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A. C. CLARKE'S SPACE ODYSSEY AND NEWTON'S LAW OF GRAVITY

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Abstract: In his famous tetralogy, *Space Odyssey*, A. C. Clarke called the calculation of a motion of a mass point in the gravitational field of the massive cuboid a classical problem of gravitational mechanics. This article presents a proposal for a solution to this problem in terms of Newton's theory of gravity. First we discuss and generalize Newton's law of gravitation. We then compare the gravitational field created by the cuboid — monolith, with the gravitational field of the homogeneous sphere. This is followed by the calculation of the shape of free fall trajectories and the solving of Newton's equations of motion, defining the motion of the mass point in the monolith's gravitational field for general initial conditions. The final section describes the procedures for calculating the shape of the monolith's equipotential surfaces. Due to the complexity of the problems, all calculations are performed in the Maple program. The results of the calculations are illustrated using both 2D and 3D graphs.

Keywords: law of gravity, gravitational force, potential, power, acceleration, equations of motion, trajectory, Maple

MSC: 68W30, 70F15, 34A34, 65Z05, 65D30

1. Introduction

In A. C. Clarke's *Space Odyssey*, namely [3] and [4], he dealt, among other things, with the motion of a spacecraft in the gravitational field of a cuboid with dimensions in the ratio of 1:4:9. He even calls the calculation of the trajectory a classical problem of gravitational mechanics. Let us try to solve this problem using classical mechanics and Newton's equations of motion.

2. Newton's law of gravitation

Let us take two mass points m_1 and m_2 . The first point is at the origin of the coordinate system, and the second point is on the coordinates $[x, y, z]$. Accord-

ing to Newton's law of gravitation, [5], the two points attract each other with a force \vec{F} :

$$\vec{F} = \frac{\kappa m_1 m_2}{x^2 + y^2 + z^2} \vec{e}_{12}, \quad \text{where} \quad \begin{array}{l} \kappa = \text{Newton's gravitational constant} \\ \vec{e}_{12} = \text{the unit vector of the line} \\ \text{connecting } m_1 \text{ and } m_2 \end{array} \quad (1)$$

This law can be generalized with a complex equation for the calculation of the gravitational pull of two homogeneous spheres of identical mass.

First, let us assume a gravitational pull of the homogeneous sphere with a mass M_1 , and radius R , with a mass point m_2 located on the coordinates $[X, 0, 0]$, $X > R$. Interestingly, it is impossible to achieve the expected result $|F| = \kappa m_2 M_1 X^{-2}$ by integration of the $|F| = \kappa m_2 \int_{M_1} X^{-2} dM_1$; it is necessary to calculate the potential energy of the mass point m_2 in the gravitational field of the sphere M_1 .

2.1. Potential energy

The potential energy of mass point m_2 in the gravitational field generated by mass point m_1 , is equal to the work required to move it from its current position to infinity. Let us assume that mass point m_2 is moved along a general, parametrically defined spatial curve $S = [x(p), y(p), z(p)]$, where $p = \text{parameter}$. If we move point m_2 by $d\vec{S}$, then it is possible using equation (1) to determine the corresponding element of work dW , according to the relationship $dW = \vec{F} \cdot d\vec{S}$,

$$dW = \kappa m_1 m_2 \int_p^\infty \frac{\frac{dR(p)}{dp}}{R(p)^2} dp \Rightarrow W = \frac{\kappa m_1 m_2}{R(p)}, \quad \text{where } R(p) = \sqrt{x(p)^2 + y(p)^2 + z(p)^2}. \quad (2)$$

If we know the potential energy W , then it is possible to determine the gravitational force using the relation $\vec{F} = -\nabla(W)$.

2.2. Gravitational force between a mass point and a homogeneous sphere

Now we can calculate the potential energy of mass point dM , on coordinates $[X, Y, Z]$ in the gravitational field of the homogeneous sphere with a radius R and mass m , which is located at the origin of the coordinate system. The position of the mass element of the sphere is entered in cylindrical coordinates as shown in Fig. 1, the density of the sphere mass $\rho = \frac{3m}{4\pi R^3}$.

$$W = \kappa dM \int_{-R}^R \left(\int_0^{\sqrt{R^2 - x^2}} \left(\int_0^{2\pi} \frac{\rho r}{\sqrt{(X-x)^2 + r^2}} d\phi \right) dr \right) dx \Rightarrow F = -\frac{\kappa m dM}{X^2}. \quad (3)$$

From Eq. (3) we can easily prove that the gravitational interaction between the homogeneous sphere and the mass point is determined by the same law as the gravitational interaction between two mass points, see Eq. (1). The generalization of two homogeneous spheres is based on the repetition of the above process. If mass point dM can move freely in the sphere's gravitational field, then the gravitational force will grant its acceleration $\vec{A} = \frac{d\vec{F}}{dM}$.

2.3. Gravitational field inside the homogeneous sphere

Let us suppose that the point P is located inside the homogeneous sphere. It is then possible to divide all the mass of the sphere that is at a greater distance from the center of the sphere S than from the point P into concentric spherical shells of elementary thickness dr . We can then pass a line through the point P which is the axis of an elementary cone with apex $d\phi$. Intersection points of this cone with a shell can be considered as two mass points, and we can prove that their gravitational forces on the point P cancel each other out. As a result, the only gravitational force affecting the point P comes from the spherical mass that is closer to the center of the sphere than the distance of the point P . The detailed computation can be found in [1]. The gravitational acceleration A_s inside and outside the homogeneous sphere with a radius R_s and mass M , equal to the mass of the monolith, is given by the following equation:

$$A_s = \frac{\kappa d M}{R_s^3} \text{ for } d \leq R_s, \quad A_s = \frac{\kappa M}{R_s^2} \text{ for } d \geq R_s, \quad \text{where } \begin{array}{l} R_s = \frac{3T}{\sqrt[3]{\pi}}, \\ T = \text{length of the shortest} \\ \text{edge of the monolith} \end{array} \quad (4)$$

3. Gravitational field of the monolith from Space Odyssey

We will calculate the monolith's gravitational acceleration $\vec{A} = [A_x, A_y, A_z]$ in the program Maple 13. To save space, we will only show the calculation for the acceleration coordinate A_x , calculations for A_y and A_z are very similar.

3.1. Monoliths gravitational force

If the acceleration vector is to be in units SI, or $[A]_{SI} = \text{m s}^{-2}$, then we must multiply it by the density of the material; let us assume that $\rho = 2000 \text{ kg m}^{-3}$ and Newton's gravitational constant $\kappa = 6.6710^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$, that is $1.33410^{-7} \text{ s}^{-2}$. Since this value is very small and it is multiplied by all members of the acceleration vector, it is not necessary to perform this multiplication, but all resulting acceleration and speed values must be multiplied by this constant in comparison with the actual acceleration values. The method also changes the speed or length of time steps. Given that this is a relative comparison of individual accelerations, velocities and displacements, these facts can be ignored.

```
> restart; with(plots): with(LinearAlgebra): R:=sqrt((X-x)^2+(Y-y)^2+(Z-z)^2):
> Lx:=9*T/2; Ly:=2*T; Lz:=T/2; Ix1:=Int((X-x)/R^3,x): Ix1:=simplify(Eval(Ix1,x=Lx)-Eval(Ix1,x=-Lx)):
> Ix1:=value(Ix1): Ix2:=Int(Ix1,y=-Ly..Ly): Ix2:=combine(value(Ix2),ln,symbolic);
```

$$Ix2 = \ln \left(\frac{(\%1 + \%4)(\%2 + \%5)}{(\%1 + \%5)(\%2 + \%4)} \right), \text{ where } \begin{array}{l} \%1 = -2Y - 4T \\ \%2 = -2Y + 4T \\ \%3 = 4X^2 + 97T^2 + 4Y^2 + 16YT + 4Z^2 + 4z^2 - 8Zz \\ \%4 = \sqrt{\%3 + 36XT} \\ \%5 = \sqrt{\%3 - 36XT} \end{array} \quad (5)$$

The final calculation of the acceleration component A_x cannot be performed by integration according to z , because this integral does not have an analytical solution. We will therefore create a procedure for A_x that will perform a numerical integration

for the coordinates $[X, Y, Z]$ on which the mass point m is located, with an accuracy of 10 significant figures. For the integration, it is necessary to enter the length of the shortest edge of the monolith, let us suppose it's $T = 10\text{km}$, as exact values are not listed in any one of the books [1-4].

```
> T:=1e5; Ax:=(a,b,c)->evalf(Int(subs(X=a,Y=b,Z=c,Ix2),z=-Lz..Lz,epsilon=10),10);
```

Now we can plot the course of the monolith's gravitational force and compare it with the gravitational field of the homogeneous sphere with the same mass and material density. The gravitational force in the direction of axis X is plotted with a red line, in the direction of axis Y it is plotted with a blue line, in the direction of axis Z it is plotted with a green line, and the homogeneous sphere is plotted with a black line. The graph in Fig. 2 shows that the course of the gravitational force surrounding the monolith is significantly different from the sphere's gravitational field.

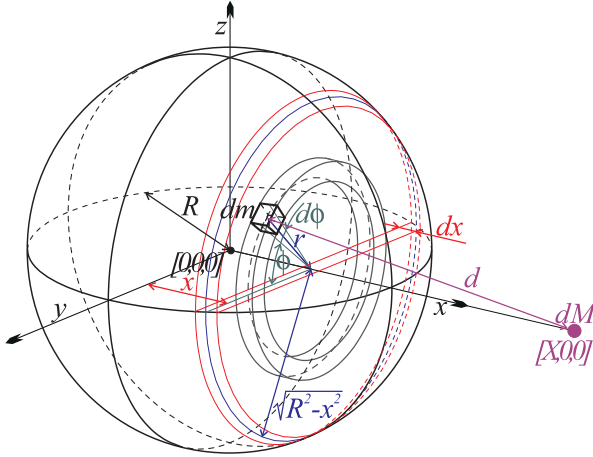


Figure 1: Gravitational interaction between sphere and dM .

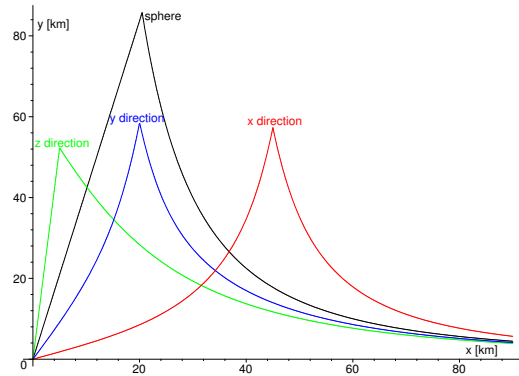


Figure 2: Gravitational force of the monolith and sphere.

4. Movement in the monolith's gravitational field

The movement of an object in the monolith's gravitational field can be solved using Newton's equations of motion written in the standard form, [5]:

$$\frac{d^2 \vec{P}(t)}{dt^2} = \vec{A}(t), \quad \vec{P}(0) = [X_0, Y_0, Z_0], \quad \left. \frac{d\vec{P}(t)}{dt} \right|_{t=0} = [V_{x_0}, V_{y_0}, V_{z_0}]. \quad (6)$$

This is a system of non-linear second order differential equations, which have no analytical solution in this case. Although the Maple program is equipped with an extensive library of numerical solvers, they cannot be used because it is necessary to calculate the gravitational acceleration in each step - see Eq. (5) and the following

text. Because this vector must be calculated by numerical integration, we cannot use the direction command `dsolve` in Maple. First, we will convert Eq. (6) to a system of first order differential equations corresponding with the fact that $\frac{d\vec{F}(t)}{dt} = \vec{V}(t)$ and $\frac{d\vec{V}(t)}{dt} = \vec{A}(t)$.

Now we can create a procedure `Step`, based on the Runge-Kutta method, which will create a numerical solution to Eq. (6) with regard to the initial conditions. This procedure is very similar to procedure described in [2]. The procedure checks the length of the spatial step. If the step is too short or long compared to the default value `DL`, the time step changes the `dt`.

```
> FV:=(t,X,Y,Z,Vx,Vy,Vz)->[Vx,Vy,Vz]; FA:=(t,a,b,c,Vx,Vy,Vz)->-
[evalf(Int(subs(X=a,Y=b,Z=0,Ix2),z=Lz..Lz,epsilon=10),10),
evalf(Int(subs(X=a,Y=b,Z=0,Iy2),x=-Lx..Lx,epsilon=10),10),
evalf(Int(subs(X=a,Y=b,Z=c,Iz2),y=-Ly..Ly,epsilon=10),10)];
> Step:=proc(nu) local pv1,pv2,pv3,pv4,vv1,vv2,vv3,vv4,DPV,d1; global n,t,dt,Pos,PV,W,VV,Tau;
pv1:=dt*evalf(FV(t,Pos[],W[])): vv1:=dt*evalf(FA(t,Pos[],W[])):
pv2:=dt*evalf(FV(t+dt/2,(Pos+pv1/2) [], (W+vv1/2) [])):
vv2:=dt*evalf(FA(t+dt/2,(Pos+pv1/2) [], (W+vv1/2) [])):
pv3:=dt*evalf(FV(t+dt/2,(Pos+pv2/2) [], (W+vv2/2) [])):
vv3:=dt*evalf(FA(t+dt/2,(Pos+pv2/2) [], (W+vv2/2) [])):
pv4:=dt*evalf(FV(t+dt,(Pos+pv3) [], (W+vv3) [])):
vv4:=dt*evalf(FA(t+dt,(Pos+pv3) [], (W+vv3) [])): #..... New Lines for StepPE
DPV:=1/6*(pv1+2*pv2+2*pv3+pv4): d1:=sqrt(add(w^2,w=DPV));
if d1>DL then dt:=dt/2; elif d1*8<DL then dt:=dt*2;
else n:=n+1; t:=t+dt; Pos:=Pos+DPV; PV:=[PV[],Pos];
W:=W+1/6*(vv1+2*vv2+2*vv3+vv4): VV:=[VV[],W];Tau:=[Tau[],t];
end if: end proc;
```

Let us suppose we let a small object fall from the resting state of $Z = 0$, from a distance of 4 times the length of the edge L_y , from places where the position angle is changed by 10° . Each time step will be stored in variable `TTau`, velocity vectors in variable `TVV` and position vectors in variable `TPV`.

```
> Nu:=9; TPV:=[]: TVV:=[]: TTau:=[]:
> for i from 0 to Nu do;
Pos:=[4*Ly*cos(pi*i/2/Nu),4*Ly*sin(pi*i/2/Nu),0]; W:=[0,0,0]; #.....Initial Conditions
t:=0; dt:=0.125; DL:=1000; n:=0; PV:=[Pos]; VV:=[W]; Tau:=[0];
while not(abs(Pos[1])<Lx and abs(Pos[2])<Ly and Pos[3]<Lz) do; Step(); end do:
TPV:=[TPV[],PV]: TVV:=[TVV[],VV]: TTau:=[TTau[],Tau];
end do;
```

The free fall trajectory for the remaining planes can be calculated in a similar manner $X = 0$ a $Y = 0$, see Fig. 3. The plotting commands are not listed here. The free fall trajectories are marked red, the trajectory tangents in the initial point of the fall are marked green, and the free fall trajectory in central gravitational field is marked brown. The figure shows that motion in the monolith's gravitational field and in the sphere's gravitational field varies considerably.

The equations of motion (6) can also be used for a general case of motion in the monolith's gravitational field. It only differs from the free fall in the choice of initial conditions. For calculating the trajectories we can use procedure `Step`. Let us suppose the motion in a plane $Z = 0$, with an initial position of $[L_x, L_y, 0]$ and the

velocity vector oriented in the direction of axis X ; we will perform the calculations for 8 velocities that are constantly increasing. Remaining Maple commands are not listed.

```
> Pos:=[Lx,Lx,0]; W:=[-i*6000,0,0]; #.....Initial Conditions .....
```

As Fig. 4 shows, the body either hits the monolith or flies around it on the equivalent of a hyperbolic orbit. We can therefore assume that finding stable orbits around the monolith can be a problem that is difficult to solve. This assumption is supported by other calculations for different initial conditions. In the previous calculation, we can simply replace the line marked as `#Initial Conditions`, with the following line:

```
> Pos:=[Lx,Lx,3*Lz]; W:[(-10600-i*200)/sqrt(2),0,(10600+i*200)/sqrt(2)]; #.....Initial Conditions
```

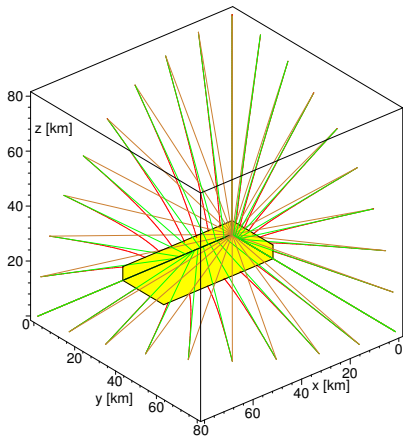


Figure 3: Free fall in the monolith's gravitational field.

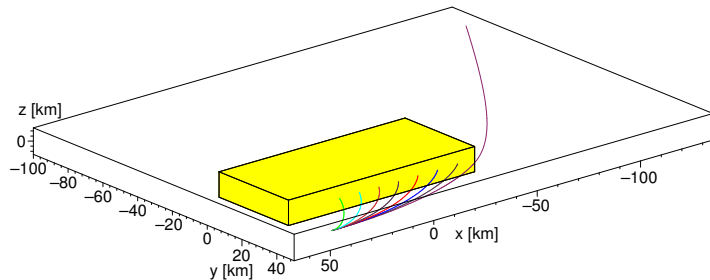


Figure 4: Motion in the plane $Z = 0$.

As Fig. 5 shows, the body moves along quite unusual trajectories, which ultimately lead it to fall on the monolith. If the initial velocity is slightly changed, the corresponding trajectories are close to each other, which is confirmed by the correctness of the calculations above.

5. Equipotential surfaces

The calculation of the shape of equipotential surfaces can be based on Eq. (2), which implies that the gravitational force is always perpendicular to the equipotential surface. We will therefore perform calculations of the shape of equipotential surfaces for the plane of symmetry of the monolith; the intersection of the plane of symmetry with the equipotential surface defines the curve that can be easily depicted thanks to the validity of Eq. (2). First, we must determine the distance of the equipotential surface from the center of the monolith. For this we can use the law of conservation

of energy. From the center of the monolith - the beginning of the coordinate system, we shoot a mass point in the direction of each axis at a known speed. The point at which its motion stops will correspond with the equipotential surface with a potential of $U = v^2/2$. The coordinates of this point can be found using the procedure `StepPE`. Since it is necessary to determine the point at which the mass point stops, it is necessary to change the lines of procedure `Step`, that follow `# New Lines for StepPE`, because it is not possible to optimize the length of the spatial step when it stops. The corresponding line of procedure `StepPE` is:

```
Pos:=Pos+1/6*(pv1+2*pv2+2*pv2+pv4): W:=W+1/6*(vv1+2*vv2+2*vv3+vv4):
```

At the stopping point, we can determine the direction of the gravitational field using numerical integration of Eq. (2), and in the plane of symmetry we can determine the direction that is perpendicular to this direction. There will be another point of the equipotential surface in this direction. This is basically determining the tangent trajectory of the direction field - solving the system of first order differential equations. A numerical solution can be found using the procedure `StepEQ`:

```
> StepEQ:=proc(0) local pv1, pv2, pv3, pv4, DPV, dl; global n, t, dt, Pos, PV, Tau;
  pv1:=dt*evalf(FA(t,Pos[])): pv2:=dt*evalf(FA(t+dt/2,(Pos+pv1/2)[])):
  pv3:=dt*evalf(FA(t+dt/2,(Pos+pv2/2)[])): pv4:=dt*evalf(FA(t+dt,(Pos+pv3)[])):
  DPV:=1/6*(pv1+2*pv2+2*pv2+pv4): dl:=sqrt(add(w^2,w=DPV));
  if dl>DL then dt:=dt/2; elif dl*8<DL then dt:=dt*2;
  else n:=n+1; t:=t+dt; DPV:=map(u->'if'(u=0,0,signum(u)*DPV[abs(u)]),0);
  Pos:=Pos+DPV; PV:=[PV[],Pos]; Tau:=[Tau[],t];
  end if: end proc;
```

The coordinates of points on equipotential surfaces can then be calculated using the following commands:

```
> TPV:=[]: i:='i':
> for i from 25 to 50 do;
  W:=[0,i*1000,0]; Pos:=[0,0,0]; VV:=[W]; t:=0; dt:=0.005; n:=0; PV:=[Pos]; Tau:=[0]; DL:=1500;
  while W[2]>0 do; StepPE(); end do;
  while abs(dt)>1e-5 do; dt:=-W[2]/FA(t,Pos[],W[])[2]; StepPE(); end do;
  PV:=[Pos];dt:=0.025; while Pos[2]>0 do; StepEQ([-2,1,0]): end do;
  TPV:=[TPV[],map(u->u[1..2],PV)];
end do;
```

The plotting commands are not listed in order to save space. The commands for calculating equipotential surfaces corresponding with remaining planes of symmetry are not listed for the same reason. The equipotential surfaces are plotted in Fig. 6.

6. Conclusion and discussion

The results of our calculations are seemingly contrary to what we know about Earth's gravitational field. Especially the free fall trajectories in Fig. 3 and the unstable orbit trajectories in Fig. 5 are quite strange. These differences can be explained quite simply. In the sphere's gravitational field the free fall trajectory is identical to the local vertical, which does not apply to the monolith.

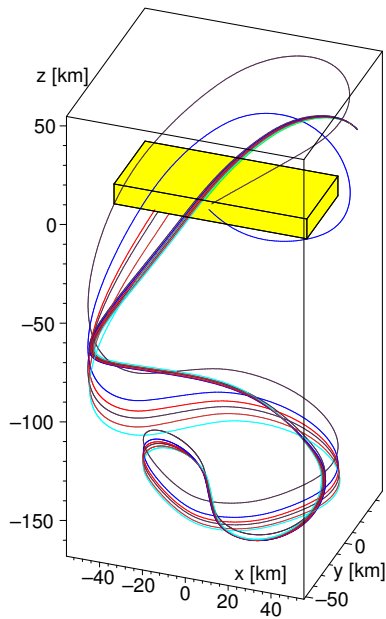


Figure 5: General orientation of the initial velocity.

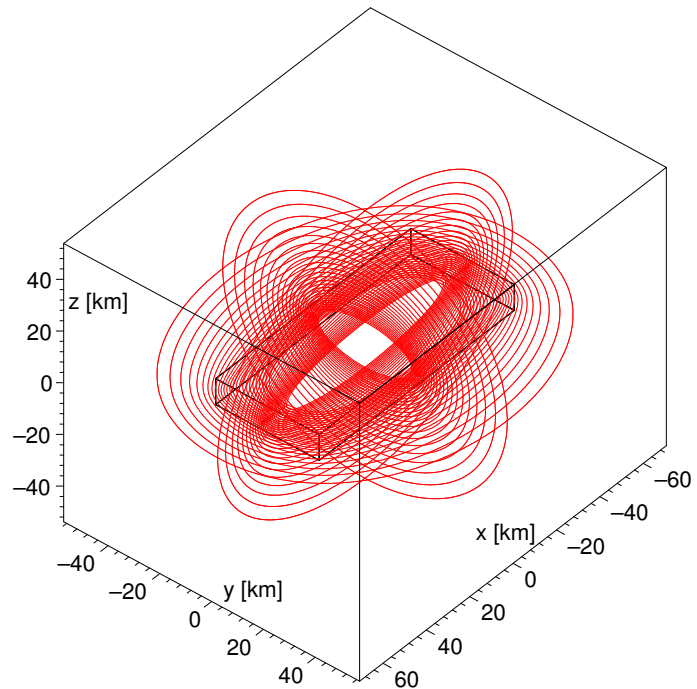


Figure 6: Equipotential levels.

Moreover, the gravitational force of the sphere's gravitational field is always directed towards the center of the object that is the source of gravity. The monolith's gravitational field does not meet this condition, which is why an object moving away from the monolith, e.g. in the direction of axis Z , can oscillate in all directions perpendicular to this axis, which is quite evident from Fig. 5.

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