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GRAPHS WITH SMALL DIAMETER DETERMINED BY THEIR D-SPECTRA

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Abstract. Let G be a connected graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$. The distance matrix $D(G)=(d_{ij})_{n\times n}$ is the matrix indexed by the vertices of G, where d_{ij} denotes the distance between the vertices v_i and v_j. Suppose that $\lambda_1(D)\geqslant\lambda_2(D)\geqslant\ldots\geqslant\lambda_n(D)$ are the distance spectrum of G. The graph G is said to be determined by its D-spectrum if with respect to the distance matrix $D(G)$, any graph having the same spectrum as G is isomorphic to G. We give the distance characteristic polynomial of some graphs with small diameter, and also prove that these graphs are determined by their D-spectra.

Keywords: distance spectrum; distance characteristic polynomial; D-spectrum determined by its D-spectrum

MSC 2010: 05C50

1. Introduction

All graphs considered here are simple, undirected and connected. Let G be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. Two vertices u and v are called adjacent if they are connected by an edge, denoted by $u\sim v$. Let $N_G(v)$ denote the neighbor set of v in G. The degree of a vertex v, written by $d_G(v)$ or $d(v)$, is the number of edges incident with v. Let X and Y be subsets of vertices of G. The induced subgraph $G[X]$ is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. We denote by $E[X,Y]$ the set of edges of G with one end in X and the other end in Y, and denote by $e[X,Y]$ their number. The distance between vertices u and v of a graph G is denoted by

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$d_G(u,v)$. The diameter of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of vertices of G. The complete product $G_1 \bigtriangledown G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. Denote by K_n, C_n, P_n and S_n the complete graph, the cycle, the path and the star, respectively, each on n vertices. Let K_n^c denote the complement of K_n.

The distance matrix $D(G) = (d_{ij})_{n\times n}$ of a connected graph G is the matrix indexed by the vertices of G, where d_{ij} denotes the distance between the vertices v_i and v_j. Let $\lambda_1(D) \geqslant \lambda_2(D) \geqslant \ldots \geqslant \lambda_n(D)$ be the spectrum of $D(G)$, that is, the distance spectrum of G. The polynomial $P_D(\lambda) = \det(\lambda I - D(G))$ is defined as the distance characteristic polynomial of a graph G. Two graphs are said to be D-cospectral if they have the same distance spectrum. A graph G is said to be determined by its D-spectrum if there is no other non-isomorphic graph D-cospectral to G.

Which graphs are determined by their spectrum seems to be a difficult and interesting problem in spectral graph theory. This question was raised by Günthard and Primas in [3]. For surveys of this question see [10], [11]. Up to now, only a few families of graphs were shown to be determined by their spectra, most of which were restricted to the adjacency, Laplacian or signless Laplacian spectra. In particular, there are much fewer results on which graphs are determined by their D-spectra. In [7], Lin et al. proved that the complete graph K_n, the complete bipartite graph K_{n_1,n_2} and the complete split graph $K_a \bigtriangledown K_b^c$ are determined by their D-spectra, and the authors proposed a conjecture that the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is determined by its D-spectrum. Recently, Jin and Zhang in [4] have confirmed the conjecture. Lin, Zhai and Gong in [8] characterized all connected graphs with $\lambda_{n-1}(D(G)) = -1$, and showed that these graphs are determined by their D-spectra. Moreover, in their paper, they also proved that the graphs with $\lambda_{n-2}(D(G)) > -1$ are determined by their distance spectra. In [6], Lin showed that connected graphs with $\lambda_n(D(G)) \geqslant -1-\sqrt{2}$ are determined by their distance spectra. Cioabă et al. in [1] affirmed that the famous friendship graph F_n^k, $k \neq 16$, is determined by its adjacency spectrum. Lu, Huang and Huang in [5] showed that all graphs with exactly two distance eigenvalues (counting multiplicity) different from -1 and -3 are determined by their D-spectra, and particularly, F_n^k is determined by its distance spectrum.

Next, we introduce a class of graphs $K_n^{n_1,n_2,\ldots,n_k}$, as shown in Figure 1.

▷ $K_n^{n_1,n_2,\ldots,n_k}\colon = (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k}) \bigtriangledown \{v\}$, where $k \geqslant 2$.

In this paper, we first show that three special classes of graphs in $K_n^{n_1,n_2,\ldots,n_k}$, that is, $K_n^h = K_n^{h-1,1,\ldots,1}$, $4 \leqslant h \leqslant n-1$, $K_n^{s,t}$, $s \geqslant 4$ and $t \geqslant 4$, and $K_n^{n_1,n_2,\ldots,n_k}$, $1 \leqslant n_i \leqslant 2$, are determined by their D-spectra. Clearly, the friendship graph F_n^k belongs to the third class.

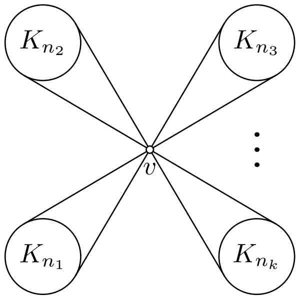

Figure 1. Graph $K_n^{n_1,n_2,\ldots,n_k}$.

Secondly, we prove that K_n^{s+t}, $s \geqslant 2$, $t \geqslant 2$ (see Figure 2), is also determined by its D-spectrum.

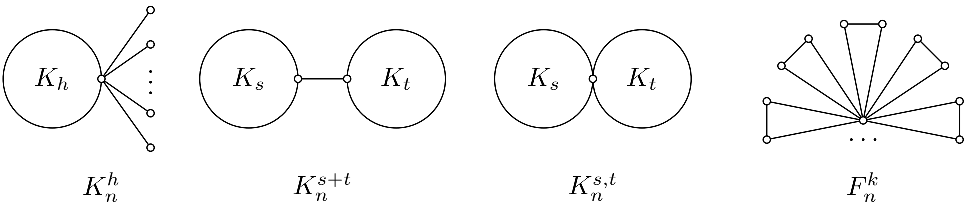

Figure 2. Graphs K_n^h, K_n^{s+t}, $K_n^{s,t}$ and F_n^k.

▷ K_n^{s+t}: the graph obtained by adding one edge joining a vertex of K_s to a vertex of K_t.

2. Preliminaries

In this section, we give some useful lemmas and results. The following lemma is the well-known Cauchy interlacing theorem.

Lemma 2.1 ([2])**.** *Let A be a Hermitian matrix of order n with eigenvalues $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \ldots \geqslant \lambda_n(A)$, and B a principal submatrix of A of order m with eigenvalues $\mu_1(B) \geqslant \mu_2(B) \geqslant \ldots \geqslant \mu_m(B)$. Then $\lambda_{n-m+i}(A) \leqslant \mu_i(B) \leqslant \lambda_i(A)$ for $i = 1, 2, \ldots, m$.*

Applying Lemma 2.1 to the distance matrix D of a graph, we have

Lemma 2.2. *Let G be a graph of order n with distance spectrum $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$, and H an induced subgraph of G on m vertices with the distance spectrum $\mu_1(H) \geqslant \mu_2(H) \geqslant \ldots \geqslant \mu_m(H)$. If $D(H)$ is a principal submatrix of $D(G)$, then $\lambda_{n-m+i}(G) \leqslant \mu_i(H) \leqslant \lambda_i(G)$ for $i = 1, 2, \ldots, m$.*

Lemma 2.3 ([7]). *Let G be a connected graph and D the distance matrix of G. Then $\lambda_n(D) = -2$ with multiplicity $n-k$ if and only if G is a complete k-partite graph for all k, $2 \leqslant k \leqslant n-1$.*

Lemma 2.4 ([12]). *Let G be a graph with order n and $d(G) = 2$. If G' has the same distance spectrum as G, then*

- ▷ $|E(G)| = |E(G')|$ *when* $d(G') = 2$;
- ▷ $|E(G)| < |E(G')|$ *when* $d(G') \geqslant 3$.

Theorem 2.5. *Let $4 \leqslant h \leqslant n-1$. The distance characteristic polynomial of K_n^h is*

$$\begin{aligned} P_D(\lambda) = (\lambda+1)^{h-2}(\lambda+2)^{n-h-1}[\lambda^3 + (h+4-2n)\lambda^2 \\ + (5-2h-2nh+2h^2-n)\lambda - nh + h^2 - 2h + 2]. \end{aligned}$$

Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be the distance spectrum of K_n^h. Then

- ▷ $\lambda_1 > 0$, $-1 < \lambda_2 < -1/2$ *and* $\lambda_3 = -1$;
- ▷ $\lambda_{n-1} \in \{-1, -2\}$ *and* $\lambda_n < -2$.

P r o o f. It is clear that the diameter of K_n^h is 2, and the distance matrix of K_n^h is

$$D = \begin{pmatrix} 0 & \ldots & 1 & 1 & 2 & \ldots & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & 0 & 1 & 2 & \ldots & 2 \\ 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ 2 & \ldots & 2 & 1 & 0 & \ldots & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & \ldots & 2 & 1 & 2 & \ldots & 0 \end{pmatrix}.$$

Then

$$\det(\lambda I - D) = \begin{vmatrix} \lambda & \ldots & -1 & -1 & -2 & \ldots & -2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \ldots & \lambda & -1 & -2 & \ldots & -2 \\ -1 & \ldots & -1 & \lambda & -1 & \ldots & -1 \\ -2 & \ldots & -2 & -1 & \lambda & \ldots & -2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & \ldots & -2 & -1 & -2 & \ldots & \lambda \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix}
\lambda-(h-2) & -1 & \dots & -1 & -1 & -2-2(n-h-1) & -2 & \dots & -2 \\
0 & \lambda+1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \lambda+1 & 0 & 0 & 0 & \dots & 0 \\
-1-(h-2) & -1 & \dots & -1 & \lambda & -1-(n-h-1) & -1 & \dots & -1 \\
-2-2(h-2) & -2 & \dots & -2 & -1 & \lambda-2(n-h-1) & -2 & \dots & -2 \\
0 & 0 & \dots & 0 & 0 & 0 & \lambda+2 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \lambda+2
\end{vmatrix} \\
&= (\lambda+1)^{h-2}(\lambda+2)^{n-h-1} \begin{vmatrix}
\lambda-(h-2) & -1 & -2-2(n-h-1) \\
-1-(h-2) & \lambda & -1-(n-h-1) \\
-2-2(h-2) & -1 & \lambda-2(n-h-1)
\end{vmatrix} \\
&= (\lambda+1)^{h-2}(\lambda+2)^{n-h-1}[\lambda^3+(h+4-2n)\lambda^2+(5-2h-2nh+2h^2-n)\lambda \\
&\quad -nh+h^2-2h+2].
\end{aligned}
$$

In the following, we will prove the remaining part of Theorem 2.5. Consider the cubic function on x

$$
f(x) = x^3+(h+4-2n)x^2+(5-2h-2nh+2h^2-n)x-nh+h^2-2h+2.
$$

From a simple calculation, we have

$$
\begin{cases}
\quad f(0) = -nh+h^2-2h+2 = -h(n-h)-(2h-2) < 0, \\
f(-\frac{1}{2}) = \frac{3}{8}-\frac{3}{4}h < 0, \\
f(-1) = h-n+nh-h^2 = (n-h)(h-1) > 0, \\
f(-2) = 6h-6n+3nh-3h^2 = (n-h)(3h-6) > 0.
\end{cases}
$$

Note that $f(x)\to\infty,\ x\to\infty$, and $f(0)<0$, so there is at least one root in $(0,\infty)$. Since $f(-1/2)<0$ and $f(-1)>0$, there is at least one root in $(-1,-1/2)$. By $f(x)\to-\infty$, $x\to-\infty$, and $f(-2)>0$, there is at least one root in $(-\infty,-2)$. Thus there is exactly one root in each of the three intervals. $\square$

Using a similar method to compute the distance characteristic polynomials of K_n^{s+t} and $K_n^{s,t}$, we have the following two results.

Theorem 2.6. *Let $s\geqslant 2$, $t\geqslant 2$ and $n=s+t$. Then the distance characteristic polynomial of K_n^{s+t} is*

$$
\begin{aligned}
P_D(\lambda) = (\lambda+1)^{n-4}[&\lambda^4+(-s-t+4)\lambda^3+(2t+2s-8st+4)\lambda^2 \\
&+(6s+6t-14st)\lambda-5st+2s+2t].
\end{aligned}
$$

Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ denote the distance spectrum of K_n^{s+t}. Then
▷ $\lambda_1 > 0$, $-1 < \lambda_2 < -1/2$ *and* $\lambda_3 = -1$;
▷ $-2 < \lambda_{n-1} < -1$ *and* $\lambda_n < -2$.

Proof. The distance matrix of K_n^{s+t} is

$$D = \begin{pmatrix} 0 & \ldots & 1 & 1 & 2 & 3 & \ldots & 3 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & 0 & 1 & 2 & 3 & \ldots & 3 \\ 1 & \ldots & 1 & 0 & 1 & 2 & \ldots & 2 \\ 2 & \ldots & 2 & 1 & 0 & 1 & \ldots & 1 \\ 3 & \ldots & 3 & 2 & 1 & 0 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & \ldots & 3 & 2 & 1 & 1 & \ldots & 0 \end{pmatrix}.$$

Similarly to the proof of Theorem 2.5, by a simple calculation, we have

$$\begin{aligned} \det(\lambda I - D) &= (\lambda+1)^{n-4} \begin{vmatrix} \lambda-(s-2) & -1 & -2 & -3-3(t-2) \\ -1-(s-2) & \lambda & -1 & -2-2(t-2) \\ -2-2(s-2) & -1 & \lambda & -1-(t-2) \\ -3-3(s-2) & -2 & -1 & \lambda-(t-2) \end{vmatrix} \\ &= (\lambda+1)^{n-4}[\lambda^4 + (-s-t+4)\lambda^3 + (2t+2s-8st+4)\lambda^2 \\ &\quad + (6s+6t-14st)\lambda - 5st + 2s + 2t]. \end{aligned}$$

Consider the quartic function on x

$$f(x) = x^4 + (-s-t+4)x^3 + (2t+2s-8st+4)x^2 + (6s+6t-14st)x - 5st + 2s + 2t.$$

Note that $(s-1)(t-1) = st - s - t + 1 > 0$, hence $st + 1 > s + t$. Then we obtain that

$$\begin{cases} f(0) = -5st + 2s + 2t < 2(st+1) - 5st = 2 - 3st < 0, \\ f(-\frac{1}{2}) = \frac{9}{16} - \frac{3}{8}s - \frac{3}{8}t < 0, \\ f(-1) = 1 - s - t + st > 0, \\ f(-2) = 6s + 6t - 9st < 6(st+1) - 9st = 6 - 3st < 0. \end{cases}$$

Note that $f(x) \to \infty$, $x \to \infty$, and $f(0) < 0$, so there is at least one root in $(0, \infty)$. Since $f(-1/2) < 0$ and $f(-1) > 0$, there is at least one root in $(-1, -1/2)$. Since $f(-1) > 0$ and $f(-2) < 0$, then there is at least one root in $(-2, -1)$. By $f(x) \to \infty$, $x \to -\infty$, and $f(-2) < 0$, there is at least one root in $(-\infty, -2)$. Thus there is exactly one root in each of the three intervals. The proof is completed. □

Theorem 2.7. *Let $s \geqslant 4$, $t \geqslant 4$ and $n = s+t-1$. Then the distance characteristic polynomial of $K_n^{s,t}$ is*

$$P_D(\lambda) = (\lambda+1)^{n-3}[\lambda^3 + (-s-t+4)\lambda^2 + (2+s+t-3st)\lambda + s+t-2st].$$

Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ denote the distance spectrum of $K_n^{s,t}$. Then

▷ $\lambda_1 > 0$, $-1 < \lambda_2 < -2/3$ *and* $\lambda_3 = -1$;

▷ $\lambda_{n-1} = -1$ *and* $\lambda_n < -2$.

Proof. The distance matrix of $K_n^{s,t}$ is

$$D = \begin{pmatrix} 0 & \ldots & 1 & 1 & 2 & \ldots & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & 0 & 1 & 2 & \ldots & 2 \\ 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ 2 & \ldots & 2 & 1 & 0 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & \ldots & 2 & 1 & 1 & \ldots & 0 \end{pmatrix}.$$

Similarly to the proof of Theorem 2.5, we have

$$\begin{aligned} \det(\lambda I - D) &= (\lambda+1)^{n-3} \begin{vmatrix} \lambda-(s-2) & -1 & -2-2(t-2) \\ -1-(s-2) & \lambda & -1-(t-2) \\ -2-2(s-2) & -1 & \lambda-(t-2) \end{vmatrix} \\ &= (\lambda+1)^{n-3}[\lambda^3 + (-s-t+4)\lambda^2 + (2+s+t-3st)\lambda + s+t-2st]. \end{aligned}$$

Consider the cubic function on x

$$f(x) = x^3 + (-s-t+4)x^2 + (2+s+t-3st)x + s+t-2st.$$

Note that $(s-1)(t-1) = st-s-t+1 > 0$, hence $st+1 > s+t$. By a simple calculation, we have

$$\begin{cases} f(0) = s+t-2st < 1-st < 0, \\ f(-\frac{2}{3}) = \frac{4}{27} - \frac{1}{9}s - \frac{1}{9}t < 0, \\ f(-1) = 1-s-t+st > 0. \end{cases}$$

Note that $f(x) \to \infty$, $x \to \infty$ and $f(0) < 0$, so there is at least one root in $(0, \infty)$. Since $f(-2/3) < 0$ and $f(-1) > 0$, there is at least one root in $(-1, -2/3)$. Since $f(-1) > 0$ and $f(x) \to -\infty$, $x \to -\infty$, there is at least one root in $(-\infty, -1)$. Thus there is exactly one root in each of the three intervals. This means that $\lambda_1 > 0$, $-1 < \lambda_2 < -2/3$, $\lambda_3 = \lambda_{n-1} = -1$ and $\lambda_n < -1$.

Obviously, the diameter of $K_n^{s,t}$ is 2, and P_3 is an induced subgraph of $K_n^{s,t}$. Moreover, $D(P_3)$ is a principal submatrix of $D(K_n^{s,t})$. It is easy to calculate that $\lambda_3(P_3) = -2$, then by Lemma 2.2, $\lambda_n(K_n^{s,t}) \leqslant \lambda_3(P_3) = -2$. Furthermore, $K_n^{s,t}$ is not a complete k-partite graph, hence by Lemma 2.3, we have $\lambda_n < -2$. □

By Theorems 2.5, 2.6 and 2.7, we obtain the following corollary.

Corollary 2.8. *No two non-isomorphic graphs of K_n^h, K_n^{s+t} and $K_n^{s,t}$ are D-cospectral.*

Proof. From the distance characteristic polynomials of K_n^h, K_n^{s+t} and $K_n^{s,t}$ for any two non-isomorphic graphs belonging to the same type, the result is obvious.

It is clear that K_n^{s+t} and $K_n^{s,t}$ have distinct distance spectra, since -1 is the distance eigenvalue of K_n^{s+t} with multiplicity $n-4$, and it is the distance eigenvalue of $K_n^{s,t}$ with multiplicity $n-3$.

Now we only need to prove that K_n^h has a distance spectrum distinct from K_n^{s+t} and $K_n^{s,t}$.

Suppose that K_n^h and K_n^{s+t} are D-cospectral. Note that -1 is the distance eigenvalue of K_n^{s+t} with multiplicity $n-4$, then -1 is also the distance eigenvalue of K_n^h with multiplicity $n-4$. On the other hand, notice that -2 is not the distance eigenvalue of K_n^{s+t}, then it follows that -2 is not the distance eigenvalue of K_n^h either, thus $n = h+1$. Then -1 is the distance eigenvalue of K_n^h with multiplicity $n-3$, a contradiction.

Assume that K_n^h and $K_n^{s,t}$ are D-cospectral. Note that -2 is not the distance eigenvalue of $K_n^{s,t}$, then it follows that -2 is not the distance eigenvalue of K_n^h either, so $n = h+1$. So we have

$$\begin{cases} P_{D(K_n^h)}(\lambda) = (\lambda+1)^{n-3}[\lambda^3 + (-n+3)\lambda^2 + (-5n+9)\lambda - 3n + 5], \\ P_{D(K_n^{s,t})}(\lambda) = (\lambda+1)^{n-3}[\lambda^3 + (-s-t+4)\lambda^2 + (2+s+t-3st)\lambda + s + t - 2st]. \end{cases}$$

Note that they have the same distance characteristic polynomial, hence

$$\begin{cases} -3n+5 = s+t-2st, \\ n = s+t-1. \end{cases}$$

Solving the two equations we get $t = 2$ or $t = n-1$, a contradiction. □

3. Main results

In this section, our first task is to show that K_n^h, K_n^{s+t} and $K_n^{s,t}$ are determined by their D-spectra. First, we give some useful graphs and their distance spectra.

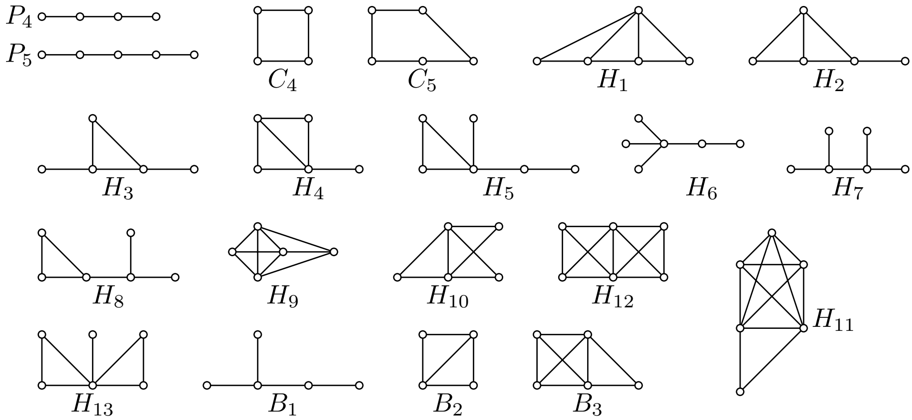

Figure 3. Graphs P_4, P_5, C_4, C_5, H_1–H_{13} and B_1–B_3.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
P_4	5.1623	−0.5858	−1.1623	−3.4142		
P_5	8.2882	−0.5578	−0.7639	−1.7304	−5.2361	
C_4	4.0000	0.0000	−2.0000	−2.0000		
C_5	6.0000	−0.3820	−0.3820	−2.6180	−2.6180	
H_1	5.2926	−0.3820	−0.7217	−1.5709	−2.6180	
H_2	6.2162	−0.4521	−1.0000	−1.1971	−3.5669	
H_3	6.6375	−0.5858	−0.8365	−1.8010	−3.4142	
H_4	5.7596	−0.5580	−0.7667	−2.0000	−2.4348	
H_5	9.3154	−0.5023	−1.0000	−1.0865	−2.3224	−4.4042
H_6	9.6702	−0.4727	−1.0566	−2.0000	−2.0000	−4.1409
H_7	10.0000	−0.4348	−1.0000	−2.0000	−2.0000	−4.5616
H_8	9.6088	−0.4931	−1.0000	−1.0924	−2.0000	−5.0233
H_9	4.4495	−0.4495	−1.0000	−1.0000	−2.0000	
H_{10}	5.3723	−0.3723	−1.0000	−2.0000	−2.0000	
H_{11}	6.1425	−0.4913	−1.0000	−1.0000	−1.0000	−2.6512
H_{12}	6.4641	−0.4641	−1.0000	−1.0000	−1.0000	−3.0000
H_{13}	7.8526	−0.6303	−1.0000	−1.0000	−2.2223	−3.0000
B_1	7.4593	−0.5120	−1.0846	−2.0000	−3.8627	
B_2	3.5616	−0.5616	−1.0000	−2.0000		
B_3	4.9018	−0.5122	−1.0000	−1.0000	−2.3896	

Next, we first show that K_n^h is determined by its D-spectrum. Let G be a graph D-cospectral to K_n^h. We call H a forbidden subgraph of G if G contains no H as an induced subgraph.

Lemma 3.1. *If G and K_n^h are D-cospectral, then C_4, C_5 and H_i, $i \in \{1, 4, 9, 10, 11, 12, 13\}$, are forbidden subgraphs of G.*

P r o o f. Let G and K_n^h have the same distance spectrum. Suppose that H is an induced subgraph of G and $H \in \{C_4, C_5, H_i, i \in \{1, 4, 9, 10, 11, 12, 13\}\}$. Note that $\operatorname{diam}(H) = 2$, obviously $D(H)$ is a principal submatrix of $D(G)$. Let $|V(H)| = m$, then by Lemma 2.2, $\lambda_2(G) \geqslant \lambda_2(H)$, $\lambda_3(G) \geqslant \lambda_3(H)$ and $\lambda_{m-1}(H) \geqslant \lambda_{n-1}(G)$. By Theorem 2.5, we know that $-1 < \lambda_2(G) < -1/2$, $\lambda_3(G) = -1$ and $\lambda_{n-1}(G) \in \{-1, -2\}$. Hence we have $\lambda_2(H) < -1/2$, $\lambda_3(H) \leqslant -1$ and $\lambda_{m-1}(H) \geqslant -2$. However, $\lambda_2 \geqslant -1/2$ for C_4, C_5 and H_i, $i \in \{1, 9, 10, 11, 12\}$; $\lambda_3 > -1$ for H_4 and $\lambda_{m-1} < -2$ for H_{13}, a contradiction. $\square$

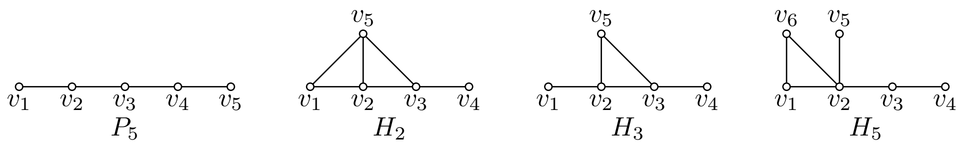

Figure 4. The labeled graphs of P_5, H_2, H_3 and H_5.

For any $S \subseteq V(G)$, let $D_G(S)$ denote the principal submatrix of $D(G)$ obtained by S.

Lemma 3.2. *If G and K_n^h are D-cospectral, then P_5 and H_i, $i \in \{2, 3, 5, 6, 7, 8\}$, are forbidden subgraphs of G.*

P r o o f. For P_5: Suppose that P_5 is an induced subgraph of G, then $d_G(v_1, v_5) \in \{2, 3, 4\}$. If $d_G(v_1, v_5) = 4$, then $D_G(\{v_1, v_2, v_3, v_4, v_5\}) = D(P_5)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_3(G) \geqslant \lambda_3(P_5) = -0.7639 > -1$, a contradiction. If $d_G(v_1, v_5) \in \{2, 3\}$, let $d_G(v_1, v_4) = a$, $d_G(v_1, v_5) = b$ and $d_G(v_2, v_5) = c$, then $a, b, c \in \{2, 3\}$. We get the principal submatrix of $D(G)$

$$D_G(\{v_1, v_2, v_3, v_4, v_5\}) = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 1 & 0 & 1 & 2 & c \\ 2 & 1 & 0 & 1 & 2 \\ a & 2 & 1 & 0 & 1 \\ b & c & 2 & 1 & 0 \end{pmatrix}.$$

By a simple calculation, we have

(a,b,c)	$(3,3,3)$	$(3,2,2)$	$(3,2,3)$	$(3,3,2)$	$(2,3,3)$	$(2,3,2)$	$(2,2,2)$	$(2,2,3)$
λ_2	-0.4348	-0.3260	0	-0.3713	-0.3713	-0.1646	-0.2909	-0.3260

By Lemma 2.2 we have $\lambda_2(G) \geqslant \lambda_2(D_G(\{v_1,v_2,v_3,v_4,v_5\})) > -1/2$. Note that $\lambda_2(G) < -1/2$, a contradiction. Hence P_5 is a forbidden subgraph of G.

For H_2: Assume that H_2 is an induced subgraph of G, then $d_G(v_1,v_4) \in \{2,3\}$. If $d_G(v_1,v_4) = 3$, then $D(H_2)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_2(G) \geqslant \lambda_2(H_2) = -0.4521 > -1/2$, a contradiction. If $d_G(v_1,v_4) = 2$, it is easy to calculate that $\lambda_2(D_G(\{v_1,v_2,v_3,v_4,v_5\})) = -0.2284 > -1/2$. By Lemma 2.2 and Theorem 2.5, we also get a contradiction. Therefore H_2 is a forbidden subgraph of G.

For H_3: Suppose that H_3 is an induced subgraph of G, then $d_G(v_1,v_4) \in \{2,3\}$. If $d_G(v_1,v_4) = 3$, then $D(H_3)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_3(G) \geqslant \lambda_3(H_3) = -0.8365 > -1$, a contradiction. If $d_G(v_1,v_4) = 2$, it is easy to check that $\lambda_2(D_G(\{v_1,v_2,v_3,v_4,v_5\})) = -0.3820 > -1/2$. By Lemma 2.2 and Theorem 2.5, we also obtain a contradiction. Hence H_3 is a forbidden subgraph of G.

For H_5: Assume that H_5 is an induced subgraph of G. If $d_G(v_1,v_4) = d_G(v_4,v_5) = d_G(v_4,v_6) = 3$, then $D(H_5)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_{n-1}(G) \leqslant \lambda_5(H_5) = -2.3224 < -2$, a contradiction. Otherwise, there exists at least one equal to 2 among $d_G(v_1,v_4), d_G(v_4,v_5)$ and $d_G(v_4,v_6)$. Without loss of generality, we may assume that $d_G(v_1,v_4) = 2$. Note that H_5 is an induced subgraph of G, hence there exists a vertex $v \in V(G) \setminus \{v_1,v_2,v_3,v_4,v_5\}$ such that $vv_1, vv_4 \in E(G)$. Then $G[vv_1v_2v_3v_4] = C_5$, $G[vv_1v_2v_3v_4] = H_1$, $G[vv_2v_3v_4] = C_4$ or $G[vv_1v_2v_3] = C_4$. By Lemma 3.1, C_4, C_5 and H_1 are forbidden subgraphs of G, a contradiction. Hence H_5 is a forbidden subgraph of G.

For H_6, H_7 and H_8: Suppose that they are induced subgraphs of G, respectively. If $D(H_6)$, $D(H_7)$ and $D(H_8)$ are principal submatrices of $D(G)$, respectively. By Lemma 2.2, $\lambda_2(G) \geqslant \lambda_2(H_i) > -1/2$ where $i \in \{6,7,8\}$, a contradiction. Otherwise, similarly to the discussion for H_5, we can also obtain the same contradictions. Thus H_6, H_7 and H_8 are forbidden subgraphs of G. □

Theorem 3.3. *The graph K_n^h is determined by its D-spectrum.*

Proof. Let G be a graph D-cospectral to K_n^h. By Lemma 3.2, P_5 is a forbidden graph of G, thus $\operatorname{diam}(G) \leqslant 3$. By $\lambda_n(G) < -2$, we have $\operatorname{diam}(G) \geqslant 2$.

Case 1: $\operatorname{diam}(G) = 3$.

If $|V(G)| = 4$, then $G = P_4$, and it is easy to check that G is not D-cospectral to K_4^3, a contradiction. Next we assume that $|V(G)| \geqslant 5$. Note that $\operatorname{diam}(G) = 3$, then there exists a diameter-path $P = u\tilde{u}\tilde{v}v$ with length 3 in G. Let $X = \{u, \tilde{u}, \tilde{v}, v\}$, hence $G[X] = P_4$. Denote by V_i, $i = 0, 1, 2, 3, 4$, the vertex subset of $V \setminus X$ whose each vertex is adjacent to i vertices of X. Clearly $V \setminus X = \bigcup_{i=0}^{4} V_i$.

Claim 1: $V_4 = \emptyset$.

Suppose not, then there exists a vertex $v_4 \in V_4$ such that $G[v_4 u\tilde{u}\tilde{v}v] = H_1$, a contradiction. Hence Claim 1 holds.

Claim 2: $V_3 = \emptyset$.

Suppose not, then there exists a vertex $v_3 \in V_3$ such that v_3 is adjacent to $\{u, \tilde{u}, \tilde{v}\}$, $\{\tilde{u}, \tilde{v}, v\}$, $\{u, \tilde{u}, v\}$ or $\{u, \tilde{v}, v\}$. Then G contains an induced subgraph H_2 or C_4, a contradiction.

Let $V_2^u = \{v_2 \in V_2\colon\ v_2u, v_2\tilde{u} \in E(G)\}$ and $V_2^v = \{v_2 \in V_2\colon\ v_2v, v_2\tilde{v} \in E(G)\}$.

Claim 3: $V_2 = V_2^u \cup V_2^v$, $G[V_2^u](G[V_2^v]) = K_{|V_2^u|}(K_{|V_2^v|})$ and $E[V_2^u, V_2^v] = \emptyset$.

For any $v_2 \in V_2$, it is impossible that v_2 is adjacent to u and v since $d_G(u, v) = 3$. If v_2 is adjacent to u and $\tilde{v}$ (or $\tilde{u}$ and v), then $G[v_2u\tilde{u}\tilde{v}] = C_4$ (or $G[v_2\tilde{u}\tilde{v}v] = C_4$), by Lemma 3.1, a contradiction. If v_2 is adjacent to $\tilde{u}$ and $\tilde{v}$, then $G[v_2u\tilde{u}\tilde{v}v] = H_3$, a contradiction. Thus $V_2 = V_2^u \cup V_2^v$. For any $v_2, v_2^\star \in V_2^u$, we then have $v_2v_2^\star \in E(G)$. Otherwise $G[v_2v_2^\star u\tilde{u}\tilde{v}] = H_4$, a contradiction. This means that $G[V_2^u] = K_{|V_2^u|}$. Similarly, $G[V_2^v] = K_{|V_2^v|}$. If $v_2v_2^\star \in E(G)$ for any $v_2 \in V_2^u$ and $v_2^\star \in V_2^v$, then $G[v_2v_2^\star\tilde{u}\tilde{v}] = C_4$, a contradiction. Hence $E[V_2^u, V_2^v] = \emptyset$.

Claim 4: $|V_1| \leqslant 1$.

Let $v_1 \in V_1$. Obviously, v_1 can only be adjacent to $\tilde{u}$ or $\tilde{v}$, otherwise $G[v_1u\tilde{u}\tilde{v}v] = P_5$, a contradiction. Now we assume that $|V_1| \geqslant 2$. Let $v_1, v_1^\star \in V_1$. If they are adjacent to the same vertex of X, then $G[v_1v_1^\star u\tilde{u}\tilde{v}v] = H_5$ or H_6, a contradiction. Otherwise, $G[v_1v_1^\star u\tilde{u}\tilde{v}v] = H_7$ or $G[v_1v_1^\star\tilde{u}\tilde{v}] = C_4$, a contradiction. Hence Claim 4 is completed.

Claim 5: Only one set of V_1 and V_2 is nonempty.

Suppose not, then there exist two vertices $v_1 \in V_1$ and $v_2 \in V_2$. Without loss of generality, we may assume that v_2 is adjacent to u and $\tilde{u}$. If v_1 is adjacent to $\tilde{u}$, then $G[v_1v_2u\tilde{u}\tilde{v}v] = H_5$ or $G[v_1v_2u\tilde{u}\tilde{v}] = H_4$, a contradiction. If v_1 is adjacent to $\tilde{v}$, then $G[v_1v_2u\tilde{u}\tilde{v}v] = H_8$ or $G[v_1v_2\tilde{u}\tilde{v}] = C_4$, a contradiction. Thus Claim 5 holds.

Claim 6: $V_0 = \emptyset$.

Suppose not, then there exists a vertex $v_0 \in V_0$ such that $v_0v^\star \in E(G)$, where $v^\star \in V_1 \cup V_2$. Then $G[v_0v^\star\tilde{u}\tilde{v}v] = P_5$ or $G[v_0v^\star u\tilde{u}\tilde{v}] = P_5$, a contradiction.

By Claims 1–6, we have $V = V_1 \cup V_2 \cup X$. If $|V_1| = 1$, then by Claim 5, $V_2 = \emptyset$. This means that $G \cong B_1$. It is easy to check that B_1 has D-spectrum distinct from K_5^h, a contradiction. So we have $V_1 = \emptyset$, then $V_2 \neq \emptyset$, and thus $G \cong K_n^{s+t}$. By

Corollary 2.8, K_n^{s+t} has D-spectrum distinct from K_n^h, a contradiction. It follows that there is no graph G with diameter 3 D-cospectral to K_n^h.

Case 2: $\mathrm{diam}(G) = 2$.

There exists a diameter-path $P = xyz$ with length 2 in G. Let $X = \{x, y, z\}$, then $G[X] = P_3$. Obviously, $V \setminus X \neq \emptyset$ since $n \geqslant 4$. Denote by V_i, $i = 0, 1, 2, 3$ the vertex subset of $V \setminus X$ whose each vertex is adjacent to i vertices of X. Clearly $V \setminus X = \bigcup\limits_{i=0}^{3} V_i$.

Claim 7: $|V_3| \leqslant 1$.

Suppose not, then there exist two vertices $v_3, v_3^\star \in V_3$. If $v_3v_3^\star \in E(G)$, then $G[v_3v_3^\star xyz] = H_9$, a contradiction. Otherwise $v_3v_3^\star \notin E(G)$, then $G[v_3v_3^\star xz] = C_4$, a contradiction. Therefore Claim 7 holds.

Let $V_{xy} = \{v_2 \in V_2\colon\ v_2x, v_2y \in E(G)\}$, $V_{yz} = \{v_2 \in V_2\colon\ v_2y, v_2z \in E(G)\}$.

Claim 8: $V_2 = V_{xy} \cup V_{yz}$, $G[V_{xy}](G[V_{yz}]) = K_{|V_{xy}|}(K_{|V_{yz}|})$, and $E[V_{xy}, V_{yz}] = \emptyset$.

For any $v_2 \in V_2$, it is impossible that v_2 is adjacent to x and z since $G[v_2xyz] = C_4$. Hence $V_2 = V_{xy} \cup V_{yz}$. For any $v_2, v_2^\star \in V_{xy}$, we then have $v_2v_2^\star \in E(G)$. Otherwise $G[v_2v_2^\star xyz] = H_4$, a contradiction. This means that $G[V_{xy}] = K_{|V_{xy}|}$. Similarly, $G[V_{yz}] = K_{|V_{yz}|}$. If $E[V_{xy}, V_{yz}] \neq \emptyset$, then there exist two vertices $v_2 \in V_{xy}$ and $v_2^\star \in V_{yz}$ such that $v_2v_2^\star \in E(G)$, and thus $G[v_2v_2^\star xyz] = H_1$, a contradiction. Hence $E[V_{xy}, V_{yz}] = \emptyset$.

Claim 9: If $v_1 \in V_1$, then v_1 must be adjacent to y.

Suppose not, then v_1 is adjacent to x or z. Without loss of generality, we may assume that $v_1x \in E(G)$. Note that $\mathrm{diam}(G) = 2$, then there exists a vertex $u \in V \setminus X$ such that $uv_1, uz \in E(G)$, and thus $u \in \bigcup\limits_{i=1}^{3} V_i$. If $u \in V_1$, then $G[uv_1xyz] = C_5$, a contradiction. If $u \in V_2$, then by Claim 8, u is adjacent to y and z, and then $G[uv_1xy] = C_4$, a contradiction. If $u \in V_3$, then $G[uv_1xyz] = H_1$, a contradiction. Thus Claim 9 holds.

Claim 10: $V_0 = \emptyset$.

Suppose not, then there exists a vertex $v_0 \in V_0$ such that v_0 is adjacent to some vertices of $V_1 \cup V_2 \cup V_3$. If v_0 is adjacent to only one vertex u of $V_1 \cup V_2 \cup V_3$, then $u \in V_3$ since $\mathrm{diam}(G) = 2$, and thus $G[v_0uxyz] = H_4$, a contradiction. So v_0 must be adjacent to at least two vertices of $V_1 \cup V_2 \cup V_3$; we always find an induced subgraph C_4 of G in each case, a contradiction. Therefore Claim 10 is obtained.

By Claim 10, $\emptyset \neq V \setminus X = \bigcup\limits_{i=1}^{3} V_i$. Next we distinguish the following four subcases.

Subcase 2.1: $V_3 \neq \emptyset$.

By Claim 7, $|V_3| = 1$. Note that H_4 and H_{10} are forbidden subgraphs of G, then $V_1 = \emptyset$. Let $V_3 = \{v_3\}$. Obviously, $v_2v_3 \in E(G)$ for each $v_2 \in V_2$. Otherwise

$G[v_2v_3xyz] = H_1$, a contradiction. If $|V_2| \leqslant 2$, i.e., there exist two vertices $v_2, v_2^\star \in V_2$, then $G[v_2v_2^\star v_3xyz] = H_{11}$ or H_{12}, a contradiction. So we have $|V_2| \leqslant 1$. If $V_2 = \emptyset$, then $G \cong B_2$, and it is easy to check that B_2 has distance spectrum distinct from K_4^3, a contradiction. If $|V_2| = 1$, then $G \cong B_3$. Clearly, B_3 is not D-cospectral to K_5^h, a contradiction.

Subcase 2.2: $V_3 = \emptyset$, $V_2 \neq \emptyset$ and $V_1 = \emptyset$.

By Claim 8, $G \cong K_n^{n-1}$ or $G \cong K_n^{s,t}$. By Corollary 2.8, $K_n^{s,t}$ and K_n^h have distinct distance spectra, a contradiction. Hence $G \cong K_n^{n-1}$.

Subcase 2.3: $V_3 = \emptyset$, $V_2 \neq \emptyset$ and $V_1 \neq \emptyset$.

For any $v_1 \in V_1$, we claim that $d(v_1) = 1$. In fact, if $d(v_1) \geqslant 2$, then there exists a vertex $v_2 \in V_2$ such that $v_1v_2 \in E(G)$, and then $G[v_1v_2xyz] = H_4$, a contradiction. Furthermore, we claim that only one set of V_{xy} and V_{yz} is nonempty. Otherwise, let $v_2 \in V_{xy}$ and $v_2^\star \in V_{yz}$, then $G[v_2v_2^\star xyz] = H_{13}$, a contradiction. Hence $G \cong K_n^h$.

Subcase 2.4: $V_3 = \emptyset$, $V_2 = \emptyset$ and $V_1 \neq \emptyset$.

Let $V_1^\star = \{v \in V_1 \colon d(v) \geqslant 2\}$. If $V_1^\star = \emptyset$, then $G \cong K_{1,n-1}$. Note that $\lambda_n(K_{1,n-1}) = -2$, then $K_{1,n-1}$ is not D-cospectral to K_n^h, a contradiction. If $V_1^\star \neq \emptyset$, we claim that $G[V_1^\star] = K_{|V_1^\star|}$. If not, there exist $u, v \in V_1^\star$ such that $uv \notin E(G)$. If there exists a vertex $w \in V_1^\star$ such that $wu, wv \in E(G)$, then $G[wuvxy] = H_4$, a contradiction. Otherwise, there exist two distinct vertices $w_1 \in V_1^\star$ and $w_2 \in V_1^\star$ such that $w_1u \in E(G)$ and $w_2v \in E(G)$, then $w_1w_2 \in E(G)$ since H_{13} is a forbidden subgraph of G. Thus $G[w_1w_2uvy] = H_1$, a contradiction. Hence $G[V_1^\star] = K_{|V_1^\star|}$, which means that $G \cong K_n^h$. $\square$

Theorem 3.4. *The graph K_n^{s+t} is determined by its D-spectrum.*

P r o o f. Let G be a graph D-cospectral to K_n^{s+t}. From Theorem 2.6, we know that $-1 < \lambda_2(G) < -1/2$, $\lambda_3(G) = -1$ and $-2 < \lambda_{n-1}(G) < -1$. Similarly to the proof of Lemmas 3.1 and 3.2, we also get P_5, C_4, C_5 and H_i, $i = 1, 2, \ldots, 13$, are forbidden subgraphs of G. Note that P_5 is a forbidden subgraph of G and $\lambda_n(G) < -2$, hence $2 \leqslant \operatorname{diam}(G) \leqslant 3$. By the above forbidden subgraphs, similarly to the proof of Theorem 3.3, we have:

- ▷ If $\operatorname{diam}(G) = 3$, then $G \cong B_1$ or $G \cong K_n^{s+t}$.
- ▷ If $\operatorname{diam}(G) = 2$, then $G \cong B_2$, $G \cong B_3$, $G \cong K_n^h$ or $G \cong K_n^{s,t}$.

From D-spectra of B_i, $i = 1, 2, 3$, and Corollary 2.8, then we must have $G \cong K_n^{s+t}$. Thus the theorem follows. $\square$

Theorem 3.5. *The graph $K_n^{s,t}$ is determined by its D-spectrum.*

P r o o f. Let G be a graph D-cospectral to $K_n^{s,t}$. By Theorem 2.7, then $-1 < \lambda_2(G) < -2/3 < -1/2$, $\lambda_3(G) = \lambda_{n-1}(G) = -1$. Hence we can still use P_5, C_4, C_5

and H_i, $i = 1, 2, \ldots, 13$, as the forbidden subgraphs of G. Note that P_5 is a forbidden subgraph of G and $\lambda_n(G) < -2$, hence $2 \leqslant \operatorname{diam}(G) \leqslant 3$. Similarly to the proof of Theorem 3.3, then:

▷ If $\operatorname{diam}(G) = 3$, then $G \cong B_1$ or $G \cong K_n^{s+t}$.

▷ If $\operatorname{diam}(G) = 2$, then $G \cong B_2$, $G \cong B_3$, $G \cong K_n^h$ or $G \cong K_n^{s,t}$.

By D-spectra of B_i, $i = 1, 2, 3$, and Corollary 2.8, then $G \cong K_n^{s,t}$. Thus $K_n^{s,t}$ is determined by its D-spectrum. □

In [9], Liu et al. give the distance characteristic polynomial of $K_n^{n_1,n_2,\ldots,n_k}$:

$$P_D(\lambda) = (\lambda+1)^{n-k-1}\left(\lambda - \sum_{i=1}^{k} \frac{n_i(2\lambda+1)}{\lambda+n_i+1}\right)\prod_{i=1}^{k}(\lambda+n_i+1).$$

Next, we will show that $K_n^{n_1,n_2,\ldots,n_k}$, $1 \leqslant n_i \leqslant 2$, is determined by its D-spectrum.

Theorem 3.6. $K_n^{n_1,n_2,\ldots,n_k}$, $1 \leqslant n_i \leqslant 2$, *is determined by its D-spectrum.*

Proof. Let $G := K_n^{n_1,n_2,\ldots,n_k}$, where $1 \leqslant n_i \leqslant 2$. Let t_1 and t_2 be two nonnegative integers with $t_1 + t_2 = k$. Suppose that $n_1 = \ldots = n_{t_1} = 1$ and $n_{t_1+1} = \ldots = n_{t_1+t_2} = 2$. Clearly, if $t_1 = 0$, then G is the friendship graph F_n^k. If $t_2 = 0$, then G is a star. Recall that the star is determined by its D-spectrum. So we assume that $t_2 \geqslant 1$. Note that the distance characteristic polynomial of G is

$$\begin{aligned}P_D(\lambda) = {}&(\lambda+1)^{n-t_1-t_2-1}(\lambda+2)^{t_1-1}(\lambda+3)^{t_2-1}(\lambda^3 + (5-4t_2-2t_1)\lambda^2 \\ &+ (6-10t_2-7t_1)\lambda - 3t_1 - 4t_2).\end{aligned}$$

Consider the cubic function

$$f(\lambda) = \lambda^3 + (5-4t_2-2t_1)\lambda^2 + (6-10t_2-7t_1)\lambda - 3t_1 - 4t_2.$$

By calculation, we have

$$\begin{cases} f(0) = -3t_1 - 4t_2 < 0, \\ f(-\frac{1}{2}) = -\frac{15}{8}, \\ f(-1) = 2t_1 + 2t_2 - 2 \geqslant 0, \\ f(-2) = 3t_1 \geqslant 0, \\ f(-3) = -10t_2 < 0. \end{cases}$$

Then the three roots of $f(\lambda) = 0$ belong to the intervals $(0, \infty)$, $[-1, -1/2)$ and $(-3, -2]$, respectively. Consequently, we have $-1 \leqslant \lambda_2(G) < -1/2$, $\lambda_3(G) = -1$ and $\lambda_n(G) = -3$.

Suppose that G' is D-cospectral to G, that is $-1 \leqslant \lambda_2(G') < -1/2$, $\lambda_3(G') = -1$ and $\lambda_n(G') = -3$. In the following, we only need to show that $G' \cong G$. It is easy to see that G' cannot contain P_4 as an induced subgraph, otherwise we would have $\lambda_n(G') \leqslant \lambda_4(P_4) = -3.4142$, which contradicts $\lambda_n(G') = -3$. Thus the diameter of G' is 2. Let $P = xyz$ be a diameter path of G'.

Claim 1: $d_{G'}(y) = n-1$. If there exists a vertex $v \in V(G')$ such that $vy \notin E(G')$, then $d_{G'}(v,y) = 2$, and thus

$$D_{G'}(\{x,y,z,v\}) = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & b \\ a & 2 & b & 0 \end{pmatrix}.$$

Then $a, b \in \{1,2\}$, and by a simple calculation we have

(a,b)	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
λ_2	0.0000	-0.3820	-0.3820	-0.6519

By Lemma 2.2, only the case $a = 2$, $b = 2$ satisfies $\lambda_2(G') < -1/2$. Thus there exists a vertex w such that the subgraph of G' induced by vertices v, w, x, y, z is T_1, T_2 or T_3 (see Figure 5). We get a principal submatrix of $D(G')$ for each case:

$$D_1 = \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

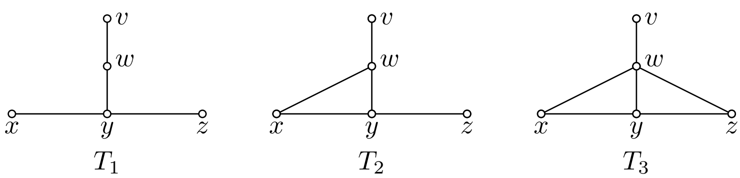

Figure 5. Graphs T_1, T_2, T_3.

A simple calculation gives $\lambda_2(D_1) = -0.2248$, $\lambda_2(D_2) = -0.3820$ and $\lambda_3(D_3) = -0.7667$. For each case, the Cauchy interlacing theorem implies $\lambda_2(G') \geqslant \lambda_2(D_1) = -0.2248$, $\lambda_2(G') \geqslant \lambda_2(D_2) = -0.3820$ and $\lambda_3(G') \geqslant \lambda_3(D_3) = -0.7667$, a contradiction. Thus Claim 1 holds.

Claim 2: $G' - y$ is the disjoint union of some cliques. According to Lemma 2.4, we obtain G' has $n-1+t_2$ edges. It follows from Claim 1 that $G'-y$ has t_2 edges. Since

$t_2 \leqslant \lfloor (n-1)/2 \rfloor$, there are at least two connected components in $G' - y$. Suppose that there is a component which is not a clique. Then we can see that H_4 is an induced subgraph of G'. Therefore $\lambda_3(G') \geqslant \lambda_3(H_4) = -0.7667$, a contradiction. Thus Claim 2 holds.

Combining Claims 1 and 2, we have $G' \cong K_1 \vee (K_{n'_1} \cup K_{n'_2} \cup \ldots \cup K_{n'_t})$. According to the distance characteristic polynomial of G and G', we have $t = k$ and $n'_i = n_i$, i.e. $G' \cong G$, as desired. $\square$

The following result follows from Theorem 3.6 immediately.

Corollary 3.7 ([5]). *The friendship graph F_n^k is determined by its D-spectrum.*

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