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GAUSSIAN DENSITY ESTIMATES FOR THE SOLUTION OF SINGULAR STOCHASTIC RICCATI EQUATIONS

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Abstract. Stochastic Riccati equation is a backward stochastic differential equation with singular generator which arises naturally in the study of stochastic linear-quadratic optimal control problems. In this paper, we obtain Gaussian density estimates for the solutions to this equation.

Keywords: stochastic Riccati equation; Malliavin calculus; density estimate

MSC 2010: 60H10, 60H07

1. Introduction

In the paper, we consider the following backward stochastic differential equation (BSDE):

$$
\begin{cases} R(t) = \xi + \displaystyle\int_t^T \Big(a(s) + b(s)R(s) + c(s)Z(s) - \frac{Z^2(s)}{R(s)} \Big)\,\mathrm{d}s - \int_t^T Z(s)\,\mathrm{d}B(s) \\ R(t) > 0, \ t \in [0,T], \end{cases} \tag{1.1}
$$

where B is a standard Brownian motion and a, b, and c are adapted and uniformly bounded stochastic processes.

Equation (1.1) is known in the literature as the stochastic Riccati equation, which arises naturally in the study of stochastic linear-quadratic (LQ) optimal control problems. More precisely, the solvability of LQ problems is reduced to proving global solvability of a stochastic Riccati equation of the form (1.1) (see, [6], [13], [12], and references therein).

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The global solvability of the Riccati equation (1.1) is well studied in [6], [13], and [12]. Because of its applications, one would like to know more information on the solution of the stochastic Riccati equation. However, to the best of our knowledge, deeper properties of the solution are scarce. Motivations of this paper come from an observation made by Dos Reis in [4], where he noted that the information on the density of the solution will provide more accurate estimates on the convergence rates of numerical schemes for BSDEs. Thus the aim of the present paper is to investigate the density of the solution to (1.1).

The study of the density of solutions is a classical topic and has many applications in the theory of stochastic differential equations (see Chapter 2 in [10]). In particular, Gaussian density estimates for various classes of stochastic equations have been extensively studied in recent years (see, e.g., [8] and references therein). However, it is surprising that only few works are devoted to this topic for the class of BSDEs. We only find in the literature the following three papers: the first one by Antonelli and Kohatsu-Higa (2005) [2], where they studied the existence and smoothness of the densities of the solution, the second one by Aboura and Bourguin (2013) [1], where they provided Gaussian estimates for the densities of the solution, and the last one by Mastrolia et al. (2015) [7], where the results are discussed in a general setting.

Since the equations considered in [1], [2], and [7] are Markovian ones, the results obtained in these papers cannot be applied to (1.1). Furthermore, the generator (or driver) of the equation (1.1) has a singularity at $R = 0$. This causes some mathematical difficulties which make the study of the density of the solution to (1.1) particularly interesting.

In this paper, we will provide Gaussian density estimates for the first component of the solution (R, Z). To get such results we will employ a Gaussian density criterion stated in the terms of Malliavin analysis. Thus the main tasks arising here are that we need to prove the Malliavin differentiability of the solution and control the boundedness of Malliavin derivatives. The difficulties coming from the singularity of the generator will be handled by using Girsanov's theorem.

The rest of the paper is organized as follows. In Section 2, we recall some fundamental concepts of Malliavin calculus and general Gaussian estimates for the density. The main results of the paper are stated and proved in Section 3.

2. Preliminaries

Let us recall some elements of stochastic calculus of variations (for more details see [10]). We suppose that $\{B(t),\ t \in [0,T]\}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filter generated by the Brownian motion B. For $h \in L^2[0,T]$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t)\,\mathrm{d}B(t).$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(B(h_1), \dots, B(h_n)), \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in L^2[0,T]$. If F has the form (2.1), we define its Malliavin derivative as the process $DF := \{D_t F,\ t \in [0,T]\}$ given by

$$D_t F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(B(h_1), \dots, B(h_n)) h_k(t).$$

We shall denote by $\mathbb{D}^{1,2}$ the space of Malliavin differentiable random variables, it is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 := E|F|^2 + \int_0^T E|D_u F|^2\,\mathrm{d}u.$$

In [8], we used Nourdin and Viens's results from [9] to obtain the following sufficient condition for a Malliavin differentiable random variable to have a density with lower and upper Gaussian bounds.

Proposition 2.1. *Let F be in $\mathbb{D}^{1,2}$ with mean zero. If there exist positive constants β_1, β_2 such that, for all $x \in \mathbb{R}$, almost surely*

$$\beta_1 \leqslant \int_0^\infty D_r F\, E[D_r F \mid \mathcal{F}_r]\,\mathrm{d}r \leqslant \beta_2,$$

then F has a density ϱ_F satisfying, for almost all $x \in \mathbb{R}$,

$$\frac{E|F|}{2\beta_2} \exp\Big(-\frac{x^2}{2\beta_1}\Big) \leqslant \varrho_F(x) \leqslant \frac{E|F|}{2\beta_1} \exp\Big(-\frac{x^2}{2\beta_2}\Big). \tag{2.2}$$

3. The main results

We first collect the conditions that were imposed in [13] to prove the existence and uniqueness of the solution:

(H_1) ξ is a $\mathcal{F}_T$-measurable random variable and there exist two positive constants m_ξ, M_ξ such that $m_\xi \leqslant \xi \leqslant M_\xi$ a.s.,

(H_2) a, b and c are $\mathbb{F}$-adapted stochastic processes and uniformly bounded on $[0,T]$ by positive constants M_a, M_b, and M_c, respectively. In addition, $a(t) \geqslant 0$ a.s. for all $t \in [0,T]$.

The next proposition refines Theorem 4.3 in [13].

Proposition 3.1. *Under the assumptions* (H_1) *and* (H_2), *the Riccati equation* (1.1) *has a unique solution* (R, Z). *Moreover, the first component of the solution is bounded uniformly in* $t \in [0,T]$:

$$k := m_\xi \mathrm{e}^{-M_b T} \leqslant R(t) \leqslant M_\xi \mathrm{e}^{M_b T} + \frac{M_a(\mathrm{e}^{M_b T} - 1)}{M_b} =: K \quad \text{a.s.} \tag{3.1}$$

and the second component of the solution satisfies the condition that $\{\int_0^t Z(s)\,\mathrm{d}B(s),\ t \in [0,T]\}$ *is a BMO-martingale.*

Proof. The existence of a unique solution (R, Z) to the Riccati equation (1.1) has been already proved by Yu in [13], Theorem 4.3. In order to finish the proof, we have to verify (3.1) and show that $\{\int_0^t Z(s)\,\mathrm{d}B(s),\ t \in [0,T]\}$ is a BMO-martingale.

Let $\alpha > 0$ be a real number that will be chosen later. Consider the following backward stochastic differential equation:

$$\begin{aligned} R_\alpha(t) = \xi + \int_t^T \Big(a(s) + b(s)R_\alpha(s) + c(s)Z_\alpha(s) - \frac{Z_\alpha^2(s)}{\alpha \vee R_\alpha(s)}\Big)\,\mathrm{d}s \\ - \int_t^T Z_\alpha(s)\,\mathrm{d}B(s), \quad t \in [0,T]. \end{aligned} \tag{3.2}$$

It is easy to verify that the equation (3.2) is a BSDE with quadratic growth. Consequently, this equation admits a unique solution (R_α, Z_α) and the stochastic process

$$\Big\{\int_0^t Z_\alpha(s)\,\mathrm{d}B(s),\ t \in [0,T]\Big\}$$

is a BMO-martingale (see, e.g., Theorems 9.6.3 and 9.6.4 in [3]). Since c is uniformly bounded, the stochastic process

$$\Big\{\int_0^t \Big(c(s) - \frac{Z_\alpha(s)}{\alpha \vee R_\alpha(s)}\Big)\,\mathrm{d}B(s),\ t \in [0,T]\Big\}$$

is also a BMO-martingale. By Theorem 2.3 in [5], the stochastic exponential

$$\exp\Bigl(\int_0^t \Bigl(c(s)-\frac{Z_\alpha(s)}{\alpha\vee R_\alpha(s)}\Bigr)\,\mathrm{d}B_s-\frac{1}{2}\int_0^t \Bigl(c(s)-\frac{Z_\alpha(s)}{\alpha\vee R_\alpha(s)}\Bigr)^2\,\mathrm{d}s\Bigr)$$

is a uniformly integrable martingale. Applying Girsanov's theorem (see, e.g., Corollary 1.2 in [5]), we obtain that the stochastic process

$$\Bigl\{B_1(t):=B(t)-\int_0^t \Bigl(c(s)-\frac{Z_\alpha(s)}{\alpha\vee R_\alpha(s)}\Bigr)\,\mathrm{d}s,\ t\in[0,T]\Bigr\}$$

is a standard Brownian motion under the probability measure Q_1, where Q_1 is defined as

$$\frac{\mathrm{d}Q_1}{\mathrm{d}P}=\exp\Bigl(\int_0^T \Bigl(c(s)-\frac{Z_\alpha(s)}{\alpha\vee R_\alpha(s)}\Bigr)\,\mathrm{d}B_s-\frac{1}{2}\int_0^T \Bigl(c(s)-\frac{Z_\alpha(s)}{\alpha\vee R_\alpha(s)}\Bigr)^2\,\mathrm{d}s\Bigr).$$

Under Q_1 equation (3.2) becomes

$$R_\alpha(t)=\xi+\int_t^T (a(s)+b(s)R_\alpha(s))\,\mathrm{d}s-\int_t^T Z_\alpha(s)\,\mathrm{d}B_1(s),\quad t\in[0,T],$$

which is a linear BSDE, its explicit solution is given by

$$R_\alpha(t)=E_{Q_1}\Bigl[\xi\mathrm{e}^{\int_t^T b(s)\,\mathrm{d}s}+\int_t^T a(s)\mathrm{e}^{\int_t^s b(u)\,\mathrm{d}u}\,\mathrm{d}s\mid\mathcal{F}_t\Bigr],\quad t\in[0,T],\tag{3.3}$$

where E_{Q_1} denotes the expectation under Q_1. By the boundedness of ξ, a, b, and c we deduce that

$$k\leqslant R_\alpha(t)\leqslant K\quad\text{a.s. }\forall\,t\in[0,T],\ \alpha>0.$$

Now we choose $\alpha=k$. Then (R_k,Z_k) satisfies equation (1.1). By the uniqueness of the solution, we conclude that $(R_k,Z_k)=(R,Z)$. So the proposition is proved. $\square$

Remark 3.1. For the reader's convenience, we recall that the local martingale of the form $\{\int_0^t u(s)\,\mathrm{d}B(s),\ t\in[0,T]\}$ is a BMO-martingale if and only if

$$\sup_{\tau\in\mathcal{T}} E\Bigl[\int_\tau^T |u(s)|^2\,\mathrm{d}s\mid\mathcal{F}_\tau\Bigr]<\infty,$$

where $\mathcal{T}$ denotes the set of all $\mathbb{F}$-stopping times.

In order to be able to apply Proposition 2.1, we need to prove that the solution of (1.1) is Malliavin differentiable. For this purpose, let us introduce the following assumptions:

(H_3) $\xi \in \mathbb{D}^{1,2}$ and there exist finite constants n_ξ, N_ξ such that for all $0 \leqslant r \leqslant T$, $0 < n_\xi \leqslant D_r\xi \leqslant N_\xi$ a.s.,

(H_4) for each $t \in [0,T]$, the random variables $a(t)$, $b(t)$, $c(t)$ belong to $\mathbb{D}^{1,2}$. Moreover, there exist finite constants N_a, N_b, and N_c such that

$$|D_r a(t)| \leqslant N_a,\ \ |D_r b(t)| \leqslant N_b,\ \ |D_r c(t)| \leqslant N_c \quad \text{a.s.}$$

for all $0 \leqslant r \leqslant t \leqslant T$.

Proposition 3.2. *Assume that* (H_1)–(H_4) *hold. Then, the unique solution* (R,Z) *of the Riccati equation* (1.1) *is Malliavin differentiable. In addition, suppose that*

$$C := \frac{2}{3}\Big(\ln\frac{K}{m_\xi} + M_bT\Big) < \frac{1}{2}. \tag{3.4}$$

Then we have the following estimate:

(3.5)
$$\begin{aligned}
&\mathrm{e}^{-M_bT}n_\xi - \frac{\mathrm{e}^{M_bT}(N_a+KN_b)T}{1-C} - \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2}\\
&\quad\leqslant D_rR(t) \leqslant \frac{\mathrm{e}^{M_bT}(N_\xi+(N_a+KN_b)T)}{1-C} + \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2} \quad \text{a.s.}
\end{aligned}$$

for all $0 \leqslant r \leqslant t \leqslant T$.

Proof. We note that the solution (R,Z) of the Riccati equation (1.1) is also the solution of the BSDE with quadratic growth (3.2). Therefore, the Malliavin differentiability of (R,Z) follows from Theorem 3.2.3 in [4]. Moreover, we have for $0 \leqslant r \leqslant t \leqslant T$,

$$\begin{aligned}
D_rR(t) = D_r\xi + \int_t^T\Big(&D_ra(s) + b(s)D_rR(s) + R(s)D_rb(s) + c(s)D_rZ(s) + Z(s)D_rc(s)\\
&- \frac{2Z(s)D_rZ(s)}{R(s)} + \frac{Z^2(s)}{R^2(s)}D_rR(s)\Big)\,\mathrm{d}s - \int_t^T D_rZ(s)\,\mathrm{d}B(s),
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathrm{d}(D_rR(t)) = -\Big[&D_ra(t) + R(t)D_rb(t) + Z(t)D_rc(t) + \Big(b(t)+\frac{Z^2(t)}{R^2(t)}\Big)D_rR(t)\\
&+ c(t)D_rZ(t) - \frac{2Z(t)D_rZ(t)}{R(t)}\Big]\,\mathrm{d}t + D_rZ(t)\,\mathrm{d}B(t), \quad t \in [r,T].
\end{aligned} \tag{3.6}$$

Equation (3.6) is a linear equation with the terminal condition $D_rR(T) = D_r\xi$, its solution is given by

$$(3.7)\quad D_rR(t) = \mathrm{e}^{\int_t^T \psi(s)\,\mathrm{d}s} D_r\xi + \int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r}\Big(D_ra(s) + R(s)D_rb(s) + Z(s)D_rc(s) + c(s)D_rZ(s) - \frac{2Z(s)D_rZ(s)}{R(s)}\Big)\,\mathrm{d}s - \int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r} D_rZ(s)\,\mathrm{d}B(s)$$

with

$$\psi(t) = b(t) + \frac{Z^2(t)}{R^2(t)}.$$

We now define the probability measure Q_2 by

$$\frac{\mathrm{d}Q_2}{\mathrm{d}P} = \exp\biggl(\int_0^T \Big(c(s) - \frac{2Z(s)}{R(s)}\Big)\,\mathrm{d}B_s - \frac{1}{2}\int_0^T \Big(c(s) - \frac{2Z(s)}{R(s)}\Big)^2\,\mathrm{d}s\biggr).$$

Under Q_2, the stochastic process

$$B_2(t) = B(t) - \int_0^t \Big(c(s) - \frac{2Z(s)}{R(s)}\Big)\,\mathrm{d}s$$

is a standard Brownian motion and (3.7) can be rewritten as

$$D_rR(t) = \mathrm{e}^{\int_t^T \psi(s)\,\mathrm{d}s} D_r\xi + \int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r}(D_ra(s) + R(s)D_rb(s) + Z(s)D_rc(s))\,\mathrm{d}s - \int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r} D_rZ(s)\,\mathrm{d}B_2(s).$$

Consequently, we can get

$$(3.8)\qquad D_rR(t) = E_{Q_2}(\mathrm{e}^{\int_t^T \psi(s)\,\mathrm{d}s} D_r\xi \mid \mathcal{F}_t) + E_{Q_2}\biggl(\int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r}[D_ra(s) + R(s)D_rb(s) + Z(s)D_rc(s)]\,\mathrm{d}s \Bigm| \mathcal{F}_t\biggr).$$

In order to be able to estimate $D_rR(t)$, we observe that under Q_2 equation (1.1) becomes

$$R(t) = \xi + \int_t^T \Big(a(s) + b(s)R(s) + \frac{Z^2(s)}{R(s)}\Big)\,\mathrm{d}s - \int_t^T Z(s)\,\mathrm{d}B_2(s), \quad t \in [0, T].$$

Applying Itô's differentiation rule to $\ln R(t)$ gives

$$\mathrm{d}(\ln R(t)) = -\Big(\frac{a(t)}{R(t)} + b(t) + \frac{Z^2(t)}{R^2(t)}\Big)\,\mathrm{d}t + \frac{Z(t)}{R(t)}\,\mathrm{d}B_2(t) - \frac{1}{2}\frac{Z^2(t)}{R^2(t)}\,\mathrm{d}t.$$

Hence,

$$E_{Q_2}\Big[\int_t^T \frac{Z^2(s)}{R^2(s)}\,\mathrm{d}s \mid \mathcal{F}_t\Big] = \frac{2}{3}E_{Q_2}\Big[\ln\frac{R(t)}{\xi} - \int_t^T \Big(\frac{a(s)}{R(s)} + b(s)\Big)\,\mathrm{d}s \mid \mathcal{F}_t\Big].$$

This, combined with (3.1) and the boundedness of a, b, and ξ, yields

$$E_{Q_2}\Big[\int_t^T \frac{Z^2(s)}{R^2(s)}\,\mathrm{d}s \mid \mathcal{F}_t\Big] \leqslant \frac{2}{3}\Big(\ln\frac{K}{m_\xi} + M_b T\Big) =: C \quad \text{a.s.} \tag{3.9}$$

Since $R^2(t) \leqslant K^2$ for all $t \in [0,T]$, we also have

$$E_{Q_2}\Big[\int_t^T Z^2(s)\,\mathrm{d}s \mid \mathcal{F}_t\Big] \leqslant K^2 C \quad \text{a.s.} \tag{3.10}$$

Furthermore, by employing the method used in the proof of Theorem 9.6.4 in [3] we can deduce from (3.9) that

$$E_{Q_2}\Big[\exp\Big(\varepsilon\int_t^T \frac{Z^2(s)}{R^2(s)}\,\mathrm{d}s\Big) \mid \mathcal{F}_t\Big] \leqslant \frac{1}{1-\varepsilon C} \quad \text{a.s.,} \tag{3.11}$$

where ε is a positive constant such that $\varepsilon C < 1$. By Hölder inequality we have

$$\begin{aligned}
&\Big|E_{Q_2}\Big(\int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r} Z(s)D_r c(s)\,\mathrm{d}s \mid \mathcal{F}_t\Big)\Big| \\
&\quad\leqslant T\mathrm{e}^{M_b T}E_{Q_2}\Big(\mathrm{e}^{\int_t^T \frac{Z^2(r)}{R^2(r)}\,\mathrm{d}r}\int_t^T |Z(s)D_r c(s)|\,\mathrm{d}s \mid \mathcal{F}_t\Big) \\
&\quad\leqslant T\mathrm{e}^{M_b T}[E_{Q_2}(\mathrm{e}^{\int_t^T 2\frac{Z^2(r)}{R^2(r)}\,\mathrm{d}r} \mid \mathcal{F}_t)]^{1/2}\Big[E_{Q_2}\Big(\Big(\int_t^T |Z(s)D_r c(s)|\,\mathrm{d}s\Big)^2 \mid \mathcal{F}_t\Big)\Big]^{1/2} \\
&\quad\leqslant T\mathrm{e}^{M_b T}N_c(T-t)^{1/2}[E_{Q_2}(\mathrm{e}^{\int_t^T 2\frac{Z^2(r)}{R^2(r)}\,\mathrm{d}r} \mid \mathcal{F}_t)]^{1/2}\Big[E_{Q_2}\Big(\int_t^T |Z(s)|^2\,\mathrm{d}s \mid \mathcal{F}_t\Big)\Big]^{1/2} \\
&\quad\leqslant \mathrm{e}^{M_b T}N_c T^{3/2}[E_{Q_2}(\mathrm{e}^{\int_t^T 2\frac{Z^2(r)}{R^2(r)}\,\mathrm{d}r} \mid \mathcal{F}_t)]^{1/2}\Big[E_{Q_2}\Big(\int_t^T |Z(s)|\,\mathrm{d}s\,\Big|^2\, \mathcal{F}_t\Big)\Big]^{1/2}.
\end{aligned}$$

We therefore can obtain from (3.10) and (3.11) with $\varepsilon = 2$ the following estimate:

$$\begin{aligned}
&\Big|E_{Q_2}\Big(\int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r} Z(s)D_r c(s)\,\mathrm{d}s \mid \mathcal{F}_t\Big)\Big| \\
&\qquad\qquad \leqslant \mathrm{e}^{M_b T}N_c K T^{3/2}\Big(\frac{1}{1-2C}\Big)^{1/2} C^{1/2} \quad \text{a.s.}
\end{aligned} \tag{3.12}$$

On the other hand, from (3.11) with $\varepsilon = 1$ we have

$$\mathrm{e}^{-M_bT} n_\xi \leqslant E_{Q_2}(\mathrm{e}^{\int_t^T \psi(s)\,\mathrm{d}s} D_r\xi|\mathcal{F}_t) \leqslant \frac{\mathrm{e}^{M_bT}N_\xi}{1-C} \quad \text{a.s.} \tag{3.13}$$

and

$$\Big|E_{Q_2}\Big(\int_t^T \mathrm{e}^{\int_t^s \psi(r)\,\mathrm{d}r}[D_r a(s) + R(s)D_r b(s)]\,\mathrm{d}s \mid \mathcal{F}_t\Big)\Big| \leqslant \frac{\mathrm{e}^{M_bT}(N_a + KN_b)T}{1-C} \quad \text{a.s.} \tag{3.14}$$

By combining (3.12), (3.13), and (3.14), we can deduce the bounds for $D_rR(t)$ as follows:

$$D_rR(t) \leqslant \frac{\mathrm{e}^{M_bT}N_\xi}{1-C} + \frac{\mathrm{e}^{M_bT}(N_a+KN_b)T}{1-C} + \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{1}{1-2C}\Big)^{1/2}C^{1/2} \quad \text{a.s.}$$

and

$$D_rR(t) \geqslant \mathrm{e}^{-M_bT}n_\xi - \frac{\mathrm{e}^{M_bT}(N_a+KN_b)T}{1-C} - \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{1}{1-2C}\Big)^{1/2}C^{1/2} \quad \text{a.s.}$$

So the proof of the proposition is complete. □

It follows from (3.5) that

$$\int_0^t (D_rR(t))^2\,\mathrm{d}r \leqslant \gamma t \quad \text{a.s.},$$

where

$$\gamma := \max\Big\{\Big[\mathrm{e}^{-M_bT}n_\xi - \frac{\mathrm{e}^{M_bT}(N_a+KN_b)T}{1-C} - \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2}\Big]^2, \Big[\frac{\mathrm{e}^{M_bT}(N_\xi + (N_a+KN_b)T)}{1-C} + \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2}\Big]^2\Big\}.$$

As a product of Proposition 3.6.2 in [11], we obtain the following corollary which provides a Gaussian upper bound for tail probabilities of $R(t)$.

Corollary 3.1. *Let assumptions* (H_1)–(H_4) *and condition* (3.4) *hold. Then, for each* $t \in (0,T)$, *we have*

$$P(R(t) \geqslant x) \leqslant \exp\Big(-\frac{(x-ER(t))^2}{2\gamma t}\Big), \quad x > ER(t),$$
$$P(R(t) \leqslant x) \leqslant \exp\Big(-\frac{(x-ER(t))^2}{2\gamma t}\Big), \quad x < ER(t).$$

We now are in a position to formulate and prove the main result of this paper.

Theorem 3.1. *Let assumptions* (H_1)–(H_4) *and condition* (3.4) *hold. We additionally assume that*

$$\mathrm{e}^{-M_bT}n_\xi - \frac{\mathrm{e}^{M_bT}(N_a+KN_b)T}{1-C} - \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2} > 0. \tag{3.15}$$

Then, for each $t\in(0,T]$, *the density* $\varrho_{R(t)}$ *of the random variable* $R(t)$ *exists and satisfies the bounds, for almost all* $x\in\mathbb{R}$,

$$\begin{aligned}\frac{E|R(t)-ER(t)|}{2\beta_2 t}\exp\Big(-\frac{(x-ER(t))^2}{2\beta_1 t}\Big) &\leqslant \varrho_{R(t)}(x)\\ &\leqslant \frac{E|R(t)-ER(t)|}{2\beta_1 t}\exp\Big(-\frac{(x-ER(t))^2}{2\beta_2 t}\Big),\end{aligned} \tag{3.16}$$

where the constants β_1, β_2 *are defined as follows*

$$\begin{aligned}\beta_1 &:= \Big[\mathrm{e}^{-M_bT}n_\xi - \frac{\mathrm{e}^{M_bT}(N_a+KN_b)T}{1-C} - \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2}\Big]^2,\\ \beta_2 &:= \Big[\frac{\mathrm{e}^{M_bT}(N_\xi+(N_a+KN_b)T)}{1-C} + \mathrm{e}^{M_bT}N_cKT^{3/2}\Big(\frac{C}{1-2C}\Big)^{1/2}\Big]^2.\end{aligned}$$

Proof. For $t\in(0,T]$, we set $F=R(t)-ER(t)$. Obviously, the random variable F has zero mean and is Malliavin differentiable with $D_rF=D_rR(t)$ for all $0\leqslant r\leqslant t\leqslant T$.

We obtain from (3.5) and (3.15) that

$$\beta_1 t \leqslant \int_0^\infty D_rF\,E[D_rF\mid\mathcal{F}_r]\,\mathrm{d}r = \int_0^t D_rF\,E[D_rF\mid\mathcal{F}_r]\,\mathrm{d}r \leqslant \beta_2 t \quad \text{a.s.}$$

Hence, in view of Proposition 2.1, the density $\varrho_F(x)$ of random variable F exists and satifies

$$\frac{E|R(t)-ER(t)|}{2\beta_2 t}\exp\Big(-\frac{x^2}{2\beta_1 t}\Big) \leqslant \varrho_F(x) \leqslant \frac{E|R(t)-ER(t)|}{2\beta_1 t}\exp\Big(-\frac{x^2}{2\beta_2 t}\Big), \quad x\in\mathbb{R}.$$

So we can finish the proof of the theorem because $\varrho_{R(t)}(x)=\varrho_F(x-ER(t))$. □

We end this paper with a remark on the class of stochastic Riccati equations with small coefficients.

Remark 3.2. For $\varepsilon\geqslant 0$, let us consider the equation

$$\begin{cases} R(t)=\xi+\displaystyle\int_t^T\Big(\varepsilon a(s)+\varepsilon b(s)R(s)+\varepsilon c(s)Z(s)-\frac{Z^2(s)}{R(s)}\Big)\,\mathrm{d}s-\int_t^T Z(s)\,\mathrm{d}B(s)\\ R(t)>0,\ t\in[0,T],\end{cases} \tag{3.17}$$

where the terminal condition ξ and coefficients a, b, c fulfil assumptions (H_1)–(H_4). When ε tends to zero, we have

$$k \to m_\xi, \quad K \to M_\xi, \quad C \to \frac{2}{3} \ln \frac{M_\xi}{m_\xi},$$

and the left-hand side of (3.15) converges to n_ξ. Thus, when ε is sufficiently small, the conditions of Theorem 3.1 are satisfied if (H_1)–(H_4) hold and $\ln \frac{M_\xi}{m_\xi} < \frac{3}{4}$.

4. Conclusion

In this paper, based on the recent advances in the theory of density estimates for a Malliavin differentiable random variable, we obtained a set of sufficient conditions that ensures that the density of the solution to stochastic Riccati equations is bounded by Gaussian densities. The results can be useful for studying the numerical approximations of the solution. We therefore partly enrich the knowledge of the theory of stochastic Riccati equations.

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