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SCAP-SUBALGEBRAS OF LIE ALGEBRAS

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Abstract. A subalgebra H of a finite dimensional Lie algebra L is said to be a SCAP-subalgebra if there is a chief series $0 = L_0 \subset L_1 \subset \ldots \subset L_t = L$ of L such that for every $i = 1, 2, \ldots, t$, we have $H + L_i = H + L_{i-1}$ or $H \cap L_i = H \cap L_{i-1}$. This is analogous to the concept of SCAP-subgroup, which has been studied by a number of authors. In this article, we investigate the connection between the structure of a Lie algebra and its SCAP-subalgebras and give some sufficient conditions for a Lie algebra to be solvable or supersolvable.

Keywords: Lie algebra; SCAP-subalgebra; chief series; solvable; supersolvable

MSC 2010: 17B05, 17B30, 17B50

1. Introduction

All Lie algebras under consideration in this article are assumed to be finite dimensional over some field Λ. We denote algebra direct sums by '$\oplus$', whereas vector space direct sums will be denoted by '$\dot{+}$'. If H is a subalgebra of a Lie algebra L, then H_L is the core (with respect to L) of H, which is the largest ideal of L contained in H. Also, let $\varphi(L)$, $N(L)$, $C_L(H)$ and $I_L(H)$ denote the Frattini ideal, the nil radical, the centralizer of H in L and the idealizer of H in L, respectively. In addition, we say H covers a chief factor A/B of L if $H + A = H + B$; likewise, H avoids A/B if $H \cap A = H \cap B$.

There has been much interest in the past in investigating the relationship between the properties of maximal subalgebras of a Lie algebra L and the structure of L. In this aspect, Towers introduced in [14] and [15], respectively, the concepts of c-ideality and the covering and avoidance properties of subalgebras of a Lie algebra L. A subalgebra H of L is called a c-ideal if there is an ideal K of L such that $L = H + K$ and $H \cap K \leqslant H_L$. Also, the subalgebra H is said to have the covering and avoidance property of L, in short, H is a CAP-subalgebra of L, if H either covers or avoids every chief factor of L. These concepts are used to give some characterizations of solvable and supersolvable Lie algebras. In particular, it is shown in [14] that a Lie algebra L is solvable if and only if its maximal subalgebras are c-ideals in L. Also, Hallahan and Overbeck in [8] proved that any Cartan subalgebra of a metanilpotent Lie algebra is a CAP-subalgebra, and Stitzinger in [11] and [12], found some kind of CAP-subalgebras in a solvable Lie algebra. In [13] Towers showed that L is supersolvable if and only if all one-dimensional subalgebras of L are CAP-subalgebras of L. Now, we present another concept which is related to the previous two.

Definition 1.1. A subalgebra H of a Lie algebra L is said to be a SCAP-subalgebra of L if there is a chief series of L such that H covers or avoids every chief factor of the series.

This is analogous to the notion of the semi-cover-avoiding subgroup of a finite group given by Fan et al. in [5] and it has since been further studied by a number of authors, including Ballester-Bolinches et al. [1] and Gue et al. [7] and Li et al. [9]. It is obvious that every CAP-subalgebra of L must be a SCAP-subalgebra of L. However, the converse is not necessarily true. The following is such an example.

Example 1.2. Let L_1, L_2 be Lie algebras over the field $\mathbb{R}$ of real numbers, in which $L_1 = \mathbb{R}t$ and $L_2 = \mathbb{R}a + \mathbb{R}b + \mathbb{R}c$ with the multiplication defined by $[a, b] = c$, $[a, c] = 0$, $[b, c] = a$. Set $L = L_1 \oplus L_2$ and assume that H is a subalgebra of L generated by $t + a$. It is easy to see that $0 \subset (\mathbb{R}a + \mathbb{R}c) \subset L_2 \subset L$ is a chief series of L and H covers L/L_2 and avoids the rest, that is, H is a SCAP-subalgebra of L. However, H is not a CAP-subalgebra of L because it does not cover or avoid the chief factor $(L_1 \oplus (\mathbb{R}a + \mathbb{R}c))/L_1$ of L.

In this article, we deal with the connection between the structure of a Lie algebra and its SCAP-subalgebras, and give some sufficient conditions for a Lie algebra to be solvable or supersolvable. Note that some results obtained here are the counterparts to well-known theorems in finite group theory. However, the Lie algebraic proofs are rather different in nature, as the group theoretic results rely on properties that do not hold in the case of Lie algebras.

2. Preliminary results

This section is devoted to some basic results which are vital in our investigation.

Lemma 2.1. *Let H be a subalgebra of a Lie algebra L. Let $0 = L_0 \subset \ldots \subset L_i \subset \ldots \subset L_j \subset \ldots \subset L_t = L$ be an ideal series of L. If H covers (avoids) L_j/L_i, then H covers (respectively, avoids) L_s/L_r for any $i \leqslant r < s \leqslant j$.*

P r o o f. Suppose first that H covers L_j/L_i. It suffices to show that $L_s \leqslant L_r + H$. Since $L_s/L_i \leqslant L_j/L_i$ and $L_j \leqslant L_i + H$, we have $L_s \leqslant L_i + H$. But $L_i \leqslant L_r$, we therefore deduce that $L_s \leqslant L_r + H$. If H avoids L_j/L_i, a similar argument shows that H avoids L_s/L_r. □

As an immediate consequence of the above lemma, we get that any c-ideal of a Lie algebra L is a SCAP-subalgebra. For, suppose that H is a c-ideal of L. Owing to ([10], Lemma 2.3 (i)), there exists an ideal K of L such that $L = H + K$ and $H \cap K = H_L$. Then $0 \leqslant H \cap K \leqslant K \leqslant L$ is an ideal series of L. Obviously, H covers both the factors $(H \cap K)/0$ and L/K, and H avoids the factor $K/(H \cap K)$. Consequently, invoking Lemma 2.1, H covers or avoids every factor of any refinement of the ideal series, and thus H is a SCAP-subalgebra of L.

Lemma 2.2. *Let L be a Lie algebra, $N \trianglelefteq L$ and $N \leqslant H \leqslant L$. If H is a* SCAP-*subalgebra of L, then H/N is a* SCAP-*subalgebra of L/N.*

P r o o f. Straightforward. □

Proposition 2.3. *A nontrivial Lie algebra L is simple if and only if it has no nontrivial proper* SCAP-*subalgebra.*

P r o o f. $\Leftarrow$: This follows from [15], Lemma 2.1 (iv).
$\Rightarrow$: Let H be a SCAP-subalgebra of L. Since $L/0$ is the unique chief factor of L, we have $H \cap L = 0$ or $L = H$, as required. □

Proposition 2.4. *Let I be an abelian minimal ideal of a Lie algebra L such that L/I is nilpotent. Then all maximal subalgebras of Cartan subalgebras of L are* SCAP-*subalgebras of L.*

P r o o f. Since L is solvable, there is a Cartan subalgebra C of L with a maximal subalgebra C_1. Then $(C+I)/I$ is a Cartan subalgebra of L/I. As L/I is nilpotent, we have $(C+I)/I = L/I$, whence $L = C+I$. Consequently, $(C_1+I)/I$ is a maximal subalgebra of L/I, implying that $(C_1+I)/I$ is an ideal of L/I. Now suppose that $L = C + L_1(C)$ is the Fitting decomposition of L with respect to C, and $u \in C$ is a regular element of L. Then the adjoint map $\operatorname{ad} u\colon\ L_1(C) \longrightarrow L_1(C)$ is nonsingular. For $x \in L_1(C)$, we can write $x = c + a$ where $c \in C$ and $a \in I$. We choose a positive integer k such that $(\operatorname{ad} u)^k(C) = 0$; then $(\operatorname{ad} u)^k(x) = (\operatorname{ad} u)^k(a)$ which lies in I. So $(\operatorname{ad} u)^k(L_1(C)) \subseteq I$, and as $(\operatorname{ad} u)^k$ is nonsingular on $L_1(C)$ we see that $L_1(C) \subseteq I$. In particular $[L_1(C), L_1(C)] = 0$. Therefore

$$[L, L_1(C)] = [C + L_1(C), L_1(C)] \subseteq L_1(C).$$

But this yields that $L_1(C)$ is an abelian ideal of L contained in I. It hence follows from the hypothesis that $L_1(C) = 0$ or $L_1(C) = I$. In the first case $L = C$, which was excluded. It is obvious that $0 < I < C_1 + I < L$ is a series of ideals of L. An easy verification shows that C_1 avoids both the factors $I/0$ and $(C_1 + I)/I$, and C_1 covers the factor $L/(C_1 + I)$. Therefore, by Lemma 2.1, C_1 is a SCAP-subalgebra of L. $\square$

3. Main results

In this section, we give some characterizations of solvable and supersolvable Lie algebras.

Theorem 3.1. *Let H be an ideal of a Lie algebra L. If every maximal subalgebra M of L satisfying $L = M + H$ is a* SCAP-*subalgebra of L, then H is solvable.*

Proof. By virtue of Proposition 2.3, the hypothesis cannot hold if L is simple and $H \neq 0$. We now consider the following two cases:

Case 1: Let H be the unique minimal ideal of L. We prove that H is nilpotent. Suppose that the map $ad_L h$ is not nilpotent for some nonzero element $h \in H$. If $L = L_0 \dot{+} L_1$ is the Fitting decomposition of L relative to the nilpotent subalgebra $\langle h \rangle$, then $L_0 \leqslant M$ for some maximal subalgebra M of L and $L_1 \leqslant H$. Evidently, $H \nleqslant M$ and so, by the assumption, M is a SCAP-subalgebra of L whence $H \cap M = 0$. We consequently have $M = L_0$ and $L_1 = H$. But this yields that $h \in M \cap H = 0$, a contradiction. So for every $h \in H$, $ad_L h$ is nilpotent and we therefore conclude that H is nilpotent.

Case 2: Let N be a minimal ideal of L with $N \neq H$. If M/N is a maximal subalgebra of L/N such that $L = M + H$, then Lemma 2.2 indicates that M/N is a SCAP-subalgebra of L/N and so applying induction on the dimension of L, it follows that $(H + N)/N$ is solvable. If $N \not< H$, then $(H + N)/N \cong H$ is solvable. Hence, suppose that all minimal ideals of L are contained in H. If K is another minimal ideal of L, then H/K is solvable and $N \cap K = 0$, so H is solvable. We therefore assume that L has a unique minimal ideal N and H/N is solvable. The same argument as in Case 1 shows that N is nilpotent and so H is solvable. The proof is complete. $\square$

Theorem 3.3 in [15] together with the above theorem imply the following corollary.

Corollary 3.2. *A Lie algebra L is solvable if and only if every maximal subalgebra of L is a* SCAP-*subalgebra of L.*

In the following, we establish, under some conditions on the field Λ, that Corollary 3.2 holds with "every maximal subalgebra" replaced by "one solvable maximal subalgebra", which is easier to find.

Theorem 3.3. *Let L be a Lie algebra over a field F which has characteristic zero, or is algebraically closed of characteristic greater than 5. Then L is solvable if and only if there is a solvable maximal subalgebra M of L such that M is a* SCAP-*subalgebra of L.*

Proof. We first prove the sufficiency of the condition. We use induction on the dimension of L. Suppose that M is a solvable maximal subalgebra of L which is a SCAP-subalgebra of L. If $M_L \neq 0$, Lemma 2.2 and induction yield that L/M_L is solvable. However, M_L is solvable and so is L. We now assume that $M_L = 0$. Since M is a SCAP-subalgebra of L, there exists a minimal ideal N of L such that $M \cap N = 0$ and then $L = M \dot{+} N$. We now conclude from [15], Proposition 3.1, that L is solvable. The converse follows from Corollary 3.2. $\square$

Theorem 3.4. *Let L be a Lie algebra over a field Λ of at least* dim L *elements. If every maximal subalgebra of L which contains the idealizer of a maximal nilpotent subalgebra of L is a* SCAP-*subalgebra of L, then L is solvable.*

Proof. If L has no proper nilpotent subalgebras, then the result trivially holds. We thus suppose that L contains a proper maximal nilpotent subalgebra, H say. If L is simple, then $I_L(H) \leqslant M$ for some maximal subalgebra M of L. By the hypothesis, M is a SCAP-subalgebra of L, contradicting Proposition 2.3. Assume that N is a minimal ideal of L, and M/N is a maximal subalgebra of L/N which contains the idealizer of a maximal nilpotent subalgebra Q/N of L/N. According to [14], Lemma 4.1, Q can be regarded as the sum $Q = C + N$ in which C is a maximal nilpotent subalgebra of L. As $I_L(C) \leqslant I_L(Q)$ and $(I_L(Q) + N)/N \leqslant I_{L/N}(Q/N) \leqslant M/N$, we have $I_L(C) \leqslant M$. It follows from the hypothesis that M is a SCAP-subalgebra of L and hence by Lemma 2.2, M/N is a SCAP-subalgebra of L/N. So, by induction L/N is solvable. As the class of all solvable Lie algebras is a saturated formation, we may assume that L has a unique minimal ideal N and L/N is solvable, and also $N(L) = 0$. Consequently, there is an element $x \in N$ such that $ad_L x$ is not nilpotent. Let $L = L_0 \dot{+} L_1$ be a Fitting decomposition relative to $K = \langle x \rangle$, in which L_0 is an Engel subalgebra of L and so contains a Cartan subalgebra of L thanks to [2], Theorem 1. Noting that $L \neq L_0$, there is a maximal subalgebra M of L such that $L_0 \leqslant M$. Thus M is a SCAP-subalgebra of L, and since $x \in M \cap N$, we must have $N \leqslant M$. But we know that $L_1 \leqslant N$, so $L = L_0 + L_1 \leqslant M$, a contradiction. Therefore N is nilpotent, implying that L is solvable. The proof is complete. $\square$

As a straightforward corollary of Theorem 3.4 we have the following corollary.

Corollary 3.5. *Let L be a Lie algebra over a field Λ of at least* dim L *elements. If every maximal subalgebra of L which contains a Cartan subalgebra of L is a* SCAP-*subalgebra of L, then L is solvable.*

P r o o f. Follows from [6], Lemma 3.2.2 and Theorem 3.4. □

Let $\mathcal{X}$ be a class of Lie algebras, in which every $L \in \mathcal{X}$ has the property that, for every quotient Lie algebra L/N, K/N is a Cartan subalgebra of L/N only if K is a Cartan subalgebra of L. The following example shows that the class $\mathcal{X}$ is nonempty.

Example 3.6. Let L be a real Lie algebra with basis $\{x_1, x_2, x_3, x_4\}$ and multiplication $[x_2, x_3] = x_4$, $[x_2, x_4] = x_3$ and $[x_3, x_4] = x_2$, and all other products being zero. Then $\langle x_1\rangle$, $\langle x_2, x_3, x_4\rangle$ are the only ideals of L and $\langle x_1, x_2\rangle$, $\langle x_1, x_3\rangle$, $\langle x_1, x_4\rangle$ are all of the Cartan subalgebras of L. We see that $\langle x_1, x_2\rangle/\langle x_1\rangle$, $\langle x_1, x_3\rangle/\langle x_1\rangle$, $\langle x_1, x_4\rangle/\langle x_1\rangle$ are the only proper Cartan subalgebras of any quotient Lie algebra of L. Furthermore, the numerators are Cartan subalgebras of L.

Theorem 3.7. *Let $L \in \mathcal{X}$ and let the ground field Λ have characteristic zero. If every Cartan subalgebra of L is a* SCAP-*subalgebra of L, then L is solvable.*

P r o o f. By the hypothesis and Proposition 2.3, if L is simple, then L has no proper Cartan subalgebras, and it therefore follows that L is nilpotent. So suppose that N is a minimal ideal of L and Q/N is a Cartan subalgebra of L/N. Then Q is a Cartan subalgebra of L and thus it is a SCAP-subalgebra of L. By induction, we see that L/N is solvable. As the class of all solvable Lie algebras is a saturated formation, we may assume that L has a unique minimal ideal N such that L/N is solvable. We claim that N is nilpotent. By [6], Corollary 3.2.10, N has a Cartan subalgebra Q. If H is a Cartan subalgebra of $L_0(Q)$, then by ([4], 2.7), H is a Cartan subalgebra of L and therefore $Q = H \cap N$. Because N is a unique minimal ideal of L, N is contained in every chief series of L. Hence, we see that either $N \leqslant H$ or $N \cap H = 0$. If $N \leqslant H$, then N is nilpotent, and if $N \cap H = 0$, then $Q = 0$, an impossibility. Therefore N is nilpotent and L is solvable. The proof is complete. □

Theorem 3.8. *Let L be a Lie algebra L over a field of characteristic zero. Then L is supersolvable if and only if there is a solvable ideal H of L such that L/H is supersolvable and every maximal subalgebra of $N(H)$ is a* SCAP-*subalgebra of L.*

P r o o f. We first prove the sufficiency of the condition. By virtue of [3], Proposition 4 and Lemma 2.2, the assumptions and the assertion are unaffected by passing to $L/(\varphi(L)\cap H)$. Hence we may assume $\varphi(L)\cap H=0$. According to [10], Proposition 2.6, we have $N(H)=B_1\oplus B_2\oplus\ldots\oplus B_n$, where B_i, $i=1,2,\ldots,n$, are minimal abelian ideals of L. We first show that $\dim(B_1)=1$. Suppose that B_1^* is any maximal subalgebra of B_1. Certainly, $M:=B_1^*+(B_2\oplus B_3\oplus\ldots\oplus B_n)$ is a maximal subalgebra of $N(H)$ and so, by the hypothesis, is a SCAP-subalgebra of L. Putting $N=B_2\oplus\ldots\oplus B_n$, one concludes from Lemma 2.2 that M/N is a SCAP-subalgebra of L/N. But this implies that there is a chief series joining N to L, say $N=L_0<L_1<\ldots<L_s=L$, such that M covers or avoids any of its chief factors. Since $N(H)\nleqslant L_0+M$, we can find a chief factor L_i/L_{i-1} such that $N(H)\leqslant L_i+M$ but $N(H)\nleqslant L_{i-1}+M$. It is readily verified that $L_i\cap M=L_{i-1}\cap M$, $N(H)=N(H)\cap(L_i+M)=(N(H)\cap L_i)+M=(B_1\cap L_i)+M$ and $N(H)\supset N(H)\cap(L_{i-1}+M)=(B_1\cap L_{i-1})+M$. Therefore, we obtain $B_1\cap L_i\neq 0$ and $B_1\cap L_{i-1}=0$. Hence $M=L_i\cap M=L_{i-1}\cap M=N$, forcing that $B_1^*=0$ and $\dim(B_1)=1$. By the same arguments as the above we may see that $\dim(B_i)=1$ for $i=2,\ldots,n$. As the factor Lie algebra $L/C_L(B_i)$ is isomorphic to a subalgebra of $\mathrm{Der}(B_i)$, it can be inferred that $L/C_L(B_i)$ and then $L/C_L(N(H))$ are supersolvable. Bearing in mind that H is solvable and $\varphi(H)=0$, see [11], Theorem 3, yields that $C_H(N(H))=N(H)$. Consequently, $L/N(H)=L/(H\cap C_L(N(H))$ is supersolvable. But since we know that $N(H)$ is a direct sum of one-dimensional ideals of L, it follows that L is supersolvable.

The necessity of the condition is easily established, for if L is supersolvable then L contains an ideal of dimension one and by [15], Proposition 2.9, every subalgebra of L is a SCAP-subalgebra. The proof of the theorem is complete. $\square$

We obtain the following corollaries which are of interest in their own account.

Corollary 3.9. *Let L be a solvable Lie algebra over a field of characteristic zero. Then L is supersolvable if and only if all maximal subalgebras of $N(L)$ are* SCAP-*subalgebras.*

Corollary 3.10. *Let L be a Lie algebra L over a field of characteristic zero. Then L is supersolvable if and only if there is a solvable ideal H of L such that L/H is supersolvable and every maximal subalgebra of $N(H)$ is a* CAP-*subalgebra of L.*

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