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Daiki Hashimoto; Takao Ohno; Tetsu Shimomura

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# BOUNDEDNESS OF GENERALIZED FRACTIONAL INTEGRAL OPERATORS ON ORLICZ SPACES NEAR $L^1$ OVER METRIC MEASURE SPACES

Daiki Hashimoto, Nagasaki, Takao Ohno, Oita,
Tetsu Shimomura, Hiroshima

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*Abstract.* We are concerned with the boundedness of generalized fractional integral operators $I_{\varrho,\tau}$ from Orlicz spaces $L^{\Phi}(X)$ near $L^1(X)$ to Orlicz spaces $L^{\Psi}(X)$ over metric measure spaces equipped with lower Ahlfors $Q$-regular measures, where $\Phi$ is a function of the form $\Phi(r) = rl(r)$ and $l$ is of log-type. We give a generalization of paper by Mizuta et al. (2010), in the Euclidean setting. We deal with both generalized Riesz potentials and generalized logarithmic potentials.

*Keywords*: Orlicz space; Riesz potential; fractional integral; metric measure space; lower Ahlfors regular

*MSC 2010*: 31B15, 46E30, 46E35

## 1. Introduction

Let $G$ be a bounded set in $\mathbb{R}^N$. O'Neil in [24] gave a sufficient condition for the boundedness of convolution operators in Orlicz spaces $L^{\Phi}(G)$ near $L^1(G)$. In this paper, we aim to give a general version of the boundedness of generalized fractional integral operators on $L^{\Phi}(X)$ near $L^1(X)$ over metric measure spaces equipped with lower Ahlfors $Q$-regular measures which are nondoubling measures, as an extension of [14] in the Euclidean setting.

We denote by $(X, d, \mu)$ a metric measure spaces, where $X$ is a bounded set, $d$ is a metric on $X$ and $\mu$ is a nonnegative complete Borel regular outer measure on $X$ which is finite in every bounded set. For simplicity, we often write $X$ instead of $(X, d, \mu)$. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball in $X$ centered

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at $x$ with radius $r$ and $d_X = \sup\{d(x,y)\colon\ x,y \in X\}$. We assume that

$$\mu(\{x\}) = 0$$

for $x \in X$ and $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity.

In the present paper, we do not postulate on $\mu$ the so-called doubling condition. Recall that a Radon measure $\mu$ is said to be doubling if there exists a constant $C_\mu > 0$ such that $\mu(B(x,2r)) \leqslant C_\mu \mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)$ $(= X)$ and $r > 0$. Otherwise $\mu$ is said to be nondoubling. In connection with the $5r$-covering lemma, the doubling condition had been a key condition in harmonic analysis. However, Nazarov, Treil and Volberg showed that the doubling condition is not necessary, by using the modified maximal operator, see [19], [20]. For non-homogeneous metric measure spaces, we refer to [12], [29]. We say that a measure $\mu$ is lower Ahlfors $Q$-regular if there exists a constant $K_0 > 0$ such that

$$\mu(B(x,r)) \geqslant K_0 r^Q \tag{1.1}$$

for all $x \in X$ and $0 < r < d_X$ (see e.g. [1], [11]). Metric measure spaces equipped with lower Ahlfors $Q$-regular measures have been studied in many articles over the past decades; see [4], [7], [9] etc. See also [21], [23] for Sobolev's inequality of Riesz potentials and [22] for Trudinger's inequality and continuity of Riesz potentials in such a metric setting. In this paper we assume that $\mu$ is lower Ahlfors $Q$-regular. Here note that if $\mu$ is a doubling measure and $d_X < \infty$, then $\mu$ is lower Ahlfors $\log_2 C_\mu$-regular since

$$\frac{\mu(B(x,r))}{\mu(B(x,d_X))} \geqslant C_\mu^{-2}\Big(\frac{r}{d_X}\Big)^{\log_2 C_\mu}$$

for all $x \in X$ and $0 < r < d_X$ (see e.g. [1], Lemma 3.3, and [9]). However, there exist lower Ahlfors measures which are nondoubling. For example, let $X_1 = \{x = (x_1, 0) \in \mathbb{R}^2\colon\ 0 \leqslant x_1 < 1\}$ and $X_2 = \{x = (x_1, x_2) \in \mathbb{R}^2\colon\ |x| < 1,\ x_1 < 0\}$ and define $(X, d, \mu) = (X_1, d_2, m_1) \cup (X_2, d_2, m_2)$, where $d_2$ denotes the 2-dimension Euclidean distance and $m_i$ denotes the $i$-dimension Lebesgue measure. It is easy to show that $\mu$ is nondoubling and lower Ahlfors 2-regular. For other examples of nondoubling metric measure spaces, see [26].

Let $\mathcal{G}$ be the set of all continuous functions from $(0,\infty)$ to itself with the doubling condition; that is, there exists a constant $c_\varphi \geqslant 1$ such that

$$\frac{1}{c_\varphi} \leqslant \frac{\varphi(r)}{\varphi(s)} \leqslant c_\varphi \quad \text{for } \frac{1}{2} \leqslant \frac{r}{s} \leqslant 2. \tag{1.2}$$

We call $c_\varphi$ the doubling constant of $\varphi$.

Let us consider the family $\mathcal{Y}$ of all continuous, increasing, convex and bijective functions from $[0,\infty)$ to itself. For $\Phi \in \mathcal{Y}$, the Orlicz space $L^{\Phi}(X)$ is defined as

$$L^{\Phi}(X) = \{f \in L^1_{\rm loc}(X)\colon \ \|f\|_{L^{\Phi}(X)} < \infty\},$$

where

$$\|f\|_{L^{\Phi}(X)} = \inf\Big\{\lambda > 0\colon \ \int_X \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,{\rm d}\mu(x) \leqslant 1\Big\}.$$

If $\Phi_1,\ \Phi_2 \in \mathcal{Y}$ and there exists a constant $C \geqslant 1$ such that $\Phi_1(C^{-1}r) \leqslant \Phi_2(r) \leqslant \Phi_1(Cr)$ for all $r > 0$, then we see easily that

$$L^{\Phi_1}(X) = L^{\Phi_2}(X)$$

with equivalent norms. Recently, there have also been a surge of activities in understanding Orlicz spaces in a general metric setting; e.g. [3], [6], [13].

Let $\varrho \in \mathcal{G}$ be a function from $(0,\infty)$ to itself with $\int_0^1 \varrho(t)\,{\rm d}t/t < \infty$. For $\tau > 2$, we define

$$I_{\varrho,\tau} f(x) = \int_X \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,{\rm d}\mu(y),$$

where $f \in L^1(X)$. See, for example, [9] and [16]. If $X = \mathbb{R}^N$ and $\varrho(r) = r^{\alpha}$ for $0 < \alpha < N$, then $I_{\varrho,\tau} f$ coincides with the usual Riesz potential $I_{\alpha} f$ of order $\alpha$. Using the operator $I_{\varrho,\tau}$, we can give a systematic proof and several new results as corollaries. We also refer the reader to [5], [8], [17] and [18] for the boundedness of $I_{\varrho} f$ in the Euclidean setting, where

$$I_{\varrho} f(x) = \int_G \frac{\varrho(|x-y|)}{|x-y|^N} f(y)\,{\rm d}y \quad (f \in L^1(G)).$$

O'Neil in [24], Theorem 5.2, gave a sufficient condition for the boundedness of convolution operators in Orlicz spaces $L^{\Phi}(G)$ near $L^1(G)$. See also Cianchi [2], page 193. He used other function spaces $M^{\Phi}$ in which $L^{\Phi}$ is a subspace (see [24], Chapter 3). In [14], we studied the boundedness of $I_{\varrho}$ from $L^{\Phi}(G)$ near $L^1(G)$ to $L^{\Psi}(G)$ and gave another sufficient condition in the Euclidean setting.

Our aim in this paper is to give a general version of the boundedness of generalized Riesz potentials $I_{\varrho,\tau} f$ from $L^{\Phi}(X)$ near $L^1(X)$ to $L^{\Psi}(X)$ over lower Ahlfors $Q$-regular metric measure spaces (Theorem 2.1 below), as an extension of [14], Theorem 7.1, in the Euclidean setting. For $L^{\Phi}$ case, the maximal function is a crucial tool by Hedberg's trick (see Hedberg [10]). In $L^{\Phi}$ near $L^1$ case, our strategy is to give an estimate of $I_{\varrho,\tau} f$ by use of a logarithmic type potential

$$\int_{\{y\in X\colon\, d(x,y)^{-\gamma} < |f(y)|\}} \frac{l_2(d(x,y)^{-1})}{\mu(B(x,\tau d(x,y)))} |f(y)|\,{\rm d}\mu(y),$$

which plays a role of maximal functions. Therefore, our proof is quite different from that of O'Neil [24].

In the last section, we show the boundedness of generalized logarithmic potentials $I_{\varrho,\tau}f$ (Theorem 3.1 below), as an extension of [14], Theorem 7.4.

For related results, see [25], [27] and [28].

Throughout this paper, let $C$ denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

## 2. Generalized Riesz potentials

Let $\mathcal{L}$ be the set of all positive continuous functions $l$ on $[0,\infty)$ for which there exists a constant $c \geqslant 1$ such that

$$c^{-1}l(r) \leqslant l(r^2) \leqslant cl(r) \quad \text{whenever } r > 0$$

and $l(r)$ is almost monotone, that is, it is either almost increasing:

$$l(r) \leqslant cl(s) \quad \text{for } 0 < r < s < \infty,$$

or almost decreasing:

$$l(s) \leqslant cl(r) \quad \text{for } 0 < r < s < \infty.$$

Here we collect the fundamental properties on functions $l \in \mathcal{L}$ (see e.g. [14] and [15]).

($\mathcal{L}1$) $l \in \mathcal{G}$ and $1/l \in \mathcal{L}$.

($\mathcal{L}2$) For all $\alpha > 0$, there exists a constant $c_\alpha \geqslant 1$ such that

$$c_\alpha^{-1}l(r) \leqslant l(r^\alpha) \leqslant c_\alpha l(r) \quad \text{for } 0 < r < \infty. \tag{2.1}$$

($\mathcal{L}3$) For each $\varepsilon > 0$, $r^\varepsilon l(r)$ is almost increasing, that is, there exists a constant $c_\varepsilon \geqslant 1$ such that

$$r^\varepsilon l(r) \leqslant c_\varepsilon s^\varepsilon l(s) \quad \text{for } 0 < r < s < \infty. \tag{2.2}$$

($\mathcal{L}4$) If $l, l_1 \in \mathcal{L}$ and $\alpha > 0$, then there exists a constant $c_\alpha \geqslant 1$ such that

$$c_\alpha{}^{-1}l(r) \leqslant l(r^\alpha l_1(r)) \leqslant c_\alpha l(r) \quad \text{for } 0 < r < \infty. \tag{2.3}$$

($\mathcal{L}5$) If $p \geqslant 1$, $l, l_1, l_2 \in \mathcal{L}$, $\Phi \in \mathcal{Y}$ and $\Phi(r) \leqslant r^p l(r) l_1(r) l_2(r)$, then there exists a constant $c > 0$ such that

$$r^{1/p} l(r)^{-1/p} l_1(r)^{-1/p} l_2(r)^{-1/p} \leqslant c\Phi^{-1}(r) \quad \text{for } 0 < r < \infty, \tag{2.4}$$

where $\Phi^{-1}(r)$ is the inverse function of $\Phi(r)$.

For $\tau > 2$, consider the generalized Riesz potential

$$I_{\varrho,\tau} f(x) = \int_X \frac{\varrho(d(x,y))}{\mu(B(x, \tau d(x,y)))} f(y) \, \mathrm{d}\mu(y),$$

where $\varrho \in \mathcal{G}$ is of the form $\varrho(r) = r^\alpha l(r^{-1})^{-1}$ with $0 < \alpha < Q$ and $l \in \mathcal{L}$.

**Theorem 2.1.** *Let $0 < \alpha < Q$ and $p = Q/(Q - \alpha)$. Let $\varrho \in \mathcal{G}$ and $\Phi \in \mathcal{Y}$ be of the form*

$$\varrho(r) = r^\alpha l(r^{-1})^{-1}$$

*and*

$$\Phi(r) = r l_1(r),$$

*where $l, l_1 \in \mathcal{L}$. Take functions $l_2 \in \mathcal{L}$ and $\Psi \in \mathcal{Y}$ satisfying*

$$\int_{d_X^{-1}}^r l_2(t) \frac{\mathrm{d}t}{t} \leqslant l_1(r) \quad \text{for } d_X^{-1} \leqslant r < \infty, \tag{2.5}$$

$$\Psi(r) \leqslant r^p l(r)^p l_1(r)^{p-1} l_2(r) \quad \text{for } 0 \leqslant r < \infty. \tag{2.6}$$

*Then there exists a constant $A > 0$ such that*

$$\|I_{\varrho,\tau} f\|_{L^\Psi(X)} \leqslant A \|f\|_{L^\Phi(X)},$$

*where the constant $A$ depends on $\tau$, $\alpha$, $Q$, $K_0$, $d_X$ and the constants appearing in ($\mathcal{L}1$)–($\mathcal{L}5$).*

As in Corollary 7.2 in [14], we have the following corollary in our setting as a special case of Theorem 2.1.

**Corollary 2.2.** *Let $0 < \alpha < Q$, $p = Q/(Q - \alpha)$. For $\alpha_1 \in \mathbb{R}$ and $\beta_1 > 0$, let*

$$\begin{aligned} \varrho(r) &= r^\alpha (\log(c + r^{-1}))^{-\alpha_1}, \\ \Phi(r) &= r(\log(c + r))^{\beta_1}, \\ \Psi(r) &= r^p (\log(c + r))^{p(\alpha_1 + \beta_1) - 1}, \end{aligned}$$

*where $c > \mathrm{e}$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then there exists a constant $A > 0$ such that*

$$\|I_{\varrho,\tau} f\|_{L^\Psi(X)} \leqslant A \|f\|_{L^\Phi(X)}.$$

**Remark 2.3.** Let $\mathbf{B} = B(0,1) \subset \mathbb{R}^N$. In Corollary 2.2 we cannot take $\beta_1 = 0$. For details, see [14], Remark 7.1.

**Remark 2.4** ([14], Remark 7.2). Let $\mathbf{B} = B(0,1) \subset \mathbb{R}^N$. Let $\alpha$, $\alpha_1$, $\beta_1$, $p$ and $\Phi$ be as in Corollary 2.2. If $\gamma > p(\alpha_1 + \beta_1) - 1$, then one can find $f \in L^\Phi(\mathbf{B})$ but

$$\int_{\mathbf{B}} |I_\varrho f(x)|^p (\log(1 + |I_\varrho f(x)|))^\gamma \,\mathrm{d}x = \infty.$$

As in Corollary 7.3 in [14], we have the following corollary in our setting as a special case of Theorem 2.1.

**Corollary 2.5.** *Let* $0 < \alpha < Q$, $p = Q/(Q-\alpha)$. *For* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $\beta_2 > 0$, *let*

$$\begin{aligned} \varrho(r) &= r^\alpha (\log(c + r^{-1}))^{-\alpha_1} (\log\log(c + r^{-1}))^{-\alpha_2}, \\ \Phi(r) &= r(\log\log(c + r))^{\beta_2}, \\ \Psi(r) &= r^p (\log(c + r))^{p\alpha_1 - 1} (\log\log(c + r))^{p(\alpha_2 + \beta_2) - 1}, \end{aligned}$$

*where* $c > \mathrm{e}^2$ *is chosen so that* $\Phi, \Psi \in \mathcal{Y}$. *Then there exists a constant* $A > 0$ *such that*

$$\|I_{\varrho,\tau} f\|_{L^\Psi(X)} \leqslant A \|f\|_{L^\Phi(X)}.$$

Proof of Theorem 2.1. We may assume that $\|f\|_{L^\Phi(X)} = 1$. Then

$$\int_X \Phi(|f(y)|) \,\mathrm{d}\mu(y) \leqslant 1.$$

Note that $l_1$ is nondecreasing since $\Phi$ is convex by our assumption.

For $0 < \gamma < \alpha$, let

$$J(x) = \int_{\{y \in X \colon d(x,y)^{-\gamma} < |f(y)|\}} \frac{l_2(d(x,y)^{-1})}{\mu(B(x, \tau d(x,y)))} |f(y)| \,\mathrm{d}\mu(y).$$

Then for $0 < \delta \leqslant d_X$, which will be determined later, we have by (2.2)

$$\begin{aligned} &\int_{B(x,\delta)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \,\mathrm{d}\mu(y) \\ &\quad \leqslant \int_{\{y \in B(x,\delta) \colon d(x,y)^{-\gamma} < |f(y)|\}} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \,\mathrm{d}\mu(y) \\ &\qquad + \int_{B(x,\delta)} \frac{d(x,y)^{\alpha-\gamma} l(d(x,y)^{-1})^{-1}}{\mu(B(x,\tau d(x,y)))} \,\mathrm{d}\mu(y) \\ &\quad \leqslant C(\delta^\alpha l(\delta^{-1})^{-1} l_2(\delta^{-1})^{-1} J(x) + \delta^{\alpha-\gamma} l(\delta^{-1})^{-1}) \end{aligned}$$

since

$$\begin{aligned}
&\int_{B(x,\delta)} \frac{d(x,y)^{\alpha-\gamma} l(d(x,y)^{-1})^{-1}}{\mu(B(x,\tau d(x,y)))} \,\mathrm{d}\mu(y) \\
&\quad = \sum_{j=1}^{\infty} \int_{B(x,\tau^{-j+1}\delta)\setminus B(x,\tau^{-j}\delta)} \frac{d(x,y)^{\alpha-\gamma} l(d(x,y)^{-1})^{-1}}{\mu(B(x,\tau d(x,y)))} \,\mathrm{d}\mu(y) \\
&\quad \leqslant C \sum_{j=1}^{\infty} \int_{B(x,\tau^{-j+1}\delta)\setminus B(x,\tau^{-j}\delta)} \frac{(\tau^{-j+1}\delta)^{\alpha-\gamma} l((\tau^{-j+1}\delta)^{-1})^{-1}}{\mu(B(x,\tau^{-j+1}\delta))} \,\mathrm{d}\mu(y) \\
&\quad \leqslant C \sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^{\alpha-\gamma} l((\tau^{-j+1}\delta)^{-1})^{-1} \leqslant \frac{C}{\log\tau} \int_0^{\delta} t^{\alpha-\gamma} l(t^{-1})^{-1} \frac{\mathrm{d}t}{t} \\
&\quad \leqslant C\delta^{\alpha-\gamma} l(\delta^{-1})^{-1}.
\end{aligned}$$

Similarly, for $\alpha < \gamma' < Q$ we obtain by (1.1), (2.1) and (2.2)

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \,\mathrm{d}\mu(y) \\
&\quad \leqslant C \int_{X\setminus B(x,\delta)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} \Big( |f(y)| \frac{l_1(|f(y)|)}{l_1(d(x,y)^{-1})} + d(x,y)^{-\gamma'} \Big) \,\mathrm{d}\mu(y) \\
&\quad \leqslant C\bigg( K_0^{-1}\tau^{-Q} \int_{X\setminus B(x,\delta)} d(x,y)^{\alpha-Q} l(d(x,y)^{-1})^{-1} l_1(d(x,y)^{-1})^{-1} \Phi(|f(y)|) \,\mathrm{d}\mu(y) \\
&\qquad + \int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\gamma'} l(d(x,y)^{-1})^{-1}}{\mu(B(x,\tau d(x,y)))} \,\mathrm{d}\mu(y) \bigg) \\
&\quad \leqslant C\bigg( \delta^{\alpha-Q} l(\delta^{-1})^{-1} l_1(\delta^{-1})^{-1} \int_X \Phi(|f(y)|) \,\mathrm{d}\mu(y) + \delta^{\alpha-\gamma'} l(\delta^{-1})^{-1} \bigg) \\
&\quad \leqslant C(\delta^{\alpha-Q} l(\delta^{-1})^{-1} l_1(\delta^{-1})^{-1} + \delta^{\alpha-\gamma'} l(\delta^{-1})^{-1})
\end{aligned}$$

since

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\gamma'} l(d(x,y)^{-1})^{-1}}{\mu(B(x,\tau d(x,y)))} \,\mathrm{d}\mu(y) \\
&\quad = \sum_{j=1}^{\infty} \int_{B(x,\tau^{j}\delta)\setminus B(x,\tau^{j-1}\delta)} \frac{d(x,y)^{\alpha-\gamma'} l(d(x,y)^{-1})^{-1}}{\mu(B(x,\tau d(x,y)))} \,\mathrm{d}\mu(y) \\
&\quad \leqslant C \sum_{j=1}^{\infty} \int_{B(x,\tau^{j}\delta)\setminus B(x,\tau^{j-1}\delta)} \frac{(\tau^{j-1}\delta)^{\alpha-\gamma'} l((\tau^{j-1}\delta)^{-1})^{-1}}{\mu(B(x,\tau^{j}\delta))} \,\mathrm{d}\mu(y) \\
&\quad \leqslant C \sum_{j=1}^{\infty} (\tau^{j-1}\delta)^{\alpha-\gamma'} l((\tau^{j-1}\delta)^{-1})^{-1} \leqslant \frac{C}{\log\tau} \int_{\delta}^{\infty} t^{\alpha-\gamma'} l(t^{-1})^{-1} \frac{\mathrm{d}t}{t} \\
&\quad \leqslant C\delta^{\alpha-\gamma'} l(\delta^{-1})^{-1}.
\end{aligned}$$

Hence, it follows from (2.2) that

$$
\begin{aligned}
|I_{\varrho,\tau} f(x)| &\leqslant \int_{B(x,\delta)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \,\mathrm{d}\mu(y) \\
&\quad + \int_{X\setminus B(x,\delta)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \,\mathrm{d}\mu(y) \\
&\leqslant C(\delta^{\alpha} l(\delta^{-1})^{-1} l_2(\delta^{-1})^{-1} J(x) + \delta^{\alpha-\gamma} l(\delta^{-1})^{-1} \\
&\quad + \delta^{\alpha-Q} l(\delta^{-1})^{-1} l_1(\delta^{-1})^{-1} + \delta^{\alpha-\gamma'} l(\delta^{-1})^{-1}) \\
&= C(\delta^{\alpha} l(\delta^{-1})^{-1} l_2(\delta^{-1})^{-1} J(x) \\
&\quad + \delta^{\alpha-Q} l(\delta^{-1})^{-1} l_1(\delta^{-1})^{-1} (\delta^{Q-\gamma} l_1(\delta^{-1}) + 1 + \delta^{Q-\gamma'} l_1(\delta^{-1}))) \\
&\leqslant C(\delta^{\alpha} l(\delta^{-1})^{-1} l_2(\delta^{-1})^{-1} J(x) + \delta^{\alpha-Q} l(\delta^{-1})^{-1} l_1(\delta^{-1})^{-1}).
\end{aligned}
$$

Now, let

$$
\delta = \min\{J(x)^{-1/Q} l_1(J(x))^{-1/Q} l_2(J(x))^{1/Q}, d_X\}.
$$

If $\delta = J(x)^{-1/Q} l_1(J(x))^{-1/Q} l_2(J(x))^{1/Q}$, then it follows from (2.3) that

$$
l(\delta^{-1}) \sim l(J(x)), \quad l_1(\delta^{-1}) \sim l_1(J(x)), \quad l_2(\delta^{-1}) \sim l_2(J(x)),
$$

so we have by (2.6) and (2.4)

$$
\begin{aligned}
|I_{\varrho,\tau} f(x)| &\leqslant C J(x)^{(Q-\alpha)/Q} l(J(x))^{-1} l_1(J(x))^{-\alpha/Q} l_2(J(x))^{-(Q-\alpha)/Q} \\
&= C J(x)^{1/p} l(J(x))^{-1} l_1(J(x))^{-(p-1)/p} l_2(J(x))^{-1/p} \\
&\leqslant C \Psi^{-1}(J(x)),
\end{aligned}
$$

where $\Psi^{-1}(r)$ is the inverse function of $\Psi(r)$. If $\delta = d_X$, then

$$
|I_{\varrho,\tau} f(x)| \leqslant C.
$$

Therefore

$$
\Psi\Big(\frac{|I_{\varrho,\tau} f(x)|}{C}\Big) \leqslant J(x) + 1.
$$

Let $j_0(y)$ be the largest nonnegative integer such that $|f(y)|^{-1/\gamma}\widetilde{\tau}^{j_0(y)-1} \leqslant d_X$ for $y \in X$, where $\widetilde{\tau} = \tau/2$. By Fubini's theorem, we obtain

$$\begin{aligned}
&\int_X J(x)\,\mathrm{d}\mu(x) \\
&= \int_X \biggl(\int_{\{x\in X\colon d(x,y)^{-\gamma}<|f(y)|\}} \frac{l_2(d(x,y)^{-1})}{\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(x)\biggr)|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant \int_X \biggl(\sum_{j=1}^{j_0(y)} \int_{K_j} \frac{l_2(d(x,y)^{-1})}{\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(x)\biggr)|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant C\int_X \biggl(\sum_{j=1}^{j_0(y)} \int_{K_j} \frac{l_2((|f(y)|^{-1/\gamma}\widetilde{\tau}^j)^{-1})}{\mu(B(x,\tau|f(y)|^{-1/\gamma}\widetilde{\tau}^{j-1}))}\,\mathrm{d}\mu(x)\biggr)|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant C\int_X \biggl(\sum_{j=1}^{j_0(y)} \int_{B(y,|f(y)|^{-1/\gamma}\widetilde{\tau}^j)} \frac{l_2((|f(y)|^{-1/\gamma}\widetilde{\tau}^j)^{-1})}{\mu(B(y,|f(y)|^{-1/\gamma}\widetilde{\tau}^j))}\,\mathrm{d}\mu(x)\biggr)|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant C\int_X \biggl(\sum_{j=1}^{j_0(y)} l_2((|f(y)|^{-1/\gamma}\widetilde{\tau}^j)^{-1})\biggr)|f(y)|\,\mathrm{d}\mu(y),
\end{aligned}$$

where

$$K_j = B(y,|f(y)|^{-1/\gamma}\widetilde{\tau}^j) \setminus B(y,|f(y)|^{-1/\gamma}\widetilde{\tau}^{j-1}).$$

By (2.5) and (2.1), we have

$$\begin{aligned}
\int_X J(x)\,\mathrm{d}\mu(x) &\leqslant C\int_X \biggl(\int_{d_X^{-1}}^{\widetilde{\tau}|f(y)|^{1/\gamma}} l_2(t)\frac{\mathrm{d}t}{t}\biggr)|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant C\int_X l_1(|f(y)|^{1/\gamma})|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant C\int_X \Phi(|f(y)|)\,\mathrm{d}\mu(y) \leqslant C.
\end{aligned}$$

Thus, this theorem is proved. □

## 3. Generalized logarithmic potentials

For $\tau > 2$, consider the generalized logarithmic potential

$$I_{\varrho,\tau}f(x) = \int_X \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,\mathrm{d}\mu(y),$$

where $\varrho \in \mathcal{G}$ is of the form $\varrho(r) = l(r^{-1})^{-1}$ with $l \in \mathcal{L}$ satisfying

$$\int_0^1 \varrho(t)\frac{\mathrm{d}t}{t} < \infty. \tag{3.1}$$

For generalized logarithmic potentials, we have the following.

**Theorem 3.1.** *Let $\varrho \in \mathcal{G}$ be of the form $\varrho(r) = l(r^{-1})^{-1}$ with $l \in \mathcal{L}$ satisfying* (3.1). *Let $\Phi \in \mathcal{Y}$ be of the form*

$$\Phi(r) = rl_1(r),$$

*where $l_1 \in \mathcal{L}$. Let $l_2, m_1, m_2, m_3, m_4$ be functions in $\mathcal{L}$ such that*

(i) *$lm_1$, $l_1/m_2$, $l/m_3$ and $l_1m_4$ are almost increasing;*
(ii) *$\int_{d_X^{-1}}^r m_1(t)\,\mathrm{d}t/t \leqslant c_1m_2(r)$ for $d_X^{-1} \leqslant r < \infty$;*
(iii) *$\int_r^\infty (m_3(t))^{-1}\,\mathrm{d}t/t \leqslant c_2/m_4(r)$ for $d_X^{-1} \leqslant r < \infty$;*
(iv) *$m_2(r)/m_1(r) + m_3(r)/m_4(r) \leqslant l_2(r)$ for $d_X^{-1} \leqslant r < \infty$,*

*where $c_1$, $c_2$ are positive constants. Take a function $\Psi \in \mathcal{Y}$ satisfying*

$$\Psi(r) \leqslant rl(r)l_1(r)l_2(r)^{-1} \quad \text{for } 0 \leqslant r < \infty.$$

*Then there exists a constant $A > 0$ such that*

$$\|I_{\varrho,\tau}f\|_{L^\Psi(X)} \leqslant A\|f\|_{L^\Phi(X)},$$

*where the constant $A$ depends on $\tau, Q, K_0, d_X$ and the constants appearing in* ($\mathcal{L}$1)–($\mathcal{L}$5) *and* (i)–(iv).

As in [14], we have the following corollaries in our setting as special cases of Theorem 3.1. For other examples, see [14].

**Corollary 3.2.** *For $\alpha_1 > 0$ and $\beta_1 > 0$, let*

$$\begin{aligned}\varrho(r) &= (\log(c + r^{-1}))^{-\alpha_1 - 1},\\ \Phi(r) &= r(\log(c + r))^{\beta_1},\\ \Psi(r) &= r(\log(c + r))^{\alpha_1 + \beta_1},\end{aligned}$$

*where $c > \mathrm{e}$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then there exists a constant $A > 0$ such that*

$$\|I_{\varrho,\tau}f\|_{L^\Psi(X)} \leqslant A\|f\|_{L^\Phi(X)}.$$

**Corollary 3.3.** *For $\alpha_1 > 0$ and $\beta_2 > 0$, let*

$$\begin{aligned}\varrho(r) &= (\log(c + r^{-1}))^{-\alpha_1 - 1},\\ \Phi(r) &= r(\log\log(c + r))^{\beta_2},\\ \Psi(r) &= r(\log(c + r))^{\alpha_1}(\log\log(c + r))^{\beta_2 - 1},\end{aligned}$$

*where $c > \mathrm{e}^2$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then there exists a constant $A > 0$ such that*

$$\|I_{\varrho,\tau} f\|_{L^{\Psi}(X)} \leqslant A\|f\|_{L^{\Phi}(X)}.$$

**Corollary 3.4.** *For $\alpha_2 > 0$, $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$, let*

$$\begin{aligned}\varrho(r) &= (\log(c + r^{-1}))^{-1}(\log\log(c + r^{-1}))^{-\alpha_2 - 1},\\ \Phi(r) &= r(\log(c + r))^{\beta_1}(\log\log(c + r))^{\beta_2},\\ \Psi(r) &= r(\log(c + r))^{\beta_1}(\log\log(c + r))^{\alpha_2 + \beta_2},\end{aligned}$$

*where $c > \mathrm{e}^2$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then there exists a constant $A > 0$ such that*

$$\|I_{\varrho,\tau} f\|_{L^{\Psi}(X)} \leqslant A\|f\|_{L^{\Phi}(X)}.$$

P r o o f of Theorem 3.1. We may assume that $\|f\|_{L^{\Phi}(X)} = 1$. Then

$$\int_X \Phi(|f(y)|)\,\mathrm{d}\mu(y) \leqslant 1.$$

Let $0 < \delta < Q$. For $x \in X$ and $0 < r < d_X$, write

$$X = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$\begin{aligned}E_0 &= \{y \in B(x,r)\colon\ |f(y)| \leqslant r^{-\delta}\},\\ E_1 &= \{y \in B(x,r)\colon\ |f(y)| > r^{-\delta},\ |f(y)| > d(x,y)^{-\delta}\},\\ E_2 &= \{y \in B(x,r)\colon\ |f(y)| > r^{-\delta},\ |f(y)| \leqslant d(x,y)^{-\delta}\},\\ E_3 &= \{y \in X \setminus B(x,r)\colon\ |f(y)| > d(x,y)^{-\delta}\},\\ E_4 &= \{y \in X \setminus B(x,r)\colon\ |f(y)| \leqslant d(x,y)^{-\delta}\}.\end{aligned}$$

Then

$$\int_{E_0} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \, \mathrm{d}\mu(y)$$

$$\leqslant r^{-\delta} \sum_{j=1}^{\infty} \int_{B(x,\tau^{-j+1}r)\setminus B(x,\tau^{-j}r)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} \, \mathrm{d}\mu(y)$$

$$\leqslant Cr^{-\delta} \sum_{j=1}^{\infty} \int_{B(x,\tau^{-j+1}r)\setminus B(x,\tau^{-j}r)} \frac{\varrho(\tau^{-j+1}r)}{\mu(B(x,\tau^{-j+1}r))} \, \mathrm{d}\mu(y)$$

$$\leqslant Cr^{-\delta} \sum_{j=1}^{\infty} \varrho(\tau^{-j+1}r) \leqslant Cr^{-\delta} \int_0^r \varrho(t) \frac{\mathrm{d}t}{t} \leqslant Cr^{-\delta}.$$

Let $j_1(r)$ be the largest integer such that $\tau^{j_1(r)-1} r \leqslant d_X$. We have

$$\int_{E_4} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \, \mathrm{d}\mu(y)$$

$$\leqslant \int_{X\setminus B(x,r)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} d(x,y)^{-\delta} \, \mathrm{d}\mu(y)$$

$$\leqslant r^{-\delta} \sum_{j=1}^{j_1(r)} \int_{B(x,\tau^{j}r)\setminus B(x,\tau^{j-1}r)} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} \, \mathrm{d}\mu(y)$$

$$\leqslant Cr^{-\delta} \sum_{j=1}^{j_1(r)} \int_{B(x,\tau^{j}r)\setminus B(x,\tau^{j-1}r)} \frac{\varrho(\tau^{j}r)}{\mu(B(x,\tau^{j}r))} \, \mathrm{d}\mu(y)$$

$$\leqslant Cr^{-\delta} \sum_{j=1}^{j_1(r)} \varrho(\tau^{j}r) \leqslant Cr^{-\delta} \int_r^{\tau d_X} \varrho(t) \frac{\mathrm{d}t}{t} \leqslant Cr^{-\delta}.$$

Noting that $l_1$ is nondecreasing by our assumption that $\Phi$ is convex, we see by (2.1), (2.2) and (1.1) that

$$\int_{E_3} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \, \mathrm{d}\mu(y)$$

$$\leqslant \int_{E_3} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \frac{l_1(|f(y)|)}{l_1(d(x,y)^{-\delta})} \, \mathrm{d}\mu(y)$$

$$\leqslant C \int_{X\setminus B(x,r)} \frac{\varrho(d(x,y))}{K_0 \tau^Q d(x,y)^Q l_1(d(x,y)^{-1})} \Phi(|f(y)|) \, \mathrm{d}\mu(y)$$

$$\leqslant \frac{C\varrho(r)}{r^Q l_1(r^{-1})} \int_{X\setminus B(x,r)} \Phi(|f(y)|) \, \mathrm{d}\mu(y) \leqslant \frac{C}{r^Q l(r^{-1}) l_1(r^{-1})}.$$

Since $r^{-\delta} \leqslant C\{r^Q l(r^{-1}) l_1(r^{-1})\}^{-1}$ by (2.2), we have

$$\int_{E_0 \cup E_3 \cup E_4} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \, \mathrm{d}\mu(y) \leqslant \frac{C}{r^Q l(r^{-1}) l_1(r^{-1})}.$$

Next, let us consider the integral over $E_1 \cup E_2$. Set

$$J(x) = J_1(x) + J_2(x),$$

where

$$J_1(x) = \int_{\widetilde{E}_1} \frac{m_1(d(x,y)^{-1})}{\mu(B(x,\tau d(x,y)))} \frac{\Phi(|f(y)|)}{m_2(|f(y)|)} \, \mathrm{d}\mu(y),$$
$$J_2(x) = \int_{\widetilde{E}_2} \frac{m_4(|f(y)|)\Phi(|f(y)|)}{\mu(B(x,\tau d(x,y))) m_3(d(x,y)^{-1})} \, \mathrm{d}\mu(y)$$

with

$$\widetilde{E}_1 = \{y \in X \colon |f(y)| > d(x,y)^{-\delta}\},$$
$$\widetilde{E}_2 = \{y \in X \colon |f(y)| \leqslant d(x,y)^{-\delta}\}.$$

We insist by assumption (iv) that

$$\begin{aligned}
&\int_{E_1} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \, \mathrm{d}\mu(y) \\
&\leqslant C \int_{E_1} \frac{1}{\mu(B(x,\tau d(x,y))) l(d(x,y)^{-1})} \frac{m_1(d(x,y)^{-1})}{m_1(d(x,y)^{-1})} |f(y)| \frac{l_1(|f(y)|)/m_2(|f(y)|)}{l_1(r^{-\delta})/m_2(r^{-\delta})} \, \mathrm{d}\mu(y) \\
&\leqslant \frac{C}{l(r^{-1}) m_1(r^{-1})} \frac{m_2(r^{-1})}{l_1(r^{-1})} J_1(x) \leqslant \frac{C l_2(r^{-1})}{l(r^{-1}) l_1(r^{-1})} J_1(x)
\end{aligned}$$

since $l m_1$ and $l_1/m_2$ are almost increasing by assumption (i), and

$$\begin{aligned}
&\int_{E_2} \frac{\varrho(d(x,y))}{\mu(B(x,\tau d(x,y)))} |f(y)| \, \mathrm{d}\mu(y) \\
&\leqslant C \int_{E_2} \frac{1}{\mu(B(x,\tau d(x,y))) l(d(x,y)^{-1})} \frac{m_3(d(x,y)^{-1})}{m_3(d(x,y)^{-1})} |f(y)| \frac{l_1(|f(y)|) m_4(|f(y)|)}{l_1(r^{-\delta}) m_4(r^{-\delta})} \, \mathrm{d}\mu(y) \\
&\leqslant C \frac{m_3(r^{-1})}{l(r^{-1})} \frac{1}{l_1(r^{-1}) m_4(r^{-1})} J_2(x) \leqslant \frac{C l_2(r^{-1})}{l(r^{-1}) l_1(r^{-1})} J_2(x)
\end{aligned}$$

since $l/m_3$ and $l_1 m_4$ are almost increasing by assumption (i). Noting from assumptions (iv) and (iii) that

$$l_2(t) \geqslant \frac{m_3(t)}{m_4(t)} \geqslant c_2^{-1} m_3(t) \int_t^{2t} \frac{1}{m_3(s)} \frac{\mathrm{d}s}{s} \geqslant C$$

for $d_X^{-1} \leqslant t < \infty$, we find

$$|I_{\varrho,\tau} f(x)| \leqslant \frac{C}{l(r^{-1})l_1(r^{-1})}\Big(\frac{1}{r^Q} + l_2(r^{-1})J(x)\Big) \leqslant \frac{Cl_2(r^{-1})}{l(r^{-1})l_1(r^{-1})}\Big(\frac{1}{r^Q} + J(x)\Big).$$

Let

$$r = \min\{J(x)^{-1/Q}, d_X\}.$$

If $r = J(x)^{-1/Q}$, then we have by (2.1) and (2.4)

$$|I_{\varrho,\tau} f(x)| \leqslant \frac{Cl_2(J(x))}{l(J(x))l_1(J(x))} J(x) \leqslant C\Psi^{-1}(J(x)).$$

If $r = d_X$, then $J(x) \leqslant {d_X}^{-Q}$ and

$$|I_{\varrho,\tau} f(x)| \leqslant C.$$

Hence

$$\Psi\Big(\frac{|I_{\varrho,\tau} f(x)|}{C}\Big) \leqslant J(x) + 1.$$

Let $j_0(y)$ be the largest nonnegative integer such that $|f(y)|^{-1/\delta}\widetilde{\tau}^{j_0(y)-1} \leqslant d_X$ for $y \in X$, where $\widetilde{\tau} = \tau/2$. By Fubini's theorem, we see that

$$\begin{aligned}
&\int_X J_1(x)\,\mathrm{d}\mu(x)\\
&\quad = \int_X \Big(\int_{\{x\in X\colon\, d(x,y)^{-\delta}<|f(y)|\}} \frac{m_1(d(x,y)^{-1})}{\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(x)\Big)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y)\\
&\quad = \int_X \Big(\sum_{j=1}^{j_0(y)} \int_{K_j} \frac{m_1(d(x,y)^{-1})}{\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(x)\Big)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y)\\
&\quad \leqslant C\int_X \Big(\sum_{j=1}^{j_0(y)} \int_{K_j} \frac{m_1((|f(y)|^{-1/\delta}\widetilde{\tau}^j)^{-1})}{\mu(B(x,\tau|f(y)|^{-1/\delta}\widetilde{\tau}^{j-1}))}\,\mathrm{d}\mu(x)\Big)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y)\\
&\quad \leqslant C\int_X \Big(\sum_{j=1}^{j_0(y)} \int_{B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^j)} \frac{m_1((|f(y)|^{-1/\delta}\widetilde{\tau}^j)^{-1})}{\mu(B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^j))}\,\mathrm{d}\mu(x)\Big)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y)\\
&\quad \leqslant C\int_X \Big(\sum_{j=1}^{j_0(y)} m_1((|f(y)|^{-1/\delta}\widetilde{\tau}^j)^{-1})\Big)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y),
\end{aligned}$$

where

$$K_j = B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^j) \setminus B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^{j-1}).$$

By assumption (ii), we have

$$\begin{aligned}\int_X J_1(x)\,\mathrm{d}\mu(x) &\leqslant C\int_X\left(\int_{d_X^{-1}}^{\widetilde{\tau}|f(y)|^{1/\delta}} m_1(t)\frac{\mathrm{d}t}{t}\right)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X m_2(|f(y)|^{1/\delta})\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X \Phi(|f(y)|)\,\mathrm{d}\mu(y)\leqslant C.\end{aligned}$$

Finally we obtain

$$\begin{aligned}&\int_X J_2(x)\,\mathrm{d}\mu(x)\\ &=\int_X\left(\int_{\{x\in X:\ d(x,y)^{-\delta}\geqslant|f(y)|\}}\frac{1}{m_3(d(x,y)^{-1})\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(x)\right)\\ &\qquad\qquad\times\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y)\\ &=\int_X\left(\sum_{j=1}^{\infty}\int_{K_j'}\frac{1}{m_3(d(x,y)^{-1})\mu(B(x,\tau d(x,y)))}\,\mathrm{d}\mu(x)\right)\\ &\qquad\qquad\times\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X\left(\sum_{j=1}^{\infty}\int_{K_j'}\frac{1}{m_3((|f(y)|^{-1/\delta}\widetilde{\tau}^{-j+1})^{-1})\mu(B(x,\tau|f(y)|^{-1/\delta}\widetilde{\tau}^{-j}))}\,\mathrm{d}\mu(x)\right)\\ &\qquad\qquad\times\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X\left(\sum_{j=1}^{\infty}\int_{B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^{-j+1})}\frac{\mathrm{d}\mu(x)}{m_3((|f(y)|^{-1/\delta}\widetilde{\tau}^{-j+1})^{-1})\mu(B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^{-j+1}))}\right)\\ &\qquad\qquad\times\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X\left(\sum_{j=1}^{\infty}\frac{1}{m_3((|f(y)|^{-1/\delta}\widetilde{\tau}^{-j+1})^{-1})}\right)\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y),\end{aligned}$$

where

$$K_j' = B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^{-j+1})\setminus B(y,|f(y)|^{-1/\delta}\widetilde{\tau}^{-j}).$$

Therefore by assumption (iii)

$$\begin{aligned}\int_X J_2(x)\,\mathrm{d}\mu(x) &\leqslant C\int_X\left(\int_{|f(y)|^{1/\delta}}^{\infty}\frac{1}{m_3(t)}\frac{\mathrm{d}t}{t}\right)\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X\frac{1}{m_4(|f(y)|^{1/\delta})}\Phi(|f(y)|)m_4(|f(y)|)\,\mathrm{d}\mu(y)\\ &\leqslant C\int_X\Phi(|f(y)|)\,\mathrm{d}\mu(y)\leqslant C.\end{aligned}$$

Thus, the conclusion follows. □

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*Authors' addresses*: Daiki Hashimoto, Nagasakihokuyodai High School, Nagasaki 851-2127, Japan; Takao Ohno, Faculty of Education, Oita University, Dannoharu Oita-city 870-1192, Japan, e-mail: `t-ohno@oita-u.ac.jp`; Tetsu Shimomura, Department of Mathematics, Graduate School of Education, Hiroshima University, 1-1-1, Kagamiyama, Higashi-Hiroshima 739-8524, Japan, e-mail: `tshimo@hiroshima-u.ac.jp`.