. Therefore, we will quickly explain it.

The shear viscosity η appears in the symmetric traceless part of the first order components of the energy-momentum tensor in the hydrodynamic limit,

$$T_{ij}^{(1)} = -2\eta\left(\partial_{(i}u_{j)} - \frac{1}{d-1}\delta_{ij}\partial^k u_k\right) - \zeta\delta_{ij}\partial^k u_k\,, \tag{3.21}$$

so it is best to look at off-diagonal dual metric perturbations h_{xy}, for which the hydrodynamic gauge/gravity prescription of [226, 227] may be applied. Again, we consider a black brane space-time in AdS_5. We then look at the Fourier components of h_{xy} and solve the equations of motion perturbatively in $\mathfrak{w}$ and $\mathfrak{q}$. These are just minimally coupled scalar equations and it is natural to use ingoing boundary conditions at the horizon. The energy-momentum tensor's retarded Green's function,

$$G^R_{\mu\nu,\rho\sigma}(\omega,q) = -i\int d^4x\, e^{-iq.x}\,\theta(t)\langle[T_{\mu\nu}(x),T_{\rho\sigma}(0)]\rangle\,, \tag{3.22}$$

may then be computed and the component we want to study is given by [227]

$$G^R_{xy,xy}(\omega,q) = -\frac{N^2T^2}{16}\left(i\,2\pi T\omega + q^2\right)\,. \tag{3.23}$$

Here we do not observe a hydrodynamic pole, although we would have got it e.g. in the channel $G^R_{tx,tx}(\omega,q)$. That pole would relate to momentum diffusion, whose diffusion constant is given by $\mathcal{D}=\eta/(\epsilon+P)$. One may simply see this momentum diffusion mode arise by rewriting the first order energy-momentum tensor as [227]

$$T_{ij}^{(1)} = -\frac{\eta}{\epsilon+P}\left(\partial_i T_{0j} + \partial_j T_{0i} - \frac{2}{d-1}\delta_{ij}\partial^k T_{0k}\right) - \frac{\zeta}{\epsilon+P}\delta_{ij}\partial^k T_{0k}\,, \tag{3.24}$$

using (3.9) to relate the momentum density to the fluid velocity $T_{0i}=(\epsilon+P)u_i$. Then one of the two modes that describe the conservation of $T_{\mu\nu}$ will be a simple diffusion mode with diffusion constant $\mathcal{D}$, the shear mode. The other one, the sound mode, has a different diffusion constant apart from another linear term in the dispersion relation $\omega = u_s q - i\mathcal{D}_s q^2$ describing the sound velocity u_s.

But considering the xy example allows for the introduction of a further computational concept, the usage of Kubo formulas. Great significance may be attached to these since they allow for a direct computation of transport coefficients from the retarded Green's function. For the shear viscosity it reads

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega}\int d^4x\, e^{i\omega x^0}\,\langle[T_{xy}(x),T_{xy}(0)]\rangle\,, \tag{3.25}$$

in striking similarity to (3.22) (for a derivation see [225]). Indeed, in $\mathcal{N} = 4$ super Yang-Mills at temperature T, it is then given by

$$\eta = \frac{\pi}{8} N^2 T^3 \,, \tag{3.26}$$

which completely agrees with the result in [130] and thus exactly corresponds to the universal $\eta/s = 1/4\pi$ result given the field theory's entropy density.

Relation to absorption cross sections

This universal result may also be directly obtained by relating the aforementioned Kubo formula for η to the absorption cross section result for a minimally coupled massless scalar like h^x_y [92]. As discussed in section 2.2.3 from the pure field theory point of view one may look at the decay of an external scalar into the fields of the D3-brane world-volume theory [165, 166]. Even more specifically, the decay of an external graviton into the energy-momentum tensor of the field theory may be initiated by a coupling

$$\int d^4x \, \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \,. \tag{3.27}$$

Then the corresponding absorption cross section may be computed via the discontinuities of branch cuts of certain correlation functions [237] and the optical theorem (see also [9] and [13]). From this, one recovers [165, 166]

$$\sigma_{\text{abs},0}(\omega) = -\frac{2\kappa^2}{\omega} \,\text{Im}\, G^R(\omega) = \frac{\kappa^2}{\omega} \int d^4x \, e^{i\omega x^0} \, \langle [T_{xy}(x), T_{xy}(0)] \rangle \,, \tag{3.28}$$

which can basically be understood as follows: The transverse graviton h_{xy} decays into T_{xy} via the coupling (3.27). Due to its non-canonical kinetic term $\sim \frac{1}{2\kappa^2} \int (\partial h^x_y)^2$ we get an external leg factor of $\sqrt{2\kappa^2}$, which is squared since for the absorption cross section of a decaying particle we need the square of the amplitude of the participating processes. It decays with an initial energy of ω, which explains the factor $1/2\omega$ from familiar absorption cross section formulae, see e.g. section 7.3 of [13]. The commutator then follows from the optical theorem or, related, the discontinuity of the amplitude at the branch cut.

The low energy limit of the absorption cross section formula (3.28) is apparently very much related to the shear viscosity's Kubo formula (3.25),

$$\eta = \frac{1}{16\pi G} \, \sigma_{\text{abs},0}(0) \,, \tag{3.29}$$

with $8\pi G = 2\kappa^2$. Since it was shown [165, 166] that the field theory absorption cross section is identical to the gravitational one and that the gravitational absorption cross section of a minimally coupled massless scalar by a black hole/black brane with transverse $O(2)$ symmetry is given by the area of its horizon [105, 238], we directly obtain the universal relation

$$\frac{\eta}{s} = \frac{1}{4\pi} \,, \tag{3.30}$$

once we use the Bekenstein-Hawking formula for the black hole's entropy

$$S = \frac{A}{4G} \,. \tag{3.31}$$

So, we have seen that studying the hydrodynamic limit of the AdS/CFT correspondence is a very interesting research area, which in particular is very useful for highlighting questions of universality of transport coefficients in strongly coupled field theories with a simple gravity dual.

We now, however, want to study a very direct map between the degrees of freedom of AdS gravity and the description of conformal fluids, the fluid/gravity correspondence.

3.2.2 The fluid/gravity correspondence

The most important object of any fluid is its energy-momentum tensor $T_{\mu\nu}$, whose dynamics we want to study via a gravitational system. Since the dual field is the graviton, it is useful to look at Einstein's equations in AdS_{d+1} space

$$R_{MN} - \frac{1}{2} R\, g_{MN} + \Lambda\, g_{MN} = 0\,, \qquad \Lambda = -\frac{d(d-1)}{2L^2} \tag{3.32}$$

where Λ is the negative cosmological constant, and try to recover, in a particular limit, the fluid equations directly from these vacuum equations of motion. This is the main idea of the fluid/gravity correspondence, originally proposed in [109] and nicely reviewed in [110, 111], which we partially follow.

It is gratifying to acknowledge that the truncation of the Kaluza-Klein spectrum of a higher-dimensional (super-)gravity theory, like type IIB supergravity, to the purely gravitational sector with equations of motion (3.32) is a *consistent* one. This innocent looking comment is actually one of the main, crucial points in the construction of the fluid/gravity correspondence. In the $N \to \infty$, $\lambda \to \infty$ limit, the complicated dynamics of type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ contains a subsector, which just describes the massless metric degrees of freedom. This subsector has a dual subsector in the $\mathcal{N} = 4$ super Yang-Mills theory, which decouples under OPEs of the single trace operators that are dual to the one-particle gravity fields. In effect, the consistency of the gravity truncation to the metric sector then guarantees that also on the field theory side there is decoupled dynamics for the single-trace operator dual to the graviton, namely the energy-momentum tensor.

From the point of view of the AdS/CFT duality, this observation is an immediate consequence of the structure of the gravitational equations of motion. But from the field theory perspective, it is far from obvious that such a decoupling should happen at large coupling. The intuition from small coupling physics, where the decoupling is not happening [111], breaks down and could not have gotten us there. Once again, the remarkable power of duality shows itself.

What we are interested in is the dynamics of the energy momentum tensor

$$T_{\mu\nu} = \epsilon\, u_\mu u_\nu + P\, \left(g_{\mu\nu} + u_\mu u_\nu\right)\,. \tag{3.33}$$

in a particular low energy limit, when the field theory is put at finite temperature. Over time and length scales bigger than the order of the mean free path, we arrive at a situation, where a *local equilibrium* situation is encountered. Temperature and fluid velocity slowly vary with respect to spatial positions

$$T \to T(x)\,, \qquad u_\mu \to u_\mu(x) \tag{3.34}$$

and by the equation of state also pressure P and energy density ϵ acquire spatial dependences via their dependence on $T(x)$. Apart from this direct generalisation, the equilibrium energy-momentum tensor (3.33), with the replacements $\epsilon \to \epsilon\,(T(x))$ etc. as just stated left implicit, also gets correction terms from higher order derivative terms, where $\Pi^{(n)}_{\mu\nu} \sim O(\partial^n)$:

$$T_{\mu\nu} = \epsilon\, u_\mu u_\nu + P\, \left(g_{\mu\nu} + u_\mu u_\nu\right) + \sum_{n=1}^{\infty} \Pi^{(n)}_{\mu\nu}\,. \tag{3.35}$$

This derivative expansion is basically an expansion in the scale of fluctuations of temperature and fluid velocity, which can roughly be parametrised as $\partial_\mu T\,, \partial_\mu u_\nu \sim O(1/L)$, in comparison to the mean free path l_m of the particles the fluid is made of. So each higher order comes at a further power of l_m/L, thus effectively at a higher derivative order.

This approach is conceptually nothing else but an effective field theory approach, in which one looks at the low-energy dynamics of the system and perturbs it by the incorporation of higher-dimensional operators, which only mildly perturb the low energy situation. However, one of the main differences as compared to e.g. chiral perturbation theory[5] is that fluid dynamics is usually formulated in terms of equations of motion and constitutive relations only. Due to dissipation a description starting from an action is notoriously difficult to formulate.

So, what are the dissipative terms at higher order in the derivative expansion? The first order terms are conventionally written as[6]

$$\Pi^{(1)}_{\mu\nu} = -2\eta\, P_\mu{}^\rho P_\nu{}^\sigma \left(\nabla_{(\rho}\, u_{\sigma)} - \frac{1}{d-1} P_{\rho\sigma} \nabla^\lambda u_\lambda \right) - \zeta P_{\mu\nu} \nabla^\rho u_\rho\,, \tag{3.36}$$

where η and ζ are the shear and bulk viscosity again, which we already met previously. Note that at first derivative order, we can only get terms $\sim \nabla_\mu u_\nu$ and no derivatives of $T(x)$, which we regroup according to their transformation under rotations perpendicular to u_μ. In (3.36), η corresponds to the symmetric traceless tensor and ζ to the singlet under such rotations.

In principle, we could also have had terms proportional to $a_\mu u_\nu$ or $a_\nu u_\mu$ with $a_\mu = u^\nu \nabla_\nu u_\mu$, or even vorticity terms $\omega_{\mu\nu} = P_\mu{}^\rho P_\nu{}^\sigma \nabla_{[\rho} u_{\sigma]}$ appear in the first order energy-momentum tensor (3.36). But these may be disregarded by choosing the *Landau frame*, which is defined by [108, 110, 225]

$$\Pi^{(k)}_{\mu\nu}\, u^\nu = 0 \qquad \text{for} \quad k \geq 1\,. \tag{3.37}$$

We may impose this condition because of an ambiguity in different notions of temperature $T(x) \to T(x) + \delta T(x)$ and fluid velocity $u_\mu(x) \to u_\mu(x) + \delta u_\mu(x)$. It defines the fluid velocity to align with energy transport, since as such it is the single time-like eigenvector of the energy-momentum tensor with the eigenvalue being the energy density

$$T_{\mu\nu} u^\nu = -\epsilon\, u_\nu\,. \tag{3.38}$$

The fluid/gravity approach now makes it possible to perturbatively derive (3.35) with (3.36) from linearised perturbations of the decoupled gravitational sector in AdS

[5] See e.g. [239] and references therein.

[6] The expression $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is a projector which projects orthogonal to the fluid velocity u_μ.

space. These are governed by Einstein's equations (3.32) only and thus effectively describe a strongly coupled fluid.

Now, how is this done? From the equilibrium properties of the AdS/CFT correspondence it is clear, that the zero'th order part of (3.35) is directly accounted for by a Schwarzschild black brane in AdS_{d+1}

$$ds^2 = -\frac{r^2}{L^2}\, f(r)\, dt^2 + \frac{r^2}{L^2}\, d\vec{x}^2 + \frac{L^2 dr^2}{r^2 f(r)}\,, \qquad f(r) = 1 - \frac{r_0^d}{r^d}\,. \tag{3.39}$$

As explained earlier in 3.2.1, we assume a large black hole, which corresponds to the deconfined phase of the strongly coupled field theory, c.f. [240]. Otherwise, the attempt to derive fluid dynamics would be predestined to fail before it actually started, e.g. thinking of the negative specific heat for small AdS black holes. Using the description of the renormalised field theory energy-momentum tensor of [229, 191, 192, 193], we directly obtain an energy-momentum tensor of fluid dynamical form, (3.35) with $u_\mu dx^\mu = -dt$, with the particular values[7] for energy density ϵ ($\sim$ ADM mass) and pressure P

$$\epsilon = \frac{3}{2\kappa_5^2}\,\frac{r_0^4}{L^5}\,, \qquad P = \frac{1}{2\kappa_5^2}\,\frac{r_0^4}{L^5}\,. \tag{3.40}$$

For generality, we perform a further boost with *constant* general boost velocity u^μ along the world-volume of our black hole.

But how does one incorporate the first and higher order corrections? In our previous hydrodynamic discussion, we have seen that by introducing space-time variations of temperature and fluid velocity (3.34) in a local equilibrium way, i.e. slowly varying in space-time, we immediately deduce the existence of correction terms of the energy-momentum tensor. Thus, from this analogy, we do the same for the gravitational background. The temperature of the field theory is related to the position of the horizon, which we denoted by r_0 in (3.39); moreover, we have the same degrees of freedom of boosts in the gravity background as in the space, the fluid is defined on, the u_μ of eq. (3.35). We thus take the gravity background (3.39) and *impose* a slowly varying space-time dependence on the *a priori constant* parameters r_0 and u_μ:

$$r_0 \to r_0(x)\,, \qquad u_\mu \to u_\mu(x)\,. \tag{3.41}$$

Clearly, on first sight, this may seem odd. The parameters r_0 and u_μ more or less by definition have to be *constant* for Einstein's equations to be satisfied. For instance, in the derivation for the AdS black brane background, one obtains r_0 as a particular integration constant in an isotropic ansatz, where the metric prefactors solely depend on the radial direction.

However, we may just take (3.41) in the boosted version of (3.39) and *impose Einstein's equations* (3.32) for the given modified background with additional linearised perturbations for which one chooses a convenient fixed gauge. The resulting equations of motion may then be solved in a linear approximation and thus we get a corrected black brane metric. Furthermore, the equations of motion constrain the space-time dependence of (3.41) in an interesting way.

But before we get into this in detail, we have to dwell on an important subtlety. After the perturbative construction, we want to obtain a space-time with a *regular*

[7] From now on, we will mostly restrict to $d = 4$ and boundary metric $g_{\mu\nu} = \eta_{\mu\nu}$ for convenience.

event horizon. This is certainly a constraint on the general gravitational space-times we could in principle obtain. However, obtaining boundary theories of hydrodynamic form is also special. *Locally*, we want to keep equilibrium, thus we also want to *locally* build up the regular future horizon of the black brane in accordance with the local equilibrium of the field theory. This is most conveniently done in a coordinate system, which directly transports local patches of the boundary via radial null geodesics into corresponding local patches on the future horizon. The coordinate system, which we refer to, is the one of ingoing Eddington-Finkelstein coordinates. They make the regularity of the horizon apparent (there is no coordinate singularity anymore) and also accomodates this tubewise building up of the horizon.

We thus transform (3.39) into ingoing Eddington-Finkelstein coordinates with time coordinate v, perform a boost to $-dv \to u_\mu dx^\mu$ and obtain

$$ds_0^2 = -2\, u_\mu dx^\mu\, dr - \frac{r^2}{L^2}\, f(r)\, u_\mu u_\nu\, dx^\mu dx^\nu + \frac{r^2}{L^2}\, (\eta_{\mu\nu} + u_\mu u_\nu)\, dx^\mu dx^\nu\,. \tag{3.42}$$

By construction, the coordinate singularity at $r = r_0$ has vanished. Now, we use the space-time dependence of (3.41) and impose Einstein's equations onto the combined system of this background and its linearised perturbations. We then solve these equations order by order. The zero'th order is clearly the background solution (3.42). At first order, we only take first derivatives of r_0 and u_μ, and so forth.

The upshot of the analysis of [109] is, that at every order, we get two kinds of equations via this procedure. First, by a linear combination of the different equations in (3.32), we get one set of equations, which is independent of the radial coordinate r and the black brane perturbations. These *constraint equations* are given in our example by

$$\begin{aligned} r_{0,i} + r_0\, \beta_{i,v} &= 0\,, \\ 3\, r_{0,v} + r_0\, \partial_i \beta_i &= 0\,, \end{aligned} \tag{3.43}$$

where with $u_\mu dx^\mu = -dv + \beta_i dx^i$ a local rest frame was chosen. Higher derivative orders do not contribute at the order we are working at.

More explicity, these two equations originate from $g^{rr}E_{ri} + g^{rv}E_{vi} = 0$ in the vector sector with respect to spatial rotations and $g^{rr}E_{rv} + g^{rv}E_{vv} = 0$ in the corresponding scalar sector, where $E_{MN} = 0$ denotes the Einstein equations. The equations (3.43) actually describe the conservation of the zero'th order boundary stress energy tensor for $\nu = i$ or $\nu = v$:

$$\partial_\mu T^{\mu\nu} = 0\,, \tag{3.44}$$

where $T_{\mu\nu}$ is given by (3.33) with energy and pressure densities (3.40) and spatial dependence $r_0 \to r_0(x)$ left implicit. At every further order in our perturbation theory, this happens again. But for looking at the next order, we first have to compute the full first order background by looking at the *dynamical equations* also. These arise by considering the equations of motion for perturbations of the whole black brane system. These equations may be looked at in different sectors according to the transformation under the spatial $SO(3)$. These sectors decouple and can therefore be looked at separately.

Say, we want to look at the tensor sector with respect to the spatial rotations. The perturbation to consider is then

$$ds^2 = ds_0^2 + \frac{r^2}{L^2}\, \alpha_{ij}(r)\, dx^i dx^j\,. \tag{3.45}$$

and the according equation of motion, using $\sigma_{ij} = \partial_{(i}\beta_{j)} - \frac{1}{3}\delta_{ij}\partial_k\beta_k$, is

$$\frac{d}{dr}\left(r^5 f(r)\,\frac{d}{dr}\,\alpha_{ij}(r)\right) = -6\,r^2L^2\,\sigma_{ij}\,. \tag{3.46}$$

This equation is representative also for the vector and scalar sectors, although the particular structure both for the differential operator and the right hand side source term changes. On the left hand side there is a homogeneous ordinary differential operator, which will reappear at every order of the perturbation theory. The right hand side contributes a source term at the order, we are working at, different at each order. Here at first order, it accomodates the only symmetric traceless combination of first order derivative terms of $r_0(x)$ and $u_\mu(x)$, the shear term $\sim \sigma_{ij}$.

We may integrate the equation of motion twice and fix the integration constants by essentially three requirements:

- imposing regularity of α_{ij} and/or its derivatives at the horizon
- fixing non-normalisable modes to zero to keep asymptotic AdS
- using redefinitions of r_0 or β_i

The first requirement may e.g. be used after integrating once. In general, the solution would indicate a pole in $\alpha'_{ij}(r)$ at the horizon, where $f(r = r_0) = 0$. This can then however be turned into a removable singularity upon a suitable choice of the integration constant. The normalisability condition is of course obtained from the holographic dictionary [7,8,9] for the particular modes looked at and the redefinitions just mean that we look at a relativistic system with a different temperature or in a different reference frame. In [109] it was shown that these boundary conditions can be inductively implemented order by order without any difficulty.

After obtaining a solution to the equations of motion with these appropriate boundary conditions, we add the perturbation term and those from the other sectors to the original metric to obtain the full first-order metric. This can then be used to obtain the first correction terms of the energy-momentum tensor. In our example, we arrive at (3.36) with

$$\eta = \frac{1}{2\kappa_5^2}\,\frac{r_0^3}{L^3}\,, \qquad \zeta = 0\,. \tag{3.47}$$

We have not shown the computation for the bulk viscosity which we would get from the scalar sector. But from general principles it is clear that it must vanish since we are describing a *conformal* field theory. The entropy density of the black brane (3.39) may easily be read off to see that the result for η fits

$$\frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{3.48}$$

as we expected. So indeed we find a quite different description of the boundary fluid, however with identical properties compared to the description of section 3.2.1.

Iteratively, one may do this order by order in the derivative expansion and therefore not only compute the first order corrections, but in principle any further higher order, although already at the second order the procedure becomes quite tedious.

So, we see that by a well defined procedure we arrive at a fluid description entirely from vacuum gravity[8] living at the boundary of AdS space. We may add different fields, go away from conformality, incorporate further conserved currents and so forth[9], but basically the story does not change: Both, the fluid dynamical conservation equations (3.44) and the constitutive relations like (3.35) and (3.36) may be derived order by order just from the gravitational equations of motion. It is fascinating to see that not only hydrodynamic equations are present in the low energy effective description of a gauge theory, but are really explicitly incorporated into the gravitational degrees of freedom itself. In many cases, this is just the low energy map which originates from AdS/CFT or general gauge/gravity duality. But having an explicit map from the one classical system to the other at our disposal constitutes an impressive explicit short-cut duality on its own.

3.2.3 Fluid/gravity and anomalies

The impressive predictive strength of the fluid/gravity correspondence was very present already in the last subsection. What we now want to briefly comment on is the explicit computation of a term in the constitutive relation of a conserved current that had not been anticipated before: In [113, 114], which we are now going to review, a term was found which was later explained to be related to quantum anomalies of the underlying microscopic field theory [115]. In these results it is apparent how much an explicitly known map like the fluid/gravity correspondence can confirm or disprove standard lore in some field. Even a well established theoretical framework like fluid mechanics sometimes has to be revised.

The main idea of [113, 114] was simple: how does the fluid/gravity map of [109] change if one incorporates another conserved current in the field theory at the AdS boundary which describes a global $U(1)$ symmetry?

From the general AdS/CFT dictionary it is clear, how to start. One needs to look at a gravitational system with a further $U(1)$ gauge boson which was readily available in terms of a consistent truncation of type IIB supergravity [198, 241]. The action for that system is [113]

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left(R + 12 - F_{MN}F^{MN} - \frac{4\kappa}{3}\epsilon^{MNPQR}A_M F_{NP} F_{QR}\right) \quad (3.49)$$

with $\kappa = \frac{1}{2\sqrt{3}}$, where apart from the usual Einstein term with negative cosmological constant an innocently looking Chern-Simons term appears[10]. But the presence of this term is actually of huge importance. One may go through the same kind of computations as explained earlier with the additional presence of scalar and vector perturbations for the gauge field, but also with another equation of motion, the Maxwell equation in curved space.

The starting black hole is the AdS Reissner-Nordström black hole, or rather black

[8] There is no bulk energy-momentum tensor on the right hand side of (3.32)!

[9] See [110, 111] and the literature referred to.

[10] We will have much more to comment on consistent truncations in general and also about this special case. This is however relegated to section 4.1. But note that the importance of the Chern-Simons for chiral anomalies was already pointed out in [8] as explained in section 2.3.

brane, with metric

$$ds^2 = -\frac{r^2}{L^2} f(r)\, dt^2 + \frac{r^2}{L^2} d\vec{x}^2 + \frac{dr^2}{r^2 f(r)}\,, \qquad f(r) = 1 - \frac{r_0^4}{r^4} + \frac{q^2}{r^6} \tag{3.50}$$

and appropriate profile for the gauge field $A_\mu(r)$. One goes to Eddington-Finkelstein coordinates, performs a boost and implements the space-time dependence (3.41) for the a priori constants black hole parameters ($\sim$ collective coordinates) r_0 and u_μ. Moreover, one imposes the additional space-time dependence for the charge

$$q \to q(x)\,. \tag{3.51}$$

Taking care of gauge redundancies one computes the constraint equations, which again take the form of conservation equations for the zero'th order energy-momentum tensor and the $U(1)$ current

$$\partial_\mu T^{\mu\nu} = 0\,, \qquad \partial_\mu J^\mu = 0\,. \tag{3.52}$$

Since we will later perform computations which reduce to these here, let us explicitly state them. The constraint equations from the Einstein equations are identical to the ones mentioned earlier (3.43), but there is also one from the Maxwell equations $\mathcal{M}_A = 0$, or more explicitly from the combination $g^{rr}\mathcal{M}_r + g^{rv}\mathcal{M}_v = 0$:

$$\begin{aligned} r_{0,i} + r_0\, \beta_{i,v} &= 0\,, \\ 3\, r_{0,v} + r_0\, \partial_i \beta_i &= 0\,, \\ q_{,v} + q\, \partial_i \beta_i &= 0 \end{aligned} \tag{3.53}$$

This last equation obviously accounts for the zero'th order current conservation, where the current is given by $J^\mu \propto q\, u^\mu$.

But fluid/gravity also gives higher-order corrections. One then computes the equations of motion for linearised perturbations of the background, solves them with the boundary conditions mentioned earlier and computes energy-momentum tensor and charge current from the holographic dictionary. We then get the corrected charge current[11]

$$J_\mu = n\, u_\mu - \mathfrak{D}\, P_\mu{}^\nu\, \mathcal{D}_\nu\, n + \xi\, l_\mu \tag{3.54}$$

in which we used the vorticity l^μ and the Weyl covariant derivative given by

$$l^\mu = \epsilon^{\nu\rho\sigma\mu} u_\nu \partial_\rho u_\sigma\,, \quad P_\mu{}^\nu \mathcal{D}_\nu q = P_\mu{}^\nu \partial_\nu q + 3(u^\nu \partial_\nu u_\mu) q \tag{3.55}$$

along with particular values for the charge density, diffusion constant and vorticity coefficient. These are given by[12]

$$n = \frac{\sqrt{3}\, q}{2\kappa_5^2}\,, \quad \mathfrak{D} = \frac{r_+^4 + r_0^4}{4\, r_0^4\, r_+}\,, \quad \xi = \frac{3\kappa\, q^2}{2\kappa_5^2\, r_0^4}\,, \tag{3.56}$$

with $\kappa = \frac{1}{2\sqrt{3}}$. Now, the upshot of the analysis is that we found a term proportional to the vorticity l_μ in (3.54). It can be traced back to the parity violating Chern-Simons term in the higher-dimensional gravity theory as the coefficient κ already indicates.

[11] We will adopt the notation of [113], although the expressions of [114] are completely equivalent.

[12] We use r_+ for the outer horizon of the Reissner-Nordström black hole.

Let us give some rough arguments how the vorticity contribution of the charge current utimately arises from the Chern-Simons term.

The original background gauge field was of usual Coulomb law like form

$$A \sim \frac{q}{r^2}\, u_\mu dx^\mu \,. \tag{3.57}$$

Now due to the fluid/gravity reasoning (3.41) the gauge field obtains spatial components from the space-time dependence of the fluid velocity[13]. Therefore, the Chern-Simons contribution to the Maxwell equation $\sim \epsilon^{ABCDE}\, F_{BC}F_{DE}$ contains vorticity contributions like in the $A = i$ component

$$\epsilon^{irvjk}\, F_{rv}F_{jk} \sim \epsilon^{vjki}\, A_v'(r)\, \big(q\,\partial_j\beta_k(x)\big) = A_v'(r)\, l^i \,. \tag{3.58}$$

These now backreact on the gauge field since they are *source* terms of the Maxwell equation. More precisely they backreact on the gauge field's first order perturbation because the source comes at first derivative order in the hydrodynamics like long wavelength approximation. After solving Maxwell's equations with the appropriate boundary conditions, this correction term is given by

$$\delta A_i \sim \frac{q^2}{r^2}\, l_i \,. \tag{3.59}$$

Note in particular, that it survives at the same order as the background gauge field when performing a large r expansion. Therefore, the same way the charge density appears in (3.54) we also get a current which is induced by the vorticity of the fluid.

On the field theory side, the term is in one to one correspondence with the anomalous non-conservation of a global $U(1)$ current given $U(1)^3$ triangle anomalies [116, 117]

$$\partial_\mu J^\mu = C\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} \tag{3.60}$$

in the presence of a background gauge field, with an appropriate coefficient C. Theories with such a microscopic anomaly *require* a vorticity contribution both to the charge current but also to the entropy current s^μ [115].

In [115], it was shown that the usual definition of an entropy current runs into trouble and does not have a manifestly positive divergence due to the anomaly term. In the usual definition

$$s^\mu = s\, u^\mu - \frac{1}{T}\, u_\nu\, \Pi^{\nu\mu}_{(1)} - \frac{\mu}{T}\, \nu^\mu_{(1)} \,, \tag{3.61}$$

where $\Pi^{\nu\mu}_{(1)}$ and $\nu^\mu_{(1)}$ are the first order corrections to the energy-momentum tensor and charge current, the *positive divergence* $\partial_\mu s^\mu \geq 0$ of this entropy current, i.e. the local version of the second law of thermodynamics, sets the requirement for positive viscosities and conductivities. But the anomaly contribution of the current (non-)conservation equation manifestly spoils this. Only by a redefinition

$$s^\mu \to s^\mu + D\, l^\mu + D_B\, B^\mu \,, \tag{3.62}$$

$$\nu^\mu_{(1)} \to \nu^\mu_{(1)} + \xi\, l^\mu + \xi_B\, B^\mu \,, \tag{3.63}$$

[13] Remember that in a local rest frame, we introduced the space-time dependence of u_μ via $u_\mu dx^\mu \to -dv + \beta_i(x)dx^i$.

where $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$, the argument may be uphold. The requirement of positive divergence then gives relations for D, D_B, ξ, ξ_B in terms of the anomaly coefficient C in (3.60).

Before we conclude, let us quickly remark on two more interesting developments in the area of anomaly related transport.

There are interesting ways how to generalise the anomaly term on the right hand side of (3.60). It may actually also receive a contribution from a mixed gauge/gravitational Chern-Simons term of the gravity theory [242, 243], roughly of the form

$$\int d^5x\sqrt{-g}\,\epsilon^{MNPQR}A_M R^A{}_{BNP} R^B{}_{AQR}\,. \tag{3.64}$$

Although one would naively assume this to be subdominant in a derivative expansion, it actually contributes at the same order as the chiral anomaly.

In the course of the derivation according to [115], we very much relied on the positivity of the divergence of the entropy current – the local form of the second law of thermodynamics. But there is also a different way how to impose conditions like equalities and/or inequalities for the coefficients of particular tensor structures of conserved currents in terms of the fluid variables. Without referring to a positive divergence entropy current, these relations follow from a partition function description of equlibrium properties only, when consistently coupling to an external gauge field or a weakly curved background metric [244, 245]. Positivity constraints then result e.g. from the positivity of the spectral function.

To summarise, in the presence of global anomalies of the underlying quantum field theory positive divergence of the entropy current or the consistent description in terms of a partition function can only be maintained via a redefinition of the entropy current itself and the incorporation of a vorticity and B field contribution to the charge current. The correction terms are unambiguously given in terms of the anomaly coefficient C, the chemical potential μ and the temperature T.

The fluid/gravity computation knows about this. When performing a computation along the usual lines outlined in [109] the Chern-Simons term in the effective action induces exactly this vorticity current. Although this fluid dynamical fact could have been found in the 1970s from pure field theory reasoning, after the importance of quantum anomalies was realised, it took decades to find this possibly observable effect. One first had to understand string theory as a theory of quantum gravity, discover the AdS/CFT correspondence and from it work out the fluid/gravity map.

3.3 Holography at finite $r' < \infty$?

In the last two sections we have seen that there are various regions where the low energy effective theory of perturbations of black holes and black branes is effectively described by a hydrodynamic theory, although the membrane paradigm theory has some unphysical properties like the negative bulk viscosity.

Firstly, we have seen that the near horizon region of a black hole can effectively be described by a hot membrane. Long wavelength perturbations of this membrane obey the equations of motion of non-relativistic fluid dynamics. Secondly, we have seen that for black holes in AdS space the fluid/gravity correspondence effectively describes the hydrodynamic limit of a large N, large λ field theory that is dual to the classical gravity construction as made rigorous in various examples by the AdS/CFT correspondence.

But also a (non-extremal) black hole in AdS space has a near horizon region which is governed by the membrane paradigm. So it suggests itself to ask if we may define a hydrodynamic theory also at every surface at constant radial coordinate r'. This hydrodynamic theory would then interpolate between the $r' = r_h + \epsilon$ membrane paradigm theory on the stretched horizon and the $r' \to \infty$ fluid. In particular, it may be possible to assign to it transport coefficients like viscosities and conductivities and analyse how they change throughout the bulk spacetime.

Partially this was done in [246], in which – motivated by the membrane paradigm – viscosities and conductivities at finite r' were considered[14]. These then obey explicit flow equations. In particular, the flow equation for the shear viscosity shows that it does *not* depend on the radial position[15]. Together with the universal membrane paradigm value, one may deduce that then also the dual gauge theory obeys this universal value

$$\frac{\eta}{s}\Big|_{r_0} = \frac{1}{4\pi}\,, \quad \partial_{r'}\left(\frac{\eta}{s}\right) = 0 \quad \Rightarrow \quad \frac{\eta}{s}\Big|_{r'\to\infty} = \frac{1}{4\pi}\,. \tag{3.65}$$

Computations which derived the field theory's shear viscosity value from an integration through the whole bulk already included [249], from which the universality was deduced by identifying the essential contributions from these integrals for general supergravity backgrounds [250]. In a sense cutting off these integrations at some finite r' would also have let to an effective membrane paradigm description at this r'.

The definition of a field theory at finite r' is for many reasons a delicate, but also highly interesting issue. The general assertion is of course that in AdS/CFT the bulk radial coordinate corresponds to the energy scale of the field theory [6, 152, 251] and section 2.1.4. One way to see this is by separating one D3-brane from the large N stack of D3-branes which originally made up the duality. This then corresponds to a Higgs effect of the field theory [6], where the length of the open strings from that D3-brane to the stack sets the energy scale in units of α'. So, in a sense the field theory at some finite r' should correspond to the field theory at some energy scale. There is of course a lot of literature on the associated renormalisation group flow, see [252] and the original works referenced therein. But this can also be thought of in a Wilsonian sense [124, 125, 126]:

In the field theory we may integrate out high energy modes between a high energy scale Λ' and a UV cutoff $\Lambda_0 \geq \Lambda'$ to define the Wilsonian effective action at energy scale Λ'. It is then proposed that we may mimic this on the gravity side also by integrating out degrees of freedom of the gravity theory in an effective radial interval $[r', \infty)$ with $r' < \infty$. Now the same way boundary conditions at the AdS boundary are imposed on the gravity fields to match the gravitational partition function with the CFT one, boundary conditions are provided at r'. The procedure of integrating out high energy modes induces double- or multi-trace operators in the effective action (this is one of the key insights!) and the independence of the cutoff scale induces flow equations for the effective action. When expanded in terms of local operators these then reduce to the β functions of the operators' couplings. In particular, in [126] it

[14] For the finite chemical potential case see [247].

[15] Already in [248] the independence of the radial variable was noted, however without boldly extending the membrane paradigm to the bulk.

was found that the aforementioned flow equations of [246] are the β-functions of the coupling of a double-trace operator that originates from the single trace operator dual to the transverse bulk graviton mode which is a scalar. Therefore the interpretation of the flow equations in [246] in terms of RG equations was made much more precise. In [253] the equivalence of these approaches and the one taken in [254] was shown. So many aspects of thinking of a field theory on a surface of finite r' seem reasonable.

However, there are also problems associated to this line of reasoning. Precision holography requires a precise holographic dictionary of renormalised quantities [191, 192,193]. The field theory UV divergences are governed by bulk divergences close to the AdS boundary [152] and the holographic renormalisation method systematically cancels these divergences by introducing counterterms that are *intrinsic* to the boundary geometry. In particular, the existence of the Fefferman-Graham expansion close to the boundary allows for the cancellation of the holographic energy-momentum tensor solely in terms of intrinsic curvature terms made from the boundary metric [229]. Away from the AdS boundary, due to the breakdown of Fefferman-Graham coordinates, it is not clear how to renormalise. There is no preferred coordinate system anymore and so the incorporation of curvature terms intrinsically defined from the induced metric on a spatial slice is ambiguous.

In the work of Brown and York [129] and follow-ups as cited in [229] the following procedure was proposed for computing the Brown-York quasi-local stress-energy tensor of a space-time, which would certainly form the starting point for holography on a radial cutoff surface: One should cancel its divergences by embedding one's system in a reference space-time and substract the Brown-York tensor of that reference system. But the existence of a well-defined reference space-time is also not clear.

Maybe part of a constructive solution to this problem can come from recent work on a map between Ricci-flat geometries and asymptotically AdS ones [255,256]. In these works flat space renormalisation may be circumvented by using the explicitly known holographic renormalisation in AdS. So far, the AdS/Ricci flat map however heavily relies on the existence of classes of solutions which can be stated in *all* dimensions. Moreover it uses what has been called generalised dimensional reduction [177], in which a lower dimensional theory is formally seen as the dimensional reduction of a higher dimensional theory with fractional dimension. But maybe the AdS/Ricci flat map can provide useful guidance and/or explicit checks.

One may also hope to address many question of the effective membrane theory at finite r' via a fluid/gravity like analysis in AdS with Dirichlet boundary conditions at finite r' [127]. In spirit this seems to be very related to the Wilsonian definition of the cut-off theory [125, 126] and was actually very much motivated by it (and additionally [124, 257]). However, a detailed analysis shows that the two approaches are related but not quite identical [127].

Technically, the computational idea is quickly summarised. In the standard quantisation within AdS/CFT the non-normalisable term in the large r expansion of the bulk field is set to zero, corresponding to a Dirichlet boundary condition. Thus, in analogy, one assumes Dirichlet boundary conditions at finite r' for the gravitational perturbations too. This very much goes along with the understanding of AdS space as a box [196]: putting Dirichlet conditions at finite r' is in a sense nothing else but confining the system into a box within the bigger AdS box. Furthermore, one assumes regularity at r', which is sometimes similar to normalisability at $r \to \infty$, and further regularity for $r < r'$, possibly at a black hole horizon, with the exception of the singularity at $r = 0$. The equations of motion along with these boundary conditions are solved and renormalised quantities are computed according to a holographic dictionary similar to the one at $r \to \infty$. We have however already pointed out the difficulty of this last step.

The imposition of Dirichlet conditions at r' changes the boundary conditions at $r \to \infty$: Dirichlet surface source and vacuum expectation value (vev) mix into boundary source and vev in a non-trivial way. If one imposes Dirichlet conditions at r', then the boundary source becomes a function of the dual operator's vev in a particular state. This can also be thought of as inducing a (non-local) multi-trace operator deformation of the field theory on the AdS boundary [127].

So, this is different from the Wilsonian way of integrating out high energy bulk modes and defining a scale dependent effective action at r'. There, also multi-trace operators play an important role, however they do not deform the field theory but only appear in the Wilsonian effective action at the lower energy scale r'. At the boundary one keeps standard boundary conditions. Nevertheless performing fluid/gravity like computations with Dirichlet conditions at a finite radial cutoff provide an interesting way to move towards a satisfying description of a theory at finite r'. The interpretation in terms of an RG flow is not quite straightforward, but still somewhat suggestive.

3.4 Blackfolds and the Dirichlet problem

The analysis of gravity with Dirichlet boundary conditions or confinement to a box may also be carried out in asympotically flat space [258, 123] in the realm of the blackfold paradigm [118, 119, 120, 121]. But before we roughly summarise the work of [123], which we are going to extend in the next chapter 4, let us quickly explain the main ideas behind the blackfold paradigm.

The wealth of different black hole/black brane solutions in higher dimensions becomes apparent when we consider two well-known facts [259]: In asymptotically flat space in $d > 4$ dimensions there exist black hole solutions with non-spherical horizon topology like black rings [202]. One may construct these by considering a black string embedded in flat space, bending it into a ring and spinning it up to balance its gravitational inpull/tendency to contract via a centrifugal repulsion. The existence of such solutions in higher dimensions shows that in higher dimensions one may circumvent a rigorous four dimensional theorem which only allows black hole solutions with spherical horizons in that dimension [260].

Another fact is that classically many uncharged higher dimensional black brane solutions [161] are unstable under long wavelength perturbations as both a quick black hole entropy argument and a more detailed linearised perturbation analysis shows [261]. It is thermodynamically favourable for the black string to decay under a long wavelength perturbation into a periodic array of black holes.

It is therefore interesting to ask which further topologies are allowed and what perturbations leave such solutions stable, trigger instabilities or how one may interpret these perturbations in terms of an effective field theory, possibly related to known physical systems. This is basically what the blackfold paradigm is about. In many ways it is also very much related to the fluid/gravity correspondence [109, 119], which it was motivated by, in the sense that a perturbative description of higher-dimensional black holes with various horizon topologies is achieved, although also for asymptotically flat space-times.

There is intrinsic and extrinsic dynamics [119]. The intrinsic world-volume perturbations are described by an effective fluid-dynamical description (as in fluid/gravity) and the extrinsic dynamics are described by geodesic embeddings in the higher dimensional flat space showing properties of black strings like e.g. elasticity. The Gregory-Laflamme instability [261] then shows up in the intrinsic hydrodynamics as a sound mode instability due to an imaginary sound velocity. So, it also captures this generic feature of higher dimensional gravity. Compared to the fluid/gravity correspondence, the two main extensions, which go beyond, are the description of perturbations in the asympotically flat part of the geometry and the description of extrinsic dynamics.

Recently, it was explicitly shown that, although anticipated before, the intrinsic dynamics of the blackfold paradigm does indeed incorporate both the fluid/gravity dynamics of asymptotically AdS space, but also the membrane paradigm/Rindler fluid description of the near-horizon zone [123].

A non-extremal asympotically flat D3-brane metric was put in a general boosted frame and the charge, temperature and boost parameters were made to depend on the world-volume coordinates as in fluid/gravity, subject to a constraint that left the number of D3-branes fixed. The equations of motion were solved in a perturbative hydrodynamics-like fashion featuring a derivative expansion in the locally equilibrated parameters subject to Dirichlet boundary conditions at a cut-off surface as in [258]. At this surface an effective fluid was described by a Brown-York type stress-energy tensor. Now, the fluid dynamical transport coefficients generically depended on the position R of that cutoff surface and an extremality parameter δ. Particular limits of these then described the fluid/gravity or membrane paradigm physics. So by going to the near-extremal, near-horizon zone the AdS throat reveals itself and by putting $R \to \infty$ in that region one recovers fluid/gravity. Or, as was written in [123]:

Membrane paradigm $\subset$ Fluid/gravity correspondence $\subset$ Blackfolds

Given the importance of the discovery of anomaly related hydrodynamic transport coefficients within the fluid/gravity context [113, 114], one may wonder if this may also be studied within the blackfold context similar to the uncharged fluid in [123]. How are the parity odd terms in the hydrodynamic constitutive relations influenced by the asymptotically flat part of the geometry? Is there a way to understand these terms via the Wilsonian holographic renormalisation program or its related Dirichlet cutoff proposal[16] given the non-renormalisation of the anomaly beyond one loop [168]? Are there interesting new scaling limits in the geometry? These are some of the questions we aim for in a description of [113, 114] in the context of the blackfold paradigm, using ideas from [127] and [123].

[16] In the pure AdS case the holographic flow equations which the anomalous terms obey and the corresponding Dirichlet problem have been worked out in [262, 263].

CHAPTER 4

Effective hydrodynamics of spinning black D3-branes

After the stage is now set, we will review consistent truncations of supergravity theory spectra in 4.1, which will lead us to the starting point of our computation. The consistent truncation of [128] provides us with a minimal setting for the description of the effective degrees of freedom of an asymptotically flat stack of spinning black D3-branes (along the fibre of a Hopf fibration), that serves as an asymptotically flat generalisation of the AdS Reissner-Nordström solution (section 4.2). Fluid/Gravity or blackfold perturbations are then set up in 4.3 and analysed according to their spatial rotational behaviour. A Brown-York like energy-momentum tensor and charge currents (with corresponding transport coefficients) show that indeed this charged generalisation provides an interpolation of membrane paradigm, fluid/gravity, cutoff surface holography and flat space holography.

The following chapter is based on so far unpublished results [2].

4.1 Consistent truncations of type IIB supergravity

The setup we aim for is the minimal generalisation of the interpolating blackfold fluid presented in [123] which additionally incorporates the feature of the described fluid to be charged.

In [123] the intrinsic hydrodynamic dynamics of a stack of D3-branes with fixed five-form charge was considered. Technically, this was done by incorporating a cutoff surface in the geometry and imposing Dirichlet boundary conditions for the fluctuating fields which then allowed for a derivation of the hydrodynamic transport coefficients of a fluid which lives on that cutoff surface. By doing so, it was possible to explicitly interpolate between the different hydrodynamic regimes already known and explained in the previous chapters.

One of the most useful conceptual aspects of the computation was the possibility to reduce type IIB supergravity on the five sphere and the subsequent consistent truncation of the fields in the low-energy effective theory to a subsector which only consisted of Einstein gravity coupled to the volume modulus of the five sphere. So, in particular, the shape of the sphere was left fixed. Although, in principle, one could have done the computation of [123] in ten dimensions, the Kaluza-Klein reduction is sensible for the following reasons: First of all, identifying the significant degrees of freedom before starting the quite elaborate computation is helpful during the computation. But on the other hand, it is also interesting for possible generalisations of the setup, where e.g. the D3-branes are not put in flat space but rather on a conical singularity as in [264] and subsequent works. It is then straightforward to see, that the results of [123] are basically not altered when the S^5 is replaced by a Sasaki-Einstein manifold being the base of the cone.

In general, the question of consistency of a truncation can be delicate and subtle, especially if one wants to retain the (pseudo-)scalar excitations, see e.g. [265]. There are simple reductions like the ones on tori, where the reduction to the massless sector is guaranteed to be consistent by simple group theory, because the massless fields are singlets of the $U(1)^n$ of the torus. But already for sphere reductions, many cases are non-trivial.

Let us show the general idea of such a complication by a simple example, taken from [265] also. Let H and L be two scalars in the lower dimensional theory after a Kaluza-Klein reduction from a higher-dimensional one; we assume H to be massive and L massless. Generically, a cubic interaction like HL^2 may be assumed and we may ask if the massive mode H can consistently be set to zero, keeping only the massless L in the low energy effective theory. From the general assumptions, the classical field equation for H has the structure

$$\Box H + m^2 H = L^2 \,, \tag{4.1}$$

where we immediately see that L sources H such that setting H to zero and keeping L indeterminate is inconsistent. So, in general, the statements about consistency of the truncation of massive fields after a Kaluza-Klein reduction very much depend on the details of the interaction terms. Statements about the consistency of truncations to massless modes are therefore usually very non-trivial statements and we are going to show a couple of examples where the detailed analysis has been undertaken. At the end of the section, as one of the main goals of the section, we are going to discuss a consistent Kaluza-Klein reduction that was found in [128] which even goes beyond the reductions to just massless fields. It even incorporates a certain part of the massive Kaluza-Klein tower, setting the rest of the massive fields consistently to zero however.

Another point which is of importance for consistent Kaluza-Klein reductions is in fact that the consistency of the reduction guarantees that the solutions to the lower-dimensional effective theory may be uplifted to a solution of the higher-dimensional theory (see e.g. [266,241]). In particular, for bottom-up models which may be described as solutions to toy-model Lagrangians this is of importance. Can the toy-model action be derived by a consistent reduction of e.g. IIB supergravity, then it is guaranteed that the solution can be lifted to a solution of IIB supergravity and then to one of IIB string theory. By this philosophy many non-trivial backgorunds of string theory may be found.

Let us now return to the Kaluza-Klein reduction which was used in [123]. The ansatz for the metric and self-dual five-form field strength was simply

$$ds^2 = ds^2(M) + e^{2\varphi}\, ds^2(S^5) \,, \tag{4.2}$$

$$F_5 = Q\left(e^{-5\varphi}\mathrm{vol}(M) + \mathrm{vol}(S^5)\right) \tag{4.3}$$

where the self-duality of the five-form has already been explicitly spelled out. The resulting low-energy effective action after reducing to five-dimensions and truncating the KK spectrum is

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}\, e^{5\varphi} \left(R + 20\, (\partial\varphi)^2 + 20 e^{-2\varphi} - 2Q^2 e^{-10\varphi} \right) . \tag{4.4}$$

This Kaluza-Klein reduction already appeared in [267, 268]. As usual, the scalar has a potential, one of whose stationary points corresponds to a negative cosmological constant such that AdS space is a solution. When applying the Kaluza-Klein ansatz to the 10 dimensional (non-extremal) D3-brane solution of IIB supergravity, the five dimensional scalar φ however obtains a specific profile which depends exclusively on the radial coordinate.

For our purposes, this minimal setup is clearly not suitable as it does not contain a gauge field which could give rise to the dynamics of a charged fluid. We therefore have to depart from this specific truncation and set out for a more general one which is appropriate for the explicit setup we are aiming for.

4.1.1 Einstein-Maxwell theory with Chern-Simons term

Guidance is provided by previous holographic studies of charged fluids. In the end, the charged fluid on a cutoff surface, which we want to derive, shall contain the charged fluid discussion in AdS space developed in [113, 114] as a specific limiting case – the usual near-horizon limit. However, we also want to retain the asymptotically flat region which, as we will see, complicates the truncation quite substantially.

The setup considered in [113, 114] is Einstein-Maxwell theory with a negative cosmological constant and Chern-Simons term for the gauge field. The fluid to be described in this setup, using the language of the fluid/gravity correspondence [109], is the hydrodynamic limit of strongly coupled $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ in the large N limit in flat Minkowski space. As $\mathcal{N} = 4$ super Yang-Mills theory has a global $SU(4)$ R-symmetry which arises from the isometry of the five-sphere in the gravity dual, one may switch on a chemical potential in the diagonal $U(1)$ of the $U(1)^3$ Cartan subalgebra of $SU(4)$ to make the fluid charged. By means of the usual holographic dictionary, this then requires a gauge field in the gravity theory; and Einstein-Maxwell theory with a negative cosmological constant is the minimal setup which encompasses this.

The setup chosen in [113, 114] stems from a truncation of IIB supergravity where in the Kaluza-Klein reduction gauge fields along the Abelian isometries of the compact space are also considered[1]. But the question which gauge fields shall be kept and which may be consistently set to zero is in general not so clear and therefore leaves some space for complication. One may start off with type IIB supergravity and reduce it on a five-sphere, where the three independent rotation angles along the five-sphere can all accomodate an independent Kaluza-Klein vector. A simplifying case is to take all gauge fields equal [198, 241]. In the notation of [269], where the five-sphere metric is given by $d\Omega_5^2 = \sum_{i=1}^3 \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right)$ and a two-sphere is parametrised by $\mu_1 = \sin\theta$, $\mu_2 = \cos\theta \sin\psi$, $\mu_3 = cos\theta \cos\psi$, the ansatz reads

$$ds^2 = ds^2(M) + \sum_{i=1}^{3} \left[d\mu_i^2 + \mu_i^2 \left(d\phi_i + \frac{1}{\sqrt{3}} A_\mu dx^\mu \right)^2 \right] \tag{4.5}$$

[1]We will take the AdS radius $L = 1$ for a while, until we reintroduce it in the next section.

together with a corresponding ansatz for the five-form

$$F_5 = (1 + \star_{10}) \left(-4\,\mathrm{vol}(M) + \frac{1}{2\sqrt{3}} \sum_{i=1}^{3} d\left(\mu_i^2\right) \wedge \left(d\phi_i + \frac{1}{\sqrt{3}} A_\mu dx^\mu \right) \wedge \star_5 dA \right) . \tag{4.6}$$

Here, the scalars were set to a constant value. Plugging the truncation ansatz into the type IIB equations of motion, we recover equations which may be reinterpreted as the equations of motion of a lower dimensional theory, Einstein-Maxwell theory in five dimensions with negative cosmological constant:

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left(R + 12 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu F_{\nu\rho} F_{\sigma\lambda} \right) \tag{4.7}$$

Here, one of the main virtues is the Chern-Simons term, which is required by supersymmetry of the underlying five dimensional $\mathcal{N} = 2$ gauged supergravity [198, 241]. It is also interesting to note, that the existence of such terms in the lower-dimensional theory is very much related to the self-duality of the five-form in ten dimensions [270]. In [113, 114] it was shown, that this term is related to a transport coefficient of the fluid previously disregarded. The explicit computation showed that in the particular case of the $\mathcal{N} = 4$ fluid, this transport coefficient is non-zero and therefore cannot be disregarded on general grounds. Its relation to chiral anomalies of the underlying microscopic theory in the general AdS/CFT context [8] and its compatibility with a positive divergence entropy current was nicely understood in [115].

Another case of a consistent truncation is the one, where only one Kaluza-Klein gauge field along an angular coordinate of the S^5 is kept in the low energy effective theory [266]. In principle, that setup does not seem to be too different from the one where we choose one gauge field along the diagonal $U(1)$ isometry ("three charges equal") of the five-sphere. However, we are going to perform the Kaluza-Klein reduction of a given ten dimensional solution soon, the rotating D3-branes. When maintaining the asymptotically flat part of that solution, we have found that using the Kaluza-Klein ansatz of [266] or [241] does not keep all modes which are switched on. This can be seen from the fact that the lower-dimensional fields after the reduction still depend on some angular coordinates of the five-sphere, which they should not in a *consistent* setup [266]. However, it is interesting to note that in the near-horizon limit of the geometry this exact procedure *does* yield a consistent truncation of the modes of the resulting AdS Reissner-Nordström branes [266, 241]. Already from this observation we may see that the region outside the AdS throat has interesting physics which imprints itself on the hydrodynamics of the cutoff surface fluid.

The main difference in the case for all three rotations equal is the symmetry enhancement of the setup as the five-sphere may be written as a Hopf fibration over $\mathbb{CP}^2$ with the $U(1)$ fibre isometry gauged. As often in physics, (enhanced) symmetry serves as a good guiding principle for extensions to aim for.

For completeness, we should mention, that the case with all Cartan rotations switched on has been analysed [241]. The lower-dimensional theory one recovers is the STU model of $\mathcal{N} = 2$ supergravity. Also, the fluid/gravity analysis around the black hole solution [271] of the STU model has been undertaken in [272]. But our upcoming analysis will already be complicated and rich enough not to ask for any further fields.

4.1.2 Consistent massive truncations of type IIB supergravity on squashed Sasaki-Einstein manifolds

The consistent truncation which will be of use for our purposes was derived in [128] and will be motivated and reviewed in the following.

One of the key aspects of the AdS/CFT correspondence is, as the name of course captures, that a gravitational theory on an AdS background is generically dual to a field theory which enjoys conformal symmetry. The question, how far this "generically" goes, is certainly one of the main objectives in research on Gauge/Gravity duality. One of the universal statements about dualities of this and other kind is however that for any pair of dual theories the symmetries on one side should be incorporated on the dual other side. These symmetries might in some sense be hidden[2], but the apparent ones usually provide a good first check of a conjectured duality.

One particular aspect of AdS/CFT realisations of Gauge/Gravity duality is found in the conformal symmetry group $SO(2, d)$ of the CFT (taking $d > 2$ space-time dimensions from now on). On the gravity side, this symmetry is represented by the isometry group of AdS space in $d + 1$ dimensions, which also happens to be $SO(2, d)$. Beyond the well established explicit constructions of AdS/CFT duality one may wonder how robust the duality is under breaking of some of its symmetries. There is a huge amount of literature on breaking various kinds of symmetries and finding explicit gravity duals for these. Most of this work is unfortunately beyond the scope of this thesis. But one particular interesting line of research is the study of systems with *non-relativistic conformal symmetry* [274, 275]. This was of great importance as a motivation for [128], so we will quickly look into this now.

It is well known, that the Galilean algebra in $d - 1$ spatial dimensions with its Hamiltonian, rotation, translation, boost and mass operators can be embedded into the Poincaré algebra in $d + 1$ space-time dimensions where the 1+1 additional dimensions are looked at in light-cone form $x^{\pm}$ (e.g. [274, 275] or more generally [25]). The Galilean algebra is then seen as the subalgebra which commutes with P_-, the momentum operator along x_-. That operator is then naturally interpreted as the mass operator $-M$ of the Galilean algebra, since (by construction) it is a central charge of the subalgebra. The Hamiltonian is represented by $-P_+$. Translations and rotations retain their trivial embedding and the Galilean boosts correspond to the additional rotation generators. The addition of an additional dilatation generator is possible and in fact crucial for the construction of the non-relativistic conformal symmetry group. In essence, the scale transformation which corresponds to this generator scales time and space in a different, generically non-relativistic way

$$x^0 \to \lambda^z x^0 \,, \qquad x^i \to \lambda x^i \,, \tag{4.8}$$

where in the condensed matter literature z is referred to as the dynamical exponent [136, p.67]. Clearly, the case $z = 2$ is special in the sense that this is the scale symmetry of the Schrödinger equation in flat space. Also, one may add another special conformal transformation generator then.

Now for CFTs with Poincaré and conformal symmetry, the generic gravity duals are gravitational theories on AdS space. So, can we also realise this non-relativistic conformal symmetry group in terms of a Gauge/Gravity dual pair? What gravitational

[2] For a particularly intriguing example see e.g. [273].

background with corresponding isometries would the theory then have to be defined on?

In [274, 275] first attempts were initiated to study holographic dualities where the field theory possesses such a scaling symmetry. A metric background with the appropriate symmetries was identified to be

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx_i^2 + 2d\xi dt}{r^2} \right) . \tag{4.9}$$

Interestingly, this background can be seen as arising from a particular solution of the equations of motion for an Einstein-Proca system with ansatz $A_+ \propto r^z$ and action

$$S = \frac{1}{2\kappa_5^2} \int d^dx dr \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \right) , \tag{4.10}$$

with the usual AdS relation of the cosmological constant to the space dimension $\Lambda = -\frac{1}{2}d(d-1)$. Apart from this, also m^2 depends on d and the cricital exponent z via $m^2 = z(z + d - 2)$ for the metric to be a solution to the equations of motion.

The gauge symmetry of the gauge boson A_μ and the Stückelberg scalar has been fixed to make the scalar vanish. Thus, the gauge boson acquires a mass and the corresponding further degree of freedom from the scalar.

Now how does this relate to string theory? The toy-model described so far is clearly in the spirit of the bottom-up approach. As always for such models, it is useful to try to embed it into a supergravity or ultimately a string theory compactification. This is not only a question of pure aesthetics but also important for several other reasons. Certainly one of the main questions is if such a theory may be incorporated into a consistent theory of quantum gravity and if proposed dualities therefore extend to a regime beyond certain mean field approximations of the field theory. Furthermore, one would ideally like to find an explicit construction of the field theory which the proposed gravity model should be dual to. A useful path for such endeavours has been the construction via intersecting brane models and their low energy effective field theory. One may furthermore wonder about the question of stability of the setup within a consistent quantum gravity framework where e.g. non-linear instabilities might spoil the proposed properties. Furthermore an embedding of such bottom-up models into string theory also helps in understanding parts of the vast string theory landscape.

The embedding of the Einstein-Proca theory under consideration was done independently in [276, 277, 128]. In particular [128] gave many details on the consistent truncation within type IIB supergravity, which we will extensively draw from in the following.

Typically, from the bottom-up perspective one starts off with an action, which enjoys the preferable symmetries and fields. Some of the parameters like masses or charges are often left arbitrary. The top-down derivation, preferably as a consistent truncation of the spectrum of a supergravity theory, usually determines these in a discrete way. For the example shown, it is peculiar to note that one aims for a consistent truncation of the spectrum of a supergravity theory which not only includes massless modes, as is most commonly studied, but where the inclusion of a massive vector is of huge importance.

In [128], two consistent truncations of type IIB supergravity were derived – both to $d = 4$, i.e. five-dimensional gravitational theories, with gauge field masses corresponding to $z = 2$ and $z = 4$. In both cases, the reduction was done on a Sasaki-Einstein[3] space Y, where it is known that in a background $AdS_5 \times Y$ there exist certain massive gauge field excitations of AdS_5 with the desired mass (e.g. for $Y = S^5$ see [171]). Then after the reduction on Y one is left with a five-dimensional action roughly of the form (4.10), which not only has AdS_5 with vanishing gauge field as a solution, the negative cosmological constant results from the positive curvature of Y, but also (4.9) with a non-vanishing profile for the gauge field.

In both cases, a particular role is played by $\eta = d\phi + P$, the one-form dual to the Reeb Killing vector $\partial/\partial\phi$ which is present in any Sasaki manifold. For the special case $Y = S^5$, the angle ϕ for example denotes simultaneous rotations along all three independent rotation angles of S^5 or, when seen as a fibration $S^5 \cong S^1 \to \mathbb{CP}^2$ the rotation angle of the S^1 fibre. P denotes the one-form which determines the closed Kähler form ω of the base space (in our example the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric) via $dP = \omega$.

In the first case, the massive gauge boson with $m^2 = 8$ stems from the anti-symmetric tensor field $B_{\mu\nu}$ in the NSNS sector, where the Kaluza-Klein reduction is performed along η via

$$B = A \wedge \eta + \theta\, \omega\,. \tag{4.11}$$

In this expression, A denotes the gauge field and θ is the Stückelberg scalar which in a particular gauge represents the massive degree of freedom of the gauge field.

But the consistent truncation which is important for our purposes is the second one. There, the reduction on a Sasaki-Einstein manifold is performed in such a way that two Kaluza-Klein gauge fields, $A_\mu^{(m)}$ from the metric and $A_\mu^{(4)}$ from the RR four-form C_4, mix into a massive and massless gauge field in the AdS_5 background. For the five-sphere compactification, the harmonic analysis with a Freund-Rubin [279] five-form background ansatz $F_5 = (1+\star_{10})\mathrm{vol}(S^5)$ was carried out in [171]. Metric and four-form fluctuations are expanded in terms of spherical harmonics of the five-sphere and the linearised equations of motion in a fixed gauge were analysed. The vector modes of the metric and the four-form mix as

$$(\delta^\mu_\nu \Box - \partial_\nu \partial^\mu) \begin{pmatrix} A_\mu^{(m)} \\ A_\mu^{(4)} \end{pmatrix} + \begin{pmatrix} \triangle_Y - 8 & 16\,\triangle_Y \\ -1 & \triangle_Y \end{pmatrix} \begin{pmatrix} A_\nu^{(m)} \\ A_\nu^{(4)} \end{pmatrix} = 0\,, \tag{4.12}$$

where $\triangle_Y$ is the Laplacian on the five-sphere Y. This system of equations can be diagonalised with eigenvalues $m^2 = (k^2 - 1)$ and $m^2 = (k+3)(k+5)$ for $k \geq 1$ and eigenvectors $A_\mu^{(m)} - 4(k+3)A_\nu^{(4)}$ and $A_\mu^{(m)} + 4(k+1)A_\nu^{(4)}$ respectively. Evidently, for the lowest level $k = 1$, we get a massless mode and one with $m^2 = 24$ in AdS units.

In [280] and [281], the inclusion of the massless gauge boson, which corresponds to the Reeb Killing vector, was lifted to a full consistent non-linear Kaluza-Klein reduction, in the generalisation to Sasaki-Einstein manifolds Y, in which the five-sphere description is only mildly modified (see references in [128]). The resulting theory obtained was $\mathcal{N} = 2$ gauged supergravity. Although the extension of our following analysis to Sasaki-Einstein compactifications is also interesting, we are going to restrict our attention to the five-sphere case. The main point of the Kaluza-Klein reduction analysed in [128] is that the second, lowest level *massive* gauge field can also be

[3] For a more general introduction to the field theory duals to string theory on $AdS_5 \times Y$ see [264,278] and the many follow-up works.

incorporated into a full consistent non-linear truncation of the type IIB supergravity modes.

For doing so, the following reduction ansatz was chosen:

$$ds^2 = ds^2(M) + e^{2U} ds^2(B_{KE}) + e^{2V}(\eta + \mathcal{A})^2\,, \tag{4.13}$$

$$\begin{aligned} F_5 = \frac{Q}{2}\big(&4e^{-4U-V}\mathrm{vol}(M) + 4e^{-4U-V}(\eta+\mathcal{A})\wedge\star_5\mathbf{A} + e^{-V}\omega\wedge\star_5\mathbb{F} \\ &+2\omega^2\wedge(\eta+\mathcal{A}) + 2\omega^2\wedge\mathbf{A} - \omega\wedge(\eta+\mathcal{A})\wedge\mathbb{F}\big)\ . \end{aligned} \tag{4.14}$$

The metric part is quite self-explanatory: $ds^2(B_{KE}) + (\eta+\mathcal{A})^2$ is the metric of the Sasaki-Einstein manifold seen as a $U(1)$ fibre bundle over the Kähler-Einstein base space B_{KE}, including the gauging of the fibre isometry. Both are multiplied with the moduli describing the volume of the fibre and the base. These can be combined into the breathing mode of the compact space and the relative squashing mode of base and fibre. The left over five-dimensional metric is not Weyl transformed into the Einstein frame and we will not perform this Weyl transformation in the following, since for our purposes there is no real need to do so. So we already expect to end up with an 5d effective action, in which the dilatons are coupled to the Ricci curvature and are not canonically normalised.

The five-form ansatz is however not so straightforward to understand. First of all, one may note that the first and second line of the ansatz already incorporate the ten-dimensional self-duality of the five-form. Each term in the lower line can be seen as the 10d Hodge dual of one in the first line; we thus restrict the description to the lower line. The term $\frac{Q}{2}\left(2\omega^2\wedge(\eta+\mathcal{A})\right)$ is the direct generalisation of the Freund-Rubin ansatz, basically being proportional to the volume form on Y and normalised such that $\int_Y F_5 = 2Q$ for compatibility with [123]. Note that by this choice we slightly adjust the normalisations of [128] to Q units of five-form charge (instead of just one). The gauge field $\mathbf{A}$ arises, as explained earlier, from the four-form $C_4 \sim \omega\wedge\eta\wedge\mathbf{A}$, where $\omega\wedge\eta$ is the natural three-form on Y. Then taking the exterior derivative $F_5 = dC_4$, replacing $\eta\to\eta+\mathcal{A}$, we see that terms like $2\omega^2\wedge\mathbf{A} - \omega\wedge(\eta+\mathcal{A})\wedge\mathbf{F}$ have to arise. But requiring that F_5 be a closed form (it is actually even exact), we see that $dF_5 \sim \ldots + 2\omega^2\wedge d(\eta+\mathcal{A}) = \ldots + 4\omega^3 + 2\omega^2\wedge\mathcal{F}$ requires the last term be replaced as

$$\omega\wedge(\eta+\mathcal{A})\wedge\mathbf{F} \to \omega\wedge(\eta+\mathcal{A})\wedge(\mathbf{F}+\mathcal{F})\,, \tag{4.15}$$

where for compactness of notation one uses $\mathbf{F}+\mathcal{F}\equiv\mathbb{F}$.

From the closure of F_5, one may in total derive four equations[4].

$$d\left(e^{-4U+V}\star_5\mathbf{A}\right) = 0 \tag{4.16}$$

$$d\left(e^{-V}\star_5\mathbb{F}\right) = -8e^{-4U+V}\star_5\mathbf{A} + 4\kappa\,\mathcal{F}\wedge\mathbb{F} \tag{4.17}$$

$$d\mathbb{F} = 0 \tag{4.18}$$

$$\mathbb{F} = \mathcal{F} + \mathbf{F} \tag{4.19}$$

The last has already been explained basically by construction of the ansatz and the first is similar to a Lorentz gauge choice for $\mathbf{A}$. Now, there is also a Maxwell equation for $\mathbb{F}$ and a simple Bianchi equation. In the Maxwell equation, it is worth emphasising the presence of the terms on the right hand side. One is basically a mass term for $\mathbf{A}$,

[4]The additional factor 4κ relative to [128] will be explained soon.

e.g. in a particular vacuum with constant U and V we do get exactly what part of the motivation was: arriving at a low-energy effective theory which incorporates a massive gauge field. Furthermore, there is a parity odd term which already raises the expectation of a Chern-Simons term in the effective action.

One may now analyse the equations of motion of IIB supergravity with vanishing fields except metric and five-form. The five-form equation of motion / Bianchi identity $dF_5 = 0$ has already been dealt with. The Einstein equation is

$$R_{MN} = \frac{5}{4 \cdot 5!} F_{MABCD} F_N^{ABCD} , \tag{4.20}$$

where we only keep the modes as indicated. The resulting equations[5] can then be reinterpreted as the equations of motion of the effective action[6]

$$\begin{aligned} S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}\, e^{4U+V} \Big[& R^{(5)} + 24\, e^{-2U} - 4\, e^{-4U+2V} - 2\, Q^2\, e^{-8U-2V} \\ & +12\, \partial_\mu U \partial^\mu U + 8\, \partial_\mu U \partial^\mu V - \frac{1}{4} e^{2V} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{8} Q^2\, e^{-4U-2V} \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} \\ & -2\, Q^2\, e^{-8U} \mathbf{A}_\mu \mathbf{A}^\mu \Big] + \frac{1}{2\kappa_5^2} \left(\frac{Q^2}{4} \right) 4\kappa \int \mathcal{A} \wedge \mathbb{F} \wedge \mathbb{F} \, . \end{aligned} \tag{4.21}$$

Before commenting on the general structure of the action, we quickly collect these equations of motion, because we will make extensive use of them.

The Einstein equation is

$$\begin{aligned} R^{(5)}_{\mu\nu} = {} & 4\left(\partial_\mu U \partial_\nu U + \nabla_\mu \partial_\nu U\right) + \left(\partial_\mu V \partial_\nu V + \nabla_\mu \partial_\nu V\right) - Q^2\, e^{-8U-2V} g_{\mu\nu} \\ & + \frac{1}{2} e^{2V} \mathcal{F}_{\mu\rho}\mathcal{F}_\nu{}^\rho + Q^2\, e^{-8U} \left(2\mathbf{A}_\mu \mathbf{A}_\nu - g_{\mu\nu} \mathbf{A}_\rho \mathbf{A}^\rho\right) \\ & + \frac{1}{16} Q^2\, e^{-4U-2V} \left(4 \mathbb{F}_{\mu\rho}\mathbb{F}_\nu{}^\rho - g_{\mu\nu} \mathbb{F}_{\rho\sigma}\mathbb{F}^{\rho\sigma}\right) . \end{aligned} \tag{4.22}$$

The (coupled) scalar equations of motion for the dilatons U and V are

$$\begin{aligned} \Box_5 U + 4\partial_\mu U \partial^\mu U + \partial_\mu U \partial^\mu V = {} & 6\, e^{-2U} - 2\, e^{-4U+2V} \\ & - Q^2\, e^{-8U-2V} - Q^2\, e^{-8U} \mathbf{A}_\mu \mathbf{A}^\mu , \end{aligned} \tag{4.23}$$

$$\begin{aligned} \Box_5 V + 4\partial_\mu U \partial^\mu V + \partial_\mu V \partial^\mu V = {} & 4\, e^{-4U+2V} - Q^2\, e^{-8U-2V} + \frac{1}{4} e^{2V} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \\ & + Q^2\, e^{-8U} \mathbf{A}_\mu \mathbf{A}^\mu - \frac{1}{16} Q^2\, e^{-4U-2V} \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} . \end{aligned} \tag{4.24}$$

Last but not least, the equation of motion for the gauge field $\mathcal{A}$, which also involves a parity odd term, is

$$d\left(e^{4U+3V} \star_5 \mathcal{F}\right) = 4\, Q^2\, e^{-4U+V} \star_5 \mathbf{A} + Q^2\, \kappa\, \mathbb{F} \wedge \mathbb{F} \, . \tag{4.25}$$

After this long detour, we have finally arrived at an action (4.21), which stems from a consistent truncation of IIB supergravity modes and possesses all the ingredients

[5] Again, we refer to the appendix of [128] for further details, however we adopt for Q units of five-form charge and correct some minor typos.

[6] For the sake of generality, we take the liberty of an additional factor κ in front of the Chern-Simons term, to keep better track of it in the analysis of the equations of motion as was done in [113] – note however the different normalisation of κ compared to that publication.

we need. The consistency of the reduction has not been explained in detail here, but it has been checked at the level of the bosonic fields in [128]. The truncation is non-supersymmetric [128], but can be extended also to supersymmetric massive truncations on Sasaki-Einstein spaces as was explicitly shown in [282, 283, 284].

We will now quickly elaborate on how the action (4.21) relates to the previously discussed theories. Indeed, it incorporates all features of those and combines them in a minimal way.

It reduces to the simple action (4.4) once the gauge fields are set to zero and the dilatons are set equal to $U = V = \varphi$. Therefore, apart from a subtlety, which we will extensively discuss at a later stage in section 4.2.2, the effective hydrodynamics of our setup reduces to [123]. More precisely, setting the charge $q \to 0$ will reduce our setup to one which is equivalent to the one studied in [123].

Also, we may reduce (4.21) to the Einstein-Maxwell theory with negative cosmological constant (4.7). For this, the second gauge field $\mathbf{A}$ needs to be set zero and the dilatons should be fixed to constant equal values. In particular, also the Chern-Simons term is recovered. Therefore, in a suitable limit, the decoupling / near horizon limit, we are going to recover the analysis of [113, 114].

Also the main features of the toy model (4.10) are captured by (4.21). Setting the scalars zero, we arrive at the right negative cosmological constant. We may further diagonalise the gauge field Lagrangian and set the massless gauge field zero. Then the massive gauge field, upon canonical normalisation, has exactly the mass $m^2 = 24$ as aimed for. Clearly, as part of the motivation, this was quickly derived in section 4.2 of [128], where also a particular solution with the sought-after non-relativistic $z = 2$ scaling symmetry was stated.

We are now ready to use the reduction ansatz explicitly for the solution of a stack of rotating D3-branes. By doing so, we will recover a solution to the effective action (4.21) which will serve as a starting point for our blackfold analysis inspired by [123].

4.2 Rotating D3-branes and their Kaluza-Klein reduction

Within type IIB supergravity, D3-branes can be understood as the black p-branes [161] (with $p = 3$) which source Ramond-Ramond five-form flux [160]. The RR field C_4 couples naturally to the $3+1$ dimensional world-volume of the D3-branes. When putting a stack of N of these into flat space there will be six transverse directions which can be seen as a radial direction and a five-sphere. The five-sphere can then be rotated in the three planes of the original six dimensional transverse space, or said differently: the $SO(6)$ isometry group of the five-sphere has a $U(1)^3$ Cartan subgroup. Therefore, D3-branes can have up to three different angular momenta l_1, l_2, l_3.

The metric for the $l_2 = l_3 = 0$ case was obtained in [285] and extended to all angular momenta non-vanishing in [286] (see also [287]) using previous results of [288]. We are going to take the results of [241] as a starting point here since it corrected some typos in the aforementioned literature. Additionally, in that work, the decoupling limit of such configurations was stated, which will be useful for us. In particular, it was shown in [241] that this near-horizon geometry can be reduced to the STU black holes [271] under a Kaluza-Klein ansatz, which in general describes the consistent truncation of ten dimensional type IIB supergravity [289] on the five-sphere [290, 171] to $\mathcal{N} = 8$, $SO(6)$ gauged supergravity [291, 292], further truncated to five dimensional $\mathcal{N} = 2$, $U(1)^3$ gauged supergravity.

The metric for the full rotating D3-brane (with all $l_i \neq 0$) is [241]

$$\begin{aligned} ds^2 = \tilde{H}^{-1/2} & \left[-\left(1 - \frac{2m}{\tilde{r}^4 \Delta}\right) dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] \\ + \tilde{H}^{1/2} & \left[\frac{\Delta d\tilde{r}^2}{H_1 H_2 H_3 - \frac{2m}{\tilde{r}^4}} + \tilde{r}^2 \sum_{i=1}^{3} H_i \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) \right. \\ & \left. - \frac{4m \cosh\alpha}{\tilde{r}^4 \tilde{H} \Delta} dt \left(\sum_{i=1}^{3} l_i \mu_i^2 d\phi_i \right) + \frac{2m}{\tilde{r}^4 \tilde{H} \Delta} \left(\sum_{i=1}^{3} l_i \mu_i^2 d\phi_i \right)^2 \right] , \end{aligned} \tag{4.26}$$

where we have used

$$\Delta(\tilde{r}) = H_1 H_2 H_3 \sum_{i=1}^{3} \frac{\mu_i^2}{H_i} , \quad \tilde{H}(\tilde{r}) = 1 + \frac{2m \sinh^2\alpha}{\tilde{r}^4 \Delta} , \quad H_i(\tilde{r}) = 1 + \frac{l_i^2}{\tilde{r}^2} . \tag{4.27}$$

The self-dual five-form $F_5 = \star F_5$ stems from $G_5 = dB_4$ via $F_5 = (1 + \star_{10})\, G_5$ and the four-form

$$B_4 = \frac{1 - \tilde{H}^{-1}}{\sinh\alpha} \left(-\cosh\alpha \, dt + \sum_{i=1}^{3} l_i \mu_i^2 d\phi_i \right) \wedge d^3x . \tag{4.28}$$

We now slightly adjust the conventions of [241] via

$$2m = r_0^4 , \quad 2m \sinh^2\alpha = L^4 \quad \Rightarrow \quad 2m \cosh\alpha = r_0^2 \sqrt{r_0^4 + L^4} \tag{4.29}$$

and take all angular momenta equal $l_i = l$. Then metric and five-form exactly fit into the Kaluza-Klein ansatz (4.13),(4.14).

To give some more details, note that we may write the five-sphere as a Hopf fibration. This is a standard generalisation of the Hopf map which works for all odd dimensional spheres $S^1 \hookrightarrow S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$. In our case, $n = 2$, it is straightforward to show ($H_i = H_1$ for all i since $l_i = l$)

$$\tilde{r}^2 \sum_{i=1}^{3} H_i \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) = \tilde{r}^2 H_1 \left(\eta^2 + 2 g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \right) , \tag{4.30}$$

where $\eta = d\phi + P$ and $g_{i\bar{j}} = \frac{1}{2} \partial_i \bar{\partial}_{\bar{j}} \mathcal{K}$ is the Fubini-Study metric on $\mathbb{CP}^2$ with Kähler potential $\mathcal{K}$ and Kähler form $\omega = d\eta/2$ given by

$$\mathcal{K} = \log\left(1 + |z_1|^2 + |z_2|^2\right) , \quad \omega = 2i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} , \quad P = \frac{i}{2} \frac{z^i d\bar{z}^i - \bar{z}^i dz^i}{1 + |\vec{z}|^2} . \tag{4.31}$$

As before, the μ_i parametrise a two-sphere (with $\sum_i \mu_i^2 = 1$). All angular coordinates of the five-sphere are then related to the complex coordinates z_1, z_2 of $\mathbb{CP}^2$ and the fibre angle ϕ by

$$|z_1|^2 = \frac{\mu_1^2}{\mu_3^2} , \quad |z_2|^2 = \frac{\mu_2^2}{\mu_3^2} , \quad \phi_1 = \phi + \arg z_1 , \quad \phi_2 = \phi + \arg z_2 , \quad \phi_3 = \phi . \tag{4.32}$$

Another easy to derive relation from this coordinate change is $\sum_i \mu_i^2 d\phi_i = \eta$.

From these relations it is clear, that the last term in (4.26) together with the ones rewritten as in (4.30) yield a relative squashing of fibre and base metrics whose respective volumes were parametrised in (4.13) by the scalars U and V. They thus receive profiles[7]

$$e^{2U} = \tilde{H}^{1/2}\tilde{r}^2 H_1\,, \qquad e^{2V} = \tilde{H}^{1/2}\left(\tilde{r}^2 H_1 + \frac{r_0^4 l^2}{\tilde{r}^4 \tilde{H}\Delta}\right) = \frac{\tilde{H}^{1/2}\tilde{r}^2}{\Delta \tilde{g}}\,. \tag{4.33}$$

where we have introduced $\tilde{g}(\tilde{r}) = \tilde{H}\left(H_1^3 \tilde{H} + \frac{r_0^4 l^2}{\tilde{r}^6}\right)^{-1}$. The (electric) gauge field profile can be read off from the off-diagonal term proportional to $dt\left(\sum_i l_i \mu_i^2 d\phi_i\right)$. It is given by

$$\mathcal{A} = -l\, r_0^2 \sqrt{r_0^4 + L^4}\left(\frac{\tilde{g}}{\tilde{r}^6 \tilde{H}}\right) dt\,. \tag{4.34}$$

Now, the only term which needs to be read off from (4.13) is the background metric. It is given by

$$ds^2(M) = \tilde{H}^{-1/2}\left[-\tilde{f}\tilde{g}\, dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + \tilde{H}^{1/2}\Delta \tilde{f}^{-1} d\tilde{r}^2 \tag{4.35}$$

where we have defined $\tilde{f}(\tilde{r}) = H_1^3 - \frac{r_0^4}{\tilde{r}^4}$. Now, we only need to check the various components of the five-form (4.14) and read off the profile of the gauge field $\mathbf{A}$. It is given by

$$\mathbf{A} = -l\, r_0^2\, L^2 \left(\frac{2}{Q}\right)\left(\frac{\tilde{f}\tilde{g}}{\tilde{r}^2 H_1 \tilde{H}}\right) dt\,. \tag{4.36}$$

The coordinates, in which we have worked so far, are the ones which directly descend from the ones in (4.26), but are not the most convenient ones e.g. for comparisons with [113, 114]. The coordinate change is however very easy to implement, namely $r^2 = \tilde{r}^2 + l^2$.

Carrying out this coordinate change, we observe, that $\tilde{f}(\tilde{r})$, $\tilde{H}(\tilde{r})$, $\tilde{g}(\tilde{r})$ are related to particular functions $f(r)$, $g(r)$, $H(r)$ which have some clear interpretation. In the new coordinate system, they are given by

$$f(r) = 1 - \frac{r_0^4}{r^4} + \frac{q^2}{r^6}\,, \quad H(r) = 1 + \frac{L^4}{r^4}\,, \quad g(r) = \frac{H(r)}{H(r) + \frac{q^2}{r^6}}\,, \tag{4.37}$$

where we reinterpreted the angular momentum parameter l as a charge q via $q \equiv l\, r_0^2$. In these coordinates, our lower-dimensional black hole solution is very much reminiscent of the AdS Reissner-Nordström solution [198, 113, 114] and, in fact, is related to it via the near-horizon limit soon to be discussed. The metric then reads

$$ds^2 = H(r)^{-1/2}\left[-f(r)g(r)\, dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + H(r)^{1/2} f(r)^{-1} dr^2\,, \tag{4.38}$$

where clearly $f(r)$ is the emblackening factor basically incorporating mass $\sim r_0^4$ and charge $\sim q$ of the Reissner-Nordström solution. $H(r)$ is the warp factor, which in

[7] Note that these acquire somewhat unusual length dimensions as in [123]. We could introduce a length scale for the radius of the Sasaki-Einstein space, but do not necessarily need to.

the near-horizon limit behaves like $H(r) \to \frac{L^4}{r^4}$ changing the asymptotics of the near-horizon geometry to asymptotically AdS.

The scalars U and V also simplify a bit

$$e^{2U} = H(r)^{1/2} r^2\,, \qquad e^{2V} = H(r)^{1/2} r^2 g(r)^{-1}\,, \tag{4.39}$$

and we can see that the newly defined $g(r)$ basically describes the relative squashing of base and fibre in the compact space we reduced on.

The gauge fields are given by

$$\mathcal{A} = -q \left(\frac{Q}{2L^2}\right)\left(\frac{g(r)}{r^6 H(r)}\right) dt\,, \quad \mathbf{A} = -q\left(\frac{2L^2}{Q}\right)\left(\frac{f(r)g(r)}{r^2 H(r)}\right) dt\,, \tag{4.40}$$

where we have reintroduced Q given by $Q = 4m \sinh\alpha \cosh\alpha$ or, as we will use it more often

$$Q = 2L^2 \sqrt{r_0^4 + L^4}\,. \tag{4.41}$$

which, in string units, is related to the number of D3-branes N via [286]

$$4\pi g_s \alpha'^2 N = \frac{Q}{2}\,. \tag{4.42}$$

We have chosen this slightly odd normalisation to keep maximal compatibility with [123].

4.2.1 The decoupling limit

Of course, it is well known that for extremal non-rotating D3-branes the near-horizon geometry is $AdS_5 \times S^5$ [293]. For the case of non-extremal rotating D3-branes as in our case, there however also exists a decoupling limit which will be important for the connection of our results (4.38),(4.39),(4.40) and their relation to [113, 114].

It is simply given by [241]

$$r_0 \to \epsilon\, r_0\,, \quad L \to L\,, \quad r \to \epsilon\, r\,, \quad x^\mu \to \epsilon^{-1} x^\mu\,, \quad q \to \epsilon^3\, q\,, \tag{4.43}$$

taking $\epsilon \to 0$, which then implies that

$$f(r) \to f(r)\,, \quad H(r) \to \frac{L^4}{\epsilon^4 r^4}\,, \quad g(r) \to 1\,. \tag{4.44}$$

As remarked in [241], the last term in (4.26) is subdominant in that limit, so that the relative squashing between base and fibre is suppressed, as can also be seen directly from (4.39) using (4.44). Also the dilatons are set to constant values $e^{2U} = e^{2V} = L^2$ accounting for the negative cosmological constant in (4.21), where we retain the AdS radius L in comparison to [113, 114].

The background profile of the gauge field $\mathbf{A}$ is suppressed by ϵ^4 (4.40), but the gauge field $\mathcal{A}$ in the limit (4.43) exactly agrees with the one in [113, 114]. For showing this, we have to remark that the different normalisation in the kinetic terms gives the relation $\frac{\sqrt{3}}{2}\mathcal{A}_\mu = A_\mu^{(B)}$, where the latter gauge field $A_\mu^{(B)}$ is the one in [113].

As mentioned earlier, the metric (4.38) does indeed reduce to the AdS Reissner-Nordström black hole, so, in total, from the near horizon scaling (4.43) in [241] we get exact agreement with the known background solution of [198, 113, 114].

4.2.2 Comments on the chargeless limit

Taking the charge q to zero of course, by construction, recovers the background solution of [123]. However, in [123], a slightly different coordinate system was chosen. For the backgrounds under consideration the coordinate change is very easy and such a standard transformation, that it almost does not seem worth mentioning at all. But in our upcoming first order fluid/gravity (or rather blackfold) analysis this slight change makes a subtle difference.

Taking conventions as in [9] for a while (i.e. the radial coordinate, which is called r in [123], is now renamed to ρ as in [9], while the r of [9] is identical with our previously used r), we have the coordinate change

$$\rho^4 = r^4 + L^4\,, \quad r_+^4 = r_0^4 + L^4\,, \quad r_- = L \tag{4.45}$$

and the warp & emblackening factors $H(r)$ and $f(r)$ basically are replaced by

$$\Delta_\pm(\rho) = 1 - \frac{r_\pm^4}{\rho^4}\,. \tag{4.46}$$

So, for example r_0 is related to the temperature of the D3-brane or, in other words, the fact that inner and outer horizons[8] $r_\pm$ do not coincide for $T \neq 0$ ($\leftrightarrow$ non-extremality).

The subtlety, which we want to emphasise, is that the coordinate change (4.45) for the radial coordinates involves the parameter L. According to the fluid/gravity and blackfold lore, we have to take such a parameter to depend on the world-volume space-time coordinates $L \to L(\sigma^a)$. We are going to take Q as in (4.41) fixed and thus replace the space-time dependence of L with the one of r_0, but implicitly then still L depends on σ^a. Then, one may define the cutoff surface as an isodilatonic surface as in [123]. To lowest order in the derivative expansion this is the same as choosing it at constant $\rho = \mathrm{P}$, because in the coordinate system of [123], in which $e^{2U} = \rho^2$ without any σ^a dependence. In the coordinate system, in which $e^{2U} = H^{1/2} r^2$, there is a σ^a dependence and isodilatonic and constant $r = R$ surfaces are not the same. Indeed, we are going to take a surface at constant $r = R$, which is then a different surface as the constant $\rho = \mathrm{P}$ one used in [123]. Some relations will thus be slightly different in the $q \to 0$ limit. We will however come back to this issue at later stages.

Apart from the subtlety just explained, we may quickly relate the expression for Q in (4.41) to the one used in [123] (citing [161]). In the chargeless limit $q \to 0$, from (4.41) we clearly have

$$Q = 2\, r_+^2\, r_-^2\,. \tag{4.47}$$

This relation is already evident from the coordinate change (4.45), but we may also use the explicit relations for $\cosh\alpha$ and $\sinh\alpha$ to outer and inner horizons; see e.g. [294], eq. (3.29),

$$\cosh^2\alpha = \frac{r_+^4}{r_H^4}\,, \quad \sinh^2\alpha = \frac{r_-^4}{r_H^4} \tag{4.48}$$

and that the horizon is at $r_H^4 = r_0^4 = 2m$ in the chargeless limit.

So, altogether, we recover the background of [123] in the limit $q \to 0$ as long as we dwell on the zero'th order background analysis. Gauge fields vanish, the dilatons (4.39) are identical and the coordinate change (4.45) is very simple to handle.

[8] Note that they are not called $\rho_\pm$ but indeed $r_\pm$ in [9].

4.3 Background for the perturbation analysis

4.3.1 Stationary background in Eddington-Finkelstein coordinates

We now would like to set up the Dirichlet problem at finite R in the spirit of the fluid/gravity correspondence and blackfold paradigm. As usual, we first transform our background (4.38),(4.40) in ingoing Eddington-Finkelstein coordinates to make the non-singular nature of the outer horizon apparent. Practically, this is implemented by a coordinate change from t to $v = t + r_\star(r)$, where the transformation is chosen in such a way that the coefficient of the dr^2 term in the metric (which signals a coordinate singularity at radii, for which $f(r) = 0$) vanishes.

For a general metric of the form

$$ds^2 = g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 + g_{xx}(r)\, d\vec{x}^2 \,, \tag{4.49}$$

the condition $r'_\star(r) = \sqrt{-g_{rr}(r)/g_{tt}(r)}$ ensures this and the transformed metric

$$ds^2 = g_{tt}(r)\, dv^2 + 2\sqrt{-g_{tt}(r)\, g_{rr}(r)}\, dv dr + g_{xx}(r)\, d\vec{x}^2 \tag{4.50}$$

is easily obtained.

For the background, this is just a different coordinate choice, which has the aforementioned well-known horizon behaviour. But for the fluctuations of the background matters are more intricate. Only those fluctuations with regular behaviour at the horizons are the ones, for which the gravity theory describes a dual fluid [295]; they are special geometries implementing cosmic censorship. Thus, by choosing Eddington-Finkelstein coordinates it is easier to implement the required regularity at the horizon for the perturbations and by this for the whole spacetime.

We may then boost along the world-volume directions $\{v, x^i\} \equiv \sigma^a$ of the brane to go from a static to a stationary spacetime by introducing the constant velocity u^a (normalised to $u^2 = -1$). The familiar projection tensor $P_{ab} = \eta_{ab} + u_a u_b$, which projects perpendicular to the velocity u^a, is readily introduced:

$$ds^2 = g_{tt}(r)\, u_a u_b\, d\sigma^a d\sigma^b - 2\sqrt{-g_{tt}(r)\, g_{rr}(r)}\, u_a\, d\sigma^a dr + g_{xx}(r)\, P_{ab}\, d\sigma^a d\sigma^b \tag{4.51}$$

For the Kaluza-Klein-reduced metric obtained earlier (4.38), we therefore get

$$ds^2 = -\frac{f(r)\, g(r)}{\sqrt{H(r)}}\, u_a u_b\, d\sigma^a d\sigma^b - 2\sqrt{g(r)}\, u_a\, d\sigma^a dr + \frac{1}{\sqrt{H(r)}} P_{ab}\, d\sigma^a d\sigma^b \,. \tag{4.52}$$

The transformation of gauge fields (4.40) is also simple. For them, we get

$$\mathcal{A} = q\left(\frac{Q}{2L^2}\right)\left(\frac{g(r)}{r^6 H(r)}\right) u_a d\sigma^a \,, \tag{4.53}$$

$$\mathbf{A} = q\left(\frac{2L^2}{Q}\right)\left(\frac{f(r)g(r)}{r^2 H(r)}\right)\left(u_a d\sigma^a - \frac{H(r)^{1/2}}{f(r)g(r)^{1/2}} dr\right) . \tag{4.54}$$

In this expression, we have used the gauge freedom of the gauge field $\mathcal{A}$ to set the component $\mathcal{A}_r$ to zero. We may however not do so for $\mathbf{A}$ since for this gauge field there is no gauge freedom left over anymore. The gauge field $\mathbf{A}$ already appears in (4.21) with an interaction term, which for stabilised dilatons would be a mass term.

If seen as an interaction or generalised mass term, in any case, the term $\sim \mathbf{A}_a\mathbf{A}^a$ appears to break the manifest gauge invariance of $\mathbf{A}$. But, as already mentioned a couple of times before, we should rather think of $\mathbf{A}$ as a normal gauge field along with an additional scalar degree of freedom θ, where the gauge transformation of $\mathbf{A}$ is accompanied by a shift in θ

$$\mathbf{A} \to \mathbf{A} + d\lambda\,, \quad \theta \to \theta - 2\lambda\,. \tag{4.55}$$

Then the action (4.21) is the one where this particular redundancy of the description was fixed to $\theta = 0$. In the Kaluza-Klein ansatz (4.14), we have not elaborated on this aspect, but we may do so analogously to (4.11): The Stückelberg scalar θ would roughly correspond to a term

$$C_4 \sim \mathbf{A} \wedge \omega \wedge \eta + \theta\, \omega \wedge \omega \tag{4.56}$$

in the Kaluza-Klein ansatz, where the gauge freedom of C_4 accounts for the transformation (4.55).

4.3.2 Long wavelength perturbations

Now, the usual logic of the fluid/gravity correspondence and blackfold paradigm asks to promote the a priori constant velocity u^a, parameters L, r_0 and charge q to *fields* depending on the world-volume coordinates σ^a.

$$u^a,\, L,\, r_0,\, q \quad \Rightarrow \quad u^a(\sigma^b),\, L(\sigma^a),\, r_0(\sigma^a),\, q(\sigma^a) \tag{4.57}$$

In general, the equations of motion for the gravitational setup will then not be satisfied anymore. However, if we expand our parameters u^a, L, r_0, q in a hydrodynamic-like derivative expansion with respect to its world-volume dependence, we may order by order in this derivative expansion *impose* the gravitational equations of motion onto the setup. This then constrains the perturbations of u^a, L, r_0, q in a particular way which can then be interpreted as the constitutive relations and current conservation equations of the familiar hydrodynamics of specific fluids. Our setup was designed to describe the single $U(1)$ charge generalisation of the setup described in [123] and in a particular limit reduce to [113, 114].

Apart from this general philosophy which we will revisit momentarily, there are some additional technical remarks to be made that either further restrict or simplify the analysis to be undertaken.

First, we would like to describe the intrinsic dynamics of a *fixed number of D3-branes*. Since the number of D3-branes Q is given in terms of L and r_0 as (4.41) [286]

$$Q = 2L^2\sqrt{L^4 + r_0^4} \tag{4.58}$$

the variations δL and δr_0 are not independent. They are rather given by

$$\delta L = -\frac{L r_0^3}{2L^4 + r_0^4}\,\delta r_0\,. \tag{4.59}$$

This was already implemented in [123], however within a different coordinate system in which the number of branes is represented as $Q = 2r_+^2 r_-^2$ [161].

Apart from this restriction of the setup, we also impose some conditions on the background, on which our perturbative analysis is performed. On the fluctuations of the various background fields we impose Dirichlet boundary conditions at a cutoff surface at finite radial slice at $r = R$ [127]. The quasi-local stress-energy tensor and charge currents at $r = R$ are then expected to be of hydrodynamic type. However, this hydrodynamic theory shall for convenience be described in *Minkowski space*. Therefore, as in [258, 123], we redefine our space-time coordinates to make this manifest[9].

$$\begin{aligned} ds_0^2 = & -\frac{f(r)\,g(r)}{f_R\,g_R}\sqrt{\frac{H_R}{H(r)}}\,u_a u_b\,d\sigma^a d\sigma^b \\ & -2\sqrt{\frac{g(r)H_R^{1/2}}{f_R\,g_R}}\,u_a\,d\sigma^a dr + \sqrt{\frac{H_R}{H(r)}}\,P_{ab}\,d\sigma^a d\sigma^b\,, \end{aligned} \tag{4.60}$$

where we have defined $f_R = f(R)$, $g_R = g(R)$ and $H_R = H(R)$. This automatically ensures that the theory on the cutoff surface $r = R$ is defined in Minkowski space.

For the dilatons U_0 and V_0 nothing changes

$$e^{2U_0} = H(r)^{1/2} r^2\,, \qquad e^{2V_0} = H(r)^{1/2} r^2 g(r)^{-1}\,, \tag{4.61}$$

but the gauge fields also undergo this coordinate rescaling. The seed gauge fields after the transformation are then

$$\mathcal{A}_0 = q\left(\frac{Q}{2L^2}\right)\left(\frac{g(r)}{r^6 H(r)}\right)\sqrt{\frac{H_R^{1/2}}{f_R\,g_R}}\,u_a d\sigma^a\,, \tag{4.62}$$

$$\mathbf{A}_0 = q\left(\frac{2L^2}{Q}\right)\left(\frac{f(r)g(r)}{r^2 H(r)}\right)\sqrt{\frac{H_R^{1/2}}{f_R\,g_R}}\left(u_a d\sigma^a - \frac{H(r)^{1/2}}{f(r)g(r)^{1/2}}dr\right)\,. \tag{4.63}$$

Note, that the dr terms in the last equations for metric and gauge fields have not been rescaled. In (4.63) the rescaling of $u_a d\sigma^a$ still factors out, because also the definition of Eddington-Finkelstein coordinates has to be modified accordingly.

As elaborated on in [258], the whole rescaling is of course not a necessity (although practical for some aspects of the problem). We could likewise compute in the background (4.38) at fixed $r = R$ without rescaling the coordinates. Then the metric h_{ab} on the world-volume of the brane is slightly different from the Minkowski metric η_{ab} (although still flat). This then also implies that the metric compatible connections $\nabla^{(\eta)}$ and $\nabla^{(h)}$ in these two descriptions (i.e. those with $\nabla_c^{(\eta)}\eta_{ab} = 0$ and $\nabla_c^{(h)} h_{ab} = 0$) are different. The difference when acting on a vector (or more generally a tensor) is given in the usual way in terms of a $(2, 1)$ tensor C^c_{ab}. So, in the end, it does not make too much of a difference to work in either of the two descriptions, c.f. appendices of [127] and [258]. One of the points one needs to be aware of during this transformation is that, for example, the change in $\nabla_a u_b$ may introduces new terms in the dissipative part of the fluid's energy momentum tensor which might not have been there for a specific fluid frame choice (e.g. Landau or Eckart frame) in one of the two descriptions. To summarise, both computations with or without

[9] We also supplement our various fields with an index 0 since these are now the fields which are the seeds for our fluid/gravity or blackfold analysis.

rescaling to Minkowski space at $r = R$ carry the same information. The one without a rescaling seems to be a bit more tractable and slightly less tedious; on the other hand the rescaling is conceptually more straightforward since we aim for a description in Minkowski space anyway.

We now proceed to compute the first order variations from (4.60). In principle, we could go even further than this; but it gets exceedingly more tedious already at the second order. For the time being, we thus restrict to first order fluctuations. We work in the vicinity of $\sigma^a = 0$ where we choose a local rest frame $u^a = \{1, 0, 0, 0\}$ as is very common in the fluid/gravity literature. Thus, the variation of the velocity to first order in the derivative expansion is given by

$$u_a d\sigma^a = -dv + \sigma^a \partial_a \beta_i dx^i\,, \quad \text{or} \quad \delta u^0 = 0 \quad \text{and} \quad \delta u^i = \sigma^a \partial_a \beta^i\,. \tag{4.64}$$

The parameters r_0, L, q are varied as

$$\delta r_0 = \sigma^a \partial_a r_0\,, \quad \delta L = \sigma^a \partial_a L\,, \quad \delta q = \sigma^a \partial_a q\,, \tag{4.65}$$

subject to keeping the number of branes fixed (4.59) as explained earlier.

Now, the total metric which will be inserted into the equations of motion is computed from the background seed metric (4.60) as

$$ds^2 = ds_0^2 + \left(\frac{\delta}{\delta u^a} ds_0^2\right)\delta u^a + \left(\frac{\delta}{\delta r_0} ds_0^2\right)\delta r_0 + \left(\frac{\delta}{\delta L} ds_0^2\right)\delta L + \left(\frac{\delta}{\delta q} ds_0^2\right)\delta q\,. \tag{4.66}$$

We proceed likewise with the gauge fields

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0 + \left(\frac{\delta}{\delta r_0}\mathcal{A}_0\right)\delta r_0 + \left(\frac{\delta}{\delta L}\mathcal{A}_0\right)\delta L + \left(\frac{\delta}{\delta q}\mathcal{A}_0\right)\delta q\,, \\ \mathbf{A} &= \mathbf{A}_0 + \left(\frac{\delta}{\delta r_0}\mathbf{A}_0\right)\delta r_0 + \left(\frac{\delta}{\delta L}\mathbf{A}_0\right)\delta L + \left(\frac{\delta}{\delta q}\mathbf{A}_0\right)\delta q\,, \end{aligned} \tag{4.67}$$

and dilatons

$$U = U_0 + \left(\frac{\delta}{\delta L}U_0\right)\delta L\,, \qquad V = V_0 + \left(\frac{\delta}{\delta L}V_0\right)\delta L + \left(\frac{\delta}{\delta q}V_0\right)\delta q\,, \tag{4.68}$$

where we have made apparent that the background profiles of the dilatons do not depend on certain parameters.

The corrections just obtained are then inserted into the equations of motion with additional fluctuations. All fluctuations may be organised according to their transformation under the spatial $SO(3)$ rotational symmetry. The metric is the only field allowing for a tensorial fluctuation, vector fluctuations may appear both in the metric as well in both gauge fields. Scalar fluctuations appear not only in the metric and gauge fields, but of course also in the dilatons. The various sectors decouple from each other and may therefore be studied independently. We first deal with the tensor sector and then solve the already quite elaborate vector sector.

4.4 Perturbations in the tensor sector of $SO(3)$

The tensor sector is as usual the easiest one to deal with. The reasons for this are not difficult to understand. Since there is only a tensorial fluctuation from the metric we do not have to deal with disentangling coupled equations of motion. Furthermore, the tensor perturbation obeys a minimally coupled scalar equation of motion (in non-Einstein frame) and is therefore relatively simple to integrate.

We start off with a tensor fluctuation in the background (4.66) with the following normalisation

$$ds_T^2 = \sqrt{\frac{H_R}{H(r)}}\,\alpha_{ij}(r)\,dx^i dx^j\,, \tag{4.69}$$

in which α_{ij} is a symmetric traceless tensor of the spatial $SO(3)$. The equation of motion for this fluctuation then is

$$\frac{d}{dr}\left(r^5 f(r)\frac{d}{dr}\alpha_{ij}\right) = -2\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\,\frac{d}{dr}\left(C_{ij} + r^5\sqrt{\frac{H(r)}{g(r)}}\sigma_{ij}\right)\,, \tag{4.70}$$

in which $\sigma_{ij} = \partial_{(i}\beta_{j)} - \frac{1}{3}\delta_{ij}\partial^k\beta_k$ is the tensorial part of the fluid/gravity perturbations described in (4.66). Evidently, it only comes from the spatial dependence of the boost parameters u^a. The equation is a minimally coupled scalar equation of motion with shear source; the left hand side may be written as $\partial_\mu\left(g^{\mu\nu}e^{4U+V}\sqrt{-g}\,\partial_\nu\,\alpha_{ij}\right)$. Note that the appearance of the dilatons is due to the non-Einstein frame. It is easy to see that this directly generalises the equivalent equation of motion of the fluid/gravity correspondence (3.46). In the near-horizon limit (4.43),(4.44), we get perfect agreement.

We may easily integrate this expression and fix the integration constant by imposing regularity of

$$\partial_r\alpha_{ij} \sim \frac{C_{ij} + r^5\sqrt{\frac{H(r)}{g(r)}}\sigma_{ij}}{f(r)} \tag{4.71}$$

at the future horizon, i.e. $C_{ij} = -r_+^5\sqrt{\frac{H(r_+)}{g(r_+)}}\sigma_{ij}$. The main reason for choosing the Eddington-Finkelstein coordinates was about making the background manifestly non-diverging at the horizon and from this to also simplify the analysis for the fluctuations. In this coordinate system the perturbations ought to be well-behaved at the horizon, too, which we just imposed on the tensorial part. Thus, it is guaranteed that the full first order solution inherits the causal structure of the horizon from the background. In the original coordinate system, this would have required a much more elaborate analysis (see e.g. [295]). The question if there is a horizon at all is not manifest. Gravitational metrics which describe hydrodynamic behaviour are singled out by this condition here.

The second integration is also straightforward and can be expressed in terms of the (original) tortoise coordinate $r_\star = \int dr f(r)^{-1}\sqrt{H(r)/g(r)}$. The corresponding integration constant is fixed by requiring that at the cutoff $r = R$ the tensor perturbation ought to vanish. Like that, we retain Minkowski space at $r = R$ also to first order in the perturbations. We then arrive at the solution

$$\alpha_{ij}(r) = -2\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\left(r_\star - R_\star + r_+^5\sqrt{\frac{H(r_+)}{g(r_+)}}\int_r^R \frac{dr'}{r'^5 f(r')}\right)\sigma_{ij}\,. \tag{4.72}$$

We see that compared to [123] not much has changed. Our solution just accounts for the appropriate generalisation of metric factors (4.37), i.e. $f(r)$ contains a charge component as compared to [123] and $g(r)$ appears in our setup since for $q \neq 0$ there is a relative squashing of base and fibre in the internal space which results in $g(r)$ being different from 1. We may thus also expect the shear viscosity η of our setup to trivially generalise the chargeless result of [123].

Also the comparison with [113, 114] or the AdS cut-off setup [263] matches perfectly. The combination of integrals and boundary conditions in (4.72) exactly reproduces their results, when we take the near-horizon limit of section 4.2.1 and put the Dirichlet surface from finite R to $R \to \infty$, the AdS boundary. So there also seems to be no obstacle with recovering their shear viscosity result too.

4.5 Perturbations in the vector sector of $SO(3)$

4.5.1 Perturbation ansatz

As already outlined, there are several perturbations of importance which transform as vectors under the spatial $SO(3)$ symmetry. First of all, there are the two kinds of vector perturbations of the metric:

$$ds_V^2 = 2\sqrt{\frac{H_R}{H(r)}}\left(1 - \frac{f(r)\,g(r)}{f_R\,g_R}\right) w_i(r)\,dx^i dv + z_i(r)\,dx^i dr\,, \tag{4.73}$$

where the way we have written the first perturbations $w_i(r)$ just resembles the way u_a appears in (4.60). This will be of importance later on, when we identify the physical information encoded in the integration constants, which we are going to get. A shift in $w_i(r)$ can then be absorbed into a redefinition of the fluid velocity, whose overall constant value is of course a free parameter, c.f. page 64.

The perturbation $z_i(r)$ does not show up anymore after the full computation of the equations of motion. Therefore it is pure gauge and may be set to zero which we henceforth do.

Apart from the metric perturbations both gauge fields contribute further vector perturbations. Since we constructed the perturbation $w_i(r)$ to appear as u_a in (4.60), equations (4.62), (4.63) tell us that we also need to incorporate them in the gauge field perturbations. We thus parametrise the further independent perturbations v_i and $\mathbf{v}_i$ as follows

$$\mathcal{A}_V = -\frac{qQ}{2L^2}\,\frac{g(r)}{r^6 H(r)}\sqrt{\frac{H_R^{1/2}}{f_R\,g_R}}\,w_i(r)\,dx^i + v_i(r)\,dx^i \tag{4.74}$$

$$\mathbf{A}_V = -\frac{2L^2 q}{Q}\,\frac{f(r)g(r)}{r^2 H(r)}\sqrt{\frac{H_R^{1/2}}{f_R\,g_R}}\,w_i(r)\,dx^i + \mathbf{v}_i(r)\,dx^i \tag{4.75}$$

Now, we plug all given perturbations into the Einstein (4.22) and Maxwell equations (4.25), (4.17) of motion. The set of three coupled second order non-homogeneous ODE's is rather complicated but may be solved exactly in a step by step procedure. Let us denote the Einstein equations by $E_{\mu\nu} = 0$, the first Maxwell equation (from (4.25)) by $\mathcal{M}_\mu = 0$ and the equation of motion derived by variation of the action with respect to $\mathbf{A}$ (from (4.17)) by $\mathbf{M}_\mu = 0$.

4.5.2 Constraint equation

The constraint equation in the vector sector may be computed directly from the Einstein equations as

$$g^{rr}E_{ri} + g^{rv}E_{vi} = 0\,. \tag{4.76}$$

Performing this already quite elaborate computation we derive the relation

$$\begin{aligned}\beta_{i,v} + \frac{q}{R^6}\left(\frac{g_R}{H_R} - \frac{1}{f_R}\right) q_i + \Bigg[\frac{2L^4 r_0^4}{R^4 f_R} - \frac{2}{f_R}\left(2L^4 + r_0^4\right) + \frac{L^4 q^2}{R^6}\left(\frac{g_R}{H_R} - \frac{3}{f_R}\right) \\ + R^4 H_R - \frac{2L^4 R^4}{r_0^4}\Bigg] \frac{r_0^3}{R^4(2L^4 + r_0^4)}\, r_{0,i} = 0\,.\end{aligned} \tag{4.77}$$

We note, that this equation is independent of r and record that we observed in the course of our computation that this would not be the case if we let δL and δr_0 vary arbitrarily from one another and restrained from imposing (4.59). In that case, it would basically mean that we would allow the number of D3-branes to vary smoothly in space-time. This is not really a sensible setup, but probably it is not surprising that an explicit dependence on r appears. In the near-horizon limit r basically corresponds to the energy scale of the field theory dual to the gravity setup. If we do not keep the number of D3-branes fixed then the appearance of an explicit energy scale dependence may not be too surprising. In any case such a formal manipulation has no clear physical interpretation and therefore it makes sense that no sensible hydrodynamic conservation equation arises, which should be independent of r.

But there is a way to interpret (4.77) in terms of conservation equation of a hydrodynamic system. Later, in section 4.7, we will construct the quasi-local stress-energy tensor $T_{\mu\nu}$ of the fluid living on the cutoff surface at $r = R$. The constraint equation (4.77) then is indeed the vectorial part of the "conservation" equation of the zero'th order stress-energy tensor. In full detail this is given by[10]

$$\begin{aligned}\kappa_5^2 \hat{\nabla}^\mu T_{\mu\nu} = {} & \left(\hat{\nabla}^\mu e^{4U+V}\right)\left(K_{\mu\nu} - h_{\mu\nu}K\right) + \hat{\nabla}_\nu\left(n^\mu \nabla_\mu e^{4U+V}\right) - \hat{\nabla}_\nu\left(4e^{3U+V} + e^{4U}\right) \\ & - 4e^{4U+V}(\hat{\nabla}_\nu U) n^\rho \nabla_\rho U - 4e^{4U+V} n^\mu h_\nu{}^\rho \nabla_\mu \nabla_\rho U \\ & - e^{4U+V}(\hat{\nabla}_\nu V) n^\rho \nabla_\rho V - e^{4U+V} n^\mu h_\nu{}^\rho \nabla_\mu \nabla_\rho V \\ & - e^{4U+V}\left(\frac{1}{2} e^{2V} n^\mu \mathcal{F}_{\mu\rho}\mathcal{F}_\nu{}^\rho + \frac{Q^2}{4} e^{-4U-2V} n^\mu \mathbb{F}_{\mu\rho}\mathbb{F}_\nu{}^\rho + 2Q^2 e^{-8U} n^\mu \mathbf{A}_\mu \mathbf{A}_\nu\right)\,.\end{aligned} \tag{4.78}$$

In this expression, $K_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma \nabla_\rho n_\sigma$ is the extrinsic curvature of the surface normal to the space-like, outward pointing normal vector n_σ, where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the projection tensor which projects parallel to the surface. $K = g^{\mu\nu}K_{\mu\nu}$ is the trace of the extrinsic curvature. $\hat{\nabla}_\mu$ is the connection on that surface compatible with $h_{\mu\nu}$, i.e. for any tensor $T^{\nu\cdots}_{\rho\cdots}$, we have $\hat{\nabla}_\mu T^{\nu\cdots}_{\rho\cdots} = h_\mu{}^\sigma h^\nu{}_\lambda h_\rho{}^\tau \cdots \nabla_\sigma T^{\lambda\cdots}_{\tau\cdots}$ (cf. Lemma 10.2.1 of [42]). Apart from the definition of the quasi-local energy-momentum tensor we used the Gauss-Codazzi equation

$$\hat{\nabla}^\mu\left(K_{\mu\nu} - h_{\mu\nu}K\right) = -h_\nu{}^\rho\, R_{\rho\sigma}\, n^\sigma \tag{4.79}$$

along with the equation of motion (4.22) to arrive at (4.78).

[10]Note the similarity to the equation (3.5) in the membrane paradigm case.

In section 4.7 we will find that the zero'th order term of the stress-energy tensor is just that of a perfect fluid with the usual energy density ϵ and pressure P terms[11], such that the left-hand side of (4.78) for $\nu = i$ simply becomes κ_5^2 times

$$(\epsilon + P)\beta_{i,v} + \hat{\nabla}_i P \,, \tag{4.80}$$

where $\epsilon \to \epsilon(x)$ and $P \to P(x)$ due to their implicit dependence on $r_0(x)$, $L(x)$ and $q(x)$.

Furthermore, it is important to remark that the constraint equation (4.77) does *not* agree with the one in [123] after taking the $q \to 0$ limit and changing the coordinate system. This may seem puzzling at first, but actually, it is exactly what one would have to expect from the different definition of the cutoff surface as already partly explained in section 4.2.2. Taking this to be an isodilatonic surface, so basically choosing $n_\mu \sim \partial_\mu \phi$ as the normal vector to the surface, in the coordinate system of [123] is the same as choosing the cutoff surface at finite $\rho = P$. In our coordinate system these two approaches disagree because of the space-time dependence of our dilatons U and V from the implicit dependence of U_0 and V_0 on $L = L(\sigma^a)$.

A different way to see this is to explicitly look at the $q \to 0$ limit of (4.78), in which also $U = V = \phi$. In that limit we note that only derivative terms of the dilaton ϕ appear on the right hand side of (4.77). Since in [123] these are set zero by construction because the projector $h_{\mu\nu}$ acts on them, we see the difference: In [123] a conserved energy-momentum tensor $\hat{\nabla}^\mu T_{\mu\nu} = 0$ is recovered, while we actually describe a forced fluid, similar to [296], in which $\hat{\nabla}^\mu T_{\mu\nu} \sim \hat{\nabla}_\nu \phi \neq 0$.

One may however modify our setup in the following sense. In the background fields (4.60) and following, we may introduce a further space-time dependence of the constant parameter $R \to R(\sigma^a) = R + \sigma^a \partial_a R + O(\partial^2)$, so basically adding a term

$$\delta\left(ds^2\right) = \left(\frac{\delta}{\delta R} ds_0^2\right) \delta R \tag{4.81}$$

into (4.66) and accordingly for the gauge fields. Then we may relate the variation δR to δL by the following reasoning. The background dilaton has the following profile $U_0(r) = \frac{1}{4}\log\left(r^4 + L^4\right)$. We may require that at $r = R$ the dilaton does not change by choosing $\delta U(R) = 0$, which then implies $\delta R = -\frac{R^3}{L^3}\delta L$. Such a condition is already implicit in the coordinate system of [123], in which the background dilaton[12] has the profile $\phi = \log \rho$. Implementing this further complication, we arrive at a constraint equation similar to (4.77), which in the limit $q \to 0$ does exactly reproduce the one of [123].

However, this procedure is not entirely convincing, since to fully capture the space-time dependence of the coordinate change (4.45), one seems to also be required to introduce $r = r(\sigma^a)$ to compensate for the space-time dependence $L(\sigma^a)$. But then the ρ coordinate system is preferred as compared to the one involving r, for which there are not too good reasons.

Also, in our setup there is more arbitrariness in choosing the cutoff surface as an isodilatonic surface. We have two dilatons and could in principle also choose any linear combination of these like the breathing or squashing mode of the Sasaki-Einstein

[11] Explicit expressions for these may be found in (4.116) and (4.117).

[12] The coordinate change was already presented in (4.45).

manifold. The fact that both dilatons U and V coincide for $q = 0$ is a fact which makes the comparison with [123] unambiguous. But for our general $q \neq 0$ setup, the ambiguity cannot be avoided.

Therefore, we avoid these complications, define our cutoff surface by $r = R$ and end up with (4.77). The downside of this is however, that e.g. in the (non-)conservation equation of the energy-momentum tensor we have to keep the many terms like $\sim h_{\mu\nu}\partial^\nu U$. Some of these terms could have been avoided by choosing e.g. $n_\mu \sim \partial_\mu U$, but then the terms like $\sim h_{\mu\nu}\partial^\nu V$ would survive anyway.

4.5.3 Dynamical equations

We are now in the position to state the dynamical equations and then solve them step by step. There are three independent dynamical equations coming from the Einstein equations (either $E_{ri} = 0$ or $E_{vi} = 0$) and the two Maxwell equations $\mathcal{M}_i = 0$ and $\mathbf{M}_i = 0$. The overall normalisation of these dynamical equations was chosen to make the check that in the near-horizon limit of the geometry (4.60) they reduce to the ones stated in [113] as simple as possible.

The one coming from the Einstein equations is

$$\begin{aligned}
0 =& \frac{r^{11}H(r)}{g(r)}\Big(f_R g_R - f(r)g(r)\Big)w_i''(r) \\
&+ 2q\,r^2\sqrt{L^4+r_0^4}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\left(3\,r^4 H(r)\,v_i'(r) + 2L^4\,\mathbf{v}_i'(r) - \frac{8L^4}{rH(r)}\,\mathbf{v}_i(r)\right) \\
&+ \frac{f_R g_R}{H(r)}\left(5L^8 r^2 + L^4\left(3q^2+10r^6\right) + r^4\left(5r^6 - q^2\right)\right)w_i'(r) \\
&+ r^4 H(r)\left(q^2 - 5r^6 - 3r_0^4 r^2\right)w_i'(r) - 2\left(\frac{f_R g_R}{\sqrt{H_R}}\right)\left(\frac{r^{11}H(r)^2}{f(r)g(r)}\right)S_{i,1}(r)\,,
\end{aligned} \tag{4.82}$$

where $S_{i,1}(r)$ is a source term completely fixed by the background. Its full expression is relegated to the appendix A. One may show that at the horizon, this source term vanishes: $S_{i,1}(r_+) = 0$; the differential equation is regular at $r = r_+$. Furthermore, we observe that the perturbation v_i only appears through its first derivative.

Next, we look at the Maxwell equation $\mathcal{M}_i = 0$:

$$\begin{aligned}
0 =& -\sqrt{3}q\sqrt{L^4+r_0^4}\,r^7 f(r)\,w_i''(r) + \sqrt{3}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\left(\frac{r^{13}f(r)H(r)}{g(r)}\right)v_i''(r) \\
&+ \sqrt{3}q\sqrt{L^4+r_0^4}\left(2\,f_R g_R\,\frac{r^2\left(L^4+3r^4\right)}{H(r)} - r^6 - 3r_0^4 r^2 + 5q^2\right)w_i'(r) \\
&+ \sqrt{3}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\Big(r^4 H(r)(3r^8 + r_0^4 r^4 - 3q^2 r^2) \\
&\qquad + 4r^{12}f(r) + q^2(r^6 + 3r_0^4 r^2 - 5q^2)\Big)v_i'(r) \\
&+ 16\sqrt{3}\,L^4\left(L^4+r_0^4\right)\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\,\frac{r^3}{H(r)}\,\mathbf{v}_i(r) - \sqrt{3}\left(\frac{f_R g_R}{H_R^{3/4}}\right)r^6 S_{i,2}(r)
\end{aligned} \tag{4.83}$$

In this equation it is noteworthy that the perturbation $\mathbf{v}_i$ appears with no derivative. So, in principle, we could use this equation for the elimination of $\mathbf{v}_i$ in the other two.

The last dynamic equation is the Maxwell equation $\mathbf{M}_i = 0$:

$$
\begin{aligned}
0 =& -\frac{\sqrt{3}\,q}{\sqrt{L^4+r_0^4}}\, r^7 f(r)\, w_i''(r) + \sqrt{3}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\, r^9 f(r)\Big(v_i''(r) + \mathbf{v}_i''(r)\Big) \\
&+ \frac{\sqrt{3}\,q}{\sqrt{L^4+r_0^4}} \Bigg(\frac{2r^6 f_R g_R}{g(r)^2} - r^6 - 3r_0^4 r^2 + 5q^2 \\
&\qquad + \frac{4r^{12} f(r) - q^2\left(3r^6 + r_0^4 r^2 - 3q^2\right)}{r^6 H(r)} \Bigg)\, g(r)\, w_i'(r) \\
&+ \sqrt{3}\sqrt{\frac{f_R g_R}{H_R^{1/2}}} \Bigg(3r^8 + r_0^4 r^4 - 3q^2 r^2 \\
&\qquad - \frac{4r^{12} f(r) + q^2\left(-5r^6 + r_0^4 r^2 + q^2\right)}{r^4 H(r)} \Bigg)\, g(r)\, \Big(v_i'(r) + \mathbf{v}_i'(r)\Big) \\
&- 8\sqrt{3}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\, \frac{r^7}{g(r)}\, \mathbf{v}_i - \sqrt{3}\left(\frac{f_R g_R}{H_R^{3/4}}\right)\left(\frac{r^{10} H(r)}{g(r)}\right) S_{i,3}(r)
\end{aligned}
\tag{4.84}
$$

Here, the appearance of $v_i(r)$ and $\mathbf{v}_i(r)$ reflects the fact that the combination $\mathbb{F} = \mathcal{F} + \mathbf{F}$ is the more natural object to consider. However, we also see a contribution proportional to $\mathbf{v}_i(r)$, i.e. without derivatives, which comes from the mass term of the massive vector field.

Looking at all dynamical equations (4.82), (4.83) and (4.84), decoupling the three perturbations and integrating the equations is not straightforward.

4.5.4 Solution of the dynamical equations

Our strategy for integrating the dynamic equations just derived will be as follows. First, we will integrate the given three equations or combinations thereof as much as possible, at first without eliminating any of the three perturbations. So, although equation (4.83) seems to suggest that we could eliminate $\mathbf{v}_i(r)$ in the other equations and then deal with only two coupled differential equations of higher than second order, we found this inconvenient.

Furthermore, in the process of integrating the equations of motion, some of the source terms will be difficult to integrate directly. Sometimes it is still possible to explicitly state the integral in terms of an elliptic function. But we will rather leave some of them unintegrated and state them only in terms of an indefinite integral keeping the integration constant arbitrary for the moment.

It turns out, that the homogeneous part of the Einstein equation (4.82) may be integrated once after solving it for $v_i'(r)$:

$$
\begin{aligned}
0 =& \frac{r^5}{g(r)}\Big(f_R g_R - f(r) g(r)\Big) w_i'(r) - 4\, r_0^4\, w_i(r) \\
&+ 6\, q\sqrt{L^4 + r_0^4}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\left(v_i(r) + \widetilde{C}_{i,1} + \frac{2\,L^4}{3r^4 H(r)}\, \mathbf{v}_i(r)\right) \\
&- 2\left(\frac{f_R g_R}{\sqrt{H_R}}\right)\int \frac{r^5 H(r)}{f(r) g(r)}\, S_{i,1}(r)\, dr\,,
\end{aligned}
\tag{4.85}
$$

where we introduced the integration constant $\widetilde{C}_{i,1}$.

The structure of the two Maxwell equations already suggests that a particular sum of them may be directly integrated without any problems. The Chern-Simons contributions are total derivatives of the Chern-Simons current and the kinetic terms are total derivatives by definition. We thus add the two Maxwell equations $\mathcal{M}_i = 0$ and $\mathbf{M}_i = 0$ in such a way that the contributions proportional to $\mathbf{A}_i$, the mass term, cancel and then integrate:

$$
\begin{aligned}
0 =& \frac{q}{2L^4\sqrt{L^4+r_0^4}} \left(\frac{r^6+3L^4r^2+q^2}{r^4}\right) w_i'(r) - \sqrt{\frac{f_R g_R}{H_R^{1/2}}}\,\mathbf{v}_i'(r) \\
&- \frac{1}{2L^4\left(L^4+r_0^4\right)} \sqrt{\frac{f_R g_R}{H_R^{1/2}}} \left(2q^2r^2H(r)+3L^8+2L^4\left(r^4+r_0^4\right)+r^8+\frac{q^4}{r^4}\right) v_i'(r) \\
&- \left(\frac{3\,q\,f_R g_R}{L^4\sqrt{L^4+r_0^4}}\right)\left(\frac{rH(r)}{f(r)g(r)}\right)\left(w_i(r)+\widetilde{C}_{i,2}\right) \\
&+ \left(\frac{f_R g_R}{H_R^{3/4}}\right)\left(\frac{rH(r)}{f(r)g(r)}\right)\int\left(\frac{S_{i,2}(r)}{2\,L^4\left(L^4+r_0^4\right)}+S_{i,3}(r)\right)dr\,.
\end{aligned}
\tag{4.86}
$$

The new integration constant appearing above is $\widetilde{C}_{i,2}$. In appendix A, we state $S_{i,2}(r)$ and $S_{i,3}(r)$ as total derivatives which is very convenient looking at (4.86).

To derive the third independent equation we combine the Einstein equation $E_{vi} = 0$ and the first Maxwell equation $\mathcal{M}_i = 0$ in such a way that the terms proportional to $\mathbf{v}_i(r)$ cancel. In addition we use (4.86) to eliminate $\mathbf{v}_i'(r)$. Note that the homogeneous part of this equation can be integrated twice. Integrated once this combination leads to

$$
\begin{aligned}
0 =& \left(\frac{\sqrt{L^4+r_0^4}}{q}\sqrt{\frac{H_R^{1/2}}{f_R g_R}}\right)\frac{d}{dr}\left[\left(\frac{f_R g_R - f(r)}{f(r)}\right)w_i(r)\right] + v_i'(r) \\
&+ 6q\left(H_R^{1/4}\sqrt{f_R g_R}\sqrt{L^4+r_0^4}\right)\widetilde{C}_{i,2}\left(\frac{1}{r^7 f(r)^2}\right) \\
&+ \frac{S_{i,4}(r)}{r^5 f(r)^2} - \frac{4r_0^4}{q}\left(H_R^{1/4}\sqrt{f_R g_R}\sqrt{L^4+r_0^4}\right)C_{i,3}\left(\frac{1}{r^5 f(r)^2}\right)\,,
\end{aligned}
\tag{4.87}
$$

in which the new integration constant $C_{i,3}$ appears and the source term $S_{i,4}$ is defined in appendix A.

Up to this point we have obtained three independent integration constants $\widetilde{C}_{i,1}$, $\widetilde{C}_{i,2}$ and $C_{i,3}$. We would like one of them, or rather a particular linear combination, to describe the freedom to shift $w_i(r)$ by a constant. This shift would correspond to a redefinition of the fluid velocity u_i and can thus be absorbed. Defining

$$
\begin{aligned}
\widetilde{C}_{i,2} &= C_{i,2} + C_{i,3}\,, \\
\widetilde{C}_{i,1} &= C_{i,1} - 4r_0^4\left(6\,q\sqrt{L^4+r_0^4}\sqrt{\frac{f_R g_R}{H_R^{1/2}}}\right)^{-1} C_{i,3}\,,
\end{aligned}
\tag{4.88}
$$

we explicitly find that $C_{i,3}$ corresponds to this shift freedom: Subject to these redefinitions, the integration constant always appears in the combination $w_i(r) + C_{i,3}$ in

(4.85), (4.86) and (4.87). Integrating (4.87) again, we obtain

$$\begin{aligned} 0 = & \frac{\sqrt{L^4+r_0^4}}{q}\sqrt{\frac{H_R^{1/2}}{f_R g_R}}\left(\frac{f_R g_R - f(r)}{f(r)}\right)\left(w_i(r)+C_{i,3}\right)+v_i(r)-C_{i,4} \\ & + 6qH_R^{1/4}\sqrt{f_R g_R}\sqrt{L^4+r_0^4}\,C_{i,2}\int\frac{1}{r^7 f(r)^2}+\int\frac{S_{i,4}(r)}{r^5 f(r)^2}\,dr\,. \end{aligned} \tag{4.89}$$

Note that in the above equations $v_i(r)$ appears as $v_i(r)+C_{i,1}$ or $v_i(r)-C_{i,4}$, i.e. also with shifts by constants. Since $v_i(r)$ describes the vector fluctuation of the gauge field $\mathcal{A}_i(r)$, we recognize that a combination of $C_{i,1}$ and $C_{i,4}$ will not have any effect on the physical observables given that only the gauge-invariant quantity $\mathcal{F}_{\mu\nu}$ is relevant. The other, linearly independent combination of $C_{i,1}$ and $C_{i,4}$ will however be important influencing the solution $w_i(r)$, as we will see shortly.

Now, we may solve the equation (4.85) for $\mathbf{v}_i(r)$ and use it to eliminate this fluctuation in (4.86). Then, one uses the expression of $v_i(r)$ from (4.89) to arrive at a second order ODE fo $w_i(r)$. Solving this would then allow for a full determination also of $v_i(r)$ and $\mathbf{v}_i(r)$ via (4.89) and (4.85).

The homogeneous part of the final ODE may be written as the following differential operator

$$\begin{aligned} 0 = & \frac{d}{dr}\Bigg[-\widetilde{C}_{i,5}+r^3\left(3r^4-r_0^4\right)^2 f(r) \\ & \times\frac{d}{dr}\left(-\widetilde{C}_{i,6}+\frac{r^4 H(r)\left[f_R g_R - f(r)g(r)\right]}{\left(3r^4-r_0^4\right)f(r)g(r)}\,w_i^{(\mathrm{hom})}(r)\right)\Bigg]\,. \end{aligned} \tag{4.90}$$

The constants $\widetilde{C}_{i,5}$ and $\widetilde{C}_{i,6}$ will thus parametrise a general solution to the homogeneous part of the ODE.

It is clear that we have to worry about regularity of

$$w_i(r)\sim\frac{1}{f_R g_R - f(r)g(r)} \tag{4.91}$$

at $r=R$ and the regularity of the entire inner derivative term $\frac{d}{dr}\big(\ldots w_i(r)\big)\sim f(r)^{-1}$ at the horizon, where $f(r_+)=0$. Imposing regularity in both cases fixes $\widetilde{C}_{i,5}$ and $\widetilde{C}_{i,6}$, as we show in the next section.

We may write the solution to this ODE as the following complicated expression.

$$\begin{aligned} w_i(r) = & -C_{i,3}-\frac{4}{3}L^4 q\sqrt{L^4+r_0^4}\left(\frac{\sqrt{f_R g_R}}{H_R^{1/4}}\right)\left(\frac{\left(3r^4-r_0^4\right)f(r)g(r)}{r^4 H(r)\left[f_R g_R - f(r)g(r)\right]}\right) \\ & \times\Bigg[-C_{i,6}+\int\frac{S_{i,5}(r)-C_{i,5}}{r^3\left(3r^4-r_0^4\right)^2 f(r)}\,dr \\ & \quad -\frac{3\sqrt{f_R g_R}}{L^4 q H_R^{1/4}\sqrt{L^4+r_0^4}}\int\left(\frac{r^3}{(3r^4-r_0^4)^2}\int\frac{r^5 H(r)}{f(r)g(r)}\,S_{i,1}(r)\,dr\right)dr \\ & \quad +\frac{3}{4L^4\left(3r^4-r_0^4\right)}\left(C_{i,1}-C_{i,4}+\int\frac{S_{i,4}(r)}{r^5 f(r)^2}\,dr\right. \end{aligned} \tag{4.92}$$

$$+6q\,\sqrt{f_R g_R}\,H_R^{1/4}\,\sqrt{L^4+r_0^4}\,C_{i,2}\int\frac{1}{r^7 f(r)^2}\,dr\Big)\Big]\,,$$

with the two solutions to the homogeneous part of the ODE parametrized by $C_{i,5} \sim \widetilde{C}_{i,5}$ and $C_{i,6} \sim \widetilde{C}_{i,6}$. In this expression, we recognise some of the structures of (4.85) and (4.89), which leads to a specific combination of source terms, integration constants and integrals. By performing an integration by parts we could easily transform the double integral into two single integrals. However, the following analysis is not simplified by this procedure, therefore we refrain from doing so. We can use this expression to compute explicit expressions for $v_i(r)$ via (4.89) and $\mathbf{v}_i(r)$ via (4.85). In particular, we may compute the first order gauge field $\mathcal{A}_V(r)$ (4.74) from these expressions.

Although it is already quite tedious, one may show that the solution (4.92) and the first order gauge field do reduce to the one in [113] for specific choices of the integration constants. The same analysis goes through for equations (4.85), (4.86) and (4.89), so we see that our solution does incorporate the one published in [113].

4.5.5 Fixing the integration constants

The next step we have to deal with is to restrict our most general solution to one which allows for the description of a sensible hydrodynamic system. This we obtain by imposing physical conditions on the perturbations which fix the integration constants in the following way:

Of the integration constants we obtained in (4.92), $C_{i,3}$ may directly be set to zero since it corresponds to a shift in the fluid velocity as already remarked earlier. We may just absorb it into a redefinition of the fluid velocity $u_i - C_{i,3} \to u_i$. The integration constants $C_{i,2}$ and $C_{i,5}$ are fixed by imposing regularity at the horizon $r = r_+$ on particular combinations of the vector fluctuations and their derivatives. $C_{i,6}$ is fixed by demanding regularity for $w_i(r)$ at $r = R$. Using this, we preserve a Minkowski metric at $r = R$ given that the off-diagonal metric component g_{vi} behaves like $g_{vi} \propto (r - R)\, w_i(r)$ for $r \approx R$, cf. equation (4.73). Since we are dealing with a charged fluid, we also have the fluid frame ambiguity choosing Landau or Eckart frame. It is convenient to choose the Landau frame which will effectively determine the combination $C_{i,1} - C_{i,4}$ in (4.92). The combination $C_{i,1} + C_{i,4}$ need not be fixed. It corresponds to a residual gauge freedom of the gauge field $\mathcal{A}$.

Fixing $C_{i,2}$ by regularity at the horizon

We may derive a relation which determines $C_{i,2}$ in terms of source terms evaluated at the horizon. The physical requirement we get this from is *regularity at the horizon* for a particular combination of the first derivatives of our perturbations $v_i'(r)$, $\mathbf{v}_i'(r)$, $w_i'(r)$.

For doing so, we take equation (4.86) and add the first derivative of (4.89) with a prefactor such that in the resulting equation the coefficient of $w_i(r)$ vanishes. From this we get an expression which only contains the first derivative terms $v_i'(r)$, $\mathbf{v}_i'(r)$, $w_i'(r)$ and further source and integration constant terms. We now require regularity at the horizon for this particular combination[13] of $v_i'(r)$, $\mathbf{v}_i'(r)$, $w_i'(r)$; since the other

[13] We require it for all these functions individually, therefore it must also hold for the combination.

terms do contain a pole $\sim 1/f(r)$ at the horizon, we require its residue to vanish. This gives us the following relation

$$\begin{aligned}0 &= 2L^4\left(L^4+r_0^4\right)\left(2r^2r_0^4-3q^2\right)\sqrt{f_Rg_R}\int\left(\frac{S_{i,2}(r)}{2\,L^4\left(L^4+r_0^4\right)}+S_{i,3}(r)\right)dr\\ &\quad -3qr^2\sqrt{H_R}\left(q\,S_{i,4}(r)+4r_0^4H_R^{1/4}\sqrt{f_Rg_R}\,\sqrt{L^4+r_0^4}\,C_{i,2}\right),\end{aligned} \tag{4.93}$$

which has to be evaluated at the horizon. To get a more compact expression, we invoke the definition of ${S_{i,4}}'(r)$ given in appendix A to eliminate the integral term. In this definition the prefactor of $S_{i,2}$ vanishes at the horizon while $S_{i,2}$ itself is regular; additionally, we have $S_{i,1}(r_+)=0$. In this way, we arrive at the relation

$$C_{i,2}=\frac{\left(2r_+^3r_0^4-3q^2r_+\right)S_{i,4}{}'\left(r_+\right)-6q^2\,S_{i,4}(r_+)}{24qr_0^4\sqrt{f_Rg_R}H_R^{1/4}\sqrt{L^4+r_0^4}}\,, \tag{4.94}$$

in which

$$\begin{aligned}&\left(2r_+^3r_0^4-3q^2r_+\right)S_{i,4}{}'\left(r_+\right)-6q^2S_{i,4}(r_+)\\ &\qquad=-\frac{48\kappa L^4q^2}{\sqrt{f_Rg_R}}\,\epsilon_{ijk}\beta_{j,k}-\frac{2\left(r_+^4+r_0^4\right)\left(3L^4+r_0^4\right)}{r_+r_0}\left(r_0\,q_i-3q\,r_{0,i}\right).\end{aligned} \tag{4.95}$$

We see that these are terms of the structure expected from the near-horizon limit [113]. We have a term which stems from the Chern-Simons term; additionally the combination $r_+^4+r_0^4\propto(1+M)$, in the notation of [113], appears as a prefactor of $(r_0\,q_i-3q\,r_{0,i})$ which in the near-horizon limit reduces to the unique first order Weyl-covariant derivative $\mathcal{D}_iq=q_i+3q\,\beta_{i,v}$ used in [113], given that in this limit the vector constraint (4.77) simplifies to $r_{0,i}+r_0\beta_{i,v}=0$. However in our full setup we get more naturally the structure in terms of q_i and $r_{0,i}$ with only the prefactor depending on R.

Fixing $C_{i,5}$ by regularity at the horizon

We may also fix the integration constant $C_{i,5}$ by imposing regularity at the horizon on a particular combination of $w_i(r)$, $w_i'(r)$ and $v_i'(r)$: Note that in (4.92) one may eliminate the particular combination of expressions involving

$$\sim\int\frac{S_{i,4}(r)}{r^5f(r)^2}\,dr\qquad\text{and}\qquad\sim C_{i,2}\int\frac{1}{r^7f(r)^2}\,dr\,, \tag{4.96}$$

which could potentially complicate considerations at the horizon, where $f(r_+)=0$, in favour of $v_i(r)$ using (4.89). Going back to the second order ODE we obtained for $w_i(r)$ alone, i.e. (4.90) along with its inhomogeneous pieces, we may use the same relation (4.89) to express part of the inhomogeneous terms in that ODE in favour of $v_i(r)$. Thus the ODE may be written as a differential equation for a combination of $w_i(r)$ and $v_i(r)$. The only potentially diverging term at $r=r_+$ in that expression then is

$$\frac{d}{dr}\left(\frac{r^4\left(f_Rg_R-f(r)g(r)\right)H(r)}{\left(-3r^4+r_0^4\right)f(r)g(r)}\,w_i(r)+(\dots)\,v_i(r)\right)\sim\frac{S_{i,5}(r)-C_{i,5}}{r^3\left(3r^4-r_0^4\right)^2f(r)}\,. \tag{4.97}$$

This would be a pole while the other terms are regular at $r = r_+$. Note in particular that the expression $S_{i,1}(r)/f(r)$ is finite as $r \to r_+$. Again, we may use this to fix an integration constant by setting the would-be residue to zero, making it a removable singularity. This time, we get

$$\begin{aligned} C_{i,5} &= S_{i,5}(r_+) \\ &= -\frac{3\left(r_+^4 + r_0^4\right)}{4L^4 r_0 r_+}\left(r_0\, q_i - 3q\, r_{0,i}\right) , \end{aligned} \tag{4.98}$$

where as in the expression for $C_{i,2}$ we see that the familiar combination $r_+^4 + r_0^4 \propto (1 + M)$ appears as a prefactor of $(r_0\, q_i - 3q\, r_{0,i})$ and only the prefactor of this expression potentially depends on R.

Fixing $C_{i,6}$ by regularity at the cutoff

For fixing $C_{i,6}$ we first trade $C_{i,1}$ and $C_{i,4}$ with $w_i(R)$ and $\mathbf{v}_i(R)$. This is of course not a physical condition imposed on them. It is only a slightly more convenient parametrisation of these integration constants. We therefore evaluate (4.85) and (4.89) at $r = R$ and reinsert the expressions we get for $C_{i,1}$ and $C_{i,4}$ into (4.92). The Dirichlet condition $v_i(R)$ drops out and some of the integrals will essentially turn into definite integrals like

$$\int \frac{1}{r^7 f(r)^2} dr \to \int_R^r \frac{1}{r'^7 f(r')^2} dr' . \tag{4.99}$$

Now we fix $C_{i,6}$ by imposing that $w_i(r)$ should have no pole at $r = R$. This is clearly a sensible physical condition which preserves Minkowski space at the cutoff surface as can be seen from the perturbation ansatz (4.73). We therefore impose that the numerator in (4.92) after we replaced $C_{i,1}$ and $C_{i,4}$ should vanish at $r = R$. From this we get

$$\begin{aligned} C_{i,6} = &\left(\frac{H_R^{1/4}\left(-2r_0^4 R^6 H_R + 3L^4 q^2 + q^2 r_0^4\right)}{4L^4 q\left(3R^4 - r_0^4\right)\sqrt{f_R g_R\left(L^4 + r_0^4\right)}}\right) w_i(R) \\ &+ \left(\frac{1}{2R^4 H_R\left(3R^4 - r_0^4\right)}\right) \mathbf{v}_i(R) + \int^R (\ldots) . \end{aligned} \tag{4.100}$$

If we reinsert this into (4.92) the last term in (4.100) sets the remaining integrals to $\int_R^r (\ldots) dr'$. So, in summary, fixing $C_{i,6}$ sets every single integral of (4.92) to $\int_R^r (\ldots) dr'$ and additionally contributes the terms explicitly spelled out in (4.100), which are proportional to $w_i(R)$ and $\mathbf{v}_i(R)$.

Landau frame choice and $\mathbf{v}_i(R)$

How do we now fix the remaining integration constant, i.e. the linear combination of $w_i(R)$ and $\mathbf{v}_i(R)$ in (4.100) which is basically equivalent to $C_{i,1} - C_{i,4}$? For doing so we may carefully extract the limit of the solution (4.92) for $r \to R$ using e.g.

$$\lim_{r \to R}\left(\frac{\int_R^r A(r')dr'}{f_R g_R - f(r)g(r)}\right) = -\frac{A(R)}{g_R\, f'(R) + f_R\, g'(R)} , \tag{4.101}$$

and therefore see that the linear combination of $w_i(R)$ and $\mathbf{v}_i(R)$ given in (4.100) can essentially be expressed in terms of $w_i(R)$ alone. But as we will see later $w_i(R)$ is uniquely determined by fixing the fluid's frame ambiguity. We are going to deal with the quasi-local energy-momentum tensor in more detail in section 4.7, but may already note that $T^{(1)}_{vi}=0$, i.e. the Landau frame, will be chosen. So, in total, the Landau frame choice determines $w_i(R)$ and from using the explicit solution (4.92), we may read off $\mathbf{v}_i(R)$ or equivalently the linear combination $C_{i,1} - C_{i,4}$. This fixes the last integration constant we need for evaluating physical quantities.

To summarize, we integrated the equations of motion in the vector sector of our setup. The metric perturbation $w_i(r)$ is read off from (4.92), from which we may deduce $v_i(r)$ and $\mathbf{v}_i(r)$ using (4.89) and (4.85). In total, we had six integration constants $C_{i,1}, \ldots, C_{i,6}$. Out of these only five linearly independent combinations appeared in (4.92). One integration constant is irrelevant since it corresponds to a shift of the gauge field $\mathcal{A}$ by a constant or in other words to the Dirichlet condition $v_i(R)$ given the perturbation ansatz (4.74). The corresponding Dirichlet condition $\mathbf{v}_i(R)$ for the other gauge field $\mathbf{A}$ however is not arbitrary because of the appearance of explicit mass terms $\mathbf{A}_\mu\mathbf{A}^\mu$ in (4.21). Effectively it is fixed by the Landau frame choice. The other constants are fixed by imposing regularity for the fluctuations at the horizon and Dirichlet cutoff surface at $r = R$ and a redefinition of the fluid velocity.

4.6 Perturbations in the scalar sector of $SO(3)$

The scalar sector is the one, about which we have the least to say at the moment, due to its complexity which even surpasses the vector sector's one.

4.6.1 Perturbation ansatz

A priori, the sector consists of seven coupled scalar perturbations. Three perturbations, $k(r)$, $j(r)$ and $h(r)$, stem from the metric. We parametrise them similar to [123] as follows

$$ds_S^2 = k(r)\, dv^2 + 2\, j(r)\, dvdr + h(r)\, dx^i dx^i\,, \tag{4.102}$$

which we might also like to rescale using the appropriate background values (4.60), but since we are not going to attempt to solve the equations for now, we leave it schematic. Of these, only two perturbations will be truely dynamical and it will be possible to pick a gauge, in which one particular linear combination is gauge fixed to zero (in addition to $g_{rr} = 0$, which has already been used above).

In [123], the perturbation of the dilaton was conveniently set to zero in this vein. This then defined the cutoff surface as an isodilatonic surface and by this expressions like (4.78) greatly simplified. But as we have already elaborated on earlier, we cannot really evade this problem in our setup by such a gauge choice.

In [113], a gauge was chosen, where the fluctuations that corresponds to $j(r)$ and $h(r)$ in our case, were basically chosen identical up to a numerical factor of 3/2. In [114], the purely spatial scalar perturbation corresponding to $h(r)$ was fixed to a constant value since it allowed for an easier decoupling of the equations of motion; also setting the off-diagonal perturbation $\sim j(r)$ constant is mentioned in that work. It is however not immediately clear, which gauge fixing would be the most convenient one in our case. This is partly due to the source terms, which – comparing to [123] or our vector sector – are expected to be quite complicated to integrate directly. However, a clever gauge choice might exactly simplify this complication and make it tractable.

Furthermore, there are the two scalar perturbations of the dilatons

$$U_S = u(r)\,, \qquad V_S = v(r)\,. \tag{4.103}$$

These comprise complications which appear to make the scalar sector in our setup significantly more difficult than the ones in [113,114] and likely even more complicated than in [123]. At most, we may gauge away only one of these (similar to [123]), but the degree of freedom appears at a different place anyway and we have no real reason which one to choose as laid out before.

Additionally, there are the gauge field perturbations. For presenting these, we recall that for $\mathcal{A}$, we have chosen an axial gauge, in which the radial component vanishes $A_r = 0$. Thus, we only get one further perturbation from $\mathcal{A}$

$$\mathcal{A}_S = a(r)\,dv\,. \tag{4.104}$$

The second gauge field $\mathbf{A}$ does not have any gauge freedom left anymore, which we could use to gauge fix $\mathbf{A}_r = 0$. We rather have to account for this degree of freedom also, which can basically be thought of as the Stückelberg scalar. Therefore, we have

$$\mathbf{A}_S = \mathbf{a}(r)\,dv + \mathbf{s}(r)\,dr \tag{4.105}$$

So, as we have seen, we have to deal with effectively seven coupled scalar perturbations after gauge fixing, which surpasses the complexity of [123, 113, 114] significantly.

4.6.2 Constraint equations

Of these seven equations, which come from the Einstein equations, the two Maxwell equations and the two dilaton equations of motion, we expect three constraint equations, which neither contain any of the perturbations just summarised nor the radial coordinate. These constraint equations encompass the "conservation" of the v component of the zero'th order energy-momentum tensor, or more precisely its non-conservation due to force terms on the right hand side as in (4.78), and similar equations which involve the zero'th order charge currents that correspond to $\mathcal{A}$ and $\mathbf{A}$. These should still be rather simple to determine and reduce to the ones in [113, 114] in the appropriate decoupling limit. Also in the $q \to 0$ limit, one should obtain an equation similar to the one in [123], but since we use a different cutoff surface, details will probably change as in the vector sector.

4.6.3 Dynamical equation

The dynamical equations in the scalar sector will be very tedious to solve. Compared to the similar project [123], we have three more perturbations which couple. But, in total, it might still be possible to reach results here. As was convincingly shown in [272], it might suffice to perform an educated guess of the solution in the scalar sector, motivated by previous analysis, and argue in favour of its uniqueness up to coordinate reparametrisations. This does not sound unreasonable also for our case. But the simplification in the setup of [272] allowed for (correctly) guessing that most of the perturbations vanish as in [113, 114], which, given the analysis of [123], seems unlikely in our case.

In solving the scalar equations of motion, previous works have repeatedly used quite nice simplification arguments. The appearing integration constants may often be reabsorbed into a redefinition of the charge q and the parameter r_0 [113]. This also seems possible in our case. Furthermore, integration constants might be fixed by requiring regularity at the horizon and at the Dirichlet cutoff $r = R$, similar to normalisability as $r \to \infty$ in the AdS case. As in the vector sector this might be difficult to realise due to the lack of proper understanding of a holographic renormalisation in this context. But what is still quite assured is that one of the integration constants cannot be fixed by such a line of reasoning, but rather by another Landau frame condition similar to the one in the vector case or in [123].

4.7 The world-volume energy-momentum tensor

We now want to give a short exposition of the energy momentum tensor on the cutoff surface. The analysis should be considered as preliminary, but we may already extract some useful information from the terms we know.

The general difficulty and the reason why it should be considered preliminary at present lies in the fact that a rigorous holographic renormalisation of such a cutoff surface stress tensor is at present not known, see however [230]. So we will restrict to the Brown-York procedure [129], which was also implemented in [123]. Clearly, the analysis of [123] provides useful guidance again; see in particular the appendix of the paper. But we will see that already extending the Brown-York procedure to our case will be considerably more difficult.

We arrive at the following expression

$$\kappa_5^2\, T_{\mu\nu} = e^{4U+V}\left(K_{\mu\nu} - K\, h_{\mu\nu}\right) + \left(n_\rho \partial^\rho e^{4U+V} - 4\, e^{3U+V} - e^{4U} + Q\right) h_{\mu\nu}\,. \tag{4.106}$$

The first terms which involve the extrinsic curvature tensor $K_{\mu\nu}$ and its trace K are not too difficult to understand: They constitute the usual terms which arise from the variation of the Gibbons-Hawking boundary term,

$$\frac{1}{\kappa_5^2}\int d^4x\, \sqrt{-h}\, e^{4U+V} K\,, \tag{4.107}$$

with respect to the induced metric $h^{\mu\nu}$.

But in the frame we are working in there is a further term. In Einstein frame, which we denote by a bar in the following notation, we may evaluate the trace of the extrinsic curvature $\bar{K} = -\bar{\nabla}_\mu n^\mu$ by using the usual simplified formula for the covariant divergence of a current n^μ. The partial derivative acting on n^μ itself will not contribute at the order we are working, so we may compute

$$\begin{aligned} \bar{K} &= -\left(e^{4U+V}\sqrt{-h}\right)^{-1} n^\mu \partial_\mu \left(e^{4U+V}\sqrt{-h}\right) \\ &= K - \left(e^{4U+V}\right)^{-1} n_\rho \partial^\rho e^{4U+V}\,. \end{aligned} \tag{4.108}$$

Computing $\sqrt{-\bar{h}}\bar{K} = e^{4U+V}\sqrt{-h}\bar{K}$ then yields the required additional term in eq. (4.106). For $\bar{K}_{\mu\nu} = K_{\mu\nu}$ itself we do not receive any contributions. These two relations are compatible with each other, because in $K_{\mu\nu}$ the indices are already projected parallel to the surface while in the trace *all* indices contribute, in particular also the terms which originate from the transverse Sasaki-Einstein space.

The last term in (4.106) subtracts the energy-density of a stack of extremal D3-branes, see again [123]. We work at a fixed number of these BPS objects[14] and only want to consider the fluctuations on top of these.

The remaining terms are the most difficult to understand and the analysis of these terms is still preliminary. The procedure of [129] asks to embed the surface of interest into a reference space-time, which we will take to be flat Minkowski space, compute the Brown-York tensor in that reference space-time and subtract it from the Brown-York tensor one actually wants to compute. So, we would like to put the fibre bundle into flat space, including its breathing and squashing modes, and compute its extrinsic curvature tensor and the corresponding trace.

Let us first look at the simplifying limit, in which U and V are independent of σ^a and equal, $U = V = \varphi$. We then get [123]

$$- 5\, e^{-\varphi}\, h_{\mu\nu} \tag{4.109}$$

since this is the very well-known formula for the extrinsic curvature of a five-sphere in Minkowski space: We may write flat space as

$$h_{ab} d\sigma^a d\sigma^b + dr^2 + r^2 d\Omega_5^2 \tag{4.110}$$

and restrict to the surface $r = e^{\boldsymbol{\varphi}} = \text{const.}$ with induced metric

$$h_{ab} d\sigma^a d\sigma^b + e^{2\boldsymbol{\varphi}} d\Omega_5^2\,. \tag{4.111}$$

The normal vector is $n_\mu = \partial_\mu\,(r - e^{\boldsymbol{\varphi}}) = \delta_{\mu r}$, since the latter term is constant. Then

$$K = -\nabla_\mu n^\mu = -r^{-5}\;\partial_r r^5\big|_{r=e^{\boldsymbol{\varphi}}} = -5\, e^{-\boldsymbol{\varphi}}\,, \tag{4.112}$$

which is then multiplied by the induced metric.

Now, if we want to generalise this to the case, in which we make φ to depend on σ^a also, we would get a surface defined by $r = e^{\varphi(\boldsymbol{r},\sigma^a)}$, in which on the right hand side the dependence on r is fixed. The normal vector is then given by

$$n_\mu = \partial_\mu\left(r - e^{\varphi(\boldsymbol{r},\sigma^a)}\right) = \delta_{\mu r} - e^{\varphi(\boldsymbol{r},\sigma^a)}\partial_\mu\varphi \tag{4.113}$$

with a simple normalisation factor α such that αn_μ has unit norm. This yields terms of the form $\partial_\mu\varphi\partial_\nu\varphi$ and $\hat{\nabla}_\mu\partial_\nu\varphi$ in the extrinsic curvature tensor and correspondingly terms $(\partial\varphi)^2 + \hat{\Box}\varphi$ in its trace. The normalisation of the normal vector factors out because the terms in which the derivative in $\nabla_\mu\,(\alpha n^\mu)$ acts on α will be proportional to n_μ and then projected out since $h_{\mu\nu}n^\nu = 0$. So, we get two-derivative contributions of the dilaton in the extrinsic curvature and therefore also in the energy-momentum tensor. This is in agreement with the general expectation from holographic renormalisation terms of a forced fluid [296]. But due to the normalisation factor $\alpha = \left(1 + e^{2\varphi}(\partial_\mu\varphi)^2\right)^{-1/2}$ these appear to be non-local in our case. In [123]

[14] Tension and four-form charge are identical due to the BPS nature [160], so we may use Q here to subtract the energy density of this Lorentz-invariant contribution.

terms were derived which *looked* non-local on first sight but were actually not since they only involved normal derivatives. Here we do get also space-time derivatives.

At the first derivative order we are working at the two-derivative terms are however subdominant anyway[15]. So we will disregard them for now and only take the terms, which do not depend on σ^a.

For the case of our fibration, we believe this argumentation must directly generalise. The only difference is however, that in (4.112) part of the space will have $r = e^U$ and the other will have $r = e^V$. This then generalises to

$$-5\,e^{-\phi} \to -4\,e^{-U} - e^{-V}\,, \tag{4.114}$$

which explains the terms stated in (4.106) taking into account the non-Einstein frame factor e^{4U+V}.

4.7.1 Energy density and pressure

From the expression of the quasi-local energy-momentum tensor (4.106), we may easily extract energy density ϵ and pressure P. As clear by construction, the zero'th order energy-momentum tensor is of ideal fluid form

$$T^{(0)}_{\mu\nu} = \epsilon\, u_\mu u_\nu + P P_{\mu\nu}\,. \tag{4.115}$$

with energy density given by

$$\begin{aligned}\kappa_5^2\,\epsilon = {} & R^4 H_R \left(1 + \frac{4}{\sqrt{g_R}}\right) - Q \\ & + \sqrt{\frac{f_R}{g_R}}\left(-3L^4 - 5R^4 H_R - \frac{5}{4}R^5 H'(R) + \frac{R^5 H_R}{2g_R} g'(R)\right)\,.\end{aligned} \tag{4.116}$$

The pressure may be calculated to be

$$\begin{aligned}P = {} & -\epsilon + \frac{\sqrt{g_R}}{R^5 R^{12}\sqrt{f_R} H_R}\left(L^8\left(2r_0^4 R^4 - 3q^2 R^2\right) + r_0^4 R^6\left(2R^6 - q^2\right)\right. \\ & \left. + L^4\left(-2q^4 + q^2\left(r_0^4 R^2 - 5R^6\right) + 4r_0^4 R^8\right)\right)\,.\end{aligned} \tag{4.117}$$

Both expressions completely agree with [123] if we take $q \to 0$ and perform the coordinate change. This was also guaranteed since at this order all expressions like the stress-energy tensor or the black brane background are manifestly equivalent. Also, in the near-horizon limit, we exactly recover the terms in (3.40),

$$\epsilon = \frac{3r_0^4}{2\kappa_5^2}\,, \qquad P = \frac{r_0^4}{2\kappa_5^2}\,, \tag{4.118}$$

if we set the AdS length to 1. This has to do with the length dimensions of the gravitational constant we have chosen as we will explain in the next subsection. The relations for energy density and pressure are also identical in the charged fluid case [113]

[15] Note that we explicitly denoted derivatives with hats, e.g. $\hat\nabla_\mu \partial_\nu \varphi$. These are the derivatives along the cut-off surface when it is embedded in *flat* space. Therefore, $\hat\Gamma^\lambda_{\mu\nu}\partial_\lambda\varphi$ will not contribute at first order, although it would if one had to replace it with the full $\Gamma^\lambda_{\mu\nu}$ of our general blackfold setup when computing the stress-energy tensor.

4.7.2 Shear viscosity

In section 4.4, we have computed the tensor perturbation by solving the corresponding equations of motion, imposing regularity at the future horizon for $\partial_r \alpha_{ij}$ and the Dirichlet condition $\alpha_{ij}(R) = 0$ to retain Minkowski space of the perturbed metric at $r = R$. This is enough information to compute the shear viscosity of the system. Although we know on general grounds that it will correspond to the usual universal value, it is an important non-trivial check if our computation with boundary conditions, the energy-momentum tensor with its regularisation and all our units are correct.

From the energy momentum tensor (4.106) with the solution (4.72) inserted we may compute the following terms in the symmetric traceless sector of the first order[16]:

$$T_{ij}^{(1)} = -\frac{1}{\kappa_5^2} H_R^{3/4} r_+^5 \sqrt{\frac{H(r_+)}{g(r_+)}} \sigma_{ij} \,. \tag{4.119}$$

This allows us to read off the shear viscosity via (3.36) to equal

$$\eta = \frac{1}{2\kappa_5^2} H_R^{3/4} r_+^5 \sqrt{\frac{H(r_+)}{g(r_+)}} \,. \tag{4.120}$$

It perfectly matches (3.47) in the near-horizon limit, in which we put the cutoff surface to the AdS boundary via $R \to \infty$.

For checking this, a quick comment about dimensions is however needed here: Note that it *looks* as if the shear viscosity (4.120) has the wrong length dimensions. In five dimensions one usually has $\left[\kappa_5^2\right] = \left[\text{length}^3\right]$. But we put the length dimensions into the dilatons as we remarked earlier, so indeed we used

$$\kappa_5^{-2} = \kappa_{10}^{-2} \, \mathrm{Vol}(S^5) \tag{4.121}$$

with a sphere of radius 1. Therefore, here it scales as $\left[\kappa_5^2\right] = \left[\text{length}^8\right]$. If we take the near-horizon, $R \to \infty$ limit of (4.120), we exactly recover (3.47), in which the gravitational constant has normal length dimensions including a factor of L^5 and for $q = 0$, we have $r_+ = r_0$.

We may use Bekenstein's formula for the black hole metric (4.60) to compute the black hole's entropy,

$$S = \frac{A}{4G} = \frac{2\pi}{\kappa_5^2} A \,. \tag{4.122}$$

But we must make sure to use it in the non-Einstein frame we are using, so also taking care of the dilaton background profiles (4.39). We then get

$$S = \frac{2\pi}{\kappa_5^2} \int d^3x \sqrt{\gamma} \, e^{4U+V} \Big|_{r_+} \tag{4.123}$$

which translates into an entropy density

$$s = \frac{2\pi}{\kappa_5^2} H_R^{3/4} r_+^5 \sqrt{\frac{H(r_+)}{g(r_+)}} \,, \tag{4.124}$$

[16]Note that we have not computed the scalar part with an expected non-vanishing bulk viscosity yet.

where again the earlier comments about length dimensions apply. From (4.120) and (4.124) we get the usual universal law (1.5)

$$\frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{4.125}$$

4.7.3 Landau frame choice

The energy-momentum tensor which we have constructed is generically not in Landau frame. While the zero'th order is, one may deduce that the first order terms are such that they make it deviate from Landau frame unless we impose a specific condition to ensure it. Using (4.106), the important component in the vector sector is

$$\begin{aligned} T^{(1)}_{vi} = &-\left(\frac{R^5 H_R^{5/4}}{\sqrt{g_R}}\right)\left(\frac{1}{2\kappa_5^2}\right)\beta_{i,v} - \left(\frac{H_R^{5/4}}{R\, f_R\sqrt{g_R}}\right)\left(\frac{q}{2\kappa_5^2}\right) q_i \\ &+\left(\frac{H_R^{1/4}}{R^5 f_R\sqrt{g_R}}\right)\left(4L^8R^2 + L^4\left(3R^2\left(r_0^4+R^4\right)-q^2\right)+2r_0^4R^6\right) \\ &\times\left(\frac{r_0^3}{2\kappa_5^2\left(2L^4+r_0^4\right)}\right) r_{0,i} \\ &+\left(\frac{\sqrt{g_R}}{\kappa_5^2 R^{12}\sqrt{f_R}H_R}\right)\left(L^8\left(2r_0^4R^4-3q^2R^2\right)+r_0^4R^6\left(2R^6-q^2\right)\right. \\ &\qquad\left.+L^4\left(-2q^4+q^2\left(r_0^4R^2-5R^6\right)+4r_0^4R^8\right)\right) w_i(R)\,. \end{aligned} \tag{4.126}$$

So, we see that by choosing $w_i(R)$ such that this component vanishes, we may remain in Landau frame. This is the condition we used to fix our last integration constant, see page 101 and following.

4.8 Charge currents

After having computed the shear viscosity of our setup we would now like to determine the physical information from the vector sector. Therefore, we compute the Dirichlet surface current according to [129] which one may read off from the boundary term after an integration by parts of the action (4.21), we have been dealing with all the time. There are two field strengths $\mathcal{F}$ and $\mathbb{F}$. In the definition of the latter, we had $\mathbb{F} = \mathcal{F}+\mathbf{F}$. But for simplicity, we would like to compute the currents which correspond to $(\mathcal{F}, \mathbb{F})$ instead of $(\mathcal{F}, \mathbf{F})$. Due to the mass term $\mathbf{A}^2$ and the different dilaton dependence of the kinetic terms, we may not diagonalise anyway, so either choice is fine. But when thinking of the near-horizon limit, this effects the normalisation.

So, the two currents are given by

$$\mathcal{J}_i = -\left(\frac{\sqrt{-\gamma}}{2\kappa_5^2}\right) e^{4U+3V} n_\mu g^{\mu\nu}\mathcal{F}_{\nu i} \tag{4.127}$$

$$\mathbb{J}_i = -\left(\frac{\sqrt{-\gamma}}{2\kappa_5^2}\right)\frac{Q^2}{2}\, e^{-V} n_\mu g^{\mu\nu}\mathbb{F}_{\nu i} - \frac{Q^2}{2\kappa_5^2}\,\kappa\,\epsilon_i^{\;\mu\nu\rho\sigma} n_\mu\,\mathcal{A}_\nu\,\mathbb{F}_{\rho\sigma}\,, \tag{4.128}$$

where it is interesting to note that only one of the currents receives a direct contribution from the Chern-Simons term. We may actually leave away $\sqrt{-\gamma}=1$ since by construction our induced metric is the Minkowski metric. The summation involves all five space-time indices. We will only focus on the current $\mathcal{J}_i$ for now.

There are two slightly different, but of course equivalent ways how to extract the charge current using the ansatz (4.74).

We may first insert the solutions for $v_i(r)$ and $w_i(r)$, which we get from (4.89) and (4.92) together with the fixed integration constants, into (4.74) to get $\mathcal{A}(r)$ and from this compute the charge current via (4.127). For this way we need all the integration constants we fixed previously and additionally need to be careful in taking the limit $r \to R$ in the end.

However, we may also use a short-cut which shows that only some of the integration constants are really important. Using (4.89) only, we may compute $\mathcal{A}(r)$ as a function of $w_i(r)$. Then the charge current involves terms $\sim w_i(r)$ and $\sim (f(r)g(r) - f_R g_R)\, w_i'(r)$, which we need to evaluate at $r = R$. For the first we may use the Landau frame condition which directly specifies $w_i(R)$; for the second we just need to use that we fixed the integration constants such that $w_i'(r)$ is regular at $r = R$. Since its prefactor vanishes, we need not care about these terms anymore. If we perform this latter computation, we thus arrive at

$$\begin{aligned} 2\kappa_5^2\,\mathcal{J}_i = {} & 6q\sqrt{L^4+r_0^4}\left(\frac{H_R^{7/4}}{f_R g_R}\right) C_{i,2} + R^2\left(\frac{H_R}{f_R g_R}\right)^{3/2} S_{i,4}(R) \\ & - \frac{\sqrt{L^4+r_0^4}}{q}\left(\frac{H_R^{3/4}}{g_R}\right)\left(R^7 H_R \frac{f'(R)}{f_R} + q^2\left(R\frac{H'(R)}{H_R} - R\frac{g'(R)}{g_R} + 6\right)\right) w_i(R) \\ & - q\sqrt{L^4+r_0^4}\left(\frac{R H_R}{\sqrt{f_R} g_R}\right)\beta_{x,v} + \sqrt{L^4+r_0^4}\left(\frac{q^2 f_R g_R - R^6 f_R H_R + q^2 H_R}{R^5 f_R^{3/2} g_R}\right) q_i \\ & - \frac{q\, r_0^3\sqrt{L^4+r_0^4}}{2L^4+r_0^4}\left(\frac{R^6 H_R\left(2R^4+5L^4\right) - 2L^4 q^2 g_R}{R^9 f_R^{1/2} g_R H_R}\right) r_{0,i} \\ & - 2q\, r_0^3\sqrt{L^4+r_0^4}\left(\frac{H_R}{R^3 f_R^{3/2} g_R}\right) r_{0,i}\,, \end{aligned} \tag{4.129}$$

from which we see that only $C_{i,2}$ and the Landau frame condition $w_i(R)$ are important. So, we use eq. (4.94) and (4.126) along with the expression for $S_{i,4}(R)$ and the vector constraint equation (4.77). The dust settles and we arrive at our final result:

$$\begin{aligned} \mathcal{J}_i = {} & -\left(\frac{H_R}{f_R g_R}\right)^{3/2}\frac{\left(3L^4+r_0^4\right)\left(r_+^4+r_0^4\right)}{4\kappa_5^2\, r_+ r_0^5}\left(r_0\, q_i - 3q\, r_{0,i}\right) \\ & - \frac{H_R^{3/2}}{R^4\left(f_R g_R\right)^2}\left(3R^4 - r_0^4\right)\frac{4\kappa L^4 q^2}{2\kappa_5^2 r_0^4}\,\epsilon_{ijk}\beta_{j,k}\,. \end{aligned} \tag{4.130}$$

The first check, we want to perform on it is again the near-horizon, large R limit (4.43), in which we expect to recover the results of [113]. Using $r_{0,i} + r_0\beta_{i,v} = 0$ for the vector constraint equation in the near horizon limit, we get

$$\mathcal{J}_i = -\frac{3L^4\left(r_+^4+r_0^4\right)}{4\kappa_5^2\, r_+ r_0^4}\left(q_i + 3q\,\beta_{i,v}\right) - \frac{6\kappa L^4 q^2}{\kappa_5^2 r_0^4}\,\epsilon_{ijk}\beta_{j,k}\,, \tag{4.131}$$

which agrees with [113] given that we have a slightly different normalisation of the gauge field and κ; see also (3.56), where for completeness we recorded the results of [113]. We have already remarked on this normalisation difference in section 4.2.1, but let us quickly repeat it here by denoting expressions in [113] with the superscript (B). Then the translation is

$$\frac{\sqrt{3}}{2}\mathcal{A}_\mu = A_\mu^{(B)}\,, \qquad \kappa = -\frac{\sqrt{3}}{2}\kappa^{(B)}\,. \tag{4.132}$$

Note furthermore, that the gauge field in [113] is not canonically normalised. So the standard formula for the charge current used in [113],

$$J^\mu = \lim_{r\to\infty}\frac{r^2}{8\pi G_5}\,A^\mu \qquad \Leftrightarrow \qquad J^\mu = \left(\frac{\sqrt{-\gamma}}{2\kappa_5^2}\right) n_\nu F^{\nu\mu}\,, \tag{4.133}$$

which is the one appropriate for a gauge field normalised as

$$\frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\,, \tag{4.134}$$

should receive an additional factor 4. Modifying the results of [113] like this, we find exact agreement with our results.

The second comparison we would like to undertake is the one with [263], in which a similar setup to ours was investigated. Einstein-Maxwell theory in asymptotically AdS space was considered and as in our case a hydrodynamic theory at a finite radial cutoff $r_c \equiv R$ was defined from perturbing the AdS-Reissner-Nordström background. The charge current J_i was unfortunately not simplified in that publication, but using their vector constraint equation, it may be shown to be equal to the following short expression:

$$\begin{aligned} J_i = &-\frac{1}{R^5\sqrt{f_R}f'(R)}\frac{2\sqrt{3}\left(r_0^4+r_+^4\right)}{g r_0 r_+}\left(r_0\, q_i - 3q\, r_{0,i}\right) \\ &- 16\kappa q^2\left(\frac{3R^4-r_0^4}{R^9 f_R f'(R)}\right)\epsilon_{ijk}\beta_{j,k}\,. \end{aligned} \tag{4.135}$$

We see that the general structure very much resembles (4.130). Yet, our setup and the one discussed in [263] are slightly different and may therefore not directly be compared, although one may be tempted to expect that the near-horizon limit of our setup should reduce to the one discussed in [263]. In the tensor sector, this assertion is correct, but due to the presence of the second gauge field in our setup, the vector sectors behave differently. The gauge field $\mathbf{A}$ modifies the bulk of the decoupling limit of our discussion as may also be seen directly from the equations of motion.

However, what both setups have in common is that the diffusion terms are proportional to $r_0\, q_i - 3q\, r_{0,i}$ with only the prefactor depending on R in an intricate way. This is noteworthy since in the near-horizon limit this corresponds to the Weyl-covariant derivative for the conformal field theory on the boundary (see appendix A of [113] and references therein). Neither we, nor [263] have a conformal field theory at finite $r = R$ though; the regulator breaks conformality. Although the bulk viscosity vanishes in the related setup [127] and in [263], the trace of the energy-momentum tensor does *not* vanish. Yet the conformal structure apparently extends also to these regions, as was already noted in a much less complex, asymptotically AdS setup by [127]. So, it is very interesting that we do also get this at finite cutoff AdS in [263] and even in our blackfold setup.

4.9 Conclusion

We have taken first steps for describing the interpolating blackfold fluid which incorporates all features from the membrane paradigm region to the fluid/gravity analysis and more simple AdS cut-off surface systems. The main virtue was the determination of the chiral anomaly related transport coefficient in (4.130), which perfectly interpolates between all known scenarios.

Furthermore, in (4.130) we have derived the diffusion constant, in which an interesting Weyl-covariant derivative structure was found. A similar result holds for the setup in [263] and was already noted in [127] though in a much simpler setup, which does not include the asymptotically flat region.

Much is left to be done. Most importantly, one would like to understand the holographic renormalisation of the setup better. This is however a very difficult goal to aim for.

What is however much simpler is to use our derived formulae and also compute the transport coefficients related to $\mathbb{J}_i$. This could be very interesting since it directly obtains contributions from the Chern-Simons term. We have all ingredients at hand and only need to use the stated formulae.

The scalar sector is the next open part which we have not much dealt with so far. It should be relatively easy to extract the constraint equations and interpret them as the v component of the "conservation" equation for the energy-momentum tensor similar to (4.78). This would then maybe also allow us to determine the speed of sound of the setup the way it was done in [123], from which Gregory-Laflamme instabilities may be seen in regions where the speed of sound gets to be negative. However, the fact that our energy-momentum tensor is not conserved will likely complicate the situation. From the solution of the dynamic equations we may compute the bulk viscosity and then see how it relates to the Buchel-bound [297]. It was already shown in [123] that the conjectured bound does not hold in general. For computing the bulk viscosity one may however rather use a different method which extracts it more straightforwardly instead of solving the complicated dynamics of the scalar sector.

CHAPTER 5

Supersymmetric hydrodynamics

In section 3.2.1, we have seen how one may compute transport and diffusion constants in the hydrodynamic limit of the AdS/CFT correspondence. We take the large N, large λ limit of e.g. $\mathcal{N} = 4$ super Yang-Mills theory and put the theory at finite temperature by describing its equilibrium state in terms of a (large) black brane within the dual AdS_5 geometry. We then perturb the theory by inserting a small source. To study the theory's response on this perturbation we seek out for discussing real time retarded Green's functions. In the hydrodynamic long wavelength, low frequency limit we may then compute diffusion or transport coefficients by analysing the Green's function's pole structure. Alternatively, we may use a Kubo formula to extract interesting information and in the case of the shear viscosity even show universality by relating the Kubo formula to a universal gravitational absorption cross section result.

The question, we want to address in this chapter is simple: Can we use this well-established technology to identify a similar hydrodynamic quantity, diffusion or transport coefficient, which has similarly universal properties?

In the following sections we will first give some more background information and motivation, where to seek for such a quantity, which might possess universality similar to η/s. After we identify a useful candidate following [106], we will compute it in arbitrary dimension in three different ways:

1. We will compute its Kubo formula and relate this to a universal absorption cross section result.

2. We will then also use the Kubo formula in the linearised analysis of gravitational perturbations to extract the desired quantity from a simple retarded Green's function.

3. Furthermore, we can extract the same results from a diffusion pole of a related retarded Green's function.

All these approaches are in principle known and have been applied in many similar situations, as we explained in section 3.2.1. However many aspects of our following analysis and its application to this particular problem is original – in particular the calculation of the specific Kubo formula and its relation to a universal absorption cross section by which a certain universality may indeed be established. But also the extension of some of the known computations of [106, 107] to arbitrary dimensions posed some intrinsic problems, which we had to solve.

The following chapter has, up to some small unpublished results and and adjustments of notation, been published in [1].

5.1 A candidate from supersymmetry

We have already in quite some depth discussed the application of holographic hydrodynamics to compute transport coefficients for strongly coupled systems, for which it has been very successful – in particular with regard to applications to real-world systems such as the quark-gluon plasma, as reviewed in [93, 298, 94]. One of the most far-reaching results obtained in this context is the universality of the ratio of shear viscosity and entropy density $\eta/s = 1/4\pi$ for theories which have a dual description in terms of Einstein gravity with unbroken rotational symmetry [250, 92, 299, 300, 93, 246].

But there are also further recent lines of investigation within gauge/gravity duality, which are concerned with the study of fermions, in which a universal result similar to η/s would be highly desirable. In particular, new results for fermionic correlators have been found in models which describe strongly coupled systems that are interesting in view of applications to condensed matter physics [137, 138, 139, 140], reviewed in [132, 131, 301]. Such systems have also been studied from a top-down perspective in [302, 303] or more recently in [304, 305].

Naturally, one would like to combine these two slightly separate research areas and set out for a search for similarly universal transport coefficients as η/s, however from fermionic correlators.

A candidate which may be computed from fermionic correlators has been studied and speculated to have universal properties in [106]. On general grounds, in the low energy, low momentum limit any quantum field theory at finite temperature may effectively be described by hydrodynamics [108]. In the hydrodynamical limit, the shear viscosity may then be calculated from the two-point function of the theory's transverse energy momentum tensor $\langle T_{xy}T_{xy}\rangle$ by applying a Kubo formula, as we showed in section 3.2.1. There is a quite natural fermionic correlator which seems to be related to this.

In supersymmetric field theories there is also the supersymmetry current S^{α}_{μ}, which belongs to the same supermultiplet as the energy-momentum tensor [100]. Therefore we may wonder if in the hydrodynamic limit of a supersymmetric field theory the two-point function of the supersymmetry current also possesses a similar universality. It has actually been argued that AdS/CFT seems to require supersymmetry to work [87, 86], since generically field theories at very large coupling $\lambda \gg 1$ (not just $\lambda \approx 1$) are unstable against the building of particle/anti-particle pairs, for which due to the very strong coupling the negative potential energy exceeds rest mass and kinetic energy (cf. the Landau pole of QED at very large coupling). According to [87, 86], only a mechanism like supersymmetry, in which the energy is bounded from below because the Hamiltonian may basically be written as the square of Hermitian supercharges, would allow for a dual stable description. On the other hand, the general dictionary (in the weak form) does not so definitely suggest supersymmetry as a requirement. In any way, it is clear that gravity duals of supersymmetric theories are certainly the best rigorously understood examples of gauge/gravity duality, so we may at present very well restrict to these.

Since supersymmetry is broken spontaneously by temperature, which we certainly need for a hydrodynamic setup, the relation between $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ and $\langle S_\mu \bar{S}_\nu \rangle$ is non-trivial. The supersymmetry generators do not annihilate the vacuum state, so $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ and $\langle S_\mu \bar{S}_\nu \rangle$ will differ beyond the simple spin structure transformation relating $T_{\mu\nu}$ and S^α_μ which one gets from the supersymmetry algebra. So, one may not simply set up a superfield correlator as in [306] and relate these two correlators and their poles to one another by expanding in $\theta, \bar{\theta}$ components as one may do in unbroken supersymmetry.

For theories at finite temperature the expectation value $\langle T_{\mu\nu} \rangle$ does not vanish, so from inspection of the Ward identity [307, 308]

$$\partial^\mu \langle T S_\mu(x) \bar{S}_\nu(0) \rangle = \delta^{(4)}(x)\, 2\, \gamma^\mu \langle T_{\mu\nu} \rangle \,, \tag{5.1}$$

we see that there must be a pole in $\langle S_\mu \bar{S}_\nu \rangle$ describing the Goldstone fermion of spontaneously broken supersymmetry [101]. This pole may be interpreted as the so-called phonino mode [102, 103, 104], a massless excitation with characteristic sound dispersion relation $\omega = v_s k - i D_s k^2$ whose appearance resembles that of the phonon. This was first established holographically for $\mathcal{N} = 4$ SYM in four dimensions in [106], where the concrete dispersion relation was computed analytically. In [106], the equations of motion of a gravitino in AdS_5 were solved to linear order in ω and k to derive the retarded Greens functions of the dual supersymmetry current operators using the prescription of [226, 227], from which the dispersion relation may be read off. This setup has been studied further in [107] using the transverse mode of the gravitino and a Kubo formula [309], also for non-vanishing chemical potentials. In three space-time dimensions, the correlator of supersymmetry currents at finite chemical potential has been studied in [310, 311, 312] and, interestingly, it was shown that no Fermi surfaces show up in the spectral functions of their setups.

The quantity which has been speculated to have similar universal properties as the shear viscosity η is the supersound diffusion constant D_s, which also appears as a particular transport coefficient in the constitutive relation of the supersymmetry current. In four dimensions, this is given by [309]

$$S^i_{\mathrm{diss}} = -D_s \nabla^i \rho - D_\sigma \sigma^{ij} \nabla_j \rho \tag{5.2}$$

with ρ the supercharge density.

However, as emphasised in [313], this supersymmetric hydrodynamics should really be understood as the low-energy effective theory of the phonino moving in the normal fluid. Since expectation values for fermionic operators vanish, one should not introduce a classical (fermionic) supercharge density in terms of which constitutive relations for the spatial parts of the supersymmetry current are expanded, as is done for normal hydrodynamics with conserved (bosonic) charges. Instead, the supercharge density has to be interpreted as the quantum phonino field itself. In addition, according to [313], the fermionic chemical potentials have to be viewed as external gravitino sources, where D_s and D_σ are interpreted as masses for the spin 3/2 and spin 1/2 components of these. This approach does not alter the form of the constitutive relations as compared to [309], but their interpretation is refined in the sense that the first derivative terms should not be thought of as dissipative parts contributing to the entropy current.

Furthermore, there is a second motivation for studying the two-point function of the supersymmetry current. Originally, the AdS/CFT correspondence has arisen from the study and comparison of classical absorption cross sections in black brane backgrounds and decay rates of external modes decaying into the worldvolume fields on Dp-branes [165,166,167], see section 2.2.3. When studying such classical absorption cross sections in detail, it had before been realised that the low-energy s-wave absorption cross section of minimally coupled massless scalars is universally given by the area of the horizon of the considered black brane background [105]. This very result was then used later to show the universality of η/s in the holographic context [92], see section 3.2.1. This proof relies on the fact that the metric fluctuation for the shear mode h_{xy} generically satisfies $\Box h_x^y = 0$. Moreover, the Kubo formula and absorption cross section formula are basically identical in the low-energy limit [130].

In [105], a further universal absorption cross section result has been given for minimally coupled massless fermions. We may therefore hope that the transverse gravitino, which is the dual mode to the supersymmetry current, also satisfies the equation of motion of a spin 1/2 fermion. Then the universality of the fermionic absorption cross section would lead to a universal relation which involves the supersound diffusion constant in very much the same way as the universality of η/s can be seen as originating from the universal absorption cross section result for scalars.

In the following, we therefore study the supersound diffusion constant D_s by computing it in various AdS/CFT setups of arbitrary dimension which include D3-, M2- and M5-brane theories. We then search for and actually show universality in the way just described.

5.2 Relation to black hole absorption cross sections

In this section we derive a new universal result for a hydrodynamic transport coefficient similar to the universality of the ratio of shear viscosity η over entropy density s. This result will be used to rederive the supersound diffusion constant D_s (at vanishing chemical potential for R charges) for the D3-brane theory [106, 107]. Moreover we extend this approach to also include the M2- and M5-brane theories. Furthermore, our approach also applies to a whole class of near-extremal non-dilatonic $p = d - 1$ branes in AdS$_{d+1}$ [314] which arise from the near-horizon limit of a class of black p-brane solutions in D dimensions [315, 293] and a subsequent sphere reduction.

5.2.1 Universal gravitational absorption cross sections

In section 3.2.1 we have already sketched the proof of universality of η/s as provided by Kovtun, Son and Starinets [92]. The essential idea involved the comparison of the field theory's Kubo formula for the shear viscosity and the gravitational absorption cross section of the minimally coupled bulk transverse graviton by the black hole/brane (3.29).

Furthermore, it was used that the transverse graviton obeys the equation of motion of a minimally coupled massless scalar field and that the absorption cross section for such fields is given by the area of the horizon of the black hole [105]

$$\sigma_{\text{abs},0}(0) = a \tag{5.3}$$

The cross-section result (5.3) had previously been proved by Das, Gibbons and Mathur [105] for arbitrary dimensional spherically symmetric black hole backgrounds,

$$ds^2 = -f(r)dt^2 + g(r)\left(dr^2 + r^2 d\Omega_p^2\right) \,. \tag{5.4}$$

For the proof, these backgrounds are assumed to have a horizon at $r = r_0$, to be asymptotically flat and non-extremal. This is a slightly unusual coordinate system compared to more standard ways to write the background e.g. of a black brane in $AdS_5 \times S^5$, in which there is no warp factor such as $g(r)$ in front of the *transverse* sphere. Nevertheless, this is just a coordinate change. The proof for (5.3) then involves solving the equation of motion of a massless minimally coupled scalar in the regions far away and close to the horizon, matching these exact asymptotic solutions and from this calculate the absorption probability of the s-wave and the absorption cross section as in [316]. This also holds for charged and/or rotating black holes [317] and can be extended to branes [238].

Furthermore, in [105] a similar result for the low-energy absorption cross section of a massless minimally coupled fermion by the black hole (5.4) has been obtained,

$$\sigma_{\text{abs},1/2}(0) = 2\, g_H^{-p/2}\, a \,, \tag{5.5}$$

where g_H is $g(r)$ of (5.4) evaluated at the horizon r_0 and the factor of 2 comes from the two helicities of the spinor. Although this is a coordinate dependent result, it is universal in the sense that it is twice the area of the horizon evaluated in the conformally related spatially flat metric $ds^2 = dr^2 + r^2 d\Omega_p^2$. Furthermore, note that this low-energy absorption cross section is determined by *horizon quantities* only!

We may use (5.5) to efficiently calculate the absorption cross sections of minimally coupled massless fermions for diverse backgrounds [318]. For example, for the four-dimensional asymptotically flat Schwarzschild geometry we may easily show $\sigma_{\text{abs},1/2} = 1/8\, \sigma_{\text{abs},0}$ in agreement with Unruh's classic result [316].

For vanishing R-charge chemical potential, the transverse gravitino satisfies the equation of motion of a minimally coupled spin 1/2 fermion, as we will later show in a particular example (5.46). We have explicitly shown this for spacetimes with metric of the form

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + h(r)d\vec{x}^2 \tag{5.6}$$

with arbitrary metric functions $f(r)$, $g(r)$ and $h(r)$. This fact is a direct consequence of gauge invariance. So, we may wonder if this simplification along with (5.5) also implies a universality relation as in [92].

What is then the relation analogous to $\eta \sim \sigma_{\text{abs},0}(0)$ (3.29) and what would be the universal quantity as in $\eta/s = 1/4\pi$?

5.2.2 Fermion absorption cross section

To obtain analogous fermionic results, let us first derive a relation for the absorption cross section $\sigma_{\text{abs},1/2}(\omega)$ of a bulk fermion similar to the scalar result (3.28), which will turn out to be very useful. The slight complication, however, in generalising (3.28) to particles with spin is that we have to specify a polarisation of the infalling particle.

From the field theory point of view, this bulk absorption may be seen as the decay of a massive particle into the world-volume theory's fields it couples to. Then the absorption cross section is just given by the standard formula of the field theory decay rate,

$$\sigma_{1/2} = \frac{1}{2\omega} \int d\Pi \, |\mathcal{M}|^2 \,, \tag{5.7}$$

where $\int d\Pi$ denotes the final state particles' momentum space integrals including the overall momentum conserving delta function. Let us take the bulk Dirac fermion Ψ to have the following kinetic term[1],

$$\frac{4}{\kappa_{d+1}^2} \int d^{d+1}x \sqrt{-g}\, \bar{\Psi} \not{D} \Psi \tag{5.8}$$

and couple its boundary values Ψ_0 and $\bar{\Psi}_0$ to a spinorial boundary operator S, which will later turn out to be a specific transverse component of the supersymmetry current, via

$$\int d^d x \left(\bar{S} P_- \Psi_0 + \bar{\Psi}_0 P_+ S \right) \,. \tag{5.9}$$

Here $P_\pm = \frac{1}{2}\left(1 \pm \gamma^d\right)$, in which γ^d denotes the gamma matrix corresponding to the radial AdS coordinate. Then for even d, $P_-\Psi_0$, $\bar{\Psi}_0 P_+$, $\bar{S}P_-$ and P_+S are chiral spinors as seen from the boundary theory, while for odd d, they are all Dirac boundary fermions [181] (see also [319]).

Then, we may use the optical theorem $2\,\mathrm{Im}\mathcal{M} = \int d\Pi\, |\mathcal{M}|^2$ and average over the polarisations for the decaying Weyl / Dirac spinors in even / odd d. Furthermore, one can use a corresponding spin sum identity $\sum u\bar{u} = \not{p} + m$. Since our decaying fermion is at rest $p^\mu = (\omega, \vec{p} = 0)$ from the point of view of the boundary theory, only the γ^0 part is left over,

$$\sigma_{\mathrm{abs},1/2}(\omega) = \frac{\kappa_{d+1}^2}{\mathrm{Tr}\,(-\gamma^0\gamma^0)}\, \mathrm{Tr}\left(-\gamma^0 \;\mathrm{Im} \int d^d x\, e^{i\omega t} \,\langle\, P_+ S(x) \bar{S}(0) P_- \,\rangle \right) \tag{5.10}$$

for even d. Likewise, in dimensions where d is odd we have

$$\sigma_{\mathrm{abs},1/2}(\omega) = \frac{\kappa_{d+1}^2}{2\;\mathrm{Tr}\,(-\gamma^0\gamma^0)}\, \mathrm{Tr}\left(-\gamma^0 \;\mathrm{Im} \int d^d x\, e^{i\omega t} \,\langle\, S(x) \bar{S}(0) \,\rangle \right) \,. \tag{5.11}$$

The given absorption cross section formulae are closely related to the Kubo formulae for a specific hydrodynamic transport coefficient, as we are about to explain now.

5.2.3 Constitutive relation and Kubo formula

A generalisation of the four-dimensional constitutive relation which relates the spatial part of the supersymmetry current S^i to the supercharge density $\rho = S^0$ is[2]

$$S^i = \frac{P}{\epsilon} \gamma^i \gamma^0 \rho - D_s \nabla^i \rho + \frac{D_\sigma}{(d-2)} \gamma^{[i}\gamma^{j]} \nabla_j \rho \,, \tag{5.12}$$

[1] The non-canonical normalisation will be explained later, following eq. (5.54).

[2] Note that, as pointed out in [313], we should really understand ρ as the phonino quantum field which then determines the supercharge density.

where ϵ and P are the energy density and pressure of the fluid. The non-dissipative part is fully determined by the d-dimensional supersymmetry algebra [320]. D_s and D_σ are transport coefficients determining the damping of a sound-like excitation, the phonino, which propagates at the speed $v_s = \frac{P}{\epsilon}$ [102, 101]. In the superconformal case we have $\gamma_\mu S^\mu = 0$, since supersymmetry relates this to $T^\mu_\mu = 0$, and therefore $D_s = D_\sigma$ and $v_s = \frac{1}{d-1}$.

For minimal supersymmetry in $d = 4$ dimensions, we may recover the Weyl form of this constitutive relation [309] by taking ρ to be a Majorana spinor in the Weyl basis. Depending on the dimension, we however take ρ to be a Dirac spinor (when d is odd), or we project to the Weyl version of (5.12) (for d even) with ρ and $\bar{\rho}$ to denote Weyl spinors. As already mentioned, this is convenient since these types of spinors are the boundary spinors inherited from Dirac spinors in $d+1$ bulk dimensions, which are the easiest to handle in general dimension. Of course this means that, depending on the dimension, the supercharge will not give rise to minimal but rather extended supersymmetry (see e.g. [310]).

We may reorder the constitutive relation (5.12) according to the spinorial representations of $O(d-1)$,

$$S^i = \frac{P}{\epsilon}\gamma^i\gamma^0\rho - D_{3/2}\left(\delta^i_j - \frac{1}{d-1}\gamma^i\gamma_j\right)\nabla^j\rho - D_{1/2}\gamma^i\not{\nabla}\rho\,, \tag{5.13}$$

where $D_{3/2} = D_s + \frac{1}{d-2}D_\sigma$ and $D_{1/2} = \frac{1}{d-1}\left(D_s - D_\sigma\right)$ are the transport coefficients corresponding to the spin 3/2 and spin 1/2 parts under $O(d-1)$ of the vector spinor $\nabla^j\rho$. This way to write the constitutive relation is completely analogous to the way the energy momentum tensor is conventionally written involving the shear and bulk viscosities η and ζ. These transport coefficients appear in front of the symmetric traceless and trace parts of $\nabla^i u^j$ (where u^j is the fluid velocity) in the first order dissipative part of the energy momentum tensor. In the conformal case, we have

$$D_s = \left(\frac{d-2}{d-1}\right)D_{3/2} \quad \text{and} \quad D_{1/2} = 0\,. \tag{5.14}$$

Basically these redefinitions are entirely equivalent to writing the usual sound attenuation in terms of shear and bulk viscosity [321, 322]. Now $D_{3/2}$ may be calculated via a Kubo formula. For even d we have

$$\epsilon D_{3/2} = \frac{2}{\mathrm{Tr}\left(-\gamma^0\gamma^0\right)}\left(\frac{1}{d-2}\right)\lim_{\omega,k\to 0}\ \mathrm{Tr}\left(-\gamma^0\delta_{ij}\ \mathrm{Im}\,\Delta^{ij}(\omega,k)\right)\,, \tag{5.15}$$

in which

$$\Delta^{ij}(\omega,k) = \int d^dx\, e^{i\omega t}\ \left\langle P_+ S^i_T(x)\bar{S}^i_T(0)P_-\right\rangle\,. \tag{5.16}$$

For odd d we get

$$\epsilon D_{3/2} = \frac{1}{\mathrm{Tr}\left(-\gamma^0\gamma^0\right)}\left(\frac{1}{d-2}\right)\lim_{\omega,k\to 0}\ \mathrm{Tr}\left(-\gamma^0\delta_{ij}\ \mathrm{Im}\,\Delta^{ij}(\omega,k)\right)\,, \tag{5.17}$$

in which this time

$$\Delta^{ij}(\omega,k) = \int d^dx\, e^{i\omega t}\ \left\langle S^i_T(x)\bar{S}^i_T(0)\right\rangle\,. \tag{5.18}$$

In both expressions, the limit $k \to 0$ is taken first as usual. The derivation is very similar to the one outlined in the appendix A of [107]: We use the solution to the current conservation equation $\partial_\mu S^\mu = 0$ involving the constitutive relation (5.12) to obtain the correlator $\langle \rho\bar{\rho}\rangle$. This then determines the correlator $\langle S^i_T \bar{\rho}\rangle$, where

$$S^i_T = \left(\delta^i_j - \frac{1}{d-1}\gamma^i\gamma_j\right) S^j = -D_{3/2}\left(\delta^i_j - \frac{1}{d-1}\gamma^i\gamma_j\right)\nabla^j\rho \qquad (5.19)$$

denotes the spin 3/2 part of the supersymmetry current under the spatial $O(d-1)$. From this we get

$$\operatorname{Im}\left(\frac{k_i}{k^2}\left\langle S^i_T\bar{\rho}\right\rangle\right) = -\left(\frac{d-2}{d-1}\right) D_{3/2}\operatorname{Re}\left\langle\rho\bar{\rho}\right\rangle \,. \qquad (5.20)$$

Using Ward identities and appropriate limits, we may then turn the current-charge correlator into the current-current correlators (5.15) and (5.17). Note that the normalisations in (5.15) and (5.17) do agree with the expectation since at vanishing $\vec{k}$, from rotational invariance, we have $\langle S^i \bar{S}^j\rangle \propto \delta^{ij}$. Therefore, $\delta^j_i\left(\delta^i_j - \frac{1}{d-1}\gamma^i\gamma_j\right) = d-2$ for $i, j = 1, \ldots, d-1$ gives the correct number of independent modes of a vector S^i_T keeping in mind the imposed constraint $\gamma_i S^i_T = 0$ from the projection. Similar reasoning regarding the purely spinorial degrees of freedom may also be applied. For these, the normalisations of the Kubo formulae are expected to involve expressions like $\operatorname{Tr}\left(-\gamma^0\gamma^0\right) = 2^{\lfloor \frac{d}{2}\rfloor}$ in the Dirac or $\frac{1}{2}\operatorname{Tr}\left(-\gamma^0\gamma^0\right)$ in the Weyl case.

At non-vanishing $\vec{k}$, one would use the projector P^T_{ij}, which projects onto the γ^i-traceless, $\vec{k}$-transverse part of a vector-spinor

$$P^T_{ij} = \delta_{ij} - \frac{1}{d-2}\left(\gamma_i - \frac{k_i \not{k}}{k^2}\right)\gamma_j - \frac{1}{(d-2)k^2}\Big((d-1)k_i - \gamma_i\not{k}\Big)k_j \,, \qquad (5.21)$$

again very similar to the shear viscosity case.

Since the gravitino couples to the supersymmetry current on the boundary, using gauge/gravity duality we may relate the gravitino absorption cross section by the brane to the retarded Green's function of the dual operator. This is then a fermionic analogue of the graviton absorption cross section considered in [166]. Similar to the way one considers transverse metric perturbations h_{xy} for the η/s case, we here focus on the gravitino modes which have spin 3/2 under the $\vec{k}$ preserving little group $O(d-2)$. These are referred to as $\eta_i = P^T_{ij}\Psi^j$. For vanishing R-charge chemical potentials these transverse gravitino components satisfy equations of motion of minimally coupled fermions similar to the transverse graviton obeying a Klein-Gordon equation. We may therefore relate the absorption cross section results (5.10) (5.11) to the Kubo formulae (5.15) and (5.17), in which $O(d-1)$ symmetry at vanishing $\vec{k}$ implies $\frac{1}{d-2}S^i_T\bar{S}^i_T = S^x_T\bar{S}^x_T \equiv S\bar{S}$. Putting these together, we obtain

$$\epsilon D_{3/2} = \frac{2}{\kappa^2_{d+1}}\,\sigma_{\text{abs},1/2}(0) = \frac{1}{4\pi G}\,\sigma_{\text{abs},1/2}(0)\,, \qquad (5.22)$$

which is the fermionic analogue of (3.29).

However the bulk fields η_i are not massless as required for the use of (5.5), but rather have mass, for instance $mL = \frac{d-1}{2}$ in AdS_{d+1}. Therefore we cannot directly relate (5.22) to the universal result (5.5), but rather need to generalise (5.5) to massive fermions.

On the other hand, the AdS fermions η_i get their mass $m \sim \frac{1}{L}$ only from (consistent) Kaluza-Klein reduction on the transverse sphere, albeit being massless from the point of view of the higher-dimensional theory. We may therefore still compute the absorption cross section in the higher-dimensional theory, directly using the $m = 0$ results (5.5). We will pursue both paths in the following subsections.

So far, we do not quite understand the field theory meaning of the right hand sides of (5.5) and therefore (5.22) in a holographic context. Since the absorption cross section is closely related to the horizon area and therefore entropy density s, we would like to divide by a quantity like s on both sides. However the right hand side of (5.5) is the area of the horizon in a conformally related spatially flat metric and not the horizon area measured in the original metric. It would be very interesting to understand this better.

5.2.4 Massive/higher dimensional absorption cross section

For computing the effect of mass on the absorption cross section result (5.5), we very closely follow [105] in their derivation of (5.5), but start off with the *massive* Dirac equation for a minimally coupled spinor field Ψ,

$$(\nabla_\mu \gamma^\mu - m)\,\Psi = 0\,. \tag{5.23}$$

It is easy using the conformal properties of the Dirac equation (under $g_{\mu\nu} \to \Omega^{-2} g_{\mu\nu}$ in d dimensions one has $(\Psi, m) \to (\Omega^{\frac{d-1}{2}}\Psi, \Omega m)$, cf. [185]) to show that in the background (5.4) this is equivalent to

$$h\gamma^i\partial_i\chi = i\omega\gamma^0\chi + mf^{1/2}\chi \tag{5.24}$$

for the spinor $\chi = f^{1/4}g^{p/4}\Psi$ and $h = \sqrt{f/g}$ and $\gamma^i\partial_i = \gamma^r[\partial_r + \frac{p}{2r}] + \frac{1}{r}(\gamma^i\nabla_i)_T$. Using a basis for spinors that satisfies $\gamma^r\lambda_n^\pm = \pm\lambda_n^\pm$ and $\gamma^0\lambda_n^\pm = \mp\lambda_n^\mp$, we may expand

$$\chi = \sum_{n=0}^{\infty} F_n(r)\lambda_n^+ + G_n(r)\lambda_n^- \tag{5.25}$$

and use the known spectrum of the Dirac operator on the sphere $\gamma^r(\gamma^i\nabla_i)_T\lambda_n^\pm = \mp(n+\frac{p}{2})\lambda_n^\pm$ (see e.g. [323]) to arrive at

$$h\left(\partial_r - \frac{n}{r}\right)F_n - mf^{1/2}F_n = i\omega G_n\,, \tag{5.26a}$$

$$h\left(\partial_r + \frac{p+n}{r}\right)G_n + mf^{1/2}G_n = i\omega F_n\,, \tag{5.26b}$$

which only slightly modifies the equations of [105]. Now, eliminating G_n one will for the $n = 0$ mode of the spinor end up with

$$h\,(\partial_r + p/r + m\sqrt{g})\,h\,(\partial_r - m\sqrt{g})\,F_0 + \omega^2 F_0 = 0\,. \tag{5.27}$$

We may now define a new coordinate via $\frac{d}{dx} = h(r)r^p\rho(r)\frac{d}{dr}$ under the condition $\rho^{-1}\partial_r\rho = 2m\sqrt{g}$ and $\rho \to 1$ as $r \to \infty$. Then defining $F_0 = \exp\left(m\int dr\sqrt{g}\right)\tilde{F}$ we have

$$\partial_x^2\tilde{F} + \omega^2 r^{2p}\rho^2\tilde{F} = 0\,, \tag{5.28}$$

which is of a suitable form to compare with the scalar case. Following the arguments in [105], so choosing an ingoing wave at the horizon, we directly obtain that the absorption cross section for a minimally coupled massive spin 1/2 fermion is given by

$$\sigma_{\mathrm{abs},1/2,\mathrm{m}} = g(r_0)^{-p/2} A \exp\left(2m \int_{\infty}^{r_0} dr \sqrt{g}\right) . \tag{5.29}$$

Note that there is no factor of 2 appearing in front compared to (5.5). For non-vanishing mass m, the low-energy absorption cross-section entirely comes from the s-wave and we may neglect the p-wave contribution. For $m = 0$, the s- and p-wave contributions to the cross-section are equal and sum up to the given factor of 2 in (5.5) (cf. figure 1 in [316] in the four-dimensional Schwarzschild case). A different way to see this is to notice that the mass terms in (5.26) roughly behave as higher angular momentum modes which distinguish between the $\lambda_n^{\pm}$ modes, although they are degenerate and contribute equally to $\sigma_{\mathrm{abs},1/2}$ for $n = 0$ in the massless limit.

5.2.5 Application to non-dilatonic black branes

We now use these results to compute the supersound diffusion constant D_s for a specific class of black brane space-times. In later sections, we will use the direct holographic methods of [106, 107] for the calculation of the same quantity. The results will agree and therefore provide a useful cross-check.

The metrics under consideration are non-dilatonic $p = d-1$ branes in AdS_{d+1} [314]:

$$ds^2 = -f(r)dt^2 + \frac{r^2}{L^2}\sum_{i=1}^{d-1} dx_i^2 + \frac{dr^2}{f(r)} , \tag{5.30}$$

in which

$$f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_0^d}{r^d}\right) , \tag{5.31}$$

and the AdS radius is L. The AdS boundary is at $r \to \infty$ and the horizon at $r = r_0$. For $d = 4, 3, 6$ the space-time represents the near-horizon limit of near-extremal D3-, M2- and M5-branes reducing the sphere. For other dimensions, these can be understood by taking the near-horizon limit of a class of near-extremal D-dimensional black p-brane solutions [315, 293] which generalise D3-, M2- and M5-branes. The Hawking temperature T, Bekenstein-Hawking entropy S and Abbott-Deser mass M of the brane (see e.g. [314]) are given by

$$T = \frac{dr_0}{4\pi L^2} , \quad S = \frac{A}{4G} = \frac{2\pi}{\kappa_{d+1}^2}\frac{r_0^{d-1}}{L^{d-1}} V_{\|} , \quad M = \frac{d-1}{2\kappa_{d+1}^2}\frac{r_0^d}{L^{d+1}} V_{\|} = \epsilon\, V_{\|} , \tag{5.32}$$

where $V_{\|}$ is the volume of the brane measured in the coordinates x_i at constant t.

Evaluating (5.29) upon suitable regularisation in the UV in the spacetime (5.31) one arrives at (note $mL = (d-1)/2$)

$$\frac{\sigma}{A} = \exp\left[2m \int_{\infty}^{r_0} dr \left(\frac{1}{\sqrt{f(r)}} - \frac{L}{r}\right)\right] = \frac{1}{4}\, 2^{\frac{2}{d}} . \tag{5.33}$$

We may also arrive at this result by evaluating (5.5) in the higher-dimensional black brane space-time before reducing on the sphere. This is due to the fact that the gravitino is massless in the higher-dimensional space-time and gains a "mass" $mL = \frac{d-1}{2}$ upon sphere reduction. We are now going to show that this approach also gives the same result (5.33).

The asymptotically flat Gibbons, Horowitz and Townsend non-dilatonic black p-branes in D space-time dimensions [315] can be written as

$$\begin{aligned} ds^2 = H(r)^{-\frac{2}{p+1}} &\left[-F(r)dt^2 + d\vec{x}_p^2\right] \\ &+ H(r)^{\frac{2}{D-p-3}} \left[F(r)^{-1}dr^2 + r^2 d\Omega_{D-p-2}^2\right] \end{aligned} \tag{5.34}$$

with $H(r) = 1 + \left(\frac{L}{r}\right)^{D-p-3}$ and $F(r) = 1 - \left(\frac{r_0}{r}\right)^{D-p-3}$. After some coordinate redefinitions, the near-horizon geometry of these reduces to

$$ds^2 = -f(r)dt^2 + \frac{r^2}{L_{\rm AdS}^2} d\vec{x}_p^2 + f(r)^{-1}dr^2 + L_{\rm Sph}^2 d\Omega_{D-p-2}^2\,, \tag{5.35}$$

where

$$f(r) = \frac{r^2}{L_{\rm AdS}^2} - \left(\frac{r_0}{L_{\rm AdS}}\right)^2 \left(\frac{r_0}{r}\right)^{p-1} \quad \text{and} \quad L = L_{\rm Sph} = \left(\frac{D-p-3}{p+1}\right) L_{\rm AdS}\,. \tag{5.36}$$

This is indeed a black p-brane in $\mathrm{AdS}_{p+2}\times \mathrm{S}^{D-p-2}$ and clearly, depending on p and the overall dimension D, it is either the near-horizon geometry of the selfdual three-brane of ten-dimensional supergravity [161] or the M2- or M5-brane of eleven-dimensional supergravity [324, 325] or the self-dual string of six-dimensional supergravity [326]. Reducing on the sphere gives (5.31).

Now, we would like to transform the relevant part of (5.34) into the form (5.4) used by Das, Gibbons and Mathur in their theorem for the low-energy absorption cross section of a spin 1/2 particle in an (asymptotically flat) black hole background. For this we need to require

$$F(r)^{-1}dr^2 + r^2 d\Omega_{D-p-2}^2 \equiv g(\tilde{r}) \left[d\tilde{r}^2 + \tilde{r}^2 d\Omega_{D-p-2}^2\right]\,. \tag{5.37}$$

So, we transform to a new "isotropic" radial coordinate, in which the space transverse to the brane is conformally flat, cf. [42, p.157] Solving this for $\tilde{r}$, we directly obtain

$$\tilde{r} = c\,r \left(1 + \sqrt{F(r)}\right)^{\frac{2}{D-p-3}}\,. \tag{5.38}$$

The requirement that we have $r = \tilde{r}$ for $r \to \infty$ yields $c = 2^{-\frac{2}{D-p-3}}$. In the $\tilde{r}$ coordinates the horizon is at $\tilde{r}_0 = c\,r_0$ with $g(\tilde{r}_0) = c^{-2}$.

Now, we would like to evaluate the absorption cross section for the spin 1/2 particle (5.5)

$$\sigma = 2\,g(\tilde{r}_0)^{-\frac{D-p-2}{2}}\,A\,. \tag{5.39}$$

It now easily follows that

$$\frac{\sigma}{A} = 2^{-2+\frac{D-p-5}{D-p-3}}\,. \tag{5.40}$$

This already gives the supersound diffusion constant calculated for the D3-, M2- and M5-branes and may be transformed directly into the result we already obtained previously (5.33).

Given a D-dimensional action which consists just of non-dilatonic Einstein gravity and the action for a p-form field, a necessary condition for the truncation of the massive modes after a sphere reduction to be consistent is given by [327]

$$(D - p - 5)\,(p - 1) = 4\,. \tag{5.41}$$

Using this we get agreement with (5.33),

$$\frac{\sigma}{A} = \frac{1}{4}\,2^{\frac{2}{d}}\,. \tag{5.42}$$

Note however that the condition (5.41) singles out D3-, M2- and M5-brane theories for integers D and p. We would have to think along the lines of generalised dimensional reduction [177] to allow for the other values. Note that to our knowledge, this has however not yet been worked out for spinors.

We are now in the position to put together our results to obtain an expression for the supersound diffusion constant D_s which may be compared to [106, 107].

Noting that for the branes (5.31) with T, A and ϵ given in (5.32), we may use the conformal relation (5.14) and (5.22) to get

$$2\,\pi T D_s = \frac{2^{2/d} d(d-2)}{2(d-1)^2}\,. \tag{5.43}$$

This result agrees with the $d = 4$ result of [106, 107] and seems to agree with the $d = 3$ result which so far has only been found numerically in [310, 312]. Furthermore, the result for D_s vanishes for $d = 2$ since (super)gravity in three dimensions has no propagating degrees of freedom.

Note that when using (5.39), the boundary diffusion constant is completely determined in terms of *horizon data* only. So, in other words, there seems to be no non-trivial bulk evolution similar to the non-evolution through the bulk of η/s. In the latter case, the independence on the radial coordinate equates the boundary field theory's result of η/s with its membrane paradigm value $1/4\pi$ [246]. However, we are dealing with a special coordinate system here, in which the flow seems to be trivial. This generically does not agree with the coordinate system in which r is the field theory's energy scale, but rather appears to be trivial in $\tilde{r}$ where $f(r)^{-1}dr^2 = \frac{d\tilde{r}^2}{\tilde{r}^2}$. It would be interesting to study the setup more intensely along the lines of [246] and [319].

5.3 Supersound diffusion constant from the transverse gravitino

We now turn our focus to the computation of the supersound diffusion constant by extending the holographic computations of [106, 107] for the D3-brane to the case of M2- and M5-brane theories. Simultaneously, we extend it to the aforementioned class of near-extremal non-dilatonic $p = d - 1$ branes in AdS_{d+1} (5.31). The chemical potentials for R-charges are taken to vanish.

We are going to present the calculation via the transversal mode (as in [107]), which requires solving a gravitino's equation of motion to 0'th order in ω and k and using a Kubo formula. The longitudinal calculation (as in [106]) is technically more difficult, since it also requires solving the equations of motion to linear order in ω and k, and will be covered in the subsequent section.

The bulk action for the linearised gravitino is given by

$$S \propto \int d^{d+1}x\sqrt{-g}\bar{\Psi}_\mu \left(\Gamma^{\mu\nu\rho}D_\nu - m\Gamma^{\mu\rho}\right)\Psi_\rho \,, \tag{5.44}$$

where the normalisation will first be unimportant.

The covariant derivative acts on spinors as $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$ where the only non-vanishing components of the spin-connection for the background (5.31) are $\omega_t^{0d} = \frac{1}{2}f'$ and $\omega_{x_i}^{di} = -\frac{\sqrt{f}}{L}$ for each $i = 1, \ldots, d-1$. Furthermore, $mL = \frac{d-1}{2}$ is required for linearised supergravity to hold in AdS_{d+1} [328] and ensures the gravitino to have the correct degrees of freedom of a massless gravitino field [329].

Using the standard gauge condition for the gravitino $\Gamma^\mu\Psi_\mu = 0$ where $\mu = 0, \ldots, d$ (which also implies $D^\mu\Psi_\mu = 0$) the Rarita-Schwinger equation may be simplified to

$$\left(\not{D} + m\right)\Psi_\mu = 0 \,. \tag{5.45}$$

These are complicated equations of motion which couple different components of the gravitino[3]. However, assuming a boundary space-time dependence $e^{-i\omega t + ikx_1}$ of the gravitino, we may use the projector (5.21) to project to the k^μ-transverse components of the gravitino which have spin 3/2 under the transverse $O(d-2)$ that preserves the boundary wave vector. These components $\eta_i = \Psi_i - \frac{1}{d-2}\gamma_i\gamma^j\Psi_j$ $(i,j \neq 1)$ decouple, so that the equation of motion for one of them in the background (5.31), call it η, reads:

$$0 = \eta' - \frac{i\omega}{f}\gamma^d\gamma^0\eta + \frac{ikL}{r\sqrt{f}}\gamma^d\gamma^1\eta + \frac{f'}{4f}\eta + \frac{d-1}{2r}\eta - \frac{m}{\sqrt{f}}\gamma^d\eta \,. \tag{5.46}$$

Note that this is exactly the equation of motion for a minimally coupled spin 1/2 fermion of mass m in the given space-time as was used heavily in the previous section.

We may now expand η in a basis of eigenspinors of γ^d and $i\gamma^1\gamma^2$ since these commute with each other. Let the basis spinors be given by

$$\gamma^d a^\pm = \pm a^\pm \,, \quad \gamma^d b^\pm = \pm b^\pm \,, \quad i\gamma^1\gamma^2 a^\pm = +a^\pm \,, \quad i\gamma^1\gamma^2 b^\pm = -b^\pm \,. \tag{5.47}$$

Then one can write

$$\eta = \eta^{a+}a^+ + \eta^{a-}a^- + \eta^{b+}b^+ + \eta^{b-}b^- \,. \tag{5.48}$$

Note that from the d-dimensional point of view, $\eta^\pm = \frac{1}{2}\left(1 \pm \gamma^d\right)\eta$ are Weyl spinors of opposite chirality when d is even. For d odd, both are d-dimensional Dirac spinors. We may additionally choose a particular set of γ matrices such that

$$\gamma^0 a^\pm = \pm a^\mp \,, \quad \gamma^0 b^\pm = \mp b^\mp \,, \quad \gamma^1 a^\pm = \pm i\, b^\mp \,, \quad \gamma^1 b^\pm = \pm i\, a^\mp \,, \tag{5.49}$$

[3]From here on, all vector-like indices are Lorentz indices, implicitly using appropriate vielbeins.

which is all compatible with the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, to reduce (5.46) to four equations for the spinors $\eta^{a\pm}$ and $\eta^{b\pm}$.

To get the retarded real-time correlator we need to solve (5.46) using ingoing boundary conditions at the horizon [226, 227]. For imposing these on our solutions, we need to look at the most singular part of (5.46) close to the horizon, where we may restrict ourselves to the + eigenspace with respect to $i\gamma^1\gamma^2$ (the − eigenspace goes analogously). Close to the horizon the solutions $\eta^{a\pm}$ are thus required to behave as

$$\eta^{a\pm} \sim (r - r_0)^{-\frac{1}{4} - \frac{i\omega}{4\pi T}} \, \eta_0^{a\pm} \, , \quad \eta_0^{a+} = \eta_0^{a-} \, . \tag{5.50}$$

Since we are going to use the Kubo formulae (5.15) (5.17) to determine the supersound diffusion constant later on, in which we need to take the limit of small frequencies and momenta, we may directly set $\omega = k = 0$ in (5.46) and straightforwardly integrate the equations for components $\eta^\pm$,

$$\eta^\pm = c^\pm f^{-1/4} r^{-\frac{d-1}{2}} \left(r^{d/2} + \sqrt{r^d - r_0^d} \right)^{\pm \frac{d-1}{d}} \, . \tag{5.51}$$

Using (5.50), we may derive the relation $c^+ = r_0^{1-d} c^-$ between the integration constants.

At the boundary, the given solutions (5.51) have the asymptotic behaviour

$$\eta^+ \sim c^+ L^{1/2} \left(\frac{2}{2^{1/d}} \right) r^{-1/2} \equiv \phi \, r^{-1/2} \, , \tag{5.52}$$

$$\eta^- \sim c^- L^{1/2} \left(\frac{2^{1/d}}{2} \right) r^{1/2-d} \equiv \chi \, r^{1/2-d} \, . \tag{5.53}$$

Clearly, the first term is a source term which couples to an operator in the boundary conformal field theory of conformal dimension $\Delta = \frac{1}{2}d + |mL|$ [183, 184] and the second is related to the operator's expectation value. At the boundary we further find $\chi = r_0^{d-1} 2^{2/d-2} \phi$.

We now insert this asymptotic behaviour into the boundary term of the gravitino [183, 184] (for spin 1/2 fermions see [181, 182]) to compute the Green's function of the dual operators. We do not need to worry about holographic renormalisation [330] here, since in the limit $\omega, k \to 0$ there are no divergences; the first covariant counterterm has to be inserted at order $O(k^\mu)$ [107]. We have

$$S_{\text{bdy}} = \frac{4}{2\kappa_{d+1}^2} \int d^d x \sqrt{-h} \, \bar{\Psi}_i h^{ij} \Psi_j \, . \tag{5.54}$$

The normalisation of this boundary term was fixed in [107] (for the AdS_5 case) by using the $T = 0$ superspace correlators in the boundary field theory. Here we proceed analogously, which will also explain the non-standard kinetic term chosen in (5.8). The supersymmetry algebra in d space-time dimensions without central extensions has the following form [320]

$$\left\{ Q, \bar{Q} \right\} = 2\gamma^\mu P_\mu \, , \tag{5.55}$$

with appropriate chiral projections $\frac{1}{2}\left(1 \pm \gamma^d\right)$ applied in even dimensions, e.g. giving $\left\{ Q_\alpha, \bar{Q}_{\dot{\beta}} \right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu$ in 4 dimensions for Weyl spinors Q_α and $\bar{Q}_{\dot{\beta}}$.

The commutation relations extend to the supercurrent multiplet which includes the energy momentum tensor and the supersymmetry current. The same applies to the R-symmetry current whose contributions will not be of importance for our argument and which we therefore suppress in the following. The commutation relations are given by

$$\left\{Q,\bar{S}_\mu\right\} = 2\gamma^\nu T_{\mu\nu}\,, \tag{5.56}$$

$$\left[\bar{Q},T_{\mu\nu}\right] = -\frac{i}{8}\partial^\rho \bar{S}_\mu\left(\gamma_\nu\gamma_\rho-\gamma_\rho\gamma_\nu\right) - \frac{i}{8}\partial^\rho \bar{S}_\nu\left(\gamma_\mu\gamma_\rho-\gamma_\rho\gamma_\mu\right)\,, \tag{5.57}$$

again with chiral projectors implicitly assumed in even dimensions (for four dimensions see [331]). We may use these to relate the two-point function of the energy-momentum tensor [332, 333] to the two-point function of the supersymmetry current [183]. The former is given by

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{C_d}{x^{2d}}\mathcal{I}_{\mu\nu,\rho\sigma}(x)\,, \tag{5.58}$$

where the normalisation in d dimensions is

$$C_d = \frac{1}{2\kappa^2_{d+1}}\frac{2d(d+1)}{(d-1)}\frac{\Gamma(d)}{\pi^{d/2}\Gamma(d/2)}\,. \tag{5.59}$$

Furthermore, for $J_{\rho\nu}(x) = \left(\eta_{\rho\nu} - 2\frac{x_\rho x_\nu}{x^2}\right)$ the tensor

$$\mathcal{I}_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}\left(J_{\mu\rho}J_{\nu\sigma}+J_{\mu\sigma}J_{\nu\rho}\right) - \frac{1}{d}\eta_{\mu\nu}\eta_{\rho\sigma} \tag{5.60}$$

represents the inversion $x_\mu \to \frac{x_\mu}{x^2}$ on symmetric traceless tensors [169]. The supersymmetry current correlator on the other hand can be written as

$$\left\langle S^+_\mu(x)\bar{S}^-_\nu(0)\right\rangle = 2a\left(\frac{\kappa^2_{d+1}}{d+1}C_d\right)\left(\delta^\rho_\mu - \frac{1}{d}\gamma_\mu\gamma^\rho\right)\frac{\gamma^\sigma x_\sigma}{x^{2d}}J_{\rho\nu}(x)\,, \tag{5.61}$$

where $S^\pm_\mu = \frac{1}{2}(1\pm\gamma^d)S_\mu$ and a represents the normalisation of the boundary action

$$S_{\text{bdy}} = a\int d^dx\sqrt{-h}\,\bar{\Psi}_i h^{ij}\Psi_j \tag{5.62}$$

we would like to determine. Using the given part of the higher-dimensional supersymmetry algebra (5.56) (5.57) and the fact that in a supersymmetric ground-state $Q|0\rangle = \bar{Q}|0\rangle = 0$, we may relate the two correlators to one another and by this determine the normalisation to be[4]

$$a = \frac{4}{2\kappa^2_{d+1}}\,. \tag{5.63}$$

This is essentially independent of the dimension and the possible Weyl character of the supercharges in even dimensions. Having fixed the boundary term normalisation, the

[4]In [107] the normalisation is stated as $\mathcal{N} = \frac{N_c^2}{\pi^2} = \frac{8}{2\kappa_5^2}$ for the AdS radius set to $L=1$. This different factor of 2 compared to (5.63) for $d=4$ seems to be compensated by the fact that $\eta_2 = \Psi_2 - \frac{1}{2}\gamma_2\left(\gamma^2\Psi_2+\gamma^3\Psi_3\right)$ and $\eta_3 = \Psi_3 - \frac{1}{2}\gamma_3\left(\gamma^2\Psi_2+\gamma^3\Psi_3\right)$ in $d=4$ are essentially the same by construction due to the vanishing spin 1/2 part identity $\gamma^2\eta_2+\gamma^3\eta_3=0$. Therefore, our results in the end nevertheless agree with [107].

normalisation of the kinetic term is not free any more although it plays no role in the solution to the equation of motion. As in the case of the Gibbons-Hawking boundary term, we may also argue the gravitino boundary term (5.54) to be present for a well defined variational principle. This imposes a relative normalisation between (5.54) and (5.44) [334, 335] which results in the specific normalisation used in (5.8) for one of the transverse gravitini.

With this normalisation, we may now finally calculate the retarded Green's function of the transverse supersymmetry current operator, which is dual to η,

$$G^R = \frac{4i}{2\kappa_{d+1}^2}\left(\frac{r_0}{L}\right)^{d-1} \operatorname{diag}\left(\frac{2^{2/d}}{4}, \ldots, \frac{2^{2/d}}{4}\right). \tag{5.64}$$

We now proceed by using the d-dimensional Kubo formula (5.15) (5.17) for the supersound diffusion constant to get

$$\epsilon D_{3/2} = \frac{1}{\kappa_{d+1}^2}\left(\frac{r_0}{L}\right)^{d-1} \frac{2^{2/d}}{2}. \tag{5.65}$$

Note that in the end formula it does not matter if we imposed a Weyl constraint in even dimensions. If so, the Green's function would have half as many entries but this would be compensated by an additional factor of two in the Kubo formula (5.15).

Using the relation between D_s and $D_{3/2}$ (5.14), the field theory's equilibrium energy density $\epsilon = M/V_{\|}$ and temperature T (5.32), we arrive at

$$2\pi T D_s = \frac{2^{2/d} d(d-2)}{2(d-1)^2}, \tag{5.66}$$

which agrees with (5.43).

5.4 Supersound diffusion constant from the phonino pole

We now determine the phonino dispersion relation

$$\omega = v_s k - i D_s k^2 \tag{5.67}$$

from the pole of the longitudinal supersymmetry current correlator, closely following and generalising the computation of [106] to d dimensions. This not only reproduces the already established result for the diffusion constant D_s (5.43) (5.66) by a further computation, but also adds additional confidence to the result since, along the way, it determines the value of the supersound velocity v_s, which agrees with the one dictated by conformal invariance $v_s = \frac{P}{\epsilon} = \frac{1}{d-1}$.

For determining the dispersion relation we need to solve part of the full set of the gauge-fixed equations of motion $(\not{D} + m)\Psi_\mu = 0$ on the gravity side to linear order in ω and k, using ingoing boundary conditions at the horizon. We then need to read off the source terms of the dual CFT operators in the expansion data at the AdS boundary from which we can then easily extract the pole of the supersymmetry current correlator.

Again we assume our gravitino to be a Dirac vector-spinor. Imposing e.g. an additional Majorana constraint in certain dimensions would basically just restrict the integration constants we encounter in the following to be real. This has no influence on the pole structure so is irrelevant for our argument.

The equations of motion are given by (again, all vector-like indices are Lorentz indices from now on, after transforming with appropriate vielbeins)

$$\begin{aligned} 0 &= \gamma^d \Psi_0' - \frac{i\omega}{f}\gamma^0\Psi_0 - \frac{f'}{2f}\gamma^0\Psi_d + \frac{f'}{4f}\gamma^d\Psi_0 \\ &\quad + \frac{ikL}{r\sqrt{f}}\gamma^1\Psi_0 + \frac{d-1}{2r}\gamma^d\Psi_0 + \frac{m}{\sqrt{f}}\Psi_0\,, \end{aligned} \tag{5.68a}$$

$$\begin{aligned} 0 &= \gamma^d \Psi_d' - \frac{i\omega}{f}\gamma^0\Psi_d - \frac{f'}{2f}\gamma^0\Psi_0 + \frac{f'}{4f}\gamma^d\Psi_d \\ &\quad + \frac{ikL}{r\sqrt{f}}\gamma^1\Psi_d + \frac{1}{r}\left(\frac{d+1}{2}\gamma^d\Psi_d + \gamma^0\Psi_0\right) + \frac{m}{\sqrt{f}}\Psi_d\,, \end{aligned} \tag{5.68b}$$

$$\begin{aligned} 0 &= \gamma^d \Psi_j' - \frac{i\omega}{f}\gamma^0\Psi_j + \frac{f'}{4f}\gamma^d\Psi_j + \frac{ikL}{r\sqrt{f}}\gamma^1\Psi_j \\ &\quad + \frac{1}{r}\gamma^j\Psi_d + \frac{d-1}{2r}\gamma^d\Psi_j + \frac{m}{\sqrt{f}}\Psi_j\,, \end{aligned} \tag{5.68c}$$

for $j = 1, \ldots, d-1$, where we will use that the massless gravitino in AdS_{d+1} has $mL = \frac{d-1}{2}$ as argued e.g. in [310, 312] following [336]. A relation, which will turn out very useful for decoupling the different components of (5.68) is derived by using the gauge condition $\gamma^\mu\Psi_\mu = 0$, taking its radial derivative and using the equations of motion given above. The calculation then gives the following constraint equation

$$\begin{aligned} 0 = &\left(\frac{f'}{2f}\gamma^d - \frac{2i\omega}{f}\gamma^0 + \frac{2ikL}{r\sqrt{f}}\gamma^1 - \frac{2m}{\sqrt{f}} + \frac{d-2}{r}\gamma^d\right)\gamma^d\Psi_d \\ &+ \frac{2ikL}{r\sqrt{f}}\Psi_1 + \left(\frac{f'}{2f}\gamma^d - \frac{2i\omega}{f}\gamma^0 - \frac{1}{r}\gamma^d\right)\gamma^0\Psi_0\,, \end{aligned} \tag{5.69}$$

which we will make frequent use of throughout the calculation.

We now start to solve the equations of motion in the hydrodynamical limit. Therefore, we expand the gravitino to first order in ω and k and solve perturbatively in these quantities:

$$\Psi_\mu = \psi_\mu + \omega\varphi_\mu + k\chi_\mu\,. \tag{5.70}$$

Let us start with the lowest order terms, where we can basically set $\omega = k = 0$ in (5.68) and (5.69). At this order the equation for Ψ_d is diagonal:

$$0 = \psi_d' + \left(\frac{3f'}{4f} + \frac{3(d-1)}{2r} - \frac{m}{\sqrt{f}}\gamma^d\right)\psi_d\,. \tag{5.71}$$

Similarly, we get

$$\psi_0' + \left(\frac{f'}{4f} + \frac{d-1}{2r} + \frac{m}{\sqrt{f}}\gamma^d\right)\psi_0 = -\frac{f'}{2f}\gamma^0\gamma^d\psi_d\,, \tag{5.72}$$

$$\psi_1' + \left(\frac{f'}{4f} + \frac{d-1}{2r} + \frac{m}{\sqrt{f}}\gamma^d\right)\psi_1 = \frac{1}{r}\gamma^1\gamma^d\psi_d\,. \tag{5.73}$$

The equation for ψ_d can be integrated directly after decomposing it analogously to (5.48) while the other ones can be solved by the method of integrating factors given the solution for ψ_d and the action of γ^0 and γ^1 on the eigenspinors $a^\pm$ and $b^\pm$ given earlier (5.47). Let us denote integration constants by a_i, b_i, c_i, d_i when integrating the zero'th order $a^\pm$, $b^\pm$ parts of the gravitino component ψ_i. In the following we use the notation $\psi_\mu = (\psi_\mu^{a+}, \psi_\mu^{a-}, \psi_\mu^{b+}, \psi_\mu^{b-})^T$.

Near the horizon we find up to $O(r-r_0)^{-1/4}$

$$\psi_d = \frac{L^{3/2} r_0^{d-3}}{(d r_0)^{3/4}} \begin{pmatrix} a_d \\ a_d \\ c_d \\ -c_d \end{pmatrix} (r-r_0)^{-3/4}\,, \tag{5.74}$$

where we have already imposed ingoing boundary conditions $\propto (r-r_0)^{-\frac{i\omega}{4\pi T}}$ at the horizon which translate into $a_d = b_d$ and $c_d = -d_d$. It will be exactly this condition for most of the other functions at all orders in ω and k, so we will not explicitly state the near-horizon analysis any more. Similar considerations for example yield

$$\psi_0 = \frac{L^{3/2} r_0^{3-d}}{(d r_0)^{3/4}} \begin{pmatrix} a_d \\ a_d \\ c_d \\ -c_d \end{pmatrix} (r-r_0)^{-3/4}\,, \tag{5.75}$$

$$\psi_1 = -i\, \frac{\sqrt{L}\, r_0^{-1/4-d}}{d^{5/4}\,(d-1)} \begin{pmatrix} 2\,(d-1)\,L\,r_0^2\,c_d - d\,r_0^d\,c_\Sigma \\ 2\,(d-1)\,L\,r_0^2\,c_d - d\,r_0^d\,c_\Sigma \\ -2\,(d-1)\,L\,r_0^2\,a_d + d\,r_0^d\,a_\Sigma \\ 2\,(d-1)\,L\,r_0^2\,a_d - d\,r_0^d\,a_\Sigma \end{pmatrix} (r-r_0)^{-1/4}\,, \tag{5.76}$$

where the matching of integration constants in ψ_d and ψ_0 is due to the constraint equation (5.69), and the integration constants a_1, c_1 for ψ_1 are written in a way to match a convenient notation that will be explained in more detail below.

The interesting AdS boundary behaviour is given by

$$\begin{pmatrix} \psi_0^{a-} \\ \psi_0^{b-} \\ \psi_1^{a-} \\ \psi_1^{b-} \end{pmatrix} = \frac{2^{(d-1)/d}\, i\,\sqrt{L}}{d-1} \begin{pmatrix} 0 \\ 0 \\ c_\Sigma \\ a_\Sigma \end{pmatrix} r^{-1/2}\,, \tag{5.77}$$

where we have only shown the source terms.

So, indeed, we do find sources for our CFT operators. In particular, it is interesting to see that there are no source terms in the time component of the gravitino, only in the longitudinal ones.

We must now proceed to the terms linear in ω and k. The strategy is exactly the same as before. We decouple the equations of motion and integrate them using the solutions given earlier by the method of integrating factors. We then obtain solutions on which we must impose ingoing boundary conditions at the horizon. Then we read off the source terms near the boundary. Although conceptionally this does not pose problems any more and it is indeed possible to decouple the equations up to a non-homogeneous, explicitly known part, the actual expressions for the integrated solutions become very complicated.

We now introduce another object which will be very handy in the actual computation of the source terms. Let us define

$$\Psi_\Sigma = (d-2)\,\gamma^1\Psi_1 - \gamma^2\Psi_2 - \ldots - \gamma^{d-1}\Psi_{d-1}\,. \tag{5.78}$$

Looking at the equations of motion (5.68), we see that its equation of motion partly decouples from the other gravitino components:

$$\begin{aligned} 0 = \Psi_\Sigma' + \frac{i\omega}{f}\gamma^0\gamma^d\Psi_\Sigma - \frac{ikL}{r\sqrt{f}}\gamma^d\left(2(d-2)\Psi_1 - \gamma^1\Psi_\Sigma\right) \\ + \frac{f'}{4f}\Psi_\Sigma + \frac{(d-1)}{2r}\Psi_\Sigma - \frac{m}{\sqrt{f}}\gamma^d\Psi_\Sigma \end{aligned} \tag{5.79}$$

We may use the solution to this field together with the gauge condition $\gamma^\mu\Psi_\mu = 0$ to solve for Ψ_1 which contains our longitudinal source terms,

$$\Psi_1 = \frac{1}{(d-1)}\left(\gamma^1\Psi_\Sigma - \gamma^1\gamma^0\Psi_0 - \gamma^1\gamma^d\Psi_d\right)\,. \tag{5.80}$$

These steps seem to be necessary computationally since the direct analytic integration of the equations of motion for φ_1 and χ_1 appears to be far too complicated even for computer algebra.

Let us begin with the calculation to first order in ω. What do the relevant equations of motion for φ_μ now look like? They are given by

$$\varphi_d' + \left(\frac{3f'}{4f} + \frac{3(d-1)}{2r} - \frac{m}{\sqrt{f}}\gamma^d\right)\varphi_d = \frac{i}{f}\gamma^0\gamma^d\psi_d - \frac{2i}{f}\psi_0\,, \tag{5.81}$$

$$\varphi_\Sigma' + \left(\frac{f'}{4f} + \frac{d-1}{2r} - \frac{m}{\sqrt{f}}\gamma^d\right)\varphi_\Sigma = -\frac{i}{f}\gamma^0\gamma^d\psi_\Sigma\,, \tag{5.82}$$

which can straightforwardly be solved by the method of integrating factors since the solutions for ψ_d, ψ_0 and ψ_Σ are known. Imposing ingoing boundary conditions at the horizon then gives a similar structure on the integration constants as before and we can read off the source terms after calculating φ_0 from (5.69) and φ_1 from (5.80). They are given by

$$\begin{pmatrix} \varphi_0^{a-} \\ \varphi_0^{b-} \\ \varphi_1^{a-} \\ \varphi_1^{b-} \end{pmatrix} = -\frac{2^{(3d+1)/d}\,L^{7/2}\,r_0^{1-d}}{d^2}\begin{pmatrix} -i\,(d-1)\,a_1 \\ i\,(d-1)\,c_1 \\ c_1 \\ a_1 \end{pmatrix} r^{-1/2}\,. \tag{5.83}$$

Now, the calculations for χ_μ are in spirit similar to the ones before. Computationally however, they are more difficult. As before, we will only give the starting point and the results. The equations of motion, which have to be solved are

$$\chi_d + \left(\frac{3f'}{4f} + \frac{3(d-1)}{2r} - \frac{m}{\sqrt{f}}\gamma^d\right)\chi_d = -\frac{iL}{r\sqrt{f}}\left(2\psi_1 + \gamma^1\gamma^d\psi_d\right)\,, \tag{5.84}$$

$$\chi_\Sigma' + \left(\frac{f'}{4f} + \frac{d-1}{2r} - \frac{m}{\sqrt{f}}\gamma^d\right)\chi_\Sigma = \frac{iL}{r\sqrt{f}}\gamma^d\left(-\gamma^1\psi_\Sigma + 2(d-2)\psi_1\right)\,. \tag{5.85}$$

One more time, we solve by using integrating factors, impose ingoing boundary conditions, calculate χ_0 and χ_1 using (5.69) and (5.80) to finally obtain

$$\begin{pmatrix} \chi_0^{a-} \\ \chi_0^{b-} \\ \chi_1^{a-} \\ \chi_1^{b-} \end{pmatrix} = \frac{2^{(d+1)/d}\, L^{5/2}\, r_0^{1-d}}{(d-1)\, d^2} \begin{pmatrix} 4\,(d-1)\, L\, c_d - d\, r_0^{d-2}\, c_\Sigma \\ 4\,(d-1)\, L\, a_d - d\, r_0^{d-2}\, a_\Sigma \\ -4\, i\,(d-1)^2\, L\, a_d \\ 4\, i\,(d-1)^2\, L\, c_d \end{pmatrix} r^{-1/2}\,. \tag{5.86}$$

In fact, we also get source terms involving additional a_Σ and c_Σ dependent terms and new integration constants. When summing up all contributions to get the source terms of the full Ψ_0 and Ψ_1, we can, as noted in Policastro's paper [106], redefine our integration constants to have ω and k dependent terms. So basically, we rename $a_\Sigma \to a'_\Sigma = a_\Sigma + \alpha\, \omega + \beta\, k$ and similarly for c_Σ since these are the integration constants at lowest order (5.77). We have already implicitly done this redefinition in the terms above and therefore only given the really relevant part of the solution.

In total, we now obtain a linear relation between Ψ_0, Ψ_1 and their boundary values of the form

$$\begin{pmatrix} \Psi_0^{a-} \\ \Psi_0^{b-} \\ \Psi_1^{a-} \\ \Psi_1^{b-} \end{pmatrix} = \mathcal{M} \begin{pmatrix} a_d \\ a_\Sigma \\ c_d \\ c_\Sigma \end{pmatrix} r^{-1/2}\,, \tag{5.87}$$

for a matrix $\mathcal{M}$ that is determined by the source terms computed in the previous two subsections. We now plan to look for non-trivial solutions to this relation, where the boundary values have poles. These poles will then show up in the CFT Green's functions and therefore may be interpreted as the phonino poles [103]. We have to compute the determinant of $\mathcal{M}$, substituting the relation

$$\omega = v_s k - i D_s k^2\,. \tag{5.88}$$

We then set the determinant to zero, and solve for v_s and D_s, to finally get

$$v_s = \frac{1}{d-1}\,, \qquad 2\pi T D_s = \frac{2^{2/d} d(d-2)}{2(d-1)^2}\,. \tag{5.89}$$

The value of the supersound velocity is the one expected from conformal invariance in d dimensions (square of the normal sound velocity). There is no imaginary part of D_s appearing. Of course, the four dimensional result, $v_s = \frac{1}{3}$ and $2\pi T D_s = \frac{4}{9}\sqrt{2}$, is exactly reproduced as for $d = 4$ our computation is just a slightly different coordinate version of [106]. It furthermore exactly agrees with (5.43) and (5.66).

5.5 Generalised dimensional reduction and Dp-branes

Before, we have derived the supersound velocity and diffusion constant for the case of asymptotically AdS black branes (5.31). However, it is not directly clear how these are related to a string or M theory background. Of course, their boundary field theories are classically conformal as, for instance, can be seen from the dimension dependence of the supersound velocity (5.89)

$$v_s = \frac{P}{\epsilon} = \frac{1}{d-1} \quad \Leftrightarrow \quad T^\mu_\mu = -\epsilon + (d-1)P = 0\,. \tag{5.90}$$

However only some of the standard p-brane backgrounds of the low-energy limit of type II supergravity and M theory, namely near-horizon D3-, M2- and M5-branes, have boundary CFTs. Since for some of the other Dp-brane backgrounds the near-horizon limit is also a decoupling limit [178] and holographic renormalisation has also been established for these [337, 338], we may wonder if it is also possible to derive the supersound diffusion constants for these backgrounds. Actually, from the conformal backgrounds (5.31) we may already learn a lot about the hydrodynamics of the field theories dual to Dp-brane backgrounds via what has been called 'generalised dimensional reduction' [177, 339]. The idea is the following:

Given the standard background of the near-horizon limit of a near-extremal p-brane relevant for string theory [161], say in string frame, one can perform a Weyl transformation to the so-called dual frame [340]. In this frame, for $p \neq 5$ the metric becomes $AdS_{p+2} \times S^{8-p}$. However, the transformation comes with the price of a non-trivial coupling of the (running) dilaton to the Einstein-Hilbert term of the action. When reducing on the sphere, we have

$$S = -L \int d^{d+1}x \, \sqrt{g} \, e^{\gamma\phi} \left[R + \beta \, (\partial\phi)^2 + C \right] \tag{5.91}$$

for some constants L, γ, β, C which depend on the worldvolume dimension of the p-brane (for concrete expressions see [338]). This action may also be obtained by dimensionally reducing $(2\sigma + 1)$-dimensional pure Einstein gravity with cosmological constant on a $(2\sigma - d)$-dimensional torus. In this reduction the dimensions are formally given in terms of a quantity σ which is defined as

$$\sigma = \frac{d}{2} - \frac{(p-3)^2}{2(p-5)} \, . \tag{5.92}$$

Note that e.g. for $p = 3$, σ is half-integer and so the 'higher-dimensional' pure gravity theory just has dimension five, Einstein gravity on AdS$_5$. For $p \neq 3$ however, σ is not necessarily half-integer any more. It has been shown in [338] however that holographic renormalisation does also make sense for arbitrary, in particular non-half-integer values of σ! From the point of view of the lower-dimensional theory's equations of motion, this is just a parameter which happens to have some meaning for the special cases of Dp-branes. Therefore, the transformation to asymptotically AdS spaces, consistent dimensional reduction and analytic continuation in σ allow determining the hydrodynamics of non-conformal branes completely by just formally looking at conformal hydrodynamics in higher dimensional AdS theories [338].

This is what we will exploit now to determine the supersound velocity v_s and diffusion constant D_s for the non-conformal Dp-branes. Since $v_s = \frac{P}{\epsilon}$, its value is already given in [177]:

$$v_s = \frac{1}{2\sigma - 1} \tag{5.93}$$

with σ given in (5.92). For D_s, we should, however, take the upcoming results with some care since for fermions the generalised dimensional reduction has so far not been worked out. We will nevertheless give some support in favour of this procedure.

The black brane space-times given earlier (5.31) are exactly the interesting class of solutions to Einstein's equations for the higher dimensional theory which is reduced on some torus. We can thus rewrite our results obtained earlier (5.43) (5.66) (5.89) by setting $d = 2\sigma$ (so that the whole bulk dimension is $2\sigma + 1 = d + 1$). We then express σ in terms of the dimension of the p-brane (5.92) and get

$$2\pi T D_s = 4 \times 2^{\frac{5-p}{7-p}} \frac{(7-p)}{(9-p)^2} \,. \tag{5.94}$$

One may also apply the methods of section 5.2: Given the near-horizon Einstein-frame metric of a Dp-brane (see e.g. [249]) with $f(r) = 1 - \left(\frac{r_0}{r}\right)^{7-p}$

$$\begin{aligned} ds_E^2 = {} & \left(\frac{r}{L}\right)^{\frac{(7-p)^2}{8}} \left(-f(r)dt^2 + dx_1^2 + \ldots + dx_p^2\right) \\ & + \left(\frac{L}{r}\right)^{\frac{(7-p)(p+1)}{8}} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_{8-p}^2\right) \end{aligned} \tag{5.95}$$

we may again use (5.5) after transforming the metric (5.95), which involves a $(8-p)$-sphere, into the form of (5.4) to arrive at

$$\frac{\sigma}{A} = \frac{1}{4} 2^{\frac{5-p}{7-p}} \,. \tag{5.96}$$

This agrees with what one would get from taking (5.33) or (5.42) and applying the generalised dimensional reduction procedure explained above. The energy densities which appear in the Kubo formulae also behave according to generalised dimensional reduction [338, 177]. Therefore, we trust our naive result (5.94) although, as we again emphasise, generalised dimensional reduction has not been properly worked out for fermions.

Of course, it would be interesting to independently check these Dp-brane results along the lines of sections 5.3 and 5.4.

5.6 Conclusion

In this chapter, we have explicitly computed the supersound diffusion constant D_s for various strongly coupled supersymmetric field theories in arbitrary dimension with specific holographic duals. Furthermore, we have connected the closely related quantity $D_{3/2}$ which appears in the constitutive relation of the field theory's supersymmetry current to a universal fermionic absorption cross section result. In essence, this relation is very similar to the connection between shear viscosity and the universal scalar absorption cross section result in the dual space-time. In the latter case the relation has a direct field theoretical interpretation since the absorption cross section is given by the area of the horizon, which determines the entropy density of the field theory. In our case, the fermionic absorption cross section is also related to the area of a horizon, however, not in the original but in a conformally related spatially flat space-time. Therefore, the universal interpretation is not quite as straightforward as for η/s, however still very striking.

Clearly, the central questions which follow from our results relate to the universal result for the transport coefficient $D_{3/2}$, namely its range of applicability and possibly its significance towards real world systems, perhaps in the context of condensed matter or phenomenology, taking into account the interpretation of the setup [309, 313].

It has been shown that the universality for η/s holds for a very wide range of possible theories with Einstein gravity dual (note also the limitations of the universality in Gauss-Bonnet gravity [99] or for anisotropic setups [341, 342, 343]). In comparison, for $D_{3/2}$, from the very beginning, we restricted to *supersymmetric* field theories with (super-)gravity dual.

Furthermore, so far we ignored the possibility (or even necessity) of the coupling of our bulk gravitino to other fields like gauge fields, which could induce chemical potentials μ in the boundary theory. Generically, $D_{3/2}$ (or the supersound diffusion constant D_s) will depend on this $D_s = D_s\left(\frac{\mu}{T}\right)$ [310, 312, 107]. Our calculations all refer to the transport coefficients' values in the limit $\mu \ll T$. Of course we would like to move away from the strict limit $\mu = 0$ and study if for non-vanishing μ the shown universality still holds. So far we cannot tell if this is possible, but it is conceivable that the methods used (especially the transverse gravitino calculation in section 5.3) are applicable for (at least) a perturbative treatment in μ/T.

For checking other limitations of the universality one could, as for η/s, break the rotational symmetry of the boundary field theory as in [341] which would probably induce a temperature dependent deviation from the universal relation.

Another point that deserves attention is that in general fermions are not minimally coupled, but their equations of motion often involve Pauli terms which we have so far ignored. Without those, we may expect some universality to remain present. However, this is not the most generic situation. The fermionic universal absorption cross section result [105] is given for minimally coupled fermions in spherically symmetric backgrounds (in arbitrary dimension) and we are not aware of direct extensions to non-minimally coupled fermions or minimally coupled fermions in rotating backgrounds. Such an extension would be important for studying e.g. the numerical results for $\mu \neq 0$ in [107] along the lines of section 5.2 using consistent truncation arguments after embedding the setup into higher dimensional rotating backgrounds [241]. Absorption cross sections of fermions in charged black hole backgrounds in four and five dimensions which do couple via Pauli terms have been studied in [344, 345]. On more general grounds, it might also be worth to study these cases from a purely gravitational point of view and see if a universal result in the form of [105] may be extracted.

In using [105], we have been somewhat generous in using the distinction between black holes and black branes. We implicitly assumed and to some degree checked that the fermionic results of [105] do hold also for the given branes. This is of course physically well motivated, but it is worth to examine this issue closer, as done for the scalar case in [238]. Actually, within our computations the contribution from additional extended brane dimensions seems to serve as the UV cutoff which we manually inserted at several places in our calculations. However [238] is more general in showing which space-times have low energy s-wave scalar absorption cross sections $\sigma_{\text{abs},0}(0) = A$ and how this may be modified for others.

For non-conformal backgrounds, it would be interesting to also have some independent direct confirmation of the result (5.94) from an explicit computation along the lines of sections 5.3 or 5.4. A Kubo formula for $D_{1/2}$ might be derived, so that we would be able to not only compute D_s but also D_σ for the non-conformal backgrounds.

It also seems possible to relate the present work to a supersymmetric extension of the fluid/gravity correspondence [109]. Initial progress in this direction for the BTZ black hole has been obtained in [346].

Furthermore, the general appearance of the phonino excitation as the Goldstone fermion of supersymmetry breaking by temperature, which was first observed within holography in [106], was given some more evidence in other dimensions. As done for the Wess-Zumino model [103] and SQED [104] it could be interesting to study this effect in other models, maybe even in simple supergravity models. For possible phenomenological implications we refer to [313].

Chapter 6

Conclusions

In this thesis we have pursued different promising approaches for a deeper understanding of the hydrodynamical limit of gauge/gravity duality.

Firstly, we have constructed a diffusion constant in a fermionic context which possesses very similar properties as the shear viscosity. Similar to the shear viscosity the supersound diffusion constant is related to a universal absorption cross section result. The various explicit computations round off a consistent picture of this result.

On first sight however there is also a difference compared to the shear viscosity case. The fermionic absorption cross section does not really have a satisfactory thermodynamic interpretation as the scalar absorption cross section. This unfortunately prevented us from going a further step similar to dividing the shear viscosity by the entropy density. But maybe one can find a deeper understanding of the fermionic gravitational absorption cross section using a map like the AdS/Ricci-flat correspondence [255, 256] or generalisations thereof.

In any case the obtained result is striking and a clear step forward to finding other universal quantities which one may derive in strongly coupled field theories with a gravity dual. In particular the fermionic nature of the computed correlator is promising with regard to condensed matter applications.

Also the blackfold analysis opens up new directions for further research. First of all, it is the most general approach for relating gravitational physics and hydrodynamics to one another and already this makes it worth studying. But the extrinsic blackfold dynamics are a further avenue which could potentially have a lot of impact on areas of physics beyond high-energy/gravitational physics.

In our work we have focussed on a concrete generalisation of the fluid/gravity analysis for charged fluids, in which many ideas come together: Most directly, it extends the simplest anomaly related transport phenomenon to this very general blackfold context. An interpolation between the various previously understood regions which relate gravtational physics to hydrodynamics, i.e. the membrane paradigm, fluid/gravity and even asymptotically flat space, is possible. The method we have used, namely implementing the Dirichlet problem at a finite radial cutoff, is furthermore a technically promising direction which sheds light on ideas of the holographic Wilsonian renormalisation group and aspects of a putative flat space holography. Thus, it is worth pursuing for understanding these aspects better and hopefully implementing them more concretely.

For doing so, the most important aspect is certainly the understanding of flat space holographic renormalisation, which we have pointed to repeatedly. The counterterms which we have inserted certainly seem to play an important role, but its incorporation was rather heuristic. It would be very interesting to understand this better.

But even with the energy-momentum tensor description we have outlined, one may study the full cutoff dependence of the chiral vortical conductivity. In our elaborate analysis we have not only recovered this, but also the usual conductivity with its cutoff dependence.

In general we see that the study of holographic hydrodynamics is still a very active and interesting field. As impressive as the relation of shear viscosity divided by entropy density is – one would like to find more relations of this kind. Maybe there are more to be found e.g. in the studies with regard to condensed matter applications.

In any case, the AdS/CFT correspondence and its various limits has already unravelled many mysteries about strongly coupled field theories and has laid out a theoretical framework that unifies and relates many seemingly disparate concepts and ideas. The general interconnection of gravitational physics and field theories is striking in so many ways that it is reasonable to expect more to be uncovered with it – may it be in an experimental realisation of a universally predicted relation or in advance on a more theoretical ground like a deeper understanding of the black hole information paradox with its recent puzzles and/or quantum gravity in general.

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Chapter A

Source terms in the vector sector

Here we collect some of the quite non-trivial source terms which appear in section 4.5.

The first source term $S_{i,1}(r)$ is the one which appears in the Einstein equation (4.82). It is given by

$$
\begin{aligned}
S_{i,1}(r) = & -\left(\frac{H_R^{1/4}}{\sqrt{f_R g_R}}\right)\left(\frac{f(r)g(r)^{3/2}}{2r^7 H(r)^{3/2}}\right)\left(3L^4r^2+2q^2+5r^6\right)\beta_{i,v} \\
& -\left(\frac{H_R^{1/4}}{\sqrt{f_R g_R}}\right)\left(\frac{f(r)g(r)^{5/2}}{2r^{13} H(r)^{5/2}}\right) q\left(2q^2-r^2\left(3L^4+r^4\right)\right)q_i \\
& -\left(\frac{f_R g_R - H_R}{R^6 H_R^{3/4} f_R\sqrt{f_R g_R}}\right)\left(\frac{f(r)g(r)^{5/2}}{2r^{13} H(r)^{5/2}}\right) \\
& \quad \times q\left(q^2\left(5L^4r^2+7r^6\right)+r^4\left(3L^8+8L^4r^4+5r^8\right)+2q^4\right)q_i \\
& +\left(\frac{H_R^{1/4}}{\sqrt{f_R g_R}}\right)\frac{L^4 r_0^3 f(r)g(r)^{5/2}}{\left(2L^4+r_0^4\right)r^{11}H(r)^{5/2}}\left(5L^4r^2+10q^2+7r^6\right)r_{0,i} \\
& -\left(\frac{f_R L^4\left(2g_R q^2 - H_R R^6\right)+2H_R^2 R^6\left(2L^4+r_0^4\right)}{R^{10} f_R^{3/2} g_R^{1/2} H_R^{7/4}\left(2L^4+r_0^4\right)}\right)\left(\frac{r_0^3 f(r)g(r)^{5/2}}{2r^{13}H(r)^{5/2}}\right) \\
& \quad \times\left(q^2\left(5L^4r^2+7r^6\right)+r^4\left(3L^8+8L^4r^4+5r^8\right)+2q^4\right)r_{0,i}
\end{aligned}
$$

In the near-horizon limit (with $R \to \infty$) the given expression hugely simplifies to

$$
-\left(\frac{3r}{2L^2}\right) f(r)\beta_{i,v}\,. \tag{A.1}
$$

With the additional prefactors in (4.82), this then exactly reproduces the source term in the Einstein equation of [113].

The first Maxwell equation $\mathcal{M}_i = 0$ (4.83) includes the source term $S_{i,2}(r)$ which may actually be written as a total derivative

$$
\begin{aligned}
S_{i,2}(r) = \frac{d}{dr}\Bigg[& -\left(\frac{\sqrt{H_R}}{f_R g_R}\right)\frac{4\kappa L^4 q^2}{r^4}\epsilon_{ijk}\beta_{j,k} - \sqrt{\frac{H_R}{f_R g_R}}\sqrt{\frac{r^2 H(r)}{g(r)}}\sqrt{L^4+r_0^4}\,q\beta_{i,v} \\
& +\left(\frac{\sqrt{H_R}}{R^6 f_R^{3/2}\sqrt{g_R}}\right)\sqrt{\frac{r^2 H(r)}{g(r)}}\sqrt{L^4+r_0^4}\,q^2 q_i
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{\frac{g_R H_R}{f_R}}\sqrt{\frac{r^2 H(r)}{g(r)}}\sqrt{L^4+r_0^4}\,q_i \\
&-2\sqrt{\frac{H_R}{f_R g_R}}\sqrt{r^2 g(r) H(r)}\sqrt{L^4+r_0^4}q_i \\
&-2\left(\frac{\sqrt{g_R}}{R^{10}\sqrt{f_R}H_R^{3/2}}\right)\sqrt{\frac{r^2 H(r)}{g(r)}}\frac{\sqrt{L^4+r_0^4}}{\left(2L^4+r_0^4\right)}L^4 q^3 r_0^3\, r_{0,i} \\
&-2\sqrt{\frac{H_R}{f_R g_R}}\sqrt{\frac{g(r)}{r^{10}H(r)}}\left(3L^4 r^2+q^2+r^6\right)\frac{\sqrt{L^4+r_0^4}}{\left(2L^4+r_0^4\right)}q r_0^3\, r_{0,i} \\
&+\left(\frac{1}{R^4\sqrt{f_R g_R H_R}}\right)\sqrt{\frac{r^2 H(r)}{g(r)}}\frac{\sqrt{L^4+r_0^4}}{\left(2L^4+r_0^4\right)}L^4 q r_0^3\, r_{0,i} \\
&-2\left(\frac{\sqrt{H_R}}{R^4 f_R^{3/2}\sqrt{g_R}}\right)\sqrt{\frac{r^2 H(r)}{g(r)}}\sqrt{L^4+r_0^4}\,q r_0^3 r_{0,i}\Bigg]
\end{aligned}
$$

In this expression, one may notice the anomaly related term $\sim \epsilon_{ijk}\beta_{j,k}$. Its near-horizon, $R\to\infty$ limit is also quite simple:

$$
\frac{16\kappa L^4}{r^5}\,q^2\epsilon_{ijk}\beta_{j,k}+\frac{L^4}{r^2}\left(q\beta_{i,v}+q_i\right) \tag{A.2}
$$

and reproduces the terms in [113].

The second Maxwell equation (4.84) includes a source term $S_{i,3}(r)$, which may also be written as a total derivative

$$
\begin{aligned}
S_{i,3}(r)=\frac{d}{dr}\Bigg[&-4\left(\frac{\sqrt{H_R}}{f_R g_R}\right)\frac{g(r)}{r^8 H(r)}\kappa q^2\epsilon_{ijk}\beta_{j,k}-\sqrt{\frac{H_R}{f_R g_R}}\sqrt{\frac{g(r)}{r^6 H(r)}}\frac{q}{\sqrt{L^4+r_0^4}}\beta_{i,v} \\
&+\left(\frac{H_R-f_R g_R}{R^6 f_R^{3/2}\sqrt{g_R H_R}}\right)\sqrt{\frac{g(r)}{r^6 H(r)}}\frac{q^2}{\sqrt{L^4+r_0^4}}q_i \\
&-\sqrt{\frac{H_R}{f_R g_R}}\frac{g(r)^{3/2}}{r^9 H(r)^{3/2}}\left(r^2 g(r)\left(L^4+r_0^4\right)+q^2\right)\frac{1}{\sqrt{L^4+r_0^4}}q_i \\
&-2\left(\frac{\sqrt{g_R}}{R^{10}\sqrt{f_R}H_R^{3/2}}\right)\sqrt{\frac{g(r)}{r^6 H(r)}}\frac{L^4 q^3 r_0^3}{\sqrt{L^4+r_0^4}\left(2L^4+r_0^4\right)}r_{0,i} \\
&+\left(\frac{1}{R^4\sqrt{f_R g_R H_R}}\right)\sqrt{\frac{g(r)}{r^6 H(r)}}\frac{L^4 q r_0^3}{\sqrt{L^4+r_0^4}\left(2L^4+r_0^4\right)}r_{0,i} \\
&-2\left(\frac{\sqrt{H_R}}{R^4 f_R^{3/2}\sqrt{g_R}}\right)\sqrt{\frac{g(r)}{r^6 H(r)}}\frac{q r_0^3}{\sqrt{L^4+r_0^4}}r_{0,i} \\
&+2\sqrt{\frac{H_R}{f_R g_R}}\frac{g(r)^{3/2}}{r^7 H(r)^{3/2}}\frac{q r_0^3}{\sqrt{L^4+r_0^4}}r_{0,i} \\
&-2\sqrt{\frac{H_R}{f_R g_R}}\frac{g(r)^{5/2}}{r^{11}H(r)^{5/2}}\frac{L^4 q r_0^3\sqrt{L^4+r_0^4}}{\left(2L^4+r_0^4\right)}r_{0,i}\Bigg]\,.
\end{aligned}
$$

In the near-horizon limit, we reproduce the source term in near-horizon limit of the first Maxwell equation, however with an additional factor of $1/L^8$.

From these source terms, we define two more source terms, which capture specific combinations and integrals of the above ones. We have already made clear, that $S_{i,2}$ and $S_{i,3}$ may be integrated. The expression one obtains like that also appears together with $S_{i,1}$ and $S_{i,2}$ in a way, which one may integrate even a further time,

$$S_{i,4}(r) = \sqrt{\frac{f_R g_R}{H_R}} \int \Bigg[-2 \left(\frac{H_R^{1/4} \sqrt{L^4 + r_0^4}}{q} \right) r^5 H(r)\, S_{i,1} - \left(\frac{f(r) g(r)}{r^2 H(r)} \right) S_{i,2}(r) \\ + \frac{4L^4 \left(L^4 + r_0^4\right)}{r^3} \int \left(\frac{S_{i,2}(r)}{2\, L^4 \left(L^4 + r_0^4\right)} + S_{i,3}(r) \right) dr \Bigg] \, dr$$

or explicitly

$$= \frac{4\kappa L^4}{r^6 \sqrt{f_R g_R}} q^2 \, \epsilon_{ijk} \beta_{j,k} + \left(\frac{g(r)^{3/2}}{r^{13} H(r)^{3/2}} \right) \left(2L^8 r^4 + 2L^4 q^2 r^2 + 4L^4 r^8 \\ - q^4 + q^2 r^6 + q^2 r_0^4 r^2 + 2r^{12}\right) \sqrt{L^4 + r_0^4}\, q_i \\ - \left(\frac{g(r)^{3/2}}{r^{13} H(r)^{3/2}} \right) \left(2L^{12} q^2 r^4 + 2L^{12} r^{10} + 6L^{12} r_0^4 r^6 + 4L^8 q^4 r^2 + 8L^8 q^2 r^8 \\ + L^8 q^2 r_0^4 r^4 + 4L^8 r^{14} + 9L^8 r_0^4 r^{10} + 7L^8 r_0^8 r^6 + 2L^4 q^6 + 6L^4 q^4 r^6 \\ - 10L^4 q^4 r_0^4 r^2 + 6L^4 q^2 r^{12} - 8L^4 q^2 r_0^4 r^8 + 12L^4 q^2 r_0^8 r^4 + 2L^4 r^{18} \\ + 2L^4 r_0^4 r^{14} + 12L^4 r_0^8 r^{10} - 3q^6 r_0^4 - 7q^4 r_0^4 r^6 + 3q^4 r_0^8 r^2 \\ - 5q^2 r_0^4 r^{12} + 8q^2 r_0^8 r^8 - r_0^4 r^{18} + 5r_0^8 r^{14}\right) \frac{\sqrt{L^4 + r_0^4}}{q r_0 \left(2L^4 + r_0^4\right)} r_{0,i} \,.$$

For actually performing the integral, one should eliminate $\beta_{i,v}$ in the integrand first using (4.77).

Its near-horizon limit is given by

$$\frac{4\kappa L^4}{r^6} q^2 \, \epsilon_{ijk} \beta_{j,k} + \frac{L^4 \left(q^2 + r^6 + 3r_0^4 r^2\right)}{q r^3} \beta_{i,v} + \frac{2L^4}{r^3} q_i \,.$$

In a slightly different combination, it also comes up in the following integral, which one may perform analytically. For actually performing the integration it is advisable to use the constraint equation (4.77) to eliminate $\beta_{i,v}$ from the integrand.

$$S_{i,5}(r) = \int \Bigg[3 \left(\frac{\sqrt{f_R g_R}}{H_R^{1/4} q L^4 \sqrt{L^4 + r_0^4}} \right) \frac{r^{11} H(r)}{g(r)} S_{i,1} \\ + 3 \sqrt{\frac{f_R g_R}{H_R}} \left(\frac{3r^4 - r_0^4}{4L^4 \left(L^4 + r_0^4\right)} \right) S_{i,2}(r) + \frac{3\left(3r^4 - r_0^4\right)}{4L^4 \left(L^4 + r_0^4\right)} \frac{d}{dr} \left(r^2 S_{i,4}(r)\right) \Bigg] \, dr \,,$$

explicitly,

$$= -\frac{3g(r)^{3/2}}{4L^4 r^9 H(r)^{3/2} \sqrt{L^4 + r_0^4}} \left(r^6 H(r) \left(L^4 \left(r^4 + r_0^4\right) - r^8 + 3r_0^4 r^4\right) \\ + q^2 \left(2L^4 r_0^4 + 2q^2 r^2 + r^8 + r_0^8\right)\right) q_i$$

$$
\frac{3\left(\frac{g(r)}{H(r)}\right)^{3/2}}{4L^4 q r^9 r_0 \sqrt{L^4+r_0^4}\left(2L^4+r_0^4\right)}\left(L^{12}\left(4q^2r^6-2r^4\left(r^8+2r_0^4r^4-3r_0^8\right)\right)\right.
$$

$$
+L^8r^2\left(8q^4r^2+q^2\left(4r^8-2r_0^4r^4+8r_0^8\right)-4r^{14}-5r_0^4r^{10}+2r_0^8r^6+7r_0^{12}r^2\right)
$$

$$
+2L^4r^2\left(2q^6+3q^4r^2\left(r^4+r_0^4\right)+q^2r_0^4\left(3r^8-5r_0^4r^4+6r_0^8\right)\right)
$$

$$
-2L^4r^8\left(r^{12}+5r_0^8r^4-6r_0^{12}\right)+r_0^4\left(-2q^6r^2-3q^4\left(r^8-r_0^8\right)\right)
$$

$$
\left.+r_0^4\left(q^2\left(8r^6r_0^8-6r^{10}r_0^4\right)+\left(r^8-6r_0^4r^4+5r_0^8\right)r^{12}\right)\right)r_{0,i}
$$

This expression reduces to

$$
-\frac{3r\left(2q^2r^2-r^8-2r_0^4r^4+3r_0^8\right)}{4L^4q}\,\beta_{i,v}-\frac{3\left(r^4+r_0^4\right)}{4L^4r}\,q_i
$$

in the near-horizon limit.

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