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# ON BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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*Abstract.* A general theorem (principle of a priori boundedness) on solvability of the boundary value problem

$$\mathrm{d}x = \mathrm{d}A(t)\cdot f(t,x), \quad h(x)=0$$

is established, where $f\colon [a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ is a vector-function belonging to the Carathéodory class corresponding to the matrix-function $A\colon [a,b]\to\mathbb{R}^{n\times n}$ with bounded total variation components, and $h\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is a continuous operator. Basing on the mentioned principle of a priori boundedness, effective criteria are obtained for the solvability of the system under the condition $x(t_1(x))=\mathcal{B}(x)\cdot x(t_2(x))+c_0$, where $t_i\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\to[a,b]$ $(i=1,2)$ and $\mathcal{B}\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ are continuous operators, and $c_0\in\mathbb{R}^n$.

*Keywords*: system of nonlinear generalized ordinary differential equations; Kurzweil-Stieltjes integral; general boundary value problem; solvability; principle of a priori boundedness

*MSC 2010*: 34K10

## 1. Statement of the problem and formulation of the results

Let $n$ be a natural number, $[a,b]$ a closed interval of real axis, $A=(a_{ik})_{i,k=1}^n\colon [a,b]\to\mathbb{R}^{n\times n}$ a matrix-function with bounded total variation components, $f$ a vector-function belonging to the Carathéodory class $\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$ corresponding to the matrix-function $A$, and $h\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ a continuous

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operator satisfying the condition

$$\sup\{\|h(x)\|\colon\ x\in \mathrm{BV}_s([a,b],\mathbb{R}^n),\ \|x\|_s\leqslant \varrho\}<\infty$$

for every $\varrho\in\,]0,\infty[$.

Consider the nonlinear system of generalized ordinary differential equations

$$\mathrm{d}x=\mathrm{d}A(t)\cdot f(t,x) \tag{1.1}$$

with the boundary condition

$$h(x)=0. \tag{1.2}$$

The theorem on the existence of a solution of the problem (1.1), (1.2) which will be given below and will be called the principle of a priori boundedness, generalizes well known Conti-Opial type theorems (see [11], [14], [21] for the case of ordinary differential equations) and supplements earlier known criteria for the solvability of nonlinear boundary value and initial problems for systems of generalized ordinary differential equations (see, e.g., [1], [3], [4], [6], [8]–[10], [12], [20], [22]–[24] and the references therein).

On the basis on the above mentioned principle of a priori boundedness, we have obtained effective criteria for the solvability of system (1.1) under the condition

$$x(t_1(x))=\mathcal{B}(x)\cdot x(t_2(x))+c_0, \tag{1.3}$$

where $t_i\colon\ \mathrm{BV}_s([a,b],\mathbb{R}^n)\to[a,b]$ $(i=1,2)$ and $\mathcal{B}\colon\ \mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ are continuous operators, and $c_0\in\mathbb{R}^n$.

Analogous and related questions are investigated in [14]–[19] (see also the references therein) for the boundary value problems for linear and nonlinear systems of ordinary differential and functional differential equations.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.g., [1]–[10], [12], [13], [20], [22], [24] and the references therein).

Throughout the paper the following notation and definitions will be used.

$\mathbb{R}=\,]-\infty,\infty[$, $\mathbb{R}_+=[0,\infty[$, $[a,b]$ $(a,b\in\mathbb{R})$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\|=\sum_{i,l=1}^{n,m}|x_{il}|;$$

$$\mathbb{R}_+^{n\times m}=\{(x_{il})_{i,l=1}^{n,m}\colon\ x_{il}\geqslant 0\ (i=1,\dots,n;\ l=1,\dots,m)\}.$$

$O_{n\times m}$ (or $O$) is the zero $n\times m$-matrix.

If $X=(x_{il})_{i,l=1}^{n,m}\in\mathbb{R}^{n\times m}$, then

$$|X|=(|x_{il}|)_{i,l=1}^{n,m}\quad\text{and}\quad \operatorname{sgn} X=(\operatorname{sgn} x_{il})_{i,l=1}^{n,m}.$$

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column $n$-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

$\langle x,y\rangle$ is the scalar product of the vectors $x$ and $y\in\mathbb{R}^n$.

If $X\in\mathbb{R}^{n\times n}$, then $\det X$ is the determinant of $X$; $I_n$ is the identity $n\times n$-matrix; $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1,\dots,\lambda_n$.

$\operatorname{var}_a^b(X)$ is the total variation of the matrix-function $X\colon [a,b]\to\mathbb{R}^{n\times m}$, i.e., the sum of total variations of its components $x_{il}$ $(i=1,\dots,n;\ l=1,\dots,m)$; $V(X)(t)=(v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(a)=0$, $v(x_{il})(t)=\operatorname{var}_a^t(x_{il})$ for $a<t\leqslant b$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X\colon [a,b]\to\mathbb{R}^{n\times m}$ at the point $t$ (we will assume $X(t)=X(a)$ for $t\leqslant a$ and $X(t)=X(b)$ for $t\geqslant b$, if necessary);

$$\Delta^- X(t)=X(t)-X(t-),\ \Delta^+X(t)=X(t+)-X(t);$$
$$\|X\|_s=\sup\{\|X(t)\|\colon\ t\in[a,b]\},\ \|X\|_v=\|X(a)\|+\operatorname{var}_a^b(X).$$

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X\colon [a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\operatorname{var}_a^b(X)<\infty$);

$\mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ is the normed space of all $X\in BV([a,b],\mathbb{R}^{n\times m})$ with the norm $\|X\|_s$;

$\mathrm{BV}_v([a,b],\mathbb{R}^{n\times m})$ is the Banach space of all $X\in BV([a,b],\mathbb{R}^{n\times m})$ with the norm $\|X\|_v$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $I\subset\mathbb{R}$ is an interval, then $C(I,\mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X\colon I\to\mathbb{R}^{n\times m}$.

If $B_1$ and $B_2$ are normed spaces, then an operator $g\colon B_1\to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x)=\lambda g(x)$$

for every $\lambda\in\mathbb{R}_+$ and $x\in B_1$.

An operator $\varphi\colon \mathrm{BV}([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is called nondecreasing if for every $x,y\in\mathrm{BV}([a,b],\mathbb{R}^n)$ such that $x(t)\leqslant y(t)$ for $t\in[a,b]$ the inequality $\varphi(x)(t)\leqslant\varphi(y)(t)$ holds for $t\in[a,b]$.

If $\alpha\colon [a,b]\to\mathbb{R}$, is a nondecreasing function, then $D_\alpha=\{t\in[a,b]\colon\ \Delta^-\alpha(t)+\Delta^+\alpha(t)\neq 0\}$.

$s_1, s_2, s_c\colon \operatorname{BV}([a,b],\mathbb{R}) \to \operatorname{BV}([a,b],\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\leqslant t} \Delta^- x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\leqslant\tau<t} \Delta^+ x(\tau) \quad \text{for } a < t \leqslant b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad \text{for } t \in [a,b].$$

If $g\colon [a,b] \to \mathbb{R}$ is a nondecreasing function, $x\colon [a,b] \to \mathbb{R}$ and $a \leqslant s < t \leqslant b$, then

$$\int_s^t x(\tau)\,\mathrm{d}g(\tau) = \int_{]s,t[} x(\tau)\,\mathrm{d}s_c(g)(\tau) + \sum_{s<\tau\leqslant t} x(\tau)\Delta^- g(\tau) + \sum_{s\leqslant\tau<t} x(\tau)\Delta^+ g(\tau),$$

where $\int_{]s,t[} x(\tau)\,\mathrm{d}s_c(g)(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$; if $a = b$, then we assume $\int_a^b x(t)\,\mathrm{d}g(t) = 0$ so that $\int_s^t x(\tau)\,\mathrm{d}g(\tau)$ is the Kurzweil-Stieltjes integral (see [20], [22], [24]);

$L([a,b],\mathbb{R};g)$ is the space of all functions $x\colon [a,b] \to \mathbb{R}$, measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<\tau\leqslant b} |x(t)|\Delta^- g(\tau) + \sum_{a\leqslant\tau<b} |x(t)|\Delta^+ g(t) < \infty,$$

with the norm

$$\|x\|_{L,g} = \int_a^b |x(t)|\,\mathrm{d}g(t).$$

If $g_j\colon [a,b] \to \mathbb{R}$ $(j = 1,2)$ are nondecreasing functions, $g(t) \equiv g_1(t) - g_2(t)$, and $x\colon [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\,\mathrm{d}g(\tau) = \int_s^t x(\tau)\,\mathrm{d}g_1(\tau) - \int_s^t x(\tau)\,\mathrm{d}g_2(\tau) \quad \text{for } a \leqslant s \leqslant t \leqslant b.$$

If $G = (g_{ik})_{i,k=1}^{l,n}\colon [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m}\colon [a,b] \to D$ such that $x_{kj} \in L([a,b],\mathbb{R};g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t \mathrm{d}G(\tau)\cdot X(\tau) = \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,\mathrm{d}g_{ik}(\tau)\right)_{i,j=1}^{l,m} \quad \text{for } a \leqslant s \leqslant t \leqslant b,$$
$$S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \ (j = 1,2) \quad \text{and} \quad S_c(G)(t) \equiv (s_c(g_{ik})(t))_{i,k=1}^{l,n}.$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $\mathrm{Car}([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m}\colon [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$

(i) the function $f_{kj}(\cdot,x)\colon [a,b] \to D_2$ is $\mu(s_c(g_{ik}))$-measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t,\cdot)\colon D_1 \to D_2$ is continuous for $\mu(s_c(g_{ik}))$-almost every $t \in [a,b]$ and for every $t \in D_{g_{ik}}$, and

$$\sup\{|f_{kj}(\cdot,x)|\colon\ x \in D_0\} \in L([a,b],\mathbb{R}; g_{ik})$$

for every compact $D_0 \subset D_1$.

If $G_j\colon [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$, and $X\colon [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t \mathrm{d}G(\tau)\cdot X(\tau) = \int_s^t \mathrm{d}G_1(\tau)\cdot X(\tau) - \int_s^t \mathrm{d}G_2(\tau)\cdot X(\tau) \quad \text{for } a \leqslant s \leqslant t \leqslant b,$$
$$S_k(G)(t) \equiv S_k(G_1)(t) - S_k(G_2)(t) \quad (k = 1,2)$$

and

$$S_c(G)(t) \equiv S_c(G_1)(t) - S_k(G_2)(t).$$

If $G_1(t) \equiv V(G)(t)$ and $G_2(t) \equiv V(G)(t) - G(t)$, then

$$L([a,b],D;G) = \bigcap_{j=1}^{2} L([a,b],D;G_j),$$
$$\mathrm{Car}([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^{2} \mathrm{Car}([a,b]\times D_1, D_2; G_j).$$

If $G(t) \equiv \mathrm{diag}(t,\dots,t)$, then we omit $G$ in the notation containing $G$.

If $X \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, $Y \in \mathrm{BV}([a,b];\mathbb{R}^{n\times m})$, and

$$\det(I_n - \Delta^- X(t)) \neq 0 \quad \text{and} \quad \det(I_n + \Delta^+ X(t)) \neq 0 \quad \text{for } t \in [a,b],$$

then

$$\begin{aligned}
\mathcal{A}(X,Y)(t,t) &= O_{n\times m} \quad \text{for } t \in [a,b],\\
\mathcal{A}(X,Y)(t,s) &= Y(t) - Y(s) + \sum_{s<\tau\leqslant t} \Delta^- X(\tau)\cdot (I_n - \Delta^- X(\tau))^{-1}\Delta^- Y(\tau)\\
&\qquad - \sum_{s\leqslant\tau<t} \Delta^+ X(\tau)\cdot (I_n + \Delta^+ X(\tau))^{-1}\Delta^+ Y(\tau) \quad \text{for } a \leqslant s < t \leqslant b,\\
\mathcal{A}(X,Y)(t,s) &= -\mathcal{A}(X,Y)(s,t) \quad \text{for } a \leqslant s < t \leqslant b.
\end{aligned}$$

The inequalities between the vectors and between the matrices are understood componentwise.

Below we assume that

$$A_1(t) \equiv V(A)(t) \quad \text{and} \quad A_2(t) \equiv V(A)(t) - A(t).$$

A vector-function $x \in \mathrm{BV}([a,b], \mathbb{R}^n)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int_s^t \mathrm{d}A(\tau) \cdot f(\tau, x(\tau)) \quad \text{for } a \leqslant s \leqslant t \leqslant b.$$

Under the solution of the problem (1.1), (1.2) we mean a solutions of the system (1.1) satisfying the boundary condition (1.2).

Let $B \in \mathrm{BV}([a,b], \mathbb{R}^{n\times n})$, $\eta\colon [a,b] \to \mathbb{R}^n$ and $q\colon \mathrm{BV}([a,b], \mathbb{R}^n_+) \to \mathrm{BV}([a,b], \mathbb{R}^n)$ be a matrix-function, a vector-function and an operator, respectively. Then by a solution of the system of generalized ordinary differential inequalities

$$\mathrm{d}x - \mathrm{d}B(t) \cdot x \leqslant \mathrm{d}\eta(t) + \mathrm{d}q(x) \qquad (\geqslant)$$

we mean a vector-function $x \in \mathrm{BV}([a,b], \mathbb{R}^n)$ such that

$$x(t) - x(s) - \int_s^t \mathrm{d}B(\tau) \cdot x(\tau) \leqslant \eta(t) - \eta(s) + q(x)(t) - q(x)(s) \qquad (\geqslant)$$
$$\text{for } a \leqslant s \leqslant t \leqslant b.$$

In addition, if the vector-function $\eta\colon [a,b] \to \mathbb{R}^n$ is nondecreasing and $g\colon \mathrm{BV}([a,b], \mathbb{R}^n_+) \to \mathrm{BV}([a,b], \mathbb{R}^n_+)$ is a positive homogeneous nondecreasing operator, then by $\Omega_{B,\eta,g}$ we denote the set of all solutions of the system

$$|\mathrm{d}x - \mathrm{d}B(t) \cdot x| \leqslant \mathrm{d}\eta(t) + \mathrm{d}g(|x|).$$

If $\eta(t) \equiv 0$ and $q$ is the trivial operator, then we omit $\eta$ and $g$ in the symbols containing them. So $\Omega_B$ is the set of all solutions of the homogeneous system of generalized differential equations

$$\mathrm{d}x = \mathrm{d}B(t) \cdot x.$$

We define

$$\alpha_l(t) = \sum_{i=1}^n v(a_{il})(t) \ (l = 1, \ldots, n) \quad \text{and} \quad \alpha(t) = \sum_{i=1}^n \alpha_i(t) \quad \text{for } t \in [a,b].$$

**Definition 1.1.** The pair $(P,l)$ of a matrix-function $P\in\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\times\mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is said to be consistent if

(i) for any fixed $x\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$ the operator $l(x,\cdot)\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is linear;

(ii) for any $z\in\mathbb{R}^n$, $x$ and $y\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$ the inequalities

$$\|P(t,z)\|\leqslant\xi(t,\|z\|),\quad \|l(x,y)\|\leqslant\xi_0(\|x\|_s)\cdot\|y\|_s$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t\in[a,b]$ and for $t\in D_\alpha$, where $\xi_0\colon\ \mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing function, and $\xi\colon\ [a,b]\times\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing in the second variable function such that $\xi(\cdot,s)\in L([a,b],\mathbb{R}_+;\alpha)$ for every $s\in\mathbb{R}_+$;

(iii) there exists a positive number $\beta$ such that for any $y\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$, $q\in L([a,b],\mathbb{R}^n;A)$ and $c_0\in\mathbb{R}^n$, for which the conditions

$$\det(I_n-\Delta^-A(t)\cdot P(t,y(t)))\neq 0\quad \text{for } t\in[a,b]$$

and

$$\det(I_n+\Delta^+A(t)\cdot P(t,y(t)))\neq 0\quad \text{for } t\in[a,b]$$

hold, an arbitrary solution $x$ of the boundary value problem

$$\mathrm{d}x=\mathrm{d}A(t)\cdot(P(t,y(t))x+q(t)),\quad l(x,y)=c_0$$

admits the estimate

$$\|y\|_s\leqslant\beta(\|c_0\|+\|q\|_{L,\alpha}). \tag{1.4}$$

**Theorem 1.1.** *Let* $A\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $f\in\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$ *and let there exist a positive number* $\varrho$ *and a consistent pair* $(P,l)$ *of a matrix-function* $P\in\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ *and a continuous operator* $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\times\mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ *such that an arbitrary solution of the problem*

$$\mathrm{d}x=\mathrm{d}A(t)\cdot(P(t,x)x+\lambda[f(t,x)-P(t,x)]x), \tag{1.5}$$

$$l(x,x)=\lambda[l(x,x)-h(x)] \tag{1.6}$$

*admits the estimate*

$$\|x\|_s\leqslant\varrho \tag{1.7}$$

*for any* $\lambda\in\,]0,1[$. *Then problem* (1.1), (1.2) *is solvable.*

**Definition 1.2.** Let $\mathcal{S} \subset \mathrm{BV}_s([a,b],\mathbb{R}^{n\times n})$, let $\mathcal{L}$ be a subset of the set of all bounded vector-functionals $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ and $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$. We say that

(i) a matrix-function $B_0 \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{E}^n_{\mathcal{S}}$ if the condition

$$\det(I_n - \Delta^- B_0(t)) \neq 0 \quad \text{and} \quad \det(I_n + \Delta^+ B_0(t)) \neq 0 \quad \text{for } t \in [a,b] \tag{1.8}$$

holds and there exists a sequence $B_k \in \mathcal{S}$ $(k = 1,2,\ldots)$ such that

$$\lim_{k\to\infty} \|B_k - B_0\|_s = 0; \tag{1.9}$$

(ii) a vector-functional $l_0\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the set $\mathcal{E}^n_{\mathcal{L}}(y)$ if there exists a sequence $l_k \in \mathcal{L}$ $(k = 1,2,\ldots)$ such that

$$\lim_{k\to\infty} l_k(y) = l_0(y). \tag{1.10}$$

**Definition 1.3.** Let $g_0\colon \mathrm{BV}([a,b],\mathbb{R}^n_+) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ be a positive homogeneous nondecreasing operator and $h_0\colon \mathrm{BV}_s([a,b],\mathbb{R}^n_+) \to \mathbb{R}^n_+$ a positive homogeneous operator. We say that the pair $(\mathcal{S},\mathcal{L})$ of a set $\mathcal{S} \subset \mathrm{BV}_s([a,b],\mathbb{R}^{n\times n})$ and a set $\mathcal{L}$ of some vector-functionals $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}^n_{g_0,h_0}$ if

(i) every operator $l \in \mathcal{L}$ is linear and continuous with respect to the norm $\|\cdot\|_s$;

(ii) there exist numbers $r_0,\ \xi_0 \in \mathbb{R}_+$ and a nondecreasing function $\varphi\colon [a,b] \to \mathbb{R}$ such that the inequalities

$$\|B(a)\| \leqslant r_0,\ \|B(t) - B(s)\| \leqslant \varphi(t) - \varphi(s) \quad \text{for } a \leqslant s < t \leqslant b \tag{1.11}$$

and

$$\|l(y)\| \leqslant \xi_0 \|y\|_s \tag{1.12}$$

are fulfilled for any $B \in \mathcal{S}$, $l \in \mathcal{L}$ and $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$;

(iii) if for $B_0 \in \mathcal{E}^n_{\mathcal{S}}$ a function $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ is a solution of the system

$$|\mathrm{d}y - \mathrm{d}B_0(t)\cdot y| \leqslant \mathrm{d}g_0(|y|) \tag{1.13}$$

under the condition

$$|l_0(y)| \leqslant h_0(|y|), \tag{1.14}$$

where $l_0 \in \mathcal{E}^n_{\mathcal{L}}(y)$, then $y(t) \equiv 0$.

If $g_0(y)(t) \equiv \int_a^t \mathrm{d}G_0(\tau) \cdot q_0(y)(\tau)$ for $y \in \mathrm{BV}([a,b],\mathbb{R}_+^n)$, where $G_0\colon [a,b] \to \mathbb{R}^n$ is a nondecreasing matrix-function, and $q_0\colon \mathrm{BV}_s([a,b],\mathbb{R}_+^n) \to \mathrm{BV}_s([a,b],\mathbb{R}_+^n)$ is a positive homogeneous operator, then we write $\mathcal{O}^n_{G_0,q_0,h_0}$ instead of $\mathcal{O}^n_{g_0,h_0}$.

**Definition 1.4.** Let $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and let $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \times \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous vector-functional. We say that the pair $(B_0,l_0)$ of the matrix-function $B_0 \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and the vector-functional $l_0\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the set $\mathcal{E}^n_{A,P,l}$ if there exists a sequence $x_k \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ $(k=1,2,\ldots)$ such that the conditions

$$\lim_{k\to\infty} \int_a^t \mathrm{d}A(\tau)\cdot P(\tau,x_k(\tau)) = B_0(t) \quad \text{uniformly on } [a,b] \tag{1.15}$$

and

$$\lim_{k\to\infty} l(x_k,y) = l(y) \quad \text{for } y \in \Omega_{B_0} \tag{1.16}$$

are valid.

**Definition 1.5.** We say that the pair $(P,l)$ of a matrix-function $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\times\mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}^n_A$ with respect to the matrix-function $A$ if

(i) for any fixed $x \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ the operator $l(x,\cdot)\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n, x$ and $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ the inequalities

$$\begin{aligned} \|P(t,z)\| &\leqslant \xi(t), \\ \|l(x,y)\| &\leqslant \xi_0\|y\|_s \end{aligned} \tag{1.17}$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t \in [a,b]$ and for $t \in D_\alpha$, where $\xi_0 \in \mathbb{R}_+$ and $\xi \in L([a,b],\mathbb{R}_+;\alpha)$;

(iii) the problem

$$\mathrm{d}y = \mathrm{d}B_0(t)\cdot y, \tag{1.18}$$

$$l_0(y) = 0 \tag{1.19}$$

has only the trivial solution for every pair $(B_0,l_0) \in \mathcal{E}^n_{A,P,l}$.

**Remark 1.1.** By (1.10) and (1.15), the condition

$$\|\Delta^- A(t)\|\cdot\xi(t) < 1 \quad \text{and} \quad \|\Delta^+ A(t)\|\cdot\xi(t) < 1 \quad \text{for } t \in [a,b]$$

guarantees condition (1.8).

**Corollary 1.1.** *Let* $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $f \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$ *and let there exist a positive number* $\varrho$ *and a pair* $(P,l) \in \mathcal{O}^n_A$ *such that an arbitrary solution of problem* (1.5), (1.6) *admits the estimate* (1.7) *for any* $\lambda \in \,]0,1[$. *Then problem* (1.1), (1.2) *is solvable.*

**Corollary 1.2.** *Let* $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $f \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$, $P \in L([a,b],\mathbb{R}^{n\times n};A)$, *and let* $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ *be a bounded linear operator such that*

$$\det(I_n - \Delta^- A(t)\cdot P(t)) \neq 0 \quad \textit{and} \quad \det(I_n + \Delta^+ A(t)\cdot P(t)) \neq 0 \quad \textit{for } t\in[a,b]$$

*and the problem*

$$\mathrm{d}y = \mathrm{d}A(t)\cdot P(t)y, \ l(y) = 0 \tag{1.20}$$

*has only the trivial solution. Let, moreover, there exist a positive number* $\varrho$ *such that an arbitrary solution of the problem*

$$\mathrm{d}x = \mathrm{d}A(t)\cdot(P(t)x + \lambda[f(t,x) - P(t)x]), \tag{1.21}$$

$$l(x) = \lambda[l(x) - h(x)] \tag{1.22}$$

*admits the estimate* (1.7) *for any* $\lambda \in \,]0,1[$. *Then problem* (1.1), (1.2) *is solvable.*

The following result is analogous to the well-known one belonging to R. Conti and Z. Opial for boundary value problems for ordinary nonlinear differential equations (see, [11], [14], [21]).

**Corollary 1.3.** *Let* $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $f \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$ *and let a pair* $(P,l)\in\mathcal{O}^n_A$ *be such that*

$$|f(t,x) - P(t,x)x| \leqslant \beta(t,\|x\|) \quad \textit{for } t\in[a,b],\ x\in\mathbb{R}^n \tag{1.23}$$

*and*

$$|h(x) - l(x,x)| \leqslant l_0(|x|) + l_1(\|x\|_s) \quad \textit{for } x \in \mathrm{BV}_s([a,b],\mathbb{R}^n), \tag{1.24}$$

*where* $\beta \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+;A)$ *is a nondecreasing in second variable vector-function,* $l_0\colon \mathrm{BV}_s([a,b],\mathbb{R}^n_+)\to\mathbb{R}^n_+$ *is a positive homogeneous continuous operator, and* $l_1 \in C(\mathbb{R}_+,\mathbb{R}^n_+)$. *Let, moreover,*

$$\lim_{k\to\infty}\frac{1}{\varrho}\int_a^b \mathrm{d}V(A)(\tau)\cdot\beta(\tau,\varrho) = O_n, \quad \lim_{\varrho\to\infty}\frac{l_1(\varrho)}{\varrho} = O_n. \tag{1.25}$$

*Then problem* (1.1), (1.2) *is solvable.*

By $Y_P(x)$ we denote the fundamental matrix of the system

$$\mathrm{d}y = \mathrm{d}A(t) \cdot P(t, x(t))y$$

for every $x \in \mathrm{BV}_s([a,b], \mathbb{R}^n)$, satisfying the condition $Y_P(x)(a) = I_n$.

**Corollary 1.4.** *Let* $A \in \mathrm{BV}([a,b], \mathbb{R}^{n\times n})$, $f \in \mathrm{Car}([a,b] \times \mathbb{R}^n, \mathbb{R}^n; A)$, $P \in \mathrm{Car}([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ *and let a continuous operator* $l\colon \mathrm{BV}_s([a,b], \mathbb{R}^n) \times \mathrm{BV}_s([a,b], \mathbb{R}^n) \to \mathbb{R}^n$, *satisfying conditions* (i) *and* (ii) *of Definition* 1.5, *be such that conditions* (1.23)–(1.25) *hold, where* $\beta \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}^n_+; A)$ *is a nondecreasing in the second variable vector-function,* $l_0\colon \mathrm{BV}_s([a,b], \mathbb{R}^n_+) \to \mathbb{R}^n_+$ *is a positive homogeneous continuous operator, and* $l_1 \in C(\mathbb{R}_+, \mathbb{R}^n_+)$. *Let, moreover,*

$$\inf\{|\det(l(x, Y_P(x)))|\colon\ x \in \mathrm{BV}_s([a,b], \mathbb{R}^n)\} > 0. \tag{1.26}$$

*Then problem* (1.1), (1.2) *is solvable.*

**Remark 1.2.** In Corollary 1.4 condition (1.26) cannot be replaced by the condition

$$\det(l(x, Y_P(x))) \neq 0 \quad \text{for } x \in \mathrm{BV}_s([a,b], \mathbb{R}^n). \tag{1.27}$$

The corresponding example for ordinary differential systems, i.e., for the case when $A(t) \equiv \mathrm{diag}(t, \dots, t)$, was constructed in [16]. Basing on this example, it is not difficult to construct analogous examples for the case when $A(t) \not\equiv \mathrm{diag}(t, \dots, t)$. Consider the scalar boundary value problem

$$\mathrm{d}x(t) = \Big(\frac{x(t)}{1 + |x(t)|} + 1\Big)\,\mathrm{d}\alpha(t), \quad x(a) = x(b),$$

where $\alpha(t) = 0$ for $a \leqslant t \leqslant c$ and $\alpha(t) = 1$ for $c < t \leqslant b$, and $c = (a+b)/2$. This problem is not solvable because $x(a) < x(b)$ for every solution $x$ of this equation. On the other hand, in this case

$$\det(l(x, Y_P(x))) = \frac{1}{1 + |x(c)|} \quad \text{for } x \in \mathrm{BV}_s([a,b], \mathbb{R}^n).$$

Therefore, all conditions of Corollary 1.4 are fulfilled except condition (1.26), instead of which condition (1.27) holds.

## 2. Auxiliary propositions

**Lemma 2.1.** *Let* $\alpha_k, \beta_k \in \mathrm{BV}_s([a,b],\mathbb{R})$ $(k=0,1,\ldots)$ *be such that*

$$\lim_{k\to\infty} \|\beta_k - \beta_0\|_s = 0,$$
$$\lim_{k\to\infty} \sup \operatorname{var}_a^b(\alpha_k) < \infty$$

*and*

$$\lim_{k\to\infty} (\alpha_k(t) - \alpha_k(a)) = \alpha_0(t) - \alpha_0(a) \quad \text{uniformly on } [a,b].$$

*Then*

$$\lim_{k\to\infty} \int_a^t \beta_k(\tau)\,\mathrm{d}\alpha_k(\tau) = \int_a^t \beta_0(\tau)\,\mathrm{d}\alpha_0(\tau) \quad \text{uniformly on } [a,b].$$

**Lemma 2.2.** *Let* $Y, Y_k \in \mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ $(k=1,2,\ldots)$ *be such that*

$$\lim_{k\to\infty} Y_k(t) = Y(t) \quad \text{for } t \in [a,b]$$

*and*

$$\|Y_k(t) - Y_k(s)\| \leqslant l_k + \|g(t) - g(s)\| \quad \text{for } a \leqslant s \leqslant t \leqslant b \ (k=1,2,\ldots),$$

*where* $l_k \geqslant 0$, $l_k \to 0$ *as* $k \to \infty$, *and* $g\colon [a,b] \to \mathbb{R}^n$ *is a nondecreasing vector-function. Then*

$$\lim_{k\to\infty} \|Y_k - Y\|_s = 0.$$

The proofs of Lemmas 2.1 and 2.2 are given in [2] and [7], respectively.

**Lemma 2.3** (Lemma on a priori estimates)**.** *Let* $g_0\colon \mathrm{BV}([a,b],\mathbb{R}_+^n) \to \mathrm{BV}([a,b],\mathbb{R}_+^n)$ *and* $h_0\colon \mathrm{BV}_s([a,b],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ *be positive homogeneous nondecreasing and continuous operators and, in addition, let* $g_0(y)\colon [a,b] \to \mathbb{R}^n$ *be a nondecreasing vector-function for* $y \in \mathrm{BV}([a,b],\mathbb{R}_+^n)$. *Let, moreover,* $(\mathcal{S},\mathcal{L}) \in \mathcal{O}^n_{g_0,h_0}$. *Then there exists a positive number* $\varrho$ *such that every solution of the problem*

$$|\mathrm{d}y - \mathrm{d}B_0(t)\cdot y| \leqslant \mathrm{d}g_0(|y|) + \mathrm{d}\eta_0(t), \tag{2.1}$$
$$|l_0(y)| \leqslant h_0(|y|) + \zeta_0 \tag{2.2}$$

*admits the estimate*

$$\|y\|_s \leqslant \varrho_0(\|\zeta_0\| + \|\eta_0(b) - \eta_0(a)\|) \tag{2.3}$$

*for every matrix-function $B_0 \in \mathcal{E}^n_{\mathcal{S}}$, vector-functional $l_0 \in \mathcal{L}$, nondecreasing vector-function $\eta_0\colon [a,b] \to \mathbb{R}^n$ and every number $\zeta_0 \in \mathbb{R}_+$.*

P r o o f. Let us assume that the statement of the lemma is not true. Then for every natural $k$ there exist a matrix-function $B_k \in \mathcal{E}^n_{\mathcal{S}}$, a linear operator $l_k \in \mathcal{L}$, a nondecreasing function $\eta_k\colon [a,b] \to \mathbb{R}^n$, a number $\zeta_k \in \mathbb{R}_+$ and a solution $y_k$ of the problem

$$|\mathrm{d}y_k - \mathrm{d}B_k(t)\cdot y_k| \leqslant \mathrm{d}g_0(|y_k|) + \mathrm{d}\eta_k(t), \tag{2.4}$$

$$|l_k(y_k)| \leqslant h_0(|y_k|) + \zeta_k \tag{2.5}$$

such that

$$\|y_k\|_s \geqslant k(\|\zeta_k\| + \|\eta_k(b) - \eta_k(a)\|). \tag{2.6}$$

According to the definition of the set $\mathcal{E}^n_{\mathcal{S}}$, for every natural $k$ there exists a sequence $B_{ki} \in \mathcal{S}$ $(i = 1,2,\ldots)$ such that

$$\lim_{i\to\infty} \|B_{ki} - B_k\|_s = 0.$$

Consequently, by Definition 1.3, the matrix-function $B_{ki}$ satisfies the inequalities (1.11) as well and so

$$\|B_k(a)\| \leqslant r_0, \quad \|B_k(t) - B_k(s)\| \leqslant \varphi(t) - \varphi(s) \quad \text{for } a \leqslant s < t \leqslant b \ (k = 1,2,\ldots). \tag{2.7}$$

Consequently, according to Helly's choice theorem and Lemma 2.2, without loss of generality we can assume that equality (1.9) holds for some matrix-function $B_0 \in \mathrm{BV}_s([a,b],\mathbb{R}^{n\times n})$. In addition, by the definition of the set $\mathcal{E}^n_{\mathcal{S}}$ we have $B_0 \in \mathcal{E}^n_{\mathcal{S}}$.

Let

$$z_k(t) = \frac{1}{\|y_k\|_s} y_k(t) \quad \text{and} \quad \tilde{\eta}_k(t) = \frac{1}{\|y_k\|_s}\eta_k(t) \quad \text{for } t \in [a,b] \ (k = 1,2,\ldots).$$

Then

$$\|z_k\|_s = 1 \quad (k = 1,2,\ldots). \tag{2.8}$$

On the other hand, by (2.4)–(2.6), for every natural $k$ we have

$$|\mathrm{d}z_k - \mathrm{d}B_k(t)\cdot z_k| \leqslant \mathrm{d}g_0(|z_k|) + \mathrm{d}\tilde{\eta}_k(t), \tag{2.9}$$

$$|l_k(z_k)| \leqslant h_0(|z_k|) + \frac{1}{k}e \tag{2.10}$$

and

$$\|\tilde\eta_k(b)-\tilde\eta_k(a)\|<\frac{1}{k}, \tag{2.11}$$

where $e$ is the vector all components of which are 1.

By the definition of solutions of generalized differential inequalities we find

$$\begin{aligned} \Bigl|z_k(t)-z_k(s)-\int_s^t \mathrm{d}B_k(t)\cdot z_k(t)\Bigr| &\leqslant \tilde\eta_k(t)-\tilde\eta_k(s)\\ &+g_0(|z_k|)(t)-g_0(|z_k|)(s)\quad \text{for } a\leqslant s\leqslant t\leqslant b\ (k=1,2,\dots)\end{aligned} \tag{2.12}$$

and

$$|z_k(t)-z_k(s)|\leqslant \tilde\eta_k(t)-\tilde\eta_k(s)+\operatorname{var}_a^b(B_k)\cdot\|z_k\|_s+g_0(|z_k|)(t)-g_0(|z_k|)(s)$$
$$\text{for } a\leqslant s\leqslant t\leqslant b\ (k=1,2,\dots).$$

From this, (2.7), (2.8) and (2.11) we have

$$\operatorname{var}_a^b(z_k)\leqslant\frac{1}{k}+\varphi(b)-\varphi(a)+g_0(e)(b)-g_0(e)(a)\quad (k=1,2,\dots)$$

and therefore, according to Helly's choice theorem and Lemma 2.2, without loss of generality we can assume that

$$\lim_{k\to\infty}\|z_k-y\|_s=0 \tag{2.13}$$

for some $y\in\mathrm{BV}([a,b],\mathbb{R}^n)$. It follows from this and (2.8) that

$$\|y\|_s=1. \tag{2.14}$$

Further, it is clear that

$$\begin{aligned}\Bigl\|\int_a^t \mathrm{d}B_k(\tau)\cdot z_k(\tau)-\int_a^t \mathrm{d}B_0(\tau)\cdot y(\tau)\Bigr\|&\\ \leqslant \Bigl\|\int_a^t \mathrm{d}B_k(\tau)\cdot z_k(\tau)-\int_a^t \mathrm{d}B_k(\tau)\cdot y(\tau)\Bigr\|&\\ +\Bigl\|\int_a^t \mathrm{d}B_k(\tau)\cdot y(\tau)-\int_a^t \mathrm{d}B_0(\tau)\cdot y(\tau)\Bigr\|&\\ \leqslant \operatorname{var}_a^b(B_k)\cdot\|z_k-y\|_s+\Bigl\|\int_a^t \mathrm{d}B_k(\tau)\cdot y(\tau)-\int_a^t \mathrm{d}B_0(\tau)\cdot y(\tau)\Bigr\|&\\ \text{for } a\leqslant t\leqslant b\ (k=1,2,\dots).&\end{aligned}$$

Applying Lemma 2.1, from this, taking into account (1.9), (2.7) and (2.13), we find

$$\lim_{k\to\infty}\int_a^t \mathrm{d}B_k(\tau)\cdot z_k(\tau)=\int_a^t \mathrm{d}B_0(\tau)\cdot y(\tau)\quad \text{uniformly for } a\leqslant t\leqslant b.$$

By this, (2.11) and (2.13), from (2.12) we conclude

$$\left|y(t)-y(s)-\int_s^t \mathrm{d}B_0(\tau)\cdot y(\tau)\right|\leqslant g_0(|y|)(t)-g_0(|y|)(s)$$
$$\text{for } a\leqslant s\leqslant t\leqslant b\ (k=1,2,\ldots),$$

i.e., $y$ is a solution of the system of generalized differential inequalities

$$|\mathrm{d}y-\mathrm{d}B_0(t)\cdot y|\leqslant \mathrm{d}g_0(|y|). \tag{2.15}$$

On the other hand, in view of (2.8) and (2.10)

$$|l_k(z_k)|\leqslant h_0(e)+\frac{1}{k}e. \tag{2.16}$$

Therefore, without loss of generality we can assume that the sequence $l_k(z_k)$ ($k=1,2,\ldots$) is convergent. Moreover, because of (1.12)

$$\|l_k(z_k)-l_k(y)\|=\|l_k(z_k-y)\|\leqslant \xi_0\|z_k-y\|_s.$$

From this and (2.13), passing to the limit in (2.16) as $k\to\infty$, we obtain that

$$l_0(y)=\lim_{k\to\infty} l_k(z_k)=\lim_{k\to\infty} l_k(y)\leqslant h_0(e).$$

Consequently, inequality (1.14) is valid.

We obtained that $y$ is a solution of problem (1.13), (1.14), where $l_0)\in\mathcal{E}^n_{\mathcal{L}}(y)$. So that, due to condition (iii) of Definition 1.3, we have $y(t)\equiv 0$. But this contradicts the condition (2.14). The lemma is proved. □

The following lemma is analogous to Lemma 2.3 for the set $\mathcal{O}^n_{A,0}$.

**Lemma 2.3′.** *Let $(P,l)\in\mathcal{O}^n_A$. Then there exists a positive number $\varrho$ such that every solution of the problem*

$$|\mathrm{d}y-\mathrm{d}B_0(t)\cdot y|\leqslant \mathrm{d}\eta_0(t),\quad |l_0(y)|\leqslant\zeta_0$$

*admits the estimate*

$$\|y\|_s\leqslant \varrho_0(\|\zeta_0\|+\|\eta_0(b)-\eta_0(a)\|)$$

*for every pair $(B_0,l_0)\in\mathcal{E}^n_{A,P,l}$, a nondecreasing function $\eta_0\colon [a,b]\to\mathbb{R}^n$ and every number $\zeta_0\in\mathbb{R}_+$.*

P r o o f. The proof of this lemma is the same as the proof of Lemma 2.3, where we assume $S=\mathcal{E}^n_S$ and $\mathcal{L}=\{l(x,\cdot)\colon\ x\in\mathrm{BV}([a,b],\mathbb{R}^n)\}$. Let $y$ and $B_0$ be the vector- and matrix-functions, respectively, appearing in the proof of Lemma 2.3. Then, in this case, the system of inequalities (2.15) coincides with the system (1.18), and the vector-inequality (1.14) coincides with the equality

$$\lim_{k\to\infty}|l(x_k,y)|=0 \tag{2.17}$$

for some sequence $x_k$ $(k=1,2,\ldots)$ from $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Let now $y_1,\ldots,y_n$ be a fundamental system os solutions of system (1.18). By condition (ii) of Definition 1.4, we can assume without loss of generality that the sequence $l(x_k,y_m)$ $(k=1,2,\ldots)$ is convergent for every $m\in\{1,\ldots,n\}$. Let an operator $l_0\colon\ \Omega_{B_0}\to\mathbb{R}^n$ be defined by

$$l_0(z)=\sum_{m=1}^n c_m\lim_{k\to\infty}l(x_k,y_m),$$

where $c_1,\ldots,c_m$ are the numbers such that $z(t)\equiv\sum\limits_{m=1}^n c_my_m(t)$. Hence, due to (2.17) and Hahn-Banach's theorem, $y$ is the solution of problem (1.18), (1.19). So that, in view of condition (iii) of Definition 1.5, we have $y(t)\equiv 0$ just as above, since $(B_0,l_0)\in\mathcal{E}^n_{A,P,l}$. □

## 3. Proofs of the main results

P r o o f of Theorem 1.1. Let $\xi$ and $\xi_0$ be the functions appearing in Definition 1.1 and corresponding to the consistent pair $(P,l)$.

Set

$$\gamma(t) = 2n^2\varrho\xi(t,2\varrho) + \sup\{\|f(t,y)\|\colon\ y\in\mathbb{R}^n, \|y\|\leqslant 2\varrho\},$$
$$\gamma_0 = 2\varrho\xi_0(2\varrho) + \sup\{\|h(y)\|\colon\ \|y\|_s\leqslant 2\varrho\},$$

(3.1) $$\sigma(s) = \begin{cases} 1 & \text{for } 0\leqslant s\leqslant\varrho,\\ 2-s/\varrho & \text{for } \varrho<s<2\varrho,\\ 0 & \text{for } s\geqslant 2\varrho;\end{cases}$$

(3.2) $$q(y)(t) = \sigma(\|y\|_s)\cdot(f(t,y(t)) - P(t,y(t))y(t)) \quad \text{for } t\in[a,b],\ y\in \mathrm{BV}([a,b],\mathbb{R}^n)$$

and

(3.3) $$c_0(y) = \sigma(\|y\|_s)\cdot(l(y,y)-h(y)) \quad \text{for } y\in\mathrm{BV}([a,b],\mathbb{R}^n).$$

Then $\gamma\in L([a.b],\mathbb{R};\alpha)$, $\gamma_0<\infty$ and the inequalities

(3.4) $$\|q(y)(t)\|\leqslant\gamma(t) \quad \text{for } y\in\mathrm{BV}([a,b],\mathbb{R}^n)$$

and

(3.5) $$\|c_0(y)\|\leqslant\gamma_0 \quad \text{for } y\in\mathrm{BV}([a,b],\mathbb{R}^n)$$

are valid for $\mu(\alpha)$-almost all $t\in[a,b]$ and for $t\in D_\alpha$.

For an arbitrary fixed $y\in\mathrm{BV}([a,b],\mathbb{R}^n)$, let us consider the linear boundary value problem

(3.6) $$\mathrm{d}x = \mathrm{d}A(t)\cdot(P(t,y(t))x + q(y)(t)),\ \ l(y,x) = c_0(y).$$

By virtue of condition (iii) of Definition 1.1, the homogeneous problem

(3.7) $$\mathrm{d}x = \mathrm{d}A(t)\cdot P(t,y(t))x, \quad l(y,x)=0$$

has only the trivial solution. Therefore, by Theorem 1.1 from [9] problem (3.6) has a unique solution $x$. In view of (1.4) and (3.4)–(3.6), this solution admits the estimate

$$\|x\|_s\leqslant\beta\biggl(\|c_0(y)\| + \int_a^b\|q(y)(t)\|\,\mathrm{d}\alpha(t)\biggr)\leqslant\beta\biggl(\gamma_0+\int_a^b\|q(y)(t)\|\,\mathrm{d}\alpha(t)\biggr)$$

and, therefore,

(3.8) $$\|x\|_s\leqslant r_0,$$

where

$$r_0 = \beta(\gamma_0 + \|\gamma\|_{L,\alpha}).$$

On the other hand, by the definition of solutions and (3.4), (3.5) we have

$$\begin{aligned}\operatorname{var}_a^b(x) &\leqslant \left\| \int_a^b \mathrm{d}A(t) \cdot (|P(t, y(t))| \cdot |x(t)| + |q(y)(t)|) \right\| \\ &\leqslant r_0 \int_a^b \|P(t, y(t))\| \,\mathrm{d}\alpha(t) + \|\gamma\|_{L,\alpha},\end{aligned}$$

i.e., by condition (ii) of Definition 1.1, we find

$$\operatorname{var}_a^b(x) \leqslant r_0 \int_a^b \xi(t, \|y\|_s) \,\mathrm{d}\alpha(t) + \|\gamma\|_{L,\alpha}. \tag{3.9}$$

Let now

$$U = \{y \in \mathrm{BV}([a,b], \mathbb{R}^n) \colon \|y\|_s \leqslant r_0, \ \operatorname{var}_a^b(y) \leqslant r_1\},$$

where

$$r_1 = r_0 \int_a^b \xi(t, \|y\|_s) \,\mathrm{d}\alpha(t) + \|\gamma\|_{L,\alpha}.$$

Let $\omega\colon \mathrm{BV}([a,b], \mathbb{R}^n) \to \mathrm{BV}([a,b], \mathbb{R}^n)$ be an operator which to every $y \in \mathrm{BV}([a,b], \mathbb{R}^n)$ assigns the solution $x$ of problem (3.6).

Let $y \in U$ and $x = \omega(y)$. Then by (3.8) and (3.9) we have

$$\|\omega(y)\|_s \leqslant r_0$$

and

$$\operatorname{var}_a^b(\omega(y)) \leqslant r_1, \tag{3.10}$$

i.e. $\omega(y) \in U$. So

$$\omega(U) \subset U. \tag{3.11}$$

It is evident that $U$ is a closed and convex subset of $\mathrm{BV}_v([a,b], \mathbb{R}^n)$.

Let us show that $\omega$ is a continuous operator with respect to the norm $\|\cdot\|_v$.

Let a sequence $y_k \in U$ $(k = 0, 1, \ldots)$ be such that

$$\lim_{k \to \infty} \|y_k - y_0\|_v = 0. \tag{3.12}$$

Then

$$\lim_{k \to \infty} \|y_k - y_0\|_s = 0$$

and by (3.3)

$$\lim_{k\to\infty} c_0(y_k) = c_0(y_0), \tag{3.13}$$

since the operators $l$ and $h$ are continuous with respect to the norm $\|\cdot\|_s$. Let now

$$\widetilde{A}_k(t) \equiv \int_a^t \mathrm{d}A(\tau)\cdot P(\tau, y_k(\tau)) \quad (k = 0, 1, \ldots).$$

Then

$$|\widetilde{A}_k(t) - \widetilde{A}_0(t)| \leqslant \int_a^b \mathrm{d}V(A)(\tau)\cdot |P(\tau, y_k(\tau)) - P(\tau, y_0(\tau))| \\ \text{for } t \in [a,b] \ (k = 1, 2, \ldots).$$

Moreover,

$$\lim_{k\to\infty} \int_a^b \mathrm{d}V(A)(\tau)\cdot |P(\tau, y_k(\tau)) - P(\tau, y_0(\tau))| = 0$$

because $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and, therefore, according to the Lebesgue theorem

$$\lim_{k\to\infty} \widetilde{A}_k(t) = \widetilde{A}_0(t) \quad \text{uniformly on } [a,b].$$

Using condition (ii) of Definition 1.1, we obtain

$$\mathrm{var}_a^b(I_n + \widetilde{A}_k) \leqslant b - a + \int_a^b \xi(\tau, r)\,\mathrm{d}\alpha(\tau),$$

where $r$ is some large enough positive number, independent of $k$. Consequently, taking into account (3.13), condition (iii) of Definition 1.1 and the fact that the vector-functionals $l(y_k,\cdot)$ $(k = 0, 1, \ldots)$ are continuous on the space $\mathrm{BV}_s([a,b],\mathbb{R}^n)$, we conclude that the conditions of Theorem 1.1 from [5] are fulfilled for $H_k(t) \equiv I_n$ $(k = 0, 1, \ldots)$. Due to this theorem,

$$\lim_{k\to\infty} \|x_k - x\|_s = 0. \tag{3.14}$$

Let us show that

$$\lim_{k\to\infty} \|x_k - x\|_v = 0. \tag{3.15}$$

We have

$$x_k(t) - x(t) - (x_k(s) - x(s)) = \int_s^t \mathrm{d}A(\tau) \cdot P(\tau, y_k(\tau))(x_k(\tau) - x(\tau))$$
$$+ \int_s^t \mathrm{d}A(\tau) \cdot (P(\tau, y_k(\tau)) - P(\tau, y(\tau))x(\tau)$$
$$+ \int_s^t \mathrm{d}A(\tau) \cdot (q(y_k)(\tau)) - q(y)(\tau)) \quad \text{for } a \leqslant s < t \leqslant b \ (k = 1, 2, \ldots).$$

From this and condition (ii) of Definition 1.1 we conclude

$$\operatorname{var}_a^b(x_k - x) \leqslant n^2 \|x_k - x\|_s \int_s^t \xi(\tau, r_0 + r_1)\,\mathrm{d}\alpha(\tau)$$
$$+ (1 + \|x\|_s) \left\| \int_s^t \mathrm{d}V(A)(\tau) \cdot |P(\tau, y_k(\tau)) - P(\tau, y(\tau)| \right\|$$
$$+ \left\| \int_s^t \mathrm{d}V(A)(\tau) \cdot (q(y_k)(\tau)) - q(y)(\tau)) \right\|$$

for any large enough natural $k$. From this, due to conditions (3.12), (3.14) and the Lebesgue theorem, condition (3.15) follows. So $\omega$ is continuous with regard to the norm $\|\cdot\|_v$.

Let us verify that the set $\omega(U)$ is precompact. Consider an arbitrary sequence of functions $y_k$ $(k = 1, 2, \ldots)$ from $U$. As above, assume $x_k = \omega(y_k)$ $(k = 1, 2, \ldots)$. Then by (3.9) and (3.10) the sequences $x_k$ $(k = 1, 2, \ldots)$ and $y_k$ $(k = 1, 2, \ldots)$ satisfy the conditions of Helly's choice theorem. Therefore, there exist vector-functions $x_0, y_0 \in \mathrm{BV}([a, b], \mathbb{R}^n)$ and a sequence of natural numbers $k_i$ $(i = 1, 2, \ldots)$ such that

$$\lim_{i\to\infty} x_{k_i}(t) = x_0(t) \quad \text{and} \quad \lim_{i\to\infty} y_{k_i}(t) = y_0(t) \quad \text{for } t \in [a, b]. \tag{3.16}$$

Taking into account these equalities and passing to the limit as $i \to \infty$ in the equalities

$$x_{k_i}(t) = x_{k_i}(a) + \int_a^t \mathrm{d}A(\tau) \cdot (P(\tau, y_{k_i}(\tau))x_{k_i}(\tau) + q(y_{k_i})(\tau)) \quad \text{for } t \in [a, b],$$

according to the Lebesgue theorem we find

$$x_0(t) = x_0(a) + \int_s^t \mathrm{d}A(\tau) \cdot (P(\tau, y_0(\tau))x_0(\tau) + q(y_0)(\tau)) \quad \text{for } t \in [a, b].$$

Hence

$$\begin{aligned}\|x_{k_i} - x_0\|_v \leqslant \|x_{k_i}(a) - x_0(a)\| \\ + \left\| \int_a^b \mathrm{d}V(A)(\tau) \cdot |P(\tau, y_{k_i}(\tau))| \cdot |x_{k_i}(\tau) - x_0(\tau)| \right\| \\ + \left\| \int_a^b \mathrm{d}V(A)(\tau) \cdot (|P(\tau, y_{k_i}(\tau)) - P(\tau, y_0(\tau))| \cdot |x_0(\tau)| + |q(y_{k_i})(\tau) - q(y_0)(\tau)|) \right\|.\end{aligned}$$

From this, using the Lebesgue theorem, in view of condition (ii) of Definition 1.1 and (3.16), we have

$$\lim_{i\to\infty} \|x_{k_i} - x_0\|_v = 0.$$

Consequently, the set $\omega(U)$ is precompact in the space $\mathrm{BV}_v([a,b],\mathbb{R}^n)$. Therefore, owing to Schauder's principle, there exists $x \in U$ such that

$$x(t) = \omega(x)(t) \quad \text{for } t \in [a,b].$$

By equalities (3.2) and (3.3), $x$ is obviously a solution of problem (1.5), (1.6), where

$$\lambda = \sigma(\|x\|_s). \tag{3.17}$$

Let us show that the function $x$ admits the estimate (1.7). Suppose the contrary. Then either

$$\varrho < \|x\|_s < 2\varrho \tag{3.18}$$

or

$$\|x\|_s \geqslant 2\varrho. \tag{3.19}$$

If we assume that inequality (3.18) is fulfilled, then because of (3.1) and (3.17) we find that $\lambda \in ]0,1[$. However, by the conditions of the theorem, in this case we conclude that estimate (1.7) holds. But this contradicts condition (3.18).

Suppose now that inequality (3.19) is fulfilled. Then by (3.1) and (3.17) we establish that $\lambda = 0$. Hence, $x$ is a solution of problem (3.7). But this is impossible because problem (3.7) has only the trivial solution. The above obtained contradiction proves the validity of estimate (1.7).

By virtue of (1.7), (3.1) and (3.17) it is clear that $\lambda = 1$. Therefore, $x$ is a solution of problem (1.1), (1.2). The theorem is proved. □

P r o o f of Corollary 1.1. Let $\mathcal{S} = \mathcal{E}^n_{A,P}$ and $\mathcal{L} = \{l(x,\cdot)\colon\ x \in \mathrm{BV}([a,b],\mathbb{R}^n)\}$. Then condition $(P,l) \in \mathcal{O}^n_A$ is equivalent to condition $(\mathcal{S},\mathcal{L}) \in \mathcal{O}^n_{0,0}$. So, due to Lemma 2.3′, the pair $(P,l)$ is consistent. Therefore, the corollary follows from Theorem 1.1. The corollary is proved. □

P r o o f of Corollary 1.2. Let the matrix-function $P$ and the linear operator $l$ be defined by $P(t,x) \equiv P(t)$ and $l(x,y) \equiv l(y)$. Then by Definition 1.5 the condition $(P,l) \in \mathcal{O}^n_A$ is fulfilled if and only if problem (1.20) has only the trivial solution. In addition, problem (1.5), (1.6) is equivalent to problem (1.21), (1.22). Therefore, Corollary 1.2 follows from Corollary 1.1. The corollary is proved. □

P r o o f of Corollary 1.3. Let $\mathcal{S}$ and $\mathcal{L}$ be the sets defined in Corollary 1.1. Then due to Lemma 2.3′ and conditions (1.23), (1.24) there exists a positive number $\varrho_0$ such that an arbitrary solution $x$ of problem (1.5), (1.6) admits the estimate

$$\|x\|_s \leqslant \lambda\varrho_0\left(\|l_1(\|x\|_s)\| + \left\|\int_a^b \mathrm{d}V(A)(\tau)\cdot\beta(\tau,\|x\|_s)\right\|\right) \tag{3.20}$$

for any $\lambda \in\, ]0,1[$.

According to condition (1.25), there exists a positive number $\varrho_1$ such that for, any $\lambda \in\, ]0,1[$,

$$\lambda\varrho_0\left(\|l_1(\varrho)\| + \left\|\int_a^b \mathrm{d}V(A)(\tau)\cdot\beta(\tau,\varrho)\right\|\right) < \varrho \quad \text{for } \varrho > \varrho_1. \tag{3.21}$$

If we assume that

$$\|x\|_s > \varrho_1,$$

then by (3.21) we find

$$\lambda\varrho_0\left(\|l_1(\|x\|_s)\| + \left\|\int_a^b \mathrm{d}V(A)(\tau)\cdot\beta(\tau,\|x\|_s)\right\|\right) < \|x\|_s,$$

which contradicts (3.20).

Hence we have

$$\|x\|_s \leqslant \varrho_1.$$

Consequently, estimate (1.7) holds for every solution $x$ of problem (1.5), (1.6). In addition, the number $\varrho_1$ does not depend on $x$. So the corollary follows from Theorem 1.1. The corollary is proved. □

P r o o f of Corollary 1.4. Let us show that problem (1.18), (1.19) has only the trivial solution for every pair $(B_0, l_0) \in \mathcal{E}^n_{A,P,l}$ appearing in Definition 1.5.

Let $x_k$ $(k = 1, 2, \ldots)$ be the sequence such that conditions (1.15) and (1.16) are valid.

Due to Theorem 1.1 from [5] we have

$$\lim_{k\to\infty} Y_P(x_k)(t) = Y_0(t) \quad \text{uniformly on } [a,b], \tag{3.22}$$

where $Y_0$ is the fundamental matrix of system (1.18) satisfying the condition $Y_0(a) = I_n$.

In view of (1.16), (3.22) and conditions (i) and (ii) of Definition 1.5 we have

$$\begin{aligned}\|l(x_k, Y_P(x_k)) - l_0(Y_0)\| &\leqslant \|l(x_k, Y_0) - l_0(Y_0)\| + \|l(x_k, Y_P(x_k) - l(x_k, Y_0)\| \\ &\leqslant \|l(x_k, Y_0) - l_0(Y_0)\| + \xi_0 \sup\{\|Y_P(x_k)(t) - Y_0(t)\|\colon\ t \in [a,b]\} \to 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } k \to \infty.\end{aligned}$$

Thus

$$\lim_{k\to\infty} l(x_k, Y_P(x_k)) = l_0(Y_0)$$

and, therefore,

$$\lim_{k\to\infty} \det(l(x_k, Y_P(x_k))) = \det(l_0(Y_0)).$$

This implies that inequality (1.26) is equivalent to the condition $\det(l_0(Y_0)) \neq 0$ for every $(B_0, l_0) \in \mathcal{E}^n_{A,P,l}$. Therefore, problem (1.18), (1.19) has only the trivial solution. So condition $(P, l) \in \mathcal{O}^A_0$ is fulfilled. Consequently, Corollary 1.4 is equivalent to Corollary 1.3. The corollary is proved. □

## 4. The theorem on solvability of problem (1.1), (1.3)

As mentioned in Section 1, we investigate problem (1.1), (1.3) under the assumption that the vector-function $f\colon\ [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Carathéodory conditions, and the operators $t_i\colon\ \mathrm{BV}_s([a,b], \mathbb{R}^n) \to \mathbb{R}^n$ $(i = 1, 2)$ and $\mathcal{B}\colon\ \mathrm{BV}_s([a,b], \mathbb{R}^n) \to \mathbb{R}^{n\times n}$ are continuous.

Let $f(t,x) = (f_l(t,x))^n_{l=1}$, $A(t) = (a_{il}(t)^n_{i,l=1}$, $a_{il}(t) \equiv a_{1il}(t) - a_{2il}(t)$, where $a_{1il}(t) \equiv v(a_{il})(t)$ and $a_{2il}(t) \equiv v(a_{il})(t) - a_{il}(t)$ $(i, l = 1, \ldots, n)$. Moreover, we assume

$$I_0 = \{t_1(x)\colon\ x \in \mathrm{BV}_s([a,b], \mathbb{R}^n)\},$$
$$\|\mathcal{B}(x)\|_0 = \max\{\|\mathcal{B}(x)y\|\colon\ y \in \mathbb{R}^n, \|y\| = 1\}.$$

Let the function $g \in L([a,b],\mathbb{R}_+;\alpha)$ be such that

$$1 - g(t)\Delta^-\alpha(t) \neq 0 \quad \text{and} \quad 1 - g(t)\Delta^+\alpha(t) \neq 0 \quad \text{for } t \in [a,b]),$$

where $\alpha$ is the function defined in Section 1. This condition guarantees the unique solvability of the Cauchy problem

$$\text{(4.1)} \quad \mathrm{d}\gamma = (-1)^j g_1(t)\gamma \, \mathrm{d}\alpha(t) \quad \text{for } t \in [a,b],\ (-1)^j(t - t_1(x)) \geqslant 0 \ (j = 1,2),$$

$$\text{(4.2)} \quad \gamma(t_1(x)) = 1$$

for every $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ and this solution $\gamma_{x,g}(t)$ is given by the formula (see [12], [13])

$$\text{(4.3)} \quad \gamma_{x,g}(t) = \begin{cases} \exp\Big(\displaystyle\int_{t_1(x)}^t g_1(\tau)\,\mathrm{d}s_c(\tau)\Big) \prod_{t_1(x)<\tau\leqslant t} (1 - g_1(\tau)\Delta^-\alpha(\tau))^{-1} \\ \quad \times \displaystyle\prod_{t_1(x)\leqslant\tau<t} (1 + g_1(\tau)\Delta^+\alpha(\tau)) \quad \text{for } t_1(x) < t \leqslant b, \\ \exp\Big(-\displaystyle\int_{t_1(x)}^t g_1(\tau)\,\mathrm{d}s_c(\tau)\Big) \prod_{t<\tau\leqslant t_1(x)} (1 + g_1(\tau)\Delta^-\alpha(\tau)) \\ \quad \times \displaystyle\prod_{t\leqslant\tau<t_1(x)} (1 - g_1(\tau)\Delta^+\alpha(\tau))^{-1} \quad \text{for } a \leqslant t < t_1(x), \\ 1 \quad \text{for } t = t_1(x). \end{cases}$$

**Theorem 4.1.** *Let $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $f \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$ and let there exist functions $g_1 \in L([a,b],\mathbb{R}_+;\alpha_{il})$ $(i,l = 1,\ldots,n)$ and $g_2 \in L([a,b],\mathbb{R}_+;\alpha_{il})$ $(i,l = 1,\ldots,n)$ such that*

$$\text{(4.4)} \quad g_1(t)\Delta^-\alpha(t) < 1 \quad \textit{and} \quad g_1(t)\Delta^+\alpha(t) < 1 \quad \textit{for } t \in [a,b];$$

$$\text{(4.5)} \quad (-1)^{m+1} f_l(t,x_1,\ldots,x_n)\,\mathrm{sgn}[(t-t_0)x_i] \leqslant g_1(t)|x_l| + g_2(t)$$

*for $\mu(s_0(a_{mil}))$-almost all $t \in [a,b]$, $t \in D_{a_{mil}}$, $t_0 \in [a,b]$, $(x_k)_{k=1}^n \in \mathbb{R}^n$ $(m = 1,2;$ $i,l = 1,\ldots,n)$;*

$$\text{(4.6)} \quad \begin{aligned} -\langle \Delta^- A(t_0)\cdot f(t_0,x), \mathrm{sgn}\,x\rangle &\leqslant (g_1(t)\|x\| + g_2(t))\Delta^-\alpha(t_0) \quad \textit{and} \\ \langle \Delta^+ A(t_0)\cdot f(t_0,x), \mathrm{sgn}\,x\rangle &\leqslant (g_1(t)\|x\| + g_2(t))\Delta^+\alpha(t_0) \\ &\qquad \textit{for } t_0 \in [a,b],\ x \in \mathbb{R}^n \quad (j = 1,2); \end{aligned}$$

*and*

$$\text{(4.7)} \quad \gamma_x(t_2(x))\cdot\|\mathcal{B}(x)\|_0 \leqslant \delta \quad \textit{for } t \in [a,b],\ x \in \mathrm{BV}([a,b],\mathbb{R}^n),$$

*where $\gamma_x(t) \equiv \gamma_{x,g}(t)$ is the function defined by equalities* (4.3).

*Then problem* (1.1), (1.3) *is solvable.*

P r o o f of Theorem 4.1. For every $x$ and $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$, we suppose

$$h(x) = x(t_1(x)) - \mathcal{B}(x)\cdot x(t_2(x)) - c_0,$$
$$P(t,x) = g_1(t)\,\mathrm{sgn}(t - t_1(x))\cdot I_n,$$
$$l(x,y) = y(t_1(x)).$$

Obviously, $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$, the operator $l\colon \mathrm{BV}_s([a,b],\mathbb{R}^n)\times \mathrm{BV}_s([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is continuous and the pair $(P,l)$ is consistent.

By Theorem 1.1, to prove Theorem 4.1 it suffices to establish a priori boundedness of solutions of the problem

$$\mathrm{d}x = \mathrm{d}A(t)\cdot((1-\lambda)P(t,x)x + \lambda f(t,x)), \tag{4.8}$$
$$x(t_1(x)) = \lambda(\mathcal{B}(x)\cdot x(t_2(x)) + c_0) \tag{4.9}$$

uniformly with respect to $\lambda \in\, ]0,1[$.

Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of problem (4.8), (4.9) for some $\lambda\in\,]0,1[$. Let $i\in\{1,\dots,n\}$ be fixed. Then

$$\begin{aligned}\mathrm{d}x_i(t) = {}&\sum_{l=1}^n (1-\lambda)g_1(t)\,\mathrm{sgn}(t-t_0))\cdot x_l(t)\,\mathrm{d}a_{il}(t)\\ &+\sum_{l=1}^n \lambda f_l(t,x_1(t),\dots,x_n(t))\,\mathrm{d}a_{il}(t),\end{aligned}$$

where $t_0 = t_1(x)$. From this, by Lemma 2.2 from [8], we have

$$\begin{aligned}\Big\{\mathrm{d}|x_i(t)| - \Big[\sum_{l=1}^n ((1-\lambda)g_1(t)\,\mathrm{sgn}(t-t_0)\cdot|x_l(t)|\,\mathrm{d}v(a_{il})(t)\\ + \lambda\,\mathrm{sgn}\,x_i(t) f_l(t,x_1(t),\dots,x_n(t))\,\mathrm{d}a_{il}(t))\Big]\Big\}\,\mathrm{sgn}(t-t_0)\leqslant 0\end{aligned} \tag{4.10}$$

and

$$\Delta^-|x_i(t_0)| \geqslant \mathrm{sgn}\,x_i(t_0)\cdot\sum_{l=1}^n f_l(t_0,x_1(t_0),\dots,x_n(t_0))\Delta^- a_{il}(t_0) \tag{4.11}$$

and

$$\Delta^+|x_i(t_0)| \leqslant \mathrm{sgn}\,x_i(t_0)\cdot\sum_{l=1}^n f_l(t_0,x_1(t_0),\dots,x_n(t_0))\Delta^+ a_{il}(t_0).$$

Using (4.5), from (4.10) we get

$$\operatorname{sgn}(t-t_0)\,\mathrm{d}|x_i(t)| \leqslant (1-\lambda)g_1(t)\sum_{l=1}^{n}|x_l(t)|\,\mathrm{d}v(a_{il})(t)$$
$$+\lambda\sum_{l=1}^{n}(g_1(t)|x_l(t)|+g_2(t))\,\mathrm{d}v(a_{il})(t)$$

and, therefore,

$$\operatorname{sgn}(t-t_0)\,\mathrm{d}|x_i(t)| \leqslant \sum_{l=1}^{n}|x_l(t)|\,\mathrm{d}v(a_{il})(t)+\lambda\sum_{l=1}^{n}g_2(t))\,\mathrm{d}v(a_{il})(t).$$

Summing over $i$ the last inequality we find

$$\operatorname{sgn}(t-t_0)\,\mathrm{d}u(t) \leqslant (g_1(t)u(t)+g_2(t))\,\mathrm{d}\alpha(t) \quad (j=1,2), \tag{4.12}$$

where $u(t)\equiv\|x(t)\|$.

On the other hand, according to (4.6), (4.11) implies

$$\Delta^{-}u(t_0) \geqslant (g_1(t_0)u(t_0)+g_2(t_0))\Delta^{-}\alpha(t_0)$$

and

$$\Delta^{+}u(t_0) \leqslant (g_1(t_0)u(t_0)+g_2(t_0))\Delta^{+}\alpha(t_0).$$

Taking into account these estimates, (4.4) and (4.12), due to Lemma 2.4 from [3] we have

$$u(t)\leqslant v(t) \quad \text{for } t\in[a,b], \tag{4.13}$$

where the function $v$ is the unique solution of the Cauchy problem

$$\mathrm{d}v=(-1)^j(g_1(t)v+g_2(t))\,\mathrm{d}\alpha(t) \quad \text{for } (-1)^j(t-t_0)\geqslant 0\ (j=1,2),\ v(t_0)=u(t_0).$$

According to the variation-of-constants formula (see Corollary III.2.14 from [24]) we get

$$v(t)=g_2(t)\operatorname{sgn}(t-t_0)+\gamma_x(t)\Big\{u(t_0)-\int_{t_0}^{t}g_2(s)\operatorname{sgn}(s-t_0)\,\mathrm{d}\gamma_x^{-1}(s)\Big\} \quad \text{for } t\in[a,b].$$

From this, using the formula of integration-by-parts (see Theorem I.4.33 from [24]) we obtain

$$(4.14)\quad v(t)=\gamma_x(t)u(t_0)+\gamma_x(t)\bigg\{\int_{t_0}^{t}\gamma_x^{-1}(s)\,\mathrm{d}g_2(s)-\sum_{t_0<s\leqslant t}\mathrm{d}_1\gamma_x^{-1}(s)\,\mathrm{d}_1g_2(s)+\sum_{t_0\leqslant s<t}\mathrm{d}_2\gamma_x^{-1}(s)\,\mathrm{d}_2g_2(s)\bigg\}\quad\text{for } t_0<t\leqslant b.$$

On the other hand, (4.1) yields the equalities

$$\mathrm{d}_j\gamma_x^{-1}(t)=-\gamma_x^{-1}(t)(1+g_1(t)\,\mathrm{d}_j\alpha(t))^{-1}g_1(t)\,\mathrm{d}_j\alpha(t)\quad\text{for } t_0\leqslant t\leqslant b.$$

Taking into account these equalities, from (4.14) we conclude that

$$v(t)=\gamma_x(t)u(t_0)+\int_{t_0}^{t}\gamma_x(t)\gamma_x^{-1}(s)\,\mathrm{d}\mathcal{A}(\tilde{g_1},g_2)(s)\quad\text{for } t_0<t\leqslant b,$$

where $\tilde{g_1}(t)=\int_{t_0}^{t}g_1(\tau)\,\mathrm{d}\alpha(\tau)$ for $t\in[a,b]$.

Analogously we show that

$$v(t)=\gamma_x(t)u(t_0)-\int_{t_0}^{t}\gamma_x(t)\gamma_x^{-1}(s)\,\mathrm{d}\mathcal{A}(\tilde{g_1},g_2)(s)\quad\text{for } a\leqslant t<t_0.$$

So, by (4.13)

$$(4.15)\qquad u(t)\leqslant\gamma_x(t)u(t_0)+\left|\int_{t_0}^{t}\gamma_x(t)\gamma_x^{-1}(s)\,\mathrm{d}\mathcal{A}(\tilde{g_1},g_2)(s)\right|\quad\text{for } a\leqslant t\leqslant b.$$

Due to (4.3) and (4.4), it is evident that the function $\gamma_x$ is nonincreasing on the interval $[a,t_0]$ and nondecreasing on the interval $[t_0,b]$. In addition, in view of (4.2), we get

$$(4.16)\qquad \gamma_x(t)\geqslant 1\quad\text{for } t\in[a,b].$$

Besides, by (4.3) and (4.4) we have

$$\ln(\gamma_x(t))=\int_{t_0}^{t}g_1(\tau)\,\mathrm{d}s_c(\tau)-\sum_{t_0<\tau\leqslant t}\ln(1-g_1(\tau)\,\mathrm{d}_1\alpha(\tau))+\sum_{t_0\leqslant\tau<t}\ln(1+g_1(\tau)\,\mathrm{d}_2\alpha(\tau))\leqslant\int_{a}^{t}g_1(\tau)\,\mathrm{d}s_c(\tau)-\sum_{a<\tau\leqslant t}\ln(1-g_1(\tau)\,\mathrm{d}_1\alpha(\tau))+\sum_{a\leqslant\tau<t}\ln(1+g_1(\tau)\,\mathrm{d}_2\alpha(\tau))\ \text{ for } t_0<t\leqslant b$$

and, consequently,

$$\ln(\gamma_x(t)) \leqslant \ln(\gamma_*(t)) \quad \text{for } t_0 \leqslant t \leqslant b, \tag{4.17}$$

where $\gamma_*$ is the unique solution of the Cauchy problem

$$\mathrm{d}\gamma(t) = g_1(t)\gamma(t)\,\mathrm{d}\alpha(t), \quad \gamma(a) = 1.$$

Analogously we show that

$$\ln(\gamma_x(t)) \leqslant \ln(\gamma^*(t)) \quad \text{for } a \leqslant t \leqslant t_0, \tag{4.18}$$

where $\gamma^*$ is the unique solution of the Cauchy problem

$$\mathrm{d}\gamma = -g_1(t)\gamma\,\mathrm{d}\alpha(t), \quad \gamma(b) = 1.$$

On the other hand, $\gamma_*$ and $\gamma^* \in \mathrm{BV}([a,b],\mathbb{R})$. From this, it follows that $\tilde{\gamma} \in \mathrm{BV}([a,b],\mathbb{R})$, where $\tilde{\gamma}(t) \equiv \max\{\gamma_*(t), \gamma^*(t)\}$. Therefore, due to (4.17) and (4.18), we have

$$\gamma_x(t) \leqslant \varrho_1 \quad \text{for } a \leqslant t \leqslant b, \tag{4.19}$$

where $\varrho_1 = \|\tilde{\gamma}\|_v$. It is evident that the number $\varrho_1$ does not depend on $x$.

By (4.15) and the equality $t_0 = t_1(x)$, it is obvious that

$$u(t_2(x)) \leqslant \gamma_x(t_2(x))u(t_1(x)) + \left|\int_{t_1(x)}^{t_2(x)} \gamma_x(t_2(x))\gamma_x^{-1}(s)\,\mathrm{d}\mathcal{A}(\alpha, g_2)(s)\right|. \tag{4.20}$$

Besides, condition (1.3) guarantees the estimate

$$u(t_1(x)) \leqslant \|\mathcal{B}(x)\|_0 \cdot u(t_2(x)) + \|c_0\|. \tag{4.21}$$

Inequality (4.20), with regard to (4.16), (4.19) and (4.21), implies

$$u(t_2(x)) \leqslant \delta u(t_2(x)) + \varrho_1(\|c_0\| + |\mathcal{A}(\alpha, g_2)(b) - \mathcal{A}(\alpha, g_2)(a)|).$$

Hence,

$$u(t_2(x)) \leqslant \varrho_2, \tag{4.22}$$

where $\varrho_2 = (1-\delta)^{-1}\varrho_1 \cdot (\|c_0\| + |\mathcal{A}(\alpha, g_2)(b) - \mathcal{A}(\alpha, g_2)(a)|)$. However, as is clear from (4.7) and (4.16),

$$\|\mathcal{B}(x)\|_0 \leqslant \delta\gamma_x^{-1}(t_2(x)) \leqslant \delta.$$

According to this inequality, (4.15), (4.21) and (4.22) imply the estimate (1.7), where $\varrho = \varrho_1(\delta\varrho_2 + \|c_0\| + |\mathcal{A}(\tilde{g_1}, g_2)(b) - \mathcal{A}(\tilde{g_1}, g_2)(a)|)$ is a positive constant which does not depend on $\lambda$ and $x$. The theorem is proved. □

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