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UNICYCLIC GRAPHS WITH BICYCLIC INVERSES

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Abstract. A graph is nonsingular if its adjacency matrix $A(G)$ is nonsingular. The inverse of a nonsingular graph G is a graph whose adjacency matrix is similar to $A(G)^{-1}$ via a particular type of similarity. Let $\mathcal{H}$ denote the class of connected bipartite graphs with unique perfect matchings. Tifenbach and Kirkland (2009) characterized the unicyclic graphs in $\mathcal{H}$ which possess unicyclic inverses. We present a characterization of unicyclic graphs in $\mathcal{H}$ which possess bicyclic inverses.

Keywords: adjacency matrix; unicyclic graph; bicyclic graph; inverse graph; perfect matching

MSC 2010: 05C50, 15A09

1. Introduction

Let G be a simple, undirected graph on n vertices. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $[i,j]$ to denote the edge between the vertices i and j. By $d_G(i)$ we denote the degree of a vertex i in G. The *adjacency matrix* $A(G)$ of G is the square symmetric matrix of size n whose (i,j)th entry a_{ij} is 1 if $[i,j] \in E(G)$ and 0 otherwise.

A graph G is *nonsingular* if $A(G)$ is nonsingular. A *perfect matching* in a graph G is a collection of vertex-disjoint edges that covers every vertex. Note that G can have more than one perfect matchings. If a graph G has a unique perfect matching, then we denote it by $\mathcal{M}$. Furthermore, when v is a vertex, we shall always use v' to denote the matching mate for v, that is, v' is the vertex for which the edge $[v,v'] \in \mathcal{M}$. A bipartite graph is nonsingular when it has a unique perfect matching, see [6]. Let $\mathcal{H}$ be the class of connected bipartite graphs with unique perfect matchings. A connected graph G is said to be a *unicyclic* graph if G has the same number of

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edges and vertices. A connected graph G of order n is said to be a *bicyclic* graph if G has $n+1$ edges. Graphs G and H are isomorphic ($G \cong H$) if one can be obtained by relabeling the vertices of the other.

In quantum chemistry, a graph known as the Hückel graph is used to model the molecular orbital energies of a hydrocarbon. Under some conditions a Hückel graph may be seen as a bipartite graph with a unique perfect matching, see [15]. This is one motivation for considering bipartite graphs with unique perfect matchings.

We say λ is an eigenvalue of G if λ is an eigenvalue of $A(G)$. We use $\sigma(G)$ to denote the spectrum of G. By P_n we denote the path on n vertices.

The notion of an inverse graph was introduced by Harary and Minc in 1976 (see [8]). A nonsingular graph G is said to be invertible if $B = A(G)^{-1}$ is a matrix with entries from $\{0,1\}$. The graph H with adjacency matrix B is called the inverse graph of G. In the same article, they proved that a connected graph G is invertible if and only if $G = P_2$.

Unfortunately, under the above definition, only one connected graph is invertible. So, we must redefine the inverse of a graph. In 1985 Godsil (see [6]) supplied another notion of an inverse graph, which generalizes the definition given by Harary and Minc.

Definition 1.1. A *signature matrix* is a diagonal matrix with diagonal entries from $\{1,-1\}$. Note that for any signature matrix S we have $S^{-1} = S$, so that two matrices A and B are signature similar if there is a signature matrix S such that $B = SAS$.

Let $G \in \mathcal{H}$. We say G has an inverse G^{+} if the matrix $A(G)^{-1}$ is signature similar to a nonnegative matrix, that is, $SA(G)^{-1}S \geqslant 0$ for some signature matrix S and G^{+} is a weighted graph associated to the matrix $SA(G)^{-1}S$. An invertible graph G is said to be a *self-inverse* graph if G is isomorphic to its inverse graph (see [6]).

There are some other notions of an inverse graph proposed in the literature. We list some of them. In 1978, Cvetković, Gutman and Simić introduced in [4] the *pseudo-inverse* of a graph. Let G be a graph. The pseudo-inverse graph $PI(G)$ of G is the graph, defined on the same vertex set as G, in which the vertices x and y are adjacent if and only if $G - x - y$ has a perfect matching. In 1988, Buckley, Doty and Harary introduced in [3] the *signed inverse* of a graph. A *signed graph* is a graph in which each edge has a positive or negative sign, see [7]. An adjacency matrix of a signed graph is symmetric and each entry is 0, 1, or -1. Let G be a nonsingular graph. The graph G has a signed inverse if $A(G)^{-1}$ is the adjacency matrix of some signed graph H. In 1990, Pavlíková and Jediný introduced in [11] another notion of the inverse of a graph. The *inverse* of a nonsingular graph with the spectrum $\lambda_1, \ldots, \lambda_n$ is a graph with the spectrum $1/\lambda_1, \ldots, 1/\lambda_n$. This type of inverse of a graph need not be unique. Henceforth, we follow the notion of inverse graph given by Godsil.

Let $\mathcal{H}_g = \{G \in \mathcal{H}\colon\ G/\mathcal{M} \text{ is bipartite}\}$ where $G/\mathcal{M}$ is the graph obtained from G by contracting each matching edge to a vertex. In [6], Godsil showed that if $G \in \mathcal{H}_g$, then G^+ exists. He posed the problem of characterizing the graphs in $\mathcal{H}$ which possess inverses. In [1], Akbari and Kirkland characterized the unicyclic graphs $G \in \mathcal{H}$ which possess inverses. In [14], Tifenbach and Kirkland supplied necessary and sufficient conditions for graphs in $\mathcal{H}$ to possess inverses, utilizing constructions derived from the graph itself. In [10], Panda and Pati supplied a more general class of graphs in $\mathcal{H}$ for which G^+ exists. The search for a characterization of graphs in $\mathcal{H}$ which possess inverses is still open.

Consider the question of characterizing the graphs in $\mathcal{H}$ which are isomorphic to their inverses. This question was posed by Godsil in 1985 for the class of graphs $\mathcal{H}_g$ and has already been answered by Simion and Cao in [12]. They showed that for any $G \in \mathcal{H}_g$, the graph $G \cong G^+$ if and only if G is a corona graph. In [14], Tifenbach and Kirkland have supplied necessary and sufficient conditions for an invertible unicyclic graph $G \in \mathcal{H}$ to be self-inverse. These conditions required constructions involving the directed graphs and the undirected interval graphs. In [13], Tifenbach supplied necessary and sufficient conditions for a graph in $\mathcal{H}$ satisfying $G \cong G^+$ via a particular isomorphism.

In [9], Panda and Pati characterized the graphs $G \in \mathcal{H}_g$ such that $G^+ \in \mathcal{H}_g$ and supplied the constructive characterization of the inverse graphs G^+ of G in $\mathcal{H}_g$.

Tifenbach and Kirkland supplied in [14] necessary and sufficient conditions for an invertible unicyclic graph $G \in \mathcal{H}$ to satisfy that G^+ is unicyclic. Having considered this result, it is natural to ask the following question. Can we characterize the unicyclic graphs G in $\mathcal{H}$ such that G^+ is bicyclic? In this article, we supply such characterization.

2. Preliminaries

In order to continue the study of the inverse graphs, the following notion of odd type and even type nonmatching edges was introduced by Panda and Pati (see [10]).

Definition 2.1 ([10])**.** Let $G \in \mathcal{H}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an *nn-alternating path* if the edges on P are alternately nonmatching and matching edges with $[u_1, u_2], [u_{2k-1}, u_{2k}] \in E(G) \setminus \mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an *mm-alternating path* if the edges on P are alternately nonmatching and matching edges with $[u_1, u_2], [u_{2k-1}, u_{2k}] \in \mathcal{M}$.

Definition 2.2 ([10])**.** Let $G \in \mathcal{H}$ and $[u,v] \in E(G) \setminus \mathcal{M}$. An *extension* at $[u,v]$ is an nn-alternating u-v-path other than $[u,v]$. An extension at $[u,v]$ is called an *even type one* or an *odd type one* if the number of nonmatching edges on that extension is even or odd, respectively.

A nonmatching edge $[u,v]$ is said to be an even type edge, if each extension at $[u,v]$ is an even type one. The nonmatching edge $[u,v]$ is said to be an odd type edge, if either there are no extensions at $[u,v]$ or each extension at $[u,v]$ is odd type.

Example 2.1 ([10])**.** In the graph G shown in Figure 1, $[i'_2, u_1, u'_1, u_2, u'_2, u_3, u'_3, i_3]$ and $[i'_2, u_1, u'_1, u_2, u'_2, v_1, v'_1, v_2, v'_2, u_3, u'_3, i_3]$ are two even type extensions at $[i'_2, i_3]$. These are the only extensions at $[i'_2, i_3]$. Hence, $[i'_2, i_3]$ is an even type edge. Every other nonmatching edge is an odd type edge.

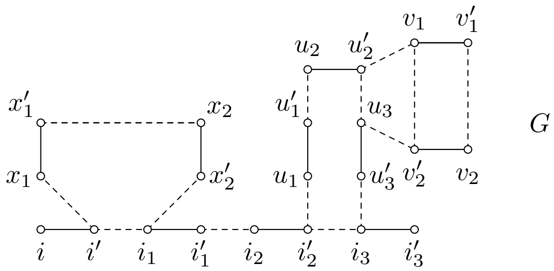

Figure 1. In all diagrams, the solid edges are matching edges.

3. Unicyclic graphs which possess bicyclic inverses

By [10], Theorem 36, a unicyclic graph G in $\mathcal{H}$ is invertible if and only if either $G \in \mathcal{H}_g$ or G has exactly one even type edge with exactly one even type extension. It is known that the inverse graph of a bipartite connected graph with a unique perfect matching is also a bipartite connected graph with a unique perfect matching. Nontheless, the inverse graph G^+ of an unweighted graph G may be weighted. In the next subsection, we characterize the unicyclic graphs G in $\mathcal{H}_g$ such that G^+ is bicyclic (that is, G^+ is an unweighted connected bicyclic graph with a unique perfect matching).

3.1. Unicyclic graphs in $\mathcal{H}_g$.

Remark 3.1. In [9], [10], the authors observed that the following points are true for any graph G in $\mathcal{H}_g$.

▷ Each nonmatching edge in G is an odd type one.

▷ If a path from i to j contains an odd or an even number of nonmatching edges, then each path from i to j must contain an odd or an even number of nonmatching edges, respectively.

▷ The inverse graph G^+ of G is unweighted if and only if there is at most one mm-alternating path from one vertex to another vertex.

To proceed further, we need the following known definitions.

Definition 3.1. A pendant vertex of a graph is a vertex of degree 1, and a quasi pendant vertex of a graph is a vertex adjacent to a pendant vertex. An edge of a graph is said to be pendant if one of its vertices is a pendant vertex.

A corona graph $G \circ K_1$ is a graph which is obtained from a graph G by adding a new pendant vertex to every vertex of G. (The term 'corona graph', in general, means the graph obtained by taking the 'corona product' of two graphs, see [5]. However, for our purpose, we only need the corona product of a graph with K_1, and we call them the corona graphs.) A *corona tree* is a corona graph which is a tree.

Theorem 3.1 ([9]). *Let $G \in \mathcal{H}_g$. Then the following conditions are equivalent.*

(1) $|\mathcal{P}_G| = |E(G^+)|$, *where $\mathcal{P}_G$ is the set of mm-alternating paths in G.*

(2) $G \cong G^+$.

(3) $G = G_1 \circ K_1$ *for some connected bipartite graph G_1.*

The following description of the inverse of the adjacency matrix of a connected, bipartite graph with a unique perfect matching is a restatement of results contained in [1], [2]. We follow the convention that a sum over an empty set is zero.

Lemma 3.1. *Let $G \in \mathcal{H}$. Let $B = [b_{ij}]$, where*

$$b_{ij} = \sum_{P \in \mathcal{P}(i,j)} (-1)^{(\|P\|-1)/2},$$

where $\mathcal{P}(i,j)$ is the set of mm-alternating i-j-paths in G and $\|P\|$ denotes the number of edges in P. Then $B = A(G)^{-1}$.

The following result supplies a necessary and sufficient condition for a unicyclic graph $G \in \mathcal{H}_g$ to be such that G^+ is bicyclic.

Theorem 3.2. *Let $G \in \mathcal{H}_g$ be unicyclic. Then G^+ is bicyclic if and only if G has exactly one mm-alternating path of length* 5.

P r o o f. Since $G \in \mathcal{H}_g$, G^+ exists. First we assume that G^+ is bicyclic. We now show that G has an mm-alternating path of length 5. Assume that G has no mm-alternating path of length 5. Then the length of each mm-alternating path in G is either 1 or 3. Using this fact one can easily show that each matching edge in G is a pendant edge. Hence G is a corona. By Theorem 3.1, $G \cong G^+$, a contradiction to the fact that G^+ is bicyclic. Thus, G has an mm-alternating path of length 5.

Suppose that G has two mm-alternating paths of length 5, say, P_1 and P_2. Since G^+ is unweighted, Remark 3.1 yields that there is at most one mm-alternating path in G from one vertex to another vertex. Then the set of end vertices of P_1 is not equal to the set of end vertices of P_2. Then by Theorem 3.1, G^+ has at least $n+2$ edges, which is not possible. Hence G has exactly one mm-alternating path of length 5.

Now we consider the converse part. Since G has exactly one mm-alternating path of length 5, the length of the other mm-alternating paths in G is at most 3. As G is unweighted, Remak 3.1 yields that there is at most one mm-alternating path in G from one vertex to another vertex. Then there are totally $n+1$ mm-alternating path in G. Using Theorem 3.1, we have G^+ has exactly $n+1$ edges. Hence G^+ is bicyclic. $\square$

Corollary 3.1. *Let $G \in \mathcal{H}_g$ be unicyclic and let G^+ be bicyclic. Assume that $[u, u', x, x', v, v']$ is the mm-alternating path in G of length 5 in G. Then*

$$d_G(x) = 2 = d_G(x') \quad \text{and} \quad d_G(v') = 1 = d_G(u).$$

P r o o f. First we show that $d_G(x) = 2$. Suppose that $d_G(2) \geqslant 3$. Let $y \neq \{u', x'\}$ be a vertex adjacent to x in G. Notice that $y \neq \{u, v'\}$, otherwise G has more than one perfect matching and $y \neq v$, otherwise G has a cycle $[x, x', v, x]$ of length 3. Then the path $[y', y, x, x', v, v']$ is also an mm-alternating path of length 5. Therefore G has at least two mm-alternating paths of length 5, a contradiction. Hence $d_G(x) = 2$. Similarly, we can show that $d_G(x') = 2$ and $d_G(v') = 1 = d_G(u)$. The proof is complete. $\square$

Lemma 3.2. *Let $G \in \mathcal{H}_g$ be unicyclic and let G^+ be bicyclic. Assume that $[u, u', x, x', v, v']$ is the mm-alternating path of length 5. Then the graph $H = G - [x, x']$ is a unicyclic corona graph.*

P r o o f. Using Corollary 3.1, we obtain $d_G(x) = 2 = d_G(x')$. There are two cases.

Case 1. The edge $[x, x']$ is not a cut edge. By Theorem 3.2, the length of each mm-alternating path in H is at most 3. Hence, H is a corona of a bipartite graph.

Case 2. The edge $[x, x']$ is a cut edge. Then H has two components H_1 and H_2. By similar arguments both H_1 and H_2 are coronas. Since G is unicyclic, one of H_1 and H_2 is a unicyclic corona graph and the other one is a corona tree. Hence, H is a unicyclic corona graph. $\square$

Definition 3.2. Let $\mathcal{F}_1$ denote the class of graphs G such that G is obtained in the following manner.

(1) Take a corona tree T. Take a disjoint union of $[u, u']$ with T.
(2) Choose two quasi-pendant vertices x and y in T such that the x-y-path in T contains an odd number of edges.
(3) Add the edges $[x, u]$ and $[y, u']$.

Example 3.1. Take a disjoint union of the corona tree T shown in Figure 2 and $[u, u']$. Take the vertices 1 and 4. Notice that the length of the 1-4-path is odd in T. Now add the edges $[u, 1]$ and $[u', 4]$. Then the resulting graph G is in $\mathcal{F}_1$.

Figure 2. Example of a graph in $\mathcal{F}_1$.

Definition 3.3. Let $\mathcal{F}_2$ denote the class of graphs G such that G is obtained in the following manner.
(1) Take a corona tree T and a unicyclic corona graph H. Take a disjoint union of $[u, u']$ with T and H.
(2) Choose two quasi-pendant vertices x in T and y in H.
(3) Add the edges $[x, u]$ and $[y, u']$.

Example 3.2. Consider the corona tree T and the unicyclic corona H shown in Figure 3. Take a disjoint union of T, H and $[u, u']$. Let 3 and 4 be two quasi-pendant vertices in T and H, respectively. Now add the edges $[u, 3]$ and $[u', 4]$. Hence the resulting graph is in $\mathcal{F}_2$.

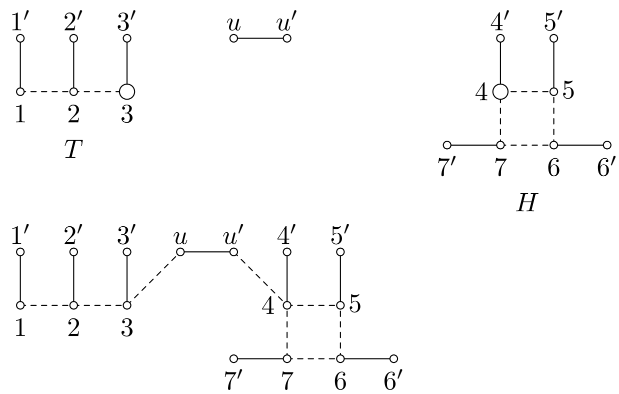

Figure 3. Example of a graph in $\mathcal{F}_2$.

Theorem 3.3. *Let $G \in \mathcal{H}_g$ be unicyclic. Then G^+ is bicyclic if and only if either $G \in \mathcal{F}_1$ or $G \in \mathcal{F}_2$.*

P r o o f. First we consider G^+ is bicyclic. Then due to Lemma 3.2, either $G \in \mathcal{F}_1$ or $G \in \mathcal{F}_2$.

Now we assume that either $G \in \mathcal{F}_1$ or $G \in \mathcal{F}_2$. First we assume that $G \in \mathcal{F}_1$. The graph G has exactly one mm-alternating path of length 5. Let $[x, x', y, y', z, z']$ be such a path. By Lemma 3.1, we have

(1) $A(G^+)_{u',v'} \neq 0$ for each nonmatching edge $[u, v]$ in G;

(2) $A(G^+)_{u,u'} \neq 0$ for each matching edge $[u, u']$ in G;

(3) $A(G^+)_{x,z'} \neq 0$.

Hence, G^+ has at least $n + 1$ edges. Now we show that there are no extra edges in G^+. Suppose that G^+ has an edge $[u, v] \neq [x, y]$ such that $[u', v']$ is not in G. Then there is an mm-alternating u-v-path in G of length more than 3, which is not possible by Theorem 3.2. Hence, G^+ has exactly $n + 1$ edges. So G^+ is bicyclic.

Similarly, if $G \in \mathcal{F}_2$, then G^+ is bicyclic. $\square$

3.2. Unicyclic graphs in $\mathcal{H} \backslash \mathcal{H}_g$. In this subsection we characterize the unicyclic graphs G in $\mathcal{H} \setminus \mathcal{H}_g$ such that G^+ is bicyclic. We recall that a unicyclic graph $G \in \mathcal{H} \setminus \mathcal{H}_g$ is invertible if and only if G has exactly one even type edge with exactly one even type extension, see [10]. The following points are true for any invertible unicyclic graph $G \in \mathcal{H} \setminus \mathcal{H}_g$.

(1) There are at most two mm-alternating paths in G from one vertex to another vertex.

(2) If there are exactly two mm-alternating paths in G from one vertex to another vertex, then one of them contains the even type edge and the other one contains the even type extension.

(3) If there are two mm-alternating paths in G from one vertex to another vertex, then one of them contains an even or odd number of nonmatching edges if and only if the other one contains, respectively, an odd or even number of nonmatching edges.

The following lemma tells us that the inverse graph G^+ for each invertible graph $G \in \mathcal{H} \setminus \mathcal{H}_g$ has at least $n - 1$ edges.

Lemma 3.3. *Let $G \in \mathcal{H} \setminus \mathcal{H}_g$ be an invertible unicyclic graph. Assume that $[x, y]$ is an even type edge in G. Then $G - [x, y]$ is isomorphic to a subgraph of G^+.*

P r o o f. Let $[u, v] \neq [x, y]$ be any nonmatching edge in G. Then there is exactly one mm-alternating u'-v'-path in G which is $[u', u, v, v']$. By Lemma 3.1, $A(G^+)_{u',v'} = 1$. For each matching edge $[u, u']$, by Lemma 3.1, we have $A(G^+)_{u',u} = 1$. Hence, for each edge $[u, v] \neq [x, y]$ in G, there is an edge $[u', v']$ in G^+. Therefore $G - [x, y]$ is isomorphic to a subgraph of G^+. The proof is complete. □

To proceed further, we need the following definition.

Definition 3.4 ([10]). A minimal path in a graph $G \in \mathcal{H}$ from a vertex u to a vertex v is an mm-alternating u-v-path which does not contain an even type extension (at some nonmatching edge in G). A simple minimal path is a minimal path which does not contain an even type edge.

Example 3.3. Consider the path $P = [i, i', x_1, x_1', x_2, x_2', i_1, i_1', i_2, i_2']$ in G shown in Figure 1. The path P contains exactly one extension $[x_1, x_1', x_2, x_2']$ which is odd type. So the path P is minimal. Each nonmatching edge on P is odd type. Hence P is a simple minimal path.

Consider the path $Q = [i_1, i_1', i_2, i_2', u_1, u_1', u_2, u_2', u_3, u_3', i_3, i_3']$ in G shown in Figure 1. The path Q contains the extension $[i_2', u_1, u_1', u_2, u_2', u_3, u_3', i_3]$ which is even type. So the path Q is an mm-alternating path but not a minimal path.

Lemma 3.4. *Let $G \in \mathcal{H} \setminus \mathcal{H}_g$ be an invertible unicyclic graph. Assume that $[x, y]$ is an even type edge in G. Then G^+ has a nonmatching edge $[u, v]$ such that $[u', v']$ is not in G if and only if G has a simple minimal u-v-path of length at least* 5.

Proof. First we assume that there is a nonmatching edge $[u, v]$ in G^+ such that $[u', v']$ is not in G. Since $[u, v]$ is in G^+, there is an mm-alternating u-v-path in G, say, $P(u, v)$. The length of $P(u, v)$ is more than 3, otherwise $[u', v']$ is in G. We now show that the path $P(u, v)$ does not contain the even type edge $[x, y]$. Suppose that $P(u, v)$ contains the even type edge $[x, y]$. Then there is another mm-alternating u-v-path in G which contains the even type extension at $[x, y]$, say, $P_1(u, v)$. Notice that the path $P(u, v)$ contains an even or odd number of nonmatching edges if and only if the path $P_1(u, v)$ has, respectively, an odd or even number of nonmatching edges. By Lemma 3.1, $A(G^+)_{u,v} = 0$, a contradiction to the fact that $[u, v]$ in G^+. Hence, the path $P(u, v)$ does not contain the even type edge $[x, y]$. Similarly, $P(u, v)$ does not contain the even type extension at $[x, y]$. Hence $P(u, v)$ is a simple minimal u-v-path in G of length at least 5. The proof is complete. □

The following theorem supplies a necessary and sufficient condition for a unicyclic graph G in $\mathcal{H} \setminus \mathcal{H}_g$ to have a bicyclic inverse.

Theorem 3.4. *Let $G \in \mathcal{H} \setminus \mathcal{H}_g$ be an invertible unicyclic graph. Then G^+ is bicyclic if and only if the graph G has exactly two simple minimal paths of length* 5.

P r o o f. First we assume that G^+ is bicyclic. Now we show that G has a simple minimal path of length 5. Assume that G has no simple minimal path of length 5. Then by virtue of Lemmas 3.3 and 3.4, G^+ has $n-1$ edges, which is a contradiction to the fact that G^+ is bicyclic. Hence G has at least one simple minimal path of length 5.

Now we show that G has exactly two simple minimal paths of length 5. Suppose that G has three such paths, say P_i for $i = 1, 2, 3$. Since G is invertible, the graph G has exactly one even type edge. Notice that the set of end vertices of P_i is not equal to the set of end vertices of P_j for $i, j = 1, 2$, and $i \neq j$. Using Lemmas 3.3 and 3.4, we get G^+ has at least $n+2$ edges, which is not possible. If G has exactly one simple minimal path, then using Lemmas 3.3 and 3.4, we have G^+ has exactly n edges, which is not possible. Then G has exactly two simple minimal paths of length 5.

We now show the converse. Since G has exactly two simple minimal paths of length 5, by Lemmas 3.3 and 3.4, G^+ has exactly $n+1$ edges. Hence, G^+ is bicyclic. $\square$

The following result is an immediate corollary of Theorem 3.4.

Corollary 3.2. *Let $G \in \mathcal{H} \setminus \mathcal{H}_g$ be an invertible unicyclic graph. Assume that G^+ is bicyclic. Then the length of the cycle in G is* 4.

P r o o f. Let C be the cycle in G. Since $G \in \mathcal{H} \setminus \mathcal{H}_g$ is an invertible unicyclic graph, the graph G has exactly one even type edge with exactly one even type extension. Let $[x, y]$ be the even type edge in G and let $Q(x, y) = [x, u_1, u_1', \ldots, u_{2k-1}, u_{2k-1}', y]$ where $k > 1$ is the even type extension at $[x, y]$. Then $C = [x, u_1, u_1', \ldots, u_{2k-1}, u_{2k-1}', y, x]$. The mm-alternating paths $[x', 1, u_1, u_1', u_2, u_2']$, $[u_1, u_1', u_2, u_2', u_3, u_3']$ and $[u_{2k-2}, u_{2k-2}', u_{2k-1}, u_{2k-1}', y, y']$ are three simple minimal paths of length 5, a contradiction to the fact that G has exactly two simple minimal paths of length 5. Then $k = 1$. Hence, the length of the cycle in G is 4. $\square$

4. Conclusions

In [6], Godsil introduced the notion of the inverse graph for bipartite graphs with unique perfect matchings (which we denoted by $\mathcal{H}$). Characterization of graphs in $\mathcal{H}$ which possess inverses is an interesting problem. In the same article Godsil supplied a class of invertible graphs (which we denoted by $\mathcal{H}_g$). In [14], the authors presented a characterization of unicyclic graphs in $\mathcal{H}$ which possess inverses. The study of the structures or characteristics of the inverses of graphs is also an interesting problem. In [14], the authors presented a characterization of the unicyclic graphs in $\mathcal{H}$ which possess unicyclic inverses. Here we have presented a characterization of the unicyclic graphs in $\mathcal{H}$ which possess bicyclic inverses. In order to do this, we have divided the unicyclic graphs in $\mathcal{H}$ into two subclasses which are $\mathcal{H}_g$ and $\mathcal{H} \setminus \mathcal{H}_g$. We have supplied a necessary and sufficient conditions for a unicyclic graph in $\mathcal{H}_g$ or $\mathcal{H} \setminus \mathcal{H}_g$ to have bicyclic inverse. Furthermore, we have found elegant constructions for unicyclic graphs in $\mathcal{H}_g$ with bicyclic inverses. However, we have not been able to find any constructions of unicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_g$ with bicyclic inverses.

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