Czechoslovak Mathematical Journal

Doo Hyun Hwang; Eunmi Pak; Changhwa Woo

Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting restricted normal Jacobi operators

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 4, 989–1004

Persistent URL: `http://dml.cz/dmlcz/146962`

Czechoslovak Mathematical Journal, 67 (142) (2017), 989–1004

REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS WITH COMMUTING RESTRICTED NORMAL JACOBI OPERATORS

Doo Hyun Hwang, Eunmi Pak, Daegu, Changhwa Woo, Samnye

Received June 7, 2016. First published August 15, 2017.

Abstract. We give a classification of Hopf real hypersurfaces in complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ with commuting conditions between the restricted normal Jacobi operator $\overline{R}_N\varphi$ and the shape operator A (or the Ricci tensor S).

Keywords: real hypersurface; complex hyperbolic two-plane Grassmannians; Hopf hypersurface; shape operator; Ricci tensor; normal Jacobi operator; commuting condition

MSC 2010: 53C40, 53C15

Introduction

A typical example of Hermitian symmetry spaces of rank 2 is the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Another one is the complex hyperbolic two-plane Grassmannian $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, the set of all complex two-dimensional linear subspaces in the indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$.

Characterizing model spaces of real hypersurfaces under certain geometric conditions is one of our main interests in the classification theory in $G_2(\mathbb{C}^{m+2})$ or $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. In this paper, we use the same geometric condition on real hypersurfaces in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ as used in $G_2(\mathbb{C}^{m+2})$ to compare the results.

$G_2(\mathbb{C}^{m+2}) = \mathrm{SU}_{2+m}/S(U_2{\cdot}U_m)$ has compact transitive group SU_{2+m}, however $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ has noncompact indefinite transitive group $\mathrm{SU}_{2,m}$. This distinction

This work was supported by Grant Proj. No. NRF-2015-R1A2A1A-01002459. The third author is supported by NRF Grant funded by the Korean Government Grant Proj. No. 2017R1C1B1010265.

gives various remarkable results. Riemannian symmetric space $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ has a remarkable geometrical structure. It is the unique noncompact, irreducible, quaternionic Kähler manifold with negative curvature.

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. Let N be a local unit normal vector field on M. Since the complex hyperbolic two-plane Grassmannian $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ has the Kähler structure J, we may define a *Reeb vector field* $\xi = -JN$ and a 1-dimensional distribution $\mathcal{C}^{\perp} = \mathrm{Span}\{\,\xi\}$.

Let $\mathcal{C}$ be the orthogonal complement of a distribution $\mathcal{C}^{\perp}$ in T_pM at $p \in M$. It is the complex maximal subbundle of TM. Thus the tangent space of M consists of the direct sum of $\mathcal{C}$ and $\mathcal{C}^{\perp}$ as follows: $T_pM = \mathcal{C} \oplus \mathcal{C}^{\perp}$. The real hypersurface M is said to be *Hopf* if $A\mathcal{C} \subset \mathcal{C}$, or equivalently, the Reeb vector field ξ is principal with principal curvature $\alpha = g(A\xi, \xi)$, where g denotes the metric. In this case, the principal curvature α is said to be a *Reeb curvature* of M.

Due to the quaternionic Kähler structure $\mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$ of $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, there naturally exist *almost contact 3-structure vector fields* $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Let $\mathcal{Q}^{\perp} = \mathrm{Span}\{\,\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent space T_pM of M at $p \in M$. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^{\perp}$ in T_pM. It is the quaternionic maximal subbundle of T_pM. Thus the tangent space of M can be split into $\mathcal{Q}$ and $\mathcal{Q}^{\perp}$ as follows: $T_pM = \mathcal{Q} \oplus \mathcal{Q}^{\perp}$.

Thus, we have considered two natural geometric conditions for real hypersurfaces in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ such that the subbundles $\mathcal{C}$ and $\mathcal{Q}$ of TM are both invariant under the shape operator. By using these geometric conditions, we will use the results of Suh in [8], Theorem 1.

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold $(\overline{M}, \overline{g})$ plays an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. It is defined by $(\overline{R}_X(Y))(p) = (\overline{R}(Y, X)X)(p)$, where $\overline{R}$ denotes the curvature tensor of $\overline{M}$ and X, Y denote any vector fields on $\overline{M}$. It is known to be a self-adjoint endomorphism on the tangent space $T_p\overline{M}$, $p \in \overline{M}$. Clearly, each tangent vector field X to $\overline{M}$ provides a Jacobi operator with respect to X. Thus the Jacobi operator on a real hypersurface M of $\overline{M}$ with respect to N is said to be a *normal Jacobi operator* and will be denoted by $\overline{R}_N$. The Riemannian curvature tensors of M and $\overline{M}$ are denoted by R and $\overline{R}$, respectively.

For a commuting problem concerned with the structure Jacobi operator R_ξ and the structure tensor φ of Hopf hypersurface M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, that is, $R_\xi\varphi A = AR_\xi\varphi$, Lee, Suh and Woo in [5] gave a characterization of real hypersurface of Tube (A) or horosphere A in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. Motivated by this result, we want to give a classification of Hopf hypersurfaces in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ whose normal Jacobi operator $\overline{R}_N$ satisfies

$$\overline{R}_N\varphi AX = A\overline{R}_N\varphi X \tag{C-1}$$

for any tangent vector field X on M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. That is, the operator $\overline{R}_N\varphi$ commutes with the shape operator A. The geometric meaning of $\overline{R}_N\varphi AX = A\overline{R}_N\varphi X$ can be explained in such a way that any eigenspace of A on $\mathcal{C}$ is invariant under $\overline{R}_N\varphi$ of M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$.

The complex maximal subbundle $\mathcal{C}$ can be split into $A(\lambda_1)\oplus A(\lambda_2)\oplus\ldots\oplus A(\lambda_k)$, where each $\{A(\lambda_j)\}_{j=1}^{j=k}$ is a principal curvature space of A with principal curvature λ_j. For any $X\in A(\lambda_j)$ on $\mathcal{C}$, (C-1) gives $A\overline{R}_N(X)=\overline{R}_N(AX)=\lambda_j X$, that is, $\overline{R}_N(X)\in A(\lambda_j)$.

In physics, space-like hypersurface with CMC has an important physical meaning in general relativity for existence and uniqueness results in the family of cosmological models including generalized Robertson Walker space time (see Alías, Romero, Sánchez [1], [2]). Moreover, it was known that such hypersurfaces play an important part in relativity, since it was noted that they can be used as initial hypersurfaces where the constraint equations can be split into a linear system and a nonlinear elliptic equation (see Latorre and Romero [4]).

Now we want to give a complete classification of Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ with $\overline{R}_N\varphi AX = A\overline{R}_N\varphi X$:

Theorem 1. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m\geqslant 3$, with $\overline{R}_N\varphi A = A\overline{R}_N\varphi$. Then M is locally congruent to one of the following:*

(i) *a tube over a totally geodesic $\mathrm{SU}_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ or*

(ii) *a horosphere in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ where the center at infinity is singular and of type $JX\in\mathfrak{J}X$.*

From the Riemannian curvature tensor R of M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ we can define the Ricci tensor S of M in such a way that

$$g(SX,Y)=\sum_{i=1}^{4m-1} g(R(e_i,X)Y,e_i),$$

where $\{e_1,\ldots,e_{4m-1}\}$ denotes a basis of the tangent space T_pM of M, $p\in M$, in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ (see [11]). Then we can consider another new commuting condition

$$\overline{R}_N\varphi SX = S\overline{R}_N\varphi X \tag{C-2}$$

for any tangent vector field X on M. That is, the operator $\overline{R}_N\varphi$ commutes with the Ricci tensor S.

Since the Ricci tensor S is also a symmetric operator, $\mathcal{C}$ is decomposed to many kinds of Einstein subspaces as follows: $S(\mu_1) \oplus S(\mu_2) \oplus \ldots \oplus S(\mu_l)$, where each $S(\mu_i) = \{X \in \mathcal{C}\colon SX = \mu_i X\}$ denotes an Einstein subspace of $\mathcal{C}$ in T_xM, $x \in M$. Then it follows that $S\overline{R}_N(X) = \overline{R}_N(SX) = \mu_j\overline{R}_N X$, that is, $\overline{R}_N(X) \in S(\mu_j)$ for any $X \in S(\mu_j)$, which means that each Einstein subspace of $\mathcal{C}$ is invariant by the normal Jacobi operator $\overline{R}_N$. It can be displaced in parallel by the normal Jacobi operator $\overline{R}_N$ along the normal direction N of M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. This means that each dimension of Einstein subspaces can be constant and cannot be contracted to a smaller dimension along the normal direction N of M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. Accordingly it follows that $\overline{R}_N\varphi\mathcal{C} = \overline{R}_N\mathcal{C} \subset \mathcal{C}$. This gives $\overline{R}_N\xi = d\xi$ for a smooth function d on M. Then also by the result due to Berndt and Suh in [3], the hypersurface M becomes CMC. In this case we also have the same physical meaning as in (C-1).

Then by [8], Theorem 1, we also give another classification related to the Ricci tensor S of M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$:

Theorem 2. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$, with $\overline{R}_N\varphi S = S\overline{R}_N\varphi$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the Reeb direction of ξ, then M is locally congruent to one of the following:*
- (i) *a tube over a totally geodesic $\mathrm{SU}_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ or*
- (ii) *a horosphere in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ where the center at infinity is singular and of type $JX \in \mathfrak{J}X$.*

We refer [5], [6], [8], [9], [10] and [11] for Riemannian geometric structures of complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$.

1. Proof of Theorem 1

Let M be a Hopf hypersurface in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ with

$$\overline{R}_N\varphi AX = A\overline{R}_N\varphi X. \tag{C-1}$$

The normal Jacobi operator $\overline{R}_N$ of M is defined by $\overline{R}_N X = \overline{R}(X, N)N$ for any tangent vector $X \in T_pM$, $p \in M$. Then for any tangent vector field X on M in

$SU_{2,m}/S(U_2{\cdot}U_m)$, from [5], (1.1), we calculate the normal Jacobi operator $\overline{R}_N$,

$$\overline{R}_N(X) = -\frac{1}{2}\Big[X + 3\eta(X)\xi + \sum_{\nu=1}^{3}\{3\eta_\nu(X)\xi_\nu - \eta_\nu(\xi)\varphi_\nu\varphi X + \eta_\nu(\xi)\eta(X)\xi_\nu + \eta_\nu(\varphi X)\varphi_\nu\xi\}\Big], \tag{1.1}$$

where α denotes the real valued function defined by $g(A\xi,\xi)$.

Now in this section, in order to prove our Theorem 1, we give important lemmas as follows:

Lemma 1.1. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$. If M satisfies the commuting condition $\overline{R}_N\varphi AX = A\overline{R}_N\varphi X$, then the Reeb vector field ξ belongs to either the maximal quaternionic subbundle $\mathcal{Q}$ or its orthogonal complement $\mathcal{Q}^\perp$.*

P r o o f. To prove our lemma, without loss of generality, ξ is written as

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{$**$}$$

where X_0 and ξ_1 are unit vectors in $\mathcal{Q}$ and $\mathcal{Q}^\perp$, respectively, and $\eta(X_0)\eta(\xi_1) \neq 0$.

Let $\mathfrak{U} = \{p \in M\colon\ \alpha(p) \neq 0\}$ be the open subset of M. Hereafter, we discuss our arguments on $\mathfrak{U}$.

From $(**)$ and $\varphi\xi = 0$, we have

$$\begin{cases} \varphi X_0 = -\eta(\xi_1)\varphi_1 X_0, \\ \varphi\xi_1 = \varphi_1\xi = \eta(X_0)\varphi_1 X_0, \\ \varphi_1\varphi X_0 = \eta_1(\xi)X_0. \end{cases} \tag{1.2}$$

From (1.1) and (1.2), we have

$$\begin{cases} \overline{R}_N(X_0) = -2\eta(X_0)\xi, \\ \overline{R}_N(\xi_1) = -2\xi - 2\eta(\xi_1)\xi_1, \\ \overline{R}_N(\varphi X_0) = 0. \end{cases} \tag{1.3}$$

From this and the fundamental formula in Section 2, $\varphi^2 X = -X + \eta(X)\xi$ for any $X \in TM$, the condition (C-1) becomes

$$\varphi AX + \sum_{\nu=1}^{3}\{3\eta_\nu(\varphi AX)\xi_\nu + \eta_\nu(\xi)\varphi_\nu AX - \eta_\nu(AX)\varphi_\nu\xi\} = A\varphi X + \sum_{\nu=1}^{3}\{3\eta_\nu(\varphi X)A\xi_\nu + \eta_\nu(\xi)A\varphi_\nu X - \eta_\nu(X)A\varphi_\nu\xi\}. \tag{1.4}$$

Moreover, if we take the symmetric part of this equation as mentioned before, we have

$$A\varphi X + \sum_{\nu=1}^{3}\{3\eta_\nu(X)A\varphi\xi_\nu + \eta_\nu(\xi)A\varphi_\nu X + g(\varphi_\nu\xi, X)A\xi_\nu\}$$
$$= \varphi AX + \sum_{\nu=1}^{3}\{3\eta_\nu(AX)\varphi\xi_\nu + \eta_\nu(\xi)\varphi_\nu AX + g(A\varphi_\nu\xi, X)\xi_\nu\}.$$

Summing up these two equations, it follows that

$$\sum_{\nu=1}^{3}\{g(A\varphi_\nu\xi, X)\xi_\nu + \eta_\nu(AX)\varphi\xi_\nu\} = \sum_{\nu=1}^{3}\{\eta_\nu(X)A\varphi_\nu\xi + g(X, \varphi_\nu\xi)A\xi_\nu\}. \tag{1.5}$$

Taking the inner product of (1.5) with ξ, we obtain $A\varphi\xi_1 = \alpha\varphi\xi_1$ which together with the elementary formulas and $(**)$ yields

$$A\varphi X_0 = \alpha\varphi X_0 \tag{1.6}$$

on $\mathfrak{U}$. Putting $X = X_0$ in (1.5), it becomes

$$\sum_{\nu=1}^{3}\{g(A\varphi_\nu\xi, X_0)\xi_\nu + \eta_\nu(AX_0)\varphi_\nu\xi\} = 0.$$

Taking the inner product of this equation with $\varphi_1\xi$, we obtain

$$\eta_1(AX_0) = 0 \quad \text{on } \mathfrak{U}. \tag{1.7}$$

On the other hand, since M is Hopf, see [5], we have

$$A\varphi AX = \frac{\alpha}{2}(A\varphi + \varphi A)X + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\xi)\varphi\xi_\nu + \eta_\nu(\xi)\eta_\nu(\varphi X)\xi\}$$
$$- \frac{1}{2}\varphi X - \frac{1}{2}\sum_{\nu=1}^{3}\{\eta_\nu(X)\varphi\xi_\nu + \eta_\nu(\varphi X)\xi_\nu + \eta_\nu(\xi)\varphi_\nu X\}. \tag{1.8}$$

We put $X = \varphi X_0$ in (1.8) and use (1.7). It follows that

$$0 = \frac{\alpha}{2}A\xi_1 + \eta_1^2(\xi)\eta(X_0)\xi - \eta(X_0)\eta_1(\xi)\xi_1 \quad \text{on } \mathfrak{U}.$$

Taking the inner product of the previous equation with ξ_1 and using (2.2), we have

$$-\eta_1(\xi)\eta^3(X_0) = 0 \quad \text{on } \mathfrak{U}.$$

It is a contradiction. Therefore, the point p must belong to $M-\mathfrak{U}$, where $M-\mathfrak{U} = \operatorname{Int}(M-\mathfrak{U}) \cup \partial(M-\mathfrak{U})$.

We consider the following two cases:

Case 1. $p \in \operatorname{Int}(M-\mathfrak{U})$. If $p \in \operatorname{Int}(M-\mathfrak{U})$, then $\alpha = 0$ on the neighborhood $\operatorname{Int}(M-\mathfrak{U})$ of p. For this case, the result was proved by the well-known equation $\operatorname{grad}\alpha = (\xi\alpha)\xi - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\varphi\xi_\nu$.

Case 2. $p \in \partial(M-\mathfrak{U})$. Since $p \in \partial(M-\mathfrak{U})$, there exists a sequence of points $p_n \to p$ with $\alpha(p) = 0$ and $\alpha(p_n) \neq 0$. Such a sequence will have an infinite subsequence where $\eta(\xi_1) = 0$ (in which case $\xi \in \mathcal{Q}$ at p, by continuity) or an infinite subsequence where $\eta(X_0) = 0$ (in which case $\xi \in \mathcal{Q}^\perp$ at p).

Now we consider only the case $p \in \partial(M-\mathfrak{U})$. Then there exists a sequence $\{p_n\} \subset \mathfrak{U}$ such that $p_n \to p$. Since $\xi(p_n) \in \mathcal{Q}$ or $\xi(p_n) \in \mathcal{Q}^\perp$, by continuity we also have $\xi(p) \in \mathcal{Q}$ or $\xi(p) \in \mathcal{Q}^\perp$.

Summing up these discussions, we get a complete proof of our lemma. □

Now let us consider Hopf hypersurfaces M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ with $\overline{R}_N\varphi A = A\overline{R}_N\varphi$. By virtue of Lemma 1.1, the Reeb vector field ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.

First we consider the case that ξ belongs to the distribution $\mathcal{Q}^\perp$.

Differentiating $\xi = \xi_1$ along any direction $X \in TM$ gives us

$$2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \varphi_1 AX - \varphi AX = 0. \tag{1.9}$$

Then, by using the symmetric (or skew-symmetric) property of the shape operator A (respectively, the structure tensor field φ), we have

$$2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\varphi_1 X - A\varphi X = 0. \tag{1.10}$$

Applying φ_1 to (1.10), we have

$$\varphi_1\varphi AX = 2\eta_3(AX)\xi_3 + 2\eta_2(AX)\xi_2 - AX + \alpha\eta(X)\xi. \tag{1.11}$$

Lemma 1.2. *Let M be a Hopf hypersurface in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$, with $\overline{R}_N\varphi A = A\overline{R}_N\varphi$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then the shape operator A commutes with the structure tensor field φ.*

Proof. Let M be a Hopf hypersurface of $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$, such that the Reeb vector field ξ is tangential to the distribution $\mathcal{Q}^\perp$ everywhere. Then the commuting condition (C-1) is equivalent to M having isometric Reeb flow.

Since $\xi \in \mathcal{Q}^\perp$, we may put $\xi = \xi_1$ for our convenience sake. Then (C-1) is equivalent to

$$\begin{aligned}\varphi AX + \varphi_1 AX + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 \\ = A\varphi X + 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\varphi_1 X.\end{aligned} \tag{1.12}$$

From (1.9), (1.10) and (1.12), we see that $(A\varphi - \varphi A)$ vanishes on $\mathfrak{U}$. Actually, Suh gave an equivalent property for the isometric Reeb flow (see [8]). By virtue of this work, we assert that the commuting condition (C-1) with respect to the normal Jacobi operator $\overline{R}_N$ on M is equivalent to the Reeb flow on M being isometric, that is, M is locally congruent to a real hypersurface of $\mathcal{T}_A$ or $\mathcal{H}_A$.

Next, if $p \in \mathrm{Int}(M - \mathfrak{U})$, we see that $\alpha(p) = 0$. From this, the equation (1.6) gives $(A\varphi - \varphi A)X(p) = 0$.

Finally, let us assume that $p \in \partial(M - \mathfrak{U})$, where $\partial(M - \mathfrak{U})$ is the boundary of $M - \mathfrak{U}$. Then there exists a subsequence $\{p_n\} \subset \mathfrak{U}$ such that $p_n \to p$. Since $(A\varphi - \varphi A)X(p_n) = 0$ on an open subset $\mathfrak{U}$ in M, by continuity we also get $(A\varphi - \varphi A)X(p) = 0$. $\square$

To summarize, it is natural that the shape operator A of M commutes with the structure tensor field φ of M under our assumption. Thus, by [7], we assert M is locally congruent to a real hypersurface of $\mathcal{T}_A$, $\mathcal{H}_A$.

By [7] we assert that M with the assumptions given in Lemma 1.2 is locally congruent to one of the following hypersurfaces:

($\mathcal{T}_A$) a tube over a totally geodesic $\mathrm{SU}_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ or,
($\mathcal{H}_A$) a horosphere in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.

From [6], we have some information related to the shape operator A of $\mathcal{T}_A$ and $\mathcal{H}_A$ as follows:

Proposition A. *Let M be a connected real hypersurface in complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \in \mathfrak{J}N$, then one of the following statements holds:*

($\mathcal{T}_A$) *M has exactly four distinct constant principal curvatures*

$$\alpha = 2\coth(2r), \quad \beta = \coth(r), \quad \lambda_1 = \tanh(r), \quad \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = \mathcal{C} \ominus \mathcal{Q}, \quad T_{\lambda_1} = E_{-1}, \quad T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).

($\mathcal{H}_A$) *M has exactly three distinct constant principal curvatures*

$$\alpha = 2, \quad \beta = 1, \quad \lambda = 0$$

with the corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1}, \quad T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\varphi\varphi_1|_{\mathcal{Q}}$ with respect to the eigenvalue $+1$ and -1, respectively.

Combining the above formulas, we conclude that

$$(\overline{R}_N\varphi)AX - A(\overline{R}_N\varphi)X = \begin{cases} 0 & \text{if } X = \xi \in T_{\alpha_1} = T_\alpha, \\ 0 & \text{if } X = \xi_l \in T_{\alpha_2} = T_\beta, \\ 0 & \text{if } X \in T_{\alpha_3} = T_\lambda, \\ 0 & \text{if } X \in T_{\alpha_4} = T_\mu. \end{cases}$$

Thus, Hopf hypersurfaces M with $\overline{R}_N\varphi AX = A\overline{R}_N\varphi X$ are locally congruent to real hypersurfaces of $\mathcal{T}_A$, $\mathcal{H}_A$ and vice versa.

Due to Lemma 1.1, let us suppose that $\xi \in \mathcal{Q}$ (i.e., $JN \perp \mathfrak{J}N$) in this section. Related to this condition, Suh in [8] proved:

Theorem B. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$, with the Reeb vector field belonging to the maximal quaternionic subbundle $\mathcal{Q}$. Then one of the following statements holds*

($\mathcal{T}_B$) *M is an open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $\mathrm{SU}_{2,2n}/S(U_2{\cdot}U_{2n})$, $m = 2n$,*

($\mathcal{H}_B$) *M is an open part of a horosphere in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or*

($\mathcal{E}$) *the normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$.*

By virtue of this result, we assert that a real hypersurface M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ satisfying the hypotheses of our main theorem is locally congruent to an open part of one of the model spaces mentioned in the above theorem. Hereafter, let us check whether the shape operator A of a model space of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ satisfies our conditions, conversely. In order to do this, let us introduce the following proposition given by [6].

Proposition C. *Let M be a connected hypersurface in* $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$. *Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \perp \mathfrak{J}N$, then one of the following statements holds:*

($\mathcal{T}_B$) *M has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures*

$$\alpha = \sqrt{2}\tanh\big(\sqrt{2}r\big), \quad \beta = \sqrt{2}\coth\big(\sqrt{2}r\big), \quad \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh\Big(\frac{1}{\sqrt{2}}r\Big), \quad \lambda_2 = \frac{1}{\sqrt{2}}\coth\Big(\frac{1}{\sqrt{2}}r\Big),$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = TM \ominus \mathcal{Q}, \quad T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under $\mathfrak{J}$ and are mapped onto each other by J. In particular, the quaternionic dimension of $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ *must be even.*

($\mathcal{H}_B$) *M has exactly three distinct constant principal curvatures*

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with the corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

($\mathcal{E}$) *M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with the corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$. Thus, the corresponding multiplicities are

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

Next, we will check whether the normal Jacobi operator $\overline{R}_N$ of a model hypersurface of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ satisfies the condition (C-1). It will turn out that the case that ξ belongs to the distribution $\mathcal{Q}$ cannot actually occur.

In order to do this, let us assume that the normal Jacobi operator $\overline{R}_N$ of $M_B\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ satisfies the condition (C-1).

Since $\xi \in \mathcal{Q}$ and $\varphi\varphi_\nu\xi = \varphi^2\xi_\nu = -\xi_\nu$, we have $\overline{R}_N(\xi_1) = -2\xi_1$ and $\overline{R}_N(\varphi\xi_1) = 0$. The tangent space of M_B can be split into

$$TM_B = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5}$$

where $T_{\alpha_1} = [\xi]$, $T_{\alpha_2} = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$, $T_{\alpha_3} = \operatorname{span}\{\varphi\xi_1, \varphi\xi_2, \varphi\xi_3\}$ and $T_{\alpha_4} \oplus T_{\alpha_5}$ is the orthogonal complement of $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$ in TM. $JT_{\alpha_5} \subset T_{\alpha_4}$ (see [9]).

Putting $X = \varphi\xi_1$ into (C-1), we have

$$\overline{R}_N\varphi A\varphi\xi_1 - A\overline{R}_N\varphi\xi_1 = 2\alpha_2\xi_1.$$

This implies that the eigenvalue α_2 vanishes, since ξ_1 is a unit tangent vector field. But in [6], we see that the eigenvalue α_2 never vanishes neither in $\mathcal{T}_B$, $\mathcal{H}_B$ nor $\mathcal{E}$, which gives us a contradiction.

Summing up these observations, we assert that the shape operator A of three model spaces $\mathcal{T}_B$, $\mathcal{H}_B$ and $\mathcal{E}$ in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ does not satisfy the condition $\overline{R}_N\varphi AX = A\overline{R}_N\varphi X$.

2. Proof of Theorem 2

In this section, by using geometric quantities in [5] and [9], we give a complete proof of Theorem 2. To prove it, we assume that M is a Hopf hypersurface in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ with (C-2), that is,

$$(\overline{R}_N\varphi)SX = S(\overline{R}_N\varphi)X. \tag{C-2}$$

From the definition of the Ricci tensor S and the fundamental formulas in [11], Section 2, we have that the Ricci tensor S of M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ is given by

$$\begin{aligned} 2SX &= -(4m+7)X + 3\eta(X)\xi + 2hAX - 2A^2X \\ &\quad + \sum_{\nu=1}^{3}\{3\eta_\nu(X)\xi_\nu - \eta_\nu(\xi)\varphi_\nu\varphi X + \eta_\nu(\varphi X)\varphi_\nu\xi + \eta(X)\eta_\nu(\xi)\xi_\nu\} \end{aligned} \tag{2.1}$$

where h denotes the trace of the shape operator A (see [11]).

Using the equations (C-2) and (2.1), we prove that the Reeb vector field ξ of M belongs to either $\mathcal{Q}$ or $\mathcal{Q}^{\perp}$.

Lemma 2.1. *Let M be a Hopf hypersurface in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$, $m \geqslant 3$, with* (C-2). *If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the Reeb direction of ξ, then the Reeb vector field ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^{\perp}$.*

Proof. In order to prove this lemma, for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^{\perp}$, we put

$$(**)\qquad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1,$$

where $\eta(X_0)\eta(\xi_1) \neq 0$ is the assumption we will disprove by contradiction.

Let $\mathfrak{U} = \{p \in M\colon\ \alpha(p) \neq 0\}$ be the open subset of M. Now we discuss our arguments on $\mathfrak{U}$ (see [5]).

By virtue of [5], Lemma 2.2, $\xi\alpha = 0$ gives $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$.

From (2.1), we have

$$(2.2)\qquad \begin{cases} S\varphi X_0 = \kappa\varphi X_0, \\ SX_0 = (-2m - 4 + 2\eta^2(X_0) + h\alpha - \alpha^2)X_0 + 2\eta_1(\xi)\eta(X_0)\xi_1, \\ S\xi_1 = (-2m - 2 + h\alpha - \alpha^2)\xi_1 + 2\eta(\xi_1)\xi, \\ S\xi = (-2m - 2 + h\alpha - \alpha^2)\xi + 2\eta_1(\xi)\xi_1, \end{cases}$$

where $\kappa := -2m - 4 + h\sigma - \sigma^2$ and $\sigma = (\alpha^2 - 2\eta^2(X_0))/\alpha$.

Putting $X = \varphi X_0$ in (C-2) and using (1.2), (1.3), (2.2), it follows that

$$-2\sigma\eta(X_0)\xi_1(\xi)\xi_1 = -2\eta(X_0)\eta_1(\xi)S\xi_1.$$

Taking the inner product with X_0 of the previous equation, we have

$$0 = -4\eta^2(X_0)\eta_1^2(\xi).$$

Therefore, p does not belong to $\mathfrak{U}$, and thus it must be $p \in (M - \mathfrak{U})$. Since $(M - \mathfrak{U}) = \operatorname{Int}(M - \mathfrak{U}) \cup \partial(M - \mathfrak{U})$, where Int and ∂ denote respectively the interior and the boundary of $M - \mathfrak{U}$, we consider the following two cases:

Case 1. $p \in \operatorname{Int}(M - \mathfrak{U})$. If $p \in \operatorname{Int}(M - \mathfrak{U})$, then $\alpha = 0$ on this neighborhood $\operatorname{Int}(M - \mathfrak{U})$ of p.

Case 2. $p \in \partial(M - \mathfrak{U})$. Since $p \in \partial(M - \mathfrak{U})$, there exists a sequence of points $p_n \to p$ with $\alpha(p) = 0$ and $\alpha(p_n) \neq 0$. Such a sequence will have an infinite subsequence where $\eta(\xi_1) = 0$ (in which case $\xi \in \mathcal{Q}$ at p, by continuity) or an infinite subsequence where $\eta(X_0) = 0$ (in which case $\xi \in \mathcal{Q}^{\perp}$ at p). Accordingly, we get a complete proof of our lemma. $\square$

Now, we shall divide our consideration into two cases when ξ belongs to either $\mathcal{Q}^{\perp}$ or $\mathcal{Q}$, respectively.

Now, we further study the case $\xi \in \mathcal{Q}^{\perp}$. We may put $\xi = \xi_1 \in \mathcal{Q}^{\perp}$ for our convenience sake.

Let M be a Hopf hypersurface in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$. If the Reeb vector field ξ belongs to $\mathcal{Q}^{\perp}$, then the Ricci tensor S commutes with the shape operator A, that is, $SA = AS$ (see [5], equation (4.7)).

Bearing in mind that $\xi = \xi_1 \in \mathcal{Q}^{\perp}$, (2.1) is simplified to

$$2SX = -(4m+7)X+7\eta(X)\xi+2\eta_2(X)\xi_2+2\eta_3(X)\xi_3-\varphi_1\varphi X+2hAX-2A^2X. \tag{2.3}$$

Here replacing X by φX in (2.3) (or applaying φ to (2.3)), we have

$$\begin{cases} 2S\varphi X = -[(4m+7)\varphi X-\varphi_1 X+2\eta_2(X)\xi_3-2\eta_3(X)\xi_2]+2hA\varphi X-2A^2\varphi X, \\ 2\varphi SX = -[(4m+7)\varphi X-\varphi_1 X+2\eta_2(X)\xi_3-2\eta_3(X)\xi_2]+2h\varphi AX-2\varphi A^2X. \end{cases} \tag{2.4}$$

On the other hand, the equations (1.9) and (2.3) give us

$$\begin{aligned} &2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \varphi_1 SX - \varphi SX \\ &\quad = (2m+4)\{2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 + \varphi X - \varphi_1 X\} := \mathrm{Rem}(X). \end{aligned} \tag{2.5}$$

Taking the symmetric part of (2.5), we obtain

$$2\eta_3(X)S\xi_2 - 2\eta_2(X)S\xi_3 + S\varphi_1 X - S\varphi X = \mathrm{Rem}(X). \tag{2.6}$$

By virtue of $SA = AS$, (2.5) and (2.6) we assert the following:

Lemma 2.2. *Let M be a Hopf hypersurface in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ with* (C-1). *If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^{\perp}$, that is, $\xi \in \mathcal{Q}^{\perp}$, then we have $S\varphi = \varphi S$.*

Proof. By (2.5) and (2.6) we have the left side of (C-2) and the right side of (C-2), respectively, as follows:

$$\begin{cases} \overline{R}_N\varphi SX = -\varphi SX + \frac{1}{2}\,\mathrm{Rem}(X), \\ S\overline{R}_N\varphi X = -S\varphi X + \frac{1}{2}\,\mathrm{Rem}(X). \end{cases} \tag{2.7}$$

Combining the equations in (2.7), we have

$$S\overline{R}_N\varphi X - \overline{R}_N\varphi SX = -S\varphi X + \varphi SX. \tag{2.8}$$

□

When $\xi \in \mathcal{Q}^\perp$, (C-2) is equivalent to (2.8). Summing up Lemmas 2.1, 2.2 and [9], Theorem, in the case of $\xi \in \mathcal{Q}^\perp$, M is a Hopf hypersurface in complex hyperbolic two-plane Grassmannians $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ satisfying (C-2), hence M is locally congruent to hypersurface $\mathcal{T}_A$ or $\mathcal{H}_A$.

Combining the above formulas, it follows that

$$(\overline{R}_N\varphi)SX - S(\overline{R}_N\varphi)X = \begin{cases} 0 & \text{if } X = \xi \in T_{\alpha_1} = T_\alpha, \\ 0 & \text{if } X = \xi_l \in T_{\alpha_2} = T_\beta, \\ 0 & \text{if } X \in T_{\alpha_3} = T_\lambda, \\ 0 & \text{if } X \in T_{\alpha_4} = T_\mu. \end{cases}$$

Thus, if $\xi\alpha = 0$, then Hopf hypersurfaces M with $\overline{R}_N\varphi SX = S\overline{R}_N\varphi X$ are locally congruent to real hypersurfaces of $\mathcal{T}_A$, $\mathcal{H}_A$ and vice versa.

When $\xi \in \mathcal{Q}$, a Hopf hypersurface M in $\mathrm{SU}_{2,m}/S(U_2{\cdot}U_m)$ is locally congruent to a hypersurface of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ by virtue of [8], Theorem.

We will now show that a hypersurface of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ cannot satisfy condition (C-2), and thus cannot occur in our situation. For this purpose, we consider a model space of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$, which will be denoted by M_B. We calculate $(\overline{R}_N\varphi)S = S(\overline{R}_N\varphi)$ for M_B. The tangent space of M_B can be split into

$$TM_B = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5},$$

where $T_{\alpha_1} = [\xi]$, $T_{\alpha_2} = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$, $T_{\alpha_3} = \operatorname{span}\{\varphi\xi_1, \varphi\xi_2, \varphi\xi_3\}$ and $T_{\alpha_4} \oplus T_{\alpha_5}$ is the orthogonal complement of $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$ in TM. Further, $JT_{\alpha_5} \subset T_{\alpha_4}$ (see [9]).

From [11], we obtain

$$SX = \begin{cases} (-2m - 2 + h\alpha_1 - \alpha_1^2)\xi & \text{if } X = \xi \in T_{\alpha_1}, \\ (-2m - 2 + h\alpha_2 - \alpha_2^2)\xi_l & \text{if } X = \xi_l \in T_{\alpha_2}, \\ (-2m - 4)\varphi\xi_l & \text{if } X = \varphi\xi_l \in T_{\alpha_3}, \\ \left(-2m - \dfrac{7}{2} + h\alpha_4 - \alpha_4^2\right)X & \text{if } X \in T_{\alpha_4}, \\ \left(-2m - \dfrac{7}{2} + h\alpha_5 - \alpha_5^2\right)X & \text{if } X \in T_{\alpha_5}, \end{cases} \tag{2.9}$$

$$\overline{R}_N(X) = \begin{cases} -2\xi & \text{if } X = \xi \in T_{\alpha_1}, \\ -2\xi_l & \text{if } X = \xi_l \in T_{\alpha_2}, \\ 0 & \text{if } X = \varphi\xi_l \in T_{\alpha_3}, \\ -\frac{1}{2}X & \text{if } X \in T_{\alpha_4}, \\ -\frac{1}{2}X & \text{if } X \in T_{\alpha_5}. \end{cases} \tag{2.10}$$

In order to check whether model spaces $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ satisfy (C-2) or not, we should verify that the following equations vanish for all cases:

$$G(X) := (\overline{R}_N\varphi)SX - S(\overline{R}_N\varphi)X. \tag{2.11}$$

Case 1. Tube $\mathcal{T}_B$. By calculation, we have $\lambda+\mu=\beta$ on $\mathcal{T}_B$. Thus we obtain $h=\alpha+3\beta+(4n-4)(\lambda+\mu)=\alpha+(4n-1)\beta$.

By putting $X=\varphi\xi_l$ into (2.11), we have

$$G(\varphi\xi_l) := (\overline{R}_N\varphi)S\varphi\xi_l - S(\overline{R}_N\varphi)\varphi\xi_l = (-4-2h\beta+2\beta^2)\xi_l. \tag{2.12}$$

Since $h=\alpha+(4n-1)\beta$, (2.12) becomes

$$0=(-8-4(2n-1)\beta^2)\xi_l. \tag{2.13}$$

Since $-8-4(2n-1)\beta^2<0$, (2.13) means $\xi_l=0$. This is a contradiction.

Case 2. Horosphere $\mathcal{H}_B$. In this case, we have $h=4n\alpha=4\sqrt{2}n$.

By putting $X=\varphi\xi_l$ into (2.11) and applying $h=4\sqrt{2}n$, we have $0=-16n\xi_l$. This means $\xi_l=0$, which is a contradiction.

Case 3. Exceptional case $\mathcal{E}$. If $\lambda=\mu$, then it is the same as Case 2. Thus we may assume $\lambda\neq\mu$. We have

$$h=4\alpha+m(\lambda)\lambda+m(\mu)\mu, \tag{2.14}$$

where $m(\lambda)\geqslant m(\mu)$ and $m(\lambda)+m(\mu)=8n-8$.

By putting $X=\varphi\xi_l\in T_\gamma$, we obtain $G(\varphi\xi_l)=-2\sqrt{2}h\xi_l$. This gives $h=0$.

For $X\in T_\mu$, $G(X)=(\mu-\lambda)(h-\mu-\lambda)\varphi X$. Since φX never vanishes, naturally we have

$$(\mu-\lambda)(h-\mu-\lambda)=0. \tag{2.15}$$

Because $\mu\neq\lambda$, $h=0$ and $\lambda=2^{-1/2}$, (2.15) should imply $\mu=-2^{-1/2}$. Moreover, since $JT_\mu\subset T_\lambda$ and $\mathfrak{J}T_\mu\subset T_\lambda$, we see that the corresponding multiplicities of the eigenvalues λ and μ satisfy $m(\lambda)\geqslant m(\mu)$. Since $m(\alpha)=4$, $m(\gamma)=3$ and $m(\lambda)+m(\mu)=8n-8$ on $\mathcal{E}$, the trace of the shape operator A denoted by h becomes $h=4\alpha+3\gamma+m(\lambda)\lambda+m(\mu)\mu=4\cdot 2^{1/2}+2^{-1/2}(m(\lambda)-m(\mu))$, which leads to a contradiction. In fact, we obtained $h=0$, which yields $(m(\lambda)-m(\mu))=-8<0$. Thus, this case does not occur.

This shows that a hypersurface of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ cannot satisfy the condition (C-2), and therefore in the situation of Theorem 2, the case $X\in\mathcal{Q}$ cannot occur. This completes the proof of Theorem 2. □

References

[1] *L. J. Alías, A. Romero, M. Sánchez*: Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes. Gen. Relativ. Gravitation *27* (1995), 71–84.

[2] *L. J. Alías, A. Romero, M. Sánchez*: Spacelike hypersurfaces of constant mean curvature in spacetimes with symmetries. Proc. Conf., Valencia, 1998 (E. Llinares Fuster et al., eds.). Publ. R. Soc. Mat. Esp. 1, Real Sociedad Matemática Española, Madrid, 2000, pp. 1–14.

[3] *J. Berndt, Y. J. Suh*: Contact hypersurfaces in Kähler manifolds. Proc. Am. Math. Soc. *143* (2015), 2637–2649.

[4] *J. M. Latorre, A. Romero*: Uniqueness of noncompact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes. Geom. Dedicata *93* (2002), 1–10.

[5] *H. Lee, Y. J. Suh, C. Woo*: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting structure Jacobi operators. Mediterr. J. Math. *13* (2016), 3389–3407.

[6] *J. D. Pérez, Y. J. Suh, C. Woo*: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting shape operator. Open Math. *13* (2015), 493–501.

[7] *Y. J. Suh*: Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv. Appl. Math. *50* (2013), 645–659.

[8] *Y. J. Suh*: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field. Adv. Appl. Math. *55* (2014), 131–145.

[9] *Y. J. Suh*: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting Ricci tensor. Int. J. Math. *26* (2015), Article ID 1550008, 26 pages.

[10] *Y. J. Suh*: Real hypersurfaces in the complex quadric with parallel Ricci tensor. Adv. Math. *281* (2015), 886–905.

[11] *Y. J. Suh, C. Woo*: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor. Math. Nachr. *287* (2014), 1524–1529.

Authors' addresses: Doo Hyun Hwang, Eunmi Pak, Department of Mathematics, Kyungpook National University, 80 Daehak-ro, Buk-gu, Daegu 702-701, Republic of Korea, e-mail: `engus0322@knu.ac.kr`, `empak@knu.ac.kr`; Changhwa Woo, Department of Mathematics Education, Woosuk University, 443 Samnyero, Samnye, 565-701 Wanju, Jeonbuk, Republic of Korea, e-mail: `legalgwch@naver.com`.