Tomasz Beberok

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THE BERGMAN KERNEL: EXPLICIT FORMULAS, DEFLATION, LU QI-KENG PROBLEM AND JACOBI POLYNOMIALS

Tomasz Beberok, Istanbul

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Abstract. We investigate the Bergman kernel function for the intersection of two complex ellipsoids $\{(z, w_1, w_2) \in \mathbb{C}^{n+2} \colon |z_1|^2 + \ldots + |z_n|^2 + |w_1|^q < 1,\ |z_1|^2 + \ldots + |z_n|^2 + |w_2|^r < 1\}$. We also compute the kernel function for $\{(z_1, w_1, w_2) \in \mathbb{C}^3 \colon |z_1|^{2/n} + |w_1|^q < 1,\ |z_1|^{2/n} + |w_2|^r < 1\}$ and show deflation type identity between these two domains. Moreover in the case that $q = r = 2$ we express the Bergman kernel in terms of the Jacobi polynomials. The explicit formulas of the Bergman kernel function for these domains enables us to investigate whether the Bergman kernel has zeros or not. This kind of problem is called a Lu Qi-Keng problem.

Keywords: Lu Qi-Keng problem; Bergman kernel; Routh-Hurwitz theorem; Jacobi polynomial

MSC 2010: 32A25, 33D70

1. Introduction

Let D be a bounded domain in $\mathbb{C}^n$. The Bergman space $L_a^2(D)$ is the space of all square integrable holomorphic functions on D. Then the Bergman kernel $K_D(z, w)$ is defined in [2] by

$$K_D(z, w) = \sum_{j=0}^{\infty} \Phi_j(z)\overline{\Phi_j(w)}, \quad (z, w) \in D \times D,$$

where $\{\Phi_j(\cdot) \colon j = 0, 1, 2, \ldots\}$ is a complete orthonormal basis for $L_a^2(D)$. It is defined for arbitrary bounded domains, but it is hard to obtain concrete representations for

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the Bergman kernel except for special cases like a Hermitian ball or polydisk. Refer to [6] for more on this topic.

In 2015 in [1] the author of this paper computed the Bergman kernel for $D_1^{q,r} = \{(z_1, w_1, w_2) \in \mathbb{C}^3\colon\ |z_1|^2 + |w_1|^q < 1,\ |z_1|^2 + |w_2|^r < 1\}$ explicitly. The goal of this paper is to extend the result in [1] to higher dimensional case, namely to $D_n^{q,r} = \{(z, w_1, w_2) \in \mathbb{C}^{n+2}\colon\ |z_1|^2 + \ldots + |z_n|^2 + |w_1|^q < 1,\ |z_1|^2 + \ldots + |z_n|^2 + |w_2|^r < 1\}$.

This paper is organized as follows. In Section 2, we compute an explicit formula of the Bergman kernel for $D_n^{q,r}$ and show the deflation identity between domains $D_n^{q,r}$ and $D_{1/n}^{q,r} = \{z = (z_1, w_1, w_2) \in \mathbb{C}^3\colon\ |z_1|^{2/n} + |w_1|^q < 1,\ |z_1|^{2/n} + |w_2|^r < 1\}$. In Section 3, we show some relation between the Bergman kernel for $D_n^{2,2}$ and the Jacobi polynomials. In Section 4, we investigate the Lu Qi-Keng problem for $D_n^{2,2}$. In the final section, we consider the Bergman kernel for $\Omega_n^r := \bigl\{(z, w) \in \mathbb{C} \times \mathbb{C}^n\colon\ |z|^2 + |w_1|^r < 1, \ldots, |z|^2 + |w_n|^r < 1\bigr\}$.

2. Bergman kernel

Let $\zeta = (z, w_1, w_2) \in \mathbb{C}^{n+2}$. Put $\Phi_\kappa(\zeta) = \zeta^\kappa = z^\alpha w_1^{\gamma_1} w_2^{\gamma_2}$ for each multi-index $\kappa = (\alpha, \gamma_1, \gamma_2)$. Since $D_n^{q,r}$ is a complete Reinhardt domain, the collection of $\{\Phi_\kappa\}$ such that each $\alpha_i \geqslant 0$ and $\gamma_j \geqslant 0$ is a complete orthogonal set for $L^2(D_n^{q,r})$.

Proposition 1. *Let $\alpha_i \in \mathbb{Z}_+$ for $i = 1, \ldots, n$ and $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$. Then we have*

$$\|z^\alpha w_1^{\gamma_1} w_2^{\gamma_2}\|^2_{L^2(D_n^{q,r})} = \frac{\pi^{n+2}\Gamma((2\gamma_1+2)/q + (2\gamma_2+2)/r + 1) \prod\limits_{i=1}^{n} \Gamma(\alpha_i + 1)}{(\gamma_2+1)(\gamma_2+1)\Gamma((2\gamma_1+2)/q + (2\gamma_2+2)/r + |\alpha| + n + 1)},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Proof.

$$\|z^\alpha w_1^{\gamma_2} w_2^{\gamma_3}\|^2_{L^2(D_n^{q,r})} = \int\limits_{D_n^{q,r}} |z|^{2\alpha} |w_1|^{2\gamma_2} |w_2|^{2\gamma_3} \, \mathrm{d}V(z) \, \mathrm{d}V(w).$$

Introducing polar coordinates $z = r\mathrm{e}^{\mathrm{i}\theta}$, $w_1 = s_1 \mathrm{e}^{\mathrm{i}\lambda_1}$, $w_2 = s_2 \mathrm{e}^{\mathrm{i}\lambda_2}$ and integrating out the angular variables we have

$$(2\pi)^{n+2} \int_{\mathrm{Re}(D_n^{q,r})} r^{2\alpha+1} s_1^{2\gamma_1+1} s_2^{2\gamma_2+1} \, \mathrm{d}V(r) \, \mathrm{d}V(s),$$

where $\mathrm{Re}(D_n^{q,r}) = \bigl\{(r, s) \in \mathbb{R}_+^n \times \mathbb{R}_+^2\colon\ \|r\|^2 + s_1^q < 1,\ \|r\|^2 + s_2^r < 1\bigr\}$. Converting to spherical coordinates in the r variable we obtain

$$(2\pi)^{n+2} \int_{S*\mathrm{Re}(D_n^{q,r})} \int_{\mathbf{S}_+^{n-1}} \varrho^{2|\alpha|+2n-1} \omega^{2\alpha+1} s_1^{2\gamma_1+1} s_2^{2\gamma_2+1} \, \mathrm{d}\sigma(\omega) \, \mathrm{d}V(s) \, \mathrm{d}\varrho,$$

where $S * \mathrm{Re}(D_n^{q,r}) = \{(\varrho, s) \in \mathbb{R}_+ \times \mathbb{R}_+^2 \colon\ \varrho^2 + s_1^q < 1,\ \varrho^2 + s_2^r < 1\}$. Using a result of D'Angelo (see Lemma 1 in [4]), we get

$$(2\pi)^{n+2}\frac{\beta(\alpha+1)}{2^{n-1}}\int_0^1\int_0^{(1-\varrho^2)^{1/q}}\int_0^{(1-\varrho^2)^{1/r}} \varrho^{2|\alpha|+2n-1}s_1^{2\gamma_1+1}s_2^{2\gamma_2+1}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}\varrho,$$

where $\beta(\alpha) = \prod_{i=1}^{n}\Gamma(\alpha_i)/\Gamma(\alpha_1+\ldots+\alpha_n)$. The next step is integrating out s_1 and s_2 variables. As a result, we have

$$\frac{(2\pi)^{n+2}\beta(\alpha+1)}{2^{n+1}(\gamma_1+1)(\gamma_2+1)}\int_0^1 \varrho^{2|\alpha|+2n-1}(1-\varrho^2)^{(2\gamma_1+2)/q+(2\gamma_2+2)/r}\,\mathrm{d}\varrho.$$

After a little calculation using the well-known fact

$$\int_0^1 x^a(1-x^p)^b\,\mathrm{d}x = \frac{\Gamma((a+1)/p)\Gamma(b+1)}{p\Gamma((a+1)/p+b+1)},$$

we obtain the desired result. □

Now we discuss the Bergman kernel for $D_n^{q,r}$.

Theorem 1. *The Bergman kernel for $D_n^{q,r}$ is given by*

$$K_{D_n^{q,r}}((z,w_1,w_2),(\eta,\xi_1,\xi_2)) = \frac{L_n^{q,r}(a,b)}{\pi^{n+2}(1-z_1\overline{\eta}_1-\ldots-z_n\overline{\eta}_n)^{2/q+2/r+n+1}},$$

where

$$a = \frac{w_1\overline{\xi}_1}{(1-z_1\overline{\eta}_1-\ldots-z_n\overline{\eta}_n)^{2/q}}, \quad b = \frac{w_2\overline{\xi}_2}{(1-z_1\overline{\eta}_1-\ldots-z_n\overline{\eta}_n)^{2/r}},$$
$$L_{n+1}^{q,r}(x,y) = (n+1)L_n^{q,r}(x,y) + \frac{2}{q}\frac{\partial}{\partial x}xL_n^{q,r}(x,y) + \frac{2}{r}\frac{\partial}{\partial y}yL_n^{q,r}(x,y)$$
$$\text{and}\quad L_1^{q,r}(x,y) = \frac{2(q(1-x)(y+1)+r(x+1)(1-y))}{qr(1-x)^3(1-y)^3}.$$

P r o o f. By Proposition 1, we have

$$K_{D_n^{q,r}}((z,w_1,w_2),(\eta,\xi_1,\xi_2)) = \frac{1}{\pi^{n+2}} \times \sum_{\alpha=0}^{\infty}\sum_{\gamma_1,\gamma_2=0}^{\infty}\frac{(\gamma_1+1)(\gamma_2+1)\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+|\alpha|+n+1)}{\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+1)\prod_{i=1}^{n}\Gamma(\alpha_i+1)}\mu^\alpha\nu_1^{\gamma_1}\nu_2^{\gamma_2},$$

where $\mu^\alpha = (z_1\overline{\eta}_1)^{\alpha_1}\dots(z_n\overline{\eta}_n)^{\alpha_n}$ and $\nu_1 = w_1\overline{\xi}_1$, $\nu_2 = w_2\overline{\xi}_2$. Summing out each α_i, we have

$$\frac{1}{\pi^{n+2}(1-\tau)^{2/q+2/r+n+1}} \times \sum_{\gamma_1,\gamma_2=0}^{\infty} \frac{(\gamma_1+1)(\gamma_2+1)\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+n+1)}{\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+1)(1-\tau)^{2\gamma_1/q+2\gamma_2/r}}\nu_1^{\gamma_1}\nu_2^{\gamma_2},$$

where $\tau = z_1\overline{\eta}_1 + \dots + z_n\overline{\eta}_n$. Now we will consider the sequence of functions $L_n^{q,r}$ defined as

$$L_n^{q,r}(x,y) = \sum_{\gamma_1,\gamma_2=0}^{\infty} \frac{(\gamma_1+1)(\gamma_2+1)\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+n+1)}{\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+1)}x^{\gamma_1}y^{\gamma_2}.$$

Using the identity $\Gamma(t+1) = t\Gamma(t)$ we easily obtain the recursion formula

$$L_{n+1}^{q,r}(x,y) = (n+1)L_n^{q,r}(x,y) + \frac{2}{q}\frac{\partial}{\partial x}xL_n^{q,r}(x,y) + \frac{2}{r}\frac{\partial}{\partial y}yL_n^{q,r}(x,y).$$

Moreover,

$$L_1^{q,r}(x,y) = \sum_{\gamma_1,\gamma_2=0}^{\infty}(\gamma_1+1)(\gamma_2+1)\Big(\frac{2\gamma_1+2}{q}+\frac{2\gamma_2+2}{r}+1\Big)x^{\gamma_1}y^{\gamma_2}.$$

Hence

$$L_1^{q,r}(x,y) = \frac{2(q(1-x)(y+1)+r(x+1)(1-y))}{qr(1-x)^3(1-y)^3},$$

which completes the proof. □

2.1. Deflation. Now we will consider the domains

$$D_{1/n}^{q,r} = \{(z,w_1,w_2)\in\mathbb{C}^3\colon\ |z|^{2/n}+|w_1|^q<1,\ |z|^{2/n}+|w_2|^r<1\}.$$

Similarly to Section 2, we have

Proposition 2. *Let $\alpha \geqslant 0$, $\gamma_1 \geqslant 0$ and $\gamma_2 \geqslant 0$. Then we have*

$$\|z^\alpha w_1^{\gamma_1}w_2^{\gamma_2}\|^2_{L^2(D_{1/n}^{q,r})} = \frac{n\pi^3\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+1)\Gamma(n\alpha+n)}{(\gamma_2+1)(\gamma_2+1)\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+n\alpha+n+1)}.$$

Therefore,

$$
\begin{aligned}
&K_{D^{q,r}_{1/n}}((0,w_1,w_2),(0,\xi_1,\xi_2))\\
&=\sum_{\gamma_1,\gamma_2=0}^{\infty}\frac{(\gamma_1+1)(\gamma_2+1)\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+n+1)}{\pi^3 n!\Gamma((2\gamma_1+2)/q+(2\gamma_2+2)/r+1)}(w_1\overline{\xi}_1)^{\gamma_1}(w_2\overline{\xi}_2)^{\gamma_2}.
\end{aligned}
$$

Hence

$$K_{D^{q,r}_{1/n}}((0,w_1,w_2),(0,\xi_1,\xi_2))=\frac{1}{\pi^3 n!}L^{q,r}_n(w_1\overline{\xi}_1,w_2\overline{\xi}_2).$$

Comparing the formulas for $K_{D^{q,r}_{1/n}}$ and $K_{D^{q,r}_n}$, we obtain the following deflation identity.

Proposition 3. *For every $n\in\mathbb{N}$ and every positive numbers q and r, we have*

$$\frac{n!}{\pi^{n-1}}K_{D^{q,r}_{1/n}}((0,w_1,w_2),(0,\xi_1,\xi_2))=K_{D^{q,r}_n}((0,\dots,0,w_1,w_2),(0,\dots,0,\xi_1,\xi_2)).$$

Note that we have some kind of deflation result similar to that obtained in [3].

3. Some representations of Bergman kernel for $D^{2,2}_n$

Jacobi polynomials are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight $(1-x)^k(1+x)^l$ on the interval $[-1,1]$. For $k,l>-1$ the Jacobi polynomials are given by the formula

$$P^{(k,l)}_d(z)=\frac{(-1)^d}{2^d d!}(1-z)^{-k}(1+z)^{-l}\frac{\partial}{\partial z^d}((1-z)^k(1+z)^l(1-z^2)^d).$$

The Jacobi polynomials are defined via the hypergeometric function as

$$P^{(k,l)}_d(z)=\frac{(k+1)_d}{d!}{}_2F_1\Big(-d,1+k+l+d;k+1,\frac{1-z}{2}\Big), \tag{3.1}$$

where $(k+1)_d$ is Pochhammer's symbol and ${}_2F_1$ is the Gaussian or ordinary hypergeometric function defined for $|z|<1$ by the power series

$${}_2F_1\,(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n.$$

The Appell series F_1 defined by

$$F_1(a;b,b';c,x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{m+n}(b)_n(b')_m}{n!m!(c)_{n+m}}x^n y^m,\quad |x|<1,\ |y|<1$$

is one of the natural two-variable extensions of the hypergeometric series ${}_2F_1$.

The following reduction formulas can be proved (for details see [5], pages 238–239):

$$F_1(a;b,b';b+b';x,y) = (1-y)^{-a}{}_2F_1(a,b;b+b';(x-y)/(1-y)), \tag{3.2}$$

$$F_1(a;b,b';c;x,x) = {}_2F_1(a,b+b';c;x). \tag{3.3}$$

Comparing the functions F_1 and $L_n^{2,2}$ it is easy to see that

$$\frac{\Gamma(3+n)}{2}F_1(3+n;2,2;3;x,y) = L_n^{2,2}(x,y).$$

The following is the main theorem of this section.

Theorem 2. *The Bergman kernel for $D_n^{2,2}$ can be expressed in the following ways:*

(i)

$$K_{D_n^{2,2}}((0,w_1,w_2),(0,\xi_1,\xi_2)) = \frac{\Gamma(3+n)}{2\pi^{n+2}}\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-1}{k}\frac{(2)_i(2)_k(w_1\overline{\xi}_1)^i(w_2\overline{\xi}_2)^k}{(3)_{i+k}(1-w_1\overline{\xi}_1)^{2+i}(1-w_2\overline{\xi}_2)^{2+k}}.$$

(ii) *For every $m\in\mathbb{N}\cup\{0\}$, we have*

$$K_{D_{2m+1}^{2,2}}((0,w_1,w_2),(0,\xi_1,\xi_2)) = \frac{C_m(2+m)(2-\nu_1-\nu_2)P_m^{(3/2,1/2)}((x'+1)/(1-x'))}{\pi^{2m+3}(1-\nu_1-\nu_2-\nu_1\nu_2)^{m+3}} - \frac{C_m(2m+1)P_m^{(3/2,-1/2)}((x'+1)/(1-x'))}{\pi^{2m+3}(1-\nu_1-\nu_2-\nu_1\nu_2)^{m+2}},$$

where $\nu_1 = w_1\overline{\xi}_1$, $\nu_2 = w_2\overline{\xi}_2$, $x' = (\nu_1-\nu_2)^2/(2-\nu_1-\nu_2)^2$ and $C_m = \Gamma(4+2m)m!/(6(\frac{5}{2})_m)$.

(iii) *For every $m\in\mathbb{N}$, we have*

$$K_{D_{2m}^{2,2}}((0,w_1,w_2),(0,\xi_1,\xi_2)) = \Gamma(3+2m)m! \times \frac{2P_m^{(3/2,-1/2)}((x'+1)/(1-x')) - (2-\nu_1-\nu_2)P_{m-1}^{(3/2,1/2)}((x'+1)/(1-x'))}{\pi^{2m+2}6(\frac{5}{2})_{m-1}(1-\nu_1-\nu_2-\nu_1\nu_2)^{m+2}}.$$

(iv) *If $w_1\overline{\xi}_1 = w_2\overline{\xi}_2$ then*

$$K_{D_n^{2,2}}((0,w_1,w_2),(0,\xi_1,\xi_2)) = \frac{\Gamma(3+n)(3+nw_1\overline{\xi}_1)}{6\pi^{n+2}(1-w_1\overline{\xi}_1)^{4+n}}.$$

Proof. For the proof of (i), if we apply the recursion formula (see [8])

$$F_1(a+n;b,b';c;x,y) = \sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-1}{k}\frac{(b)_i(b')_k}{(c)_{i+k}} \times x^iy^kF_1(a+i+k;b+i,b'+k;c+i+k;x,y),$$

then using the formula

$$F_1(3+i+k;2+i,2+k;3+i+k;x,y)=\frac{1}{(1-x)^{2+i}(1-y)^{2+k}}$$

we obtain (i).

In order to prove the second statement we need the well-known contiguous relation

$$\begin{aligned}cF_1(a;b,b';c;x,y)-(c-a)F_1(a;b,b';c+1;x,y)\\ -aF_1(a+1;b,b';c+1;x,y)=0.\end{aligned}$$

It follows that

$$\begin{aligned}F_1(4+2m;2,2;3;x,y)= &-\frac{2m+1}{3}F_1(2m+4;2,2;4;x,y)\\ &+\frac{4+2m}{3}F_1(2m+5;2,2;4;x,y).\end{aligned}$$

By (3.2), we have

$$\begin{aligned}F_1(4+2m;2,2;3;x,y)= &-\frac{2m+1}{3(1-y)^{4+2m}}{}_2F_1\Big(2m+4,2;4;\frac{x-y}{1-y}\Big)\\ &+\frac{4+2m}{3(1-y)^{5+2m}}{}_2F_1\Big(2m+5,2;4;\frac{x-y}{1-y}\Big).\end{aligned}$$

Then by (for details see [5], page 66)

$${}_2F_1(a,b;2b;z)=\Big(1-\frac{z}{2}\Big)^{-a}{}_2F_1\Big(\frac{a}{2},\frac{a+1}{2};b+\frac{1}{2};\Big(\frac{z}{(2-z)}\Big)^2\Big),$$

we have

$$\begin{aligned}&F_1(4+2m;2,2;3;x,y)\\ &\quad=-\frac{(2m+1)2^{4+2m}}{3(2-x-y)^{4+2m}}{}_2F_1\Big(m+2,m+\frac{5}{2};\frac{5}{2};(x')^2\Big)\\ &\quad\quad+\frac{(4+2m)2^{5+2m}}{3(2-x-y)^{5+2m}}{}_2F_1\Big(m+\frac{5}{2},m+3;\frac{5}{2};(x')^2\Big),\end{aligned}$$

where $x'=(x-y)^2/(2-x-y)^2$. Next by (for details see [5], page 64)

$$\begin{aligned}{}_2F_1(a,b;c;z)=&\ (1-z)^{-a}\,{}_2F_1\Big(a,c-b;c;\frac{z}{z-1}\Big)\\ =&\ (1-z)^{-b}{}_2F_1\Big(c-a,b;c;\frac{z}{z-1}\Big),\end{aligned}$$

we can easily get

$$F_1(4+2m;2,2;3;x,y)$$
$$= -\frac{2m+1}{3(1-x-y-xy)^{m+2}}\,{}_2F_1\Big(m+2,-m;\frac{5}{2};\Big(\frac{x'}{x'-1}\Big)^2\Big)$$
$$+\frac{(2+m)(2-x-y)}{3(1-x-y-xy)^{m+3}}\,{}_2F_1\Big(m+\frac{5}{2},m+3;\frac{5}{2};\Big(\frac{x'}{x'-1}\Big)^2\Big).$$

Finally, by (3.1) we obtain (ii). The formula (iii) can be obtained by the same method and we omit the details. Now we will prove (iv). This is a formal exercise but we include it for completeness: From (3.3)

$$F_1(3+n;2,2;3;x,y) = {}_2F_1(3+n,4;3;x).$$

Now it is relatively easy to compute the following result:

$${}_2F_1(3+n,4;3;x) = \frac{3+nx}{3(1-x)^{4+n}}.$$

Thus we have proved (iv). □

4. Lu Qi-Keng problem

The explicit formula of the Bergman kernel function for the domains $D_n^{q,r}$ enables us to investigate whether the Bergman kernel has zeros in $D_n^{q,r}\times D_n^{q,r}$ or not. We will call this kind of problem a Lu Qi-Keng problem. If the Bergman kernel for a bounded domain has no zeros, then the domain will be called a Lu Qi-Keng domain.

By Theorem 1.2 from [1] if $n=1$, then $D_n^{q,r}$ is a Lu Qi-Keng domain for all positive real numbers q and r. Combining the deflation identity (Proposition 3) and Proposition 4.4 from [1], if $n=2$ and $q=r$, then $D_2^{r,r}$ is not a Lu Qi-Keng domain. Using the same method as in [1] we will prove that it is also true for $n=3$.

Denote

$$G(x,y) = 3r^3(1-x)^3(1-y)^3 + 22r^2(1-x)^2(1-y)^2(1-xy)$$
$$+24r(1-x)(1-y)(x(2x+1)y^2+(x-8)xy+x+y+2)$$
$$+8(1-xy)(x^2y(4y+7)+x^2+y^2+xy(7y-38)+7x+7y+4).$$

Since

$$G(x,y) = \frac{r^3(1-x)^5(1-y)^5}{2}L_3^{r,r}(x,y),$$

the Bergman kernel $K_{D_3^{r,r}}$ has a zero inside $D_2^{r,r} \times D_2^{r,r}$ if the polynomial $G(\varepsilon x, \varepsilon y)$ does not satisfy the stability property (see [1], for details) for some $0 < \varepsilon < 1$. Following Šiljak and Stipanović, see [7], we consider the polynomial

$$z^3 G\Big(\mathrm{e}^{\mathrm{i}\eta}, \frac{1}{z}\Big) = d(z) = d_3 z^3 + d_2 z^2 + d_1 z + d_0,$$

where

$$\begin{aligned}
d_0 &= 3r^3(-1+t)^3 - 22r^2(-1+t)^2 t + 24rt(-1-t+2t^2) - 8t(1+7t+4t^2),\\
d_1 &= 8 - 9r^3(-1+t)^3 + 336t^2 - 56t^3 + 22r^2(-1+t)^2(1+2t)\\
&\quad - 24r(1-10t+8t^2+t^3),\\
d_2 &= 9r^3(-1+t)^3 - 22r^2(2-3t+t^3) - 8(-7+42t+t^3)\\
&\quad - 24r(1+8t-10t^2+t^3),\\
d_3 &= 22r^2(-1+t)^2 - 3r^3(-1+t)^3 - 24r(-2+t+t^2) + 8(4+7t+t^2)\\
&\text{and} \quad t = \mathrm{e}^{\mathrm{i}\eta}.
\end{aligned}$$

With the polynomial $d(z)$ we associate the Schur-Cohn 3×3 matrix

$$M(\mathrm{e}^{\mathrm{i}\eta}) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix},$$

$d_{jk} = \sum_{l=1}^{j} (d_{m-j+l}\overline{d}_{m-k+l} - \overline{d}_{j-l} d_{k-l})$, where $1 \leqslant j \leqslant k$. The matrix $M(\mathrm{e}^{\mathrm{i}\eta})$ is defined when $j > k$ to become Hermitian. After some calculation (with help of a computer program Maple or Mathematica), we have

$$\det M(\mathrm{e}^{\mathrm{i}\eta}) = -130459631616 r^3 \sin^{12}\Big(\frac{\eta}{2}\Big)\Big(\sum_{n=0}^{3} g_n(r)\cos(n\eta)\Big),$$

where

$$\begin{aligned}
g_0(r) &= 26624 - 24672r^2 + 15724r^4 - 2430r^6,\\
g_1(r) &= 12288 + 22496r^2 - 20822r^4 + 3645r^6,\\
g_2(r) &= -512 + 2208r^2 + 5012r^4 - 1458r^6,\\
g_3(r) &= r^2(-32 + 86r^2 + 243r^4).
\end{aligned}$$

Since

$$\sum_{n=0}^{3} g_n(r) = 38400,$$

it is easy to see that for every $r>0$ there exists η such that $\det M(\mathrm{e}^{\mathrm{i}\eta})<0$. Hence there exists $1>\varepsilon>0$ such that the polynomial $G(\varepsilon x,\varepsilon y)$ does not satisfy the stability property (see [7]). As a consequence of the above consideration, we have the following corollary

Corollary 1. *For any $r>0$, the domains $D_3^{r,r}$ and $D_{1/3}^{r,r}$ are not Lu Qi-Keng.*

Now we will study the Lu Qi-Keng problem for $D_n^{2,2}$ in the case when $n>3$. By representation (iv) from Theorem 2

$$K_{D_n^{2,2}}((0,w_1,w_1),(0,\xi_1,\xi_1))=\frac{\Gamma(3+n)(3+nw_1\overline{\xi}_1)}{6\pi^{n+2}(1-w_1\overline{\xi}_1)^{4+n}}.$$

Hence

$$K_{D_n^{2,2}}\left(\left(0,\mathrm{i}\sqrt{\frac{3}{n}},\mathrm{i}\sqrt{\frac{3}{n}}\right),\left(0,-\mathrm{i}\sqrt{\frac{3}{n}},-\mathrm{i}\sqrt{\frac{3}{n}}\right)\right)=0.$$

For brevity, we shall summarize these last statements by

Proposition 4. *The domain $D_n^{2,2}$ is Lu Qi-Keng if and only if $n=1$.*

At the end of this section we would like to present the following relations between zeros of the Bergman kernel for the domains $D_n^{q,r}$ and $D_{1/n}^{q,r}$.

Proposition 5. *For any positive real numbers q and r*

(i) *if $K_{D_n^{q,r}}$ has zeros, then $K_{D_{1/n}^{q,r}}$ also has zeros,*
(ii) *if $D_{1/n}^{q,r}$ is a Lu Qi-Keng domain, then $D_n^{q,r}$ is a Lu Qi-Keng domain.*

P r o o f. Note that the zero set is a bi-holomorphic invariant object. Hence any point $(z,w_1,w_2)\in D_n^{q,r}$ can be mapped equivalently onto the form $(0,\widetilde{w_1},\widetilde{w_2})$ by the automorphism of the $D_n^{q,r}$

$$D_n^{q,r}\ni(z,w_1,w_2)\mapsto\left(\Psi_a(z),\frac{(1-\|a\|^2)^{1/q}}{(1-\langle z,a\rangle)^{2/q}}w_1,\frac{(1-\|a\|^2)^{1/r}}{(1-\langle z,a\rangle)^{2/r}}w_2\right)\in\mathbb{C}^{n+2},$$

where

$$\Psi_a(z)=\frac{\left(\dfrac{\langle z,a\rangle}{1+\sqrt{1-\|a\|^2}}-1\right)a+z\sqrt{1-\|a\|^2}}{1-\langle z,a\rangle}.$$

Therefore, we need only to consider the zeroes restricted to $\{0\}\times\mathbb{D}\times\mathbb{D}$, where $\mathbb{D}:=\{z\in\mathbb{C}\colon\ |z|<1\}$. The results now follow from Proposition 3. □

5. Additional results

Our purpose in this section is to consider domains Ω_n^r defined for every positive real number r by

$$\Omega_n^r := \left\{(z,w) \in \mathbb{C} \times \mathbb{C}^n \colon \ |z|^2 + |w_1|^r < 1, \ldots, |z|^2 + |w_n|^r < 1\right\}.$$

The reader can see that the following proposition is completely analogous to the results presented earlier.

Proposition 6. *For $j = 1, 2, \ldots, n$ let $\beta_j \geqslant 0$. Then for $\alpha \geqslant 0$ we have*

$$\|z^\alpha w^\beta\|^2_{L^2(\Omega_n^r)} = \frac{\pi^{n+1}\Gamma\Big(\dfrac{2}{r}\Big(\sum\limits_{j=1}^n \beta_j + n\Big) + 1\Big)\Gamma(\alpha+1)}{\Gamma\Big(\dfrac{2}{r}\Big(\sum\limits_{j=1}^n \beta_j + n\Big) + \alpha + 2\Big)\prod\limits_{j=1}^n (\beta_j + 1)}.$$

Now we will give an explicit formula for the kernel $K_{\Omega_n^r}$ of Ω_n^r.

Theorem 3. *If $w, \zeta \in \mathbb{D}^n$, then the Bergman kernel for Ω_n^r is*

$$\pi^{n+1} K_{\Omega_n^r}((0,w),(0,\zeta)) = \prod_{k=1}^n (1-\nu_k)^{-2} + \sum_{k=1}^n \frac{2(1+\nu_k)}{r(1-\nu_1)^2 \ldots (1-\nu_n)^2(1-\nu_k)},$$

where $\nu_k = w_k\overline{\zeta}_k$ for $k = 1, \ldots, n$. Moreover, if $\nu_1 = \nu_2 = \ldots = \nu_n$, then

$$\pi^{n+1} K_{\Omega_n^r}((0,w),(0,\zeta)) = \frac{(2n-r)\nu_1 + 2n + r}{r(1-\nu_1)^{2n+1}}.$$

P r o o f. As a consequence of Proposition 6, we have

$$\pi^{n+1} K_{\Omega_n^r}((0,w),(0,\zeta)) = \sum_{\beta_1,\ldots,\beta_n \geqslant 0}^{\infty} \frac{\Gamma\Big(\dfrac{2}{r}\Big(\sum\limits_{j=1}^n \beta_j + n\Big) + 2\Big)\prod\limits_{j=1}^n (\beta_j + 1)}{\Gamma\Big(\dfrac{2}{r}\Big(\sum\limits_{j=1}^n \beta_j + n\Big) + 1\Big)} \nu^\beta,$$

where $\nu^\beta = \nu_1^{\beta_1} \ldots \nu_n^{\beta_n}$. Then, using the fact that $a\Gamma(a) = \Gamma(a+1)$,

$$\pi^{n+1} K_{\Omega_n^r}((0,w),(0,\zeta)) = \sum_{\beta_1,\ldots,\beta_n \geqslant 0}^{\infty} \frac{2}{r}\Big(\beta_1 + \ldots + \beta_n + n + \frac{r}{2}\Big)\prod_{j=1}^n (\beta_j + 1)\nu^\beta.$$

Hence, the kernel is

$$\frac{2}{r\pi^{n+1}}\sum_{k=0}^{n}\sum_{\beta_1,\ldots,\beta_n\geqslant 0}^{\infty}(\beta_k+1)\prod_{j=1}^{n}(\beta_j+1)\nu^{\beta},$$

where $\beta_0 = r/2-1$. After some calculation using the formulas

$$\sum_{m=0}^{\infty}(m+1)x^m = (1-x)^{-2} \quad \text{and} \quad \sum_{m=0}^{\infty}(m+1)^2x^m = \frac{1+x}{(1-x)^3},$$

we obtain the desired formula. □

5.1. Zeros of Bergman kernels on Ω_n^r. In this subsection, we will prove that the Bergman kernel function of Ω_n^r for any natural number n and positive real number r is zero-free.

Proposition 7. *For any positive integer n, the domain Ω_n^r is a Lu Qi-Keng domain.*

Proof. It is obvious that the Bergman kernel for Ω_1^r has no zeros. The Bergman kernel function of Ω_n^r has zeros if and only if

$$\frac{r}{2}+\sum_{k=1}^{n}\frac{1+\nu_k}{1-\nu_k} = n+\frac{r}{2}+\sum_{k=1}^{n}\frac{2\nu_k}{1-\nu_k} = 0$$

for some $\nu_1,\ldots,\nu_n$ such that $|\nu_k|<1$ for each $k=1,\ldots,n$. Since $|\nu_k|<1$ we have

$$\operatorname{Re}\Big(\frac{2\nu_k}{1-\nu_k}\Big) > -1,$$

where $\operatorname{Re}(\xi)$ is the real part of the complex number ξ. Thus

$$\operatorname{Re}\Big(n+\frac{r}{2}+\sum_{k=1}^{n}\frac{2\nu_k}{1-\nu_k}\Big) > 0.$$

Hence, we see that

$$\frac{r}{2}+\sum_{k=1}^{n}\frac{1+\nu_k}{1-\nu_k} \neq 0,$$

if $(\nu_1,\ldots,\nu_n)\in\mathbb{D}^n$. The proof is therefore complete. □

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Author's address: Tomasz Beberok, Sabanci University, Orta Mahalle, Universite Caddesi No: 27, Lojmanlari G7-102, Tuzla, 34956 Istanbul, Turkey, e-mail: `tbeberok@ar.krakow.pl`.