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SOME REMARKS ON THE NAVIER-STOKES EQUATIONS WITH REGULARITY IN ONE DIRECTION

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Abstract. We review the developments of the regularity criteria for the Navier-Stokes equations, and make some further improvements.

Keywords: regularity criteria; Navier-Stokes equations

MSC 2010: 35B65, 35Q30, 76D03

1. Introduction

The homogeneous incompressible fluid flow is governed by the following Navier-Stokes equations (NSE):

$$
\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{1.1}
$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, π is a scalar pressure, $\boldsymbol{u}_0$ is the prescribed initial velocity field satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0 = 0$, and

$$
\partial_t \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad (\boldsymbol{u}\cdot\nabla) = \sum_{i=1}^{3} u_i \partial_i.
$$

The global existence of a weak solution to the evolutionary NSE (1.1) was established by Leray [12] and Hopf [8] long time ago. However, the issue of its regularity

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and uniqueness remains open up to now. Pioneered by Serrin [19], we began studying the regularity criterion for the NSE (1.1); that is, finding a sufficient condition for the smoothness of the solution. The classical Prodi-Serrin conditions (see [6], [17], [19]) says that if

$$\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3 \leqslant q \leqslant \infty, \tag{1.2}$$

then the solution is regular on $(0,T)$.

This was generalized by Beirão da Veiga [1] by considering the velocity gradient or vorticity,

$$\nabla \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2} \leqslant q \leqslant \infty. \tag{1.3}$$

Notice that the case $q \in [\frac{3}{2}, 3)$ follows directly from (1.2) and the Sobolev inequality.

In view of the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$, it is natural to ask whether or not we can reduce (1.2) and (1.3) to its partial components. One way is to consider regularity criteria involving only one velocity component, which were done in [3], [9], [10], [13], [25], [27], and the references therein. Another way is to study the possible components reduction of $\nabla \boldsymbol{u}$ to ∇u_3, see for instance [4], [5], [7], [10], [16], [20], [22], [24], [26], [27]; or to $\partial_3 \boldsymbol{u}$, see for example [2], [11], [15], [14], [23]. In [14], Penel and Pokorný showed that if

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{3}{2}, \quad 2 \leqslant q \leqslant \infty, \tag{1.4}$$

then the solution is smooth. This was improved by Kukavica-Ziane [11] to be

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{9}{4} \leqslant q \leqslant 3. \tag{1.5}$$

Later on, Cao [2] employed multiplicative Sobolev inequalities

$$1 \leqslant q < \infty \Rightarrow \|f\|_{L^{3q}} \leqslant C\|\partial_1 f\|_{L^2}^{1/3}\|\partial_2 f\|_{L^2}^{1/3}\|\partial_3 f\|_{L^q}^{1/3} \tag{1.6}$$

and

$$1 \leqslant q < \infty \Rightarrow \|f\|_{L^{5q}} \leqslant C\|\partial_1 (f^2)\|_{L^2}^{1/5}\|\partial_2 (f^2)\|_{L^2}^{1/5}\|\partial_3 f\|_{L^q}^{1/5} \tag{1.7}$$

to get the following extended regularity condition:

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{27}{16} \leqslant q \leqslant \frac{5}{2}. \tag{1.8}$$

In a recent paper, Zhang [23] generalized (1.7), and improved (1.5) and (1.8), simultaneously. Precisely, he showed that the condition

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad 1.56207 \approx \frac{3\sqrt{37}}{4}-3 \leqslant q \leqslant 3 \tag{1.9}$$

could ensure the regularity of the solution.

Combining the progress listed above, we see that the state of the regularity criterion for the Navier-Stokes equations is the following:

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\begin{cases} 2, & \dfrac{3\sqrt{37}}{4}-3 \leqslant q \leqslant 3, \\ \dfrac{3}{2}, & 3 < q \leqslant \infty. \end{cases} \tag{1.10}$$

Consequently, two questions appeared naturally. One is whether we could establish regularity conditions via $\partial_3 \boldsymbol{u}$ for the space integrability index

$$q \in \Big(\frac{3}{2}, \frac{3\sqrt{37}}{4}-3\Big).$$

Second is to improve the scaling dimension $\frac{3}{2}$ for $q \in [3,\infty)$.

In this paper, we shall prove the following two regularity criteria:

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{8}{5}+\frac{3}{5q}, \quad 4 \leqslant q \leqslant \infty, \tag{1.11}$$

and

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{14}{11}+\frac{9}{11q}, \quad \frac{5}{2} \leqslant q \leqslant \infty. \tag{1.12}$$

Whence, the up-to-date status of this topic is the following smoothness condition:

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\begin{cases} 2, & \dfrac{3\sqrt{37}}{4}-3 \leqslant q \leqslant 3, \\ \dfrac{14}{11}+\dfrac{9}{11q}, & 3 < q < \dfrac{18}{5}, \\ \dfrac{3}{2}, & \dfrac{18}{5} \leqslant q < 4, \\ \dfrac{8}{5}+\dfrac{3}{5q}, & 4 \leqslant q \leqslant \infty. \end{cases} \tag{1.13}$$

Before stating the main result, let us recall the weak solution of (1.1) in the sense of Leray and Hopf, see for instance [18], Definitions 3.3 and 4.9.

Definition 1.1. Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{u}_0 = 0$, $T > 0$. A measurable $\mathbb{R}^3$-valued function $\boldsymbol{u}$ defined on $[0,T]\times\mathbb{R}^3$ is said to be a weak solution to (1.1) if

(1) $\boldsymbol{u} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$;

(2) $(1.1)_1$ and $(1.1)_2$ hold in the sense of distributions, i.e.

$$\int_0^t\int_{\mathbb{R}^3} \boldsymbol{u}\cdot[\partial_t\boldsymbol{\varphi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\varphi}]\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{R}^3}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0)\,\mathrm{d}x = \int_0^T\int_{\mathbb{R}^3}\nabla\boldsymbol{u}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t$$

for each $\boldsymbol{\varphi} \in C_c^\infty([0,T)\times\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{\varphi} = 0$, where $A:B = \sum_{i,j=1}^{3} a_{ij}b_{ij}$ for 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$, and

$$\int_0^T\int_{\mathbb{R}^3}\boldsymbol{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = 0$$

for each $\psi \in C_c^\infty(\mathbb{R}^3\times[0,T))$;

(3) the strong energy inequality, that is,

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + 2\int_s^t \|\nabla\boldsymbol{u}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqslant \|\boldsymbol{u}(s)\|_{L^2}^2 \quad \forall\, s < t < T, \tag{1.14}$$

holds for $s = 0$ and almost all times $s \in (0,T)$.

Now, our main result reads:

Theorem 1.2. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{u}_0 = 0$, $T > 0$. Assume that $\boldsymbol{u}$ is a weak solution to* (1.1) *on $[0,T]$ with initial data $\boldsymbol{u}_0$. If*

$$\partial_3\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = \frac{8}{5} + \frac{3}{5q}, \quad 4 \leqslant q \leqslant \infty, \tag{1.15}$$

or

$$\partial_3\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = \frac{14}{11} + \frac{9}{11q}, \quad \frac{5}{2} \leqslant q \leqslant \infty, \tag{1.16}$$

then the solution $\boldsymbol{u}$ is smooth on $(0,T]\times\mathbb{R}^3$.

The proof of Theorem 1.2 will be provided in Section 2. Before doing that, let us recall an interesting combinatoric regularity condition by Penel and Pokorný [14], Theorem 1 (a):

$$\begin{aligned} &u_3 \in L^{2s/(s-3)}(0,T;L^s(\mathbb{R}^3)), \quad 3 < s \leqslant \infty; \\ &\partial_3 u_1,\ \partial_3 u_2 \in L^{2q/(2q-3)}(0,T;L^q(\mathbb{R}^3)), \quad \frac{3}{2} < q \leqslant \infty. \end{aligned} \tag{1.17}$$

Also, two inequalities concerning the pressure are recalled (see [14], Lemma 1):

$$\|\pi\|_{L^r} \leqslant C\|\boldsymbol{u}\|_{L^{2r}}^2, \quad \|\partial_3\pi\|_{L^r} \leqslant C\sum_{i,j=1}^{3}\|\partial_3(u_iu_j)\|_{L^r}, \quad 1<r<\infty. \tag{1.18}$$

2. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2.

For any $\varepsilon \in (0,T)$, since $\boldsymbol{u} \in L^2(0,T;H^1(\mathbb{R}^3))$ and (1.14) holds for almost all times $s \in (0,T)$, we may find a $\delta \in (0,\varepsilon)$ such that

$$\boldsymbol{u}(\delta) = \boldsymbol{u}(\cdot,\delta) \in L^2(\mathbb{R}^3),\ \nabla\boldsymbol{u}(\delta) \in L^2(\mathbb{R}^3) \Rightarrow \boldsymbol{u}(\delta) \in L^3(\mathbb{R}^3),$$

as well as

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + 2\int_\delta^t \|\nabla\boldsymbol{u}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqslant \|\boldsymbol{u}(\delta)\|_{L^2}^2 \quad \forall\,\delta<t<T.$$

Take this $\boldsymbol{u}(\delta)$ as initial data, there exists an $\tilde{\boldsymbol{u}} \in C([\delta,\Gamma^*),H^1(\mathbb{R}^3)) \cap L^2(0,\Gamma^*;H^2(\mathbb{R}^3))$, where $[\delta,\Gamma^*)$ is the life span of the unique strong solution, see [21]. Moreover, $\tilde{\boldsymbol{u}} \in C^\infty(\mathbb{R}^3\times(\delta,\Gamma^*))$. According to the uniqueness result, $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ on $[\delta,\Gamma^*)$. If $\Gamma^* \geqslant T$, we have already that $\boldsymbol{u} \in C^\infty(\mathbb{R}^3\times(0,T))$, due to the arbitrariness of $\varepsilon \in (0,T)$. In the case when $\Gamma^* < T$, our strategy is to show that $u_3 \in L^3(\delta,\Gamma^*;L^9(\mathbb{R}^3))$. Then by the fact that

$$\begin{aligned}
&(1.15) \Rightarrow \partial_3\boldsymbol{u} \in L^{5q/(2(2q-3))}(\delta,\Gamma^*;L^q(\mathbb{R}^3)) \subset L^{2q/(2q-3)}(\delta,\Gamma^*;L^q(\mathbb{R}^3)),\\
&(1.16) \Rightarrow \partial_3\boldsymbol{u} \in L^{11q/(7q-12)}(\delta,\Gamma^*;L^q(\mathbb{R}^3)) \subset L^{2q/(2q-3)}(\delta,\Gamma^*;L^q(\mathbb{R}^3)),
\end{aligned}$$

we may conclude the proof by invoking (1.17).

Case I. (1.15) *holds.* Multiplying the equation of u_3 in (1.1) by $|u_3|u_3$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\||u_3|^{3/2}\|_{L^2}^2 + \frac{4}{9}\|\nabla|u_3|^{3/2}\|_{L^2}^2 + \int_{\mathbb{R}^3}|u_3|\cdot|\nabla u_3|^2\,\mathrm{d}x\\
= -\int_{\mathbb{R}^3}\partial_3\pi|u_3|u_3\,\mathrm{d}x \equiv I.
\end{aligned} \tag{2.1}$$

We just need to estimate I:

$$
\begin{aligned}
(2.2)\qquad I &\leqslant \|\partial_3\pi\|_{L^3}\|u_3\|_{L^3}^2 \quad \text{(Hölder inequality)}\\
&\leqslant C\|\boldsymbol{u}\|_{L^{3q/(q-3)}}\|\partial_3\boldsymbol{u}\|_{L^q}\||u_3|^{3/2}\|_{L^2}^{4/3} \quad \text{(by (1.18))}\\
&\leqslant C\|\boldsymbol{u}\|_{L^2}^{2(q-4)/(3q-2)}\|\boldsymbol{u}\|_{L^{3q}}^{(q+6)/(3q-2)}\|\partial_3\boldsymbol{u}\|_{L^q}\||u_3|^{3/2}\|_{L^2}^{4/3}\\
&\quad \text{(interpolation inequality)}\\
&\leqslant C\|\partial_1\boldsymbol{u}\|_{L^2}^{(q+6)/(3(3q-2))}\|\partial_2\boldsymbol{u}\|_{L^2}^{(q+6)/(3(3q-2))}\|\partial_3\boldsymbol{u}\|_{L^q}^{(q+6)/(3(3q-2))}\\
&\qquad\qquad \times \|\partial_3\boldsymbol{u}\|_{L^q}\||u_3|^{3/2}\|_{L^2}^{4/3}\\
&\quad \text{(by Definition 1.1 item 1, and (1.6))}\\
&\leqslant C\|\nabla\boldsymbol{u}\|_{L^2}^{2(q+6)/(3(3q-2))}\|\partial_3\boldsymbol{u}\|_{L^q}^{10q/(3(3q-2))}\||u_3|^{3/2}\|_{L^2}^{4/3}\\
&\leqslant C(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\partial_3\boldsymbol{u}\|_{L^q}^{5q/(4q-6)})\||u_3|^{3/2}\|_{L^2}^{4/3} \quad \text{(Young inequality).}
\end{aligned}
$$

Plugging (2.2) into (2.1), we find

$$
\frac{\mathrm{d}}{\mathrm{d}t}\||u_3|^{3/2}\|_{L^2}^2 \leqslant C(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\partial_3\boldsymbol{u}\|_{L^q}^{5q/(2(2q-3))})\||u_3|^{3/2}\|_{L^2}^{4/3}.
$$

Dividing both sides by $\||u_3|^{3/2}\|_{L^2}^{4/3}$ and integrating with respect to t, we deduce by assumption (1.15) that

$$
\|u_3\|_{L^\infty(\delta,\Gamma^*;L^3(\mathbb{R}^3))} \leqslant C.
$$

Integrating (2.1) then yields

$$
\|u_3\|_{L^3(\delta,\Gamma^*;L^9(\mathbb{R}^3))} = \||u_3|^{3/2}\|_{L^2(\delta,\Gamma^*;L^6(\mathbb{R}^3))} \leqslant C\|\nabla|u_3|^{3/2}\|_{L^2(\delta,\Gamma^*;L^2(\mathbb{R}^3))} \leqslant C,
$$

as desired.

Case II. (1.16) *holds.* We should dominate I in another way:

$$
\begin{aligned}
(2.3)\qquad I &\leqslant C\int_{\mathbb{R}^3}|\pi|\cdot|\partial_3u_3|\cdot|u_3|\,\mathrm{d}x\\
&\leqslant C\|\pi\|_{L^{3q/(2q-3)}}\|\partial_3u_3\|_{L^q}\|u_3\|_{L^3} \quad \text{(Hölder inequality)}\\
&\leqslant C\|\boldsymbol{u}\|_{L^{6q/(2q-3)}}^2\|\partial_3u_3\|_{L^q}\|u_3\|_{L^3} \quad \text{(by (1.18))}\\
&\leqslant C\|\boldsymbol{u}\|_{L^2}^{2(2q-5)/(3q-2)}\|\boldsymbol{u}\|_{L^{3q}}^{2(q+3)/(3q-2)}\|\partial_3u_3\|_{L^q}\||u_3|^{3/2}\|_{L^2}^{2/3}\\
&\quad \text{(interpolation inequality)}\\
&\leqslant C\|\partial_1\boldsymbol{u}\|_{L^2}^{2(q+3)/(3(3q-2))}\|\partial_2\boldsymbol{u}\|_{L^2}^{2(q+3)/(3(3q-2))}\|\partial_3\boldsymbol{u}\|_{L^q}^{2(q+3)/(3(3q-2))}\\
&\qquad\qquad \times \|\partial_3u_3\|_{L^q}\||u_3|^{3/2}\|_{L^2}^{2/3}\\
&\quad \text{(by Definition 1.1 item 1, and (1.6))}\\
&\leqslant C\|\nabla\boldsymbol{u}\|_{L^2}^{4(q+3)/(3(3q-2))}\|\partial_3\boldsymbol{u}\|_{L^q}^{11q/(3(3q-2))}\||u_3|^{3/2}\|_{L^2}^{2/3}\\
&\leqslant C(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\partial_3\boldsymbol{u}\|_{L^q}^{11q/(7q-12)})\||u_3|^{3/2}\|_{L^2}^{2/3} \quad \text{(Young inequality).}
\end{aligned}
$$

Further, we may proceed as above to conclude the proof of Theorem 1.2 under condition (1.16).

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