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LOWER BOUNDS FOR THE LARGEST EIGENVALUE OF THE GCD MATRIX ON $\{1, 2, \ldots, n\}$

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Dedicated to the memory of Miroslav Fiedler

Abstract. Consider the $n \times n$ matrix with (i, j)'th entry $\gcd(i, j)$. Its largest eigenvalue λ_n and sum of entries s_n satisfy $\lambda_n > s_n/n$. Because s_n cannot be expressed algebraically as a function of n, we underestimate it in several ways. In examples, we compare the bounds so obtained with one another and with a bound from S. Hong, R. Loewy (2004). We also conjecture that $\lambda_n > 6\pi^{-2} n \log n$ for all n. If n is large enough, this follows from F. Balatoni (1969).

Keywords: eigenvalue bounds; greatest common divisor matrix

MSC 2010: 15A42, 15B36, 11A05

1. Introduction

Given $n > 1$, let $\mathbf{A}_n = (a_{ij})$ be the greatest common divisor (gcd) matrix on $\{1, 2, \ldots, n\}$, that is, $a_{ij} = \gcd(i, j)$, $i, j = 1, 2, \ldots, n$. Let λ_n be its largest eigenvalue and s_n the sum of its entries. Denote by $\mathbf{e}_n$ the n-vector with each entry one. Applying the Rayleigh quotient and noting that $\mathbf{e}_n$ is not an eigenvector corresponding to λ_n, we have

$$\lambda_n > \frac{\mathbf{e}_n^T \mathbf{A}_n \mathbf{e}_n}{\mathbf{e}_n^T \mathbf{e}_n} = \frac{s_n}{n} =: l_n, \tag{1}$$

see [5], Theorem 4.2.2. The lower bound for the largest eigenvalue of a Hermitian matrix, obtained in this way, is often quite good if the matrix is positive definite and (entrywise) positive. Because $\mathbf{A}_n$ is positive definite, see [3], Theorem 2, we are motivated to a closer look at l_n.

The study of gcd matrices traces back to Smith in [7] but did not attract much attention until recent decades. Hong and Loewy in [4] may be regarded as initiators of studying eigenstructures of gcd and related matrices. For a brief historical survey on this topic with references, see Altınışık et al. [1].

Because s_n cannot be expressed algebraically as a function of n, we underestimate it; then we are actually studying lower bounds for l_n. The simplest way is to replace all off-diagonal entries of $\mathbf{A}_n$ by 1; let $\mathbf{B}_n = (b_{ij})$ be the matrix so obtained. Since the sum of its entries is

$$\frac{n(n+1)}{2} + n(n-1) = \frac{3n^2 - n}{2} =: t_n,$$

we have

$$\lambda_n > \frac{t_n}{n} = \frac{3n-1}{2} =: u_n.$$

Our task is to find for λ_n better bounds than u_n. Because we are interested also in asymptotic bounds, we will first (Section 2) take a look at the asymptotics of λ_n and l_n. Thereafter (Sections 3–7) we will improve u_n. We will take a suitable nonzero and (entrywise) nonnegative matrix $\mathbf{E}_n = (e_{ij})$ with the following properties:

(i) Its all diagonal entries are zero.

(ii) Its all off-diagonal entries satisfy $b_{ij} + e_{ij} \leqslant a_{ij}$.

(iii) The sum of its entries, denoted by τ_n, is easy to calculate.

Then

$$s_n \geqslant t_n + \tau_n > t_n,$$

which implies, by (1),

$$\lambda_n > u_n + \frac{\tau_n}{n} > u_n. \tag{2}$$

Different choices of $\mathbf{E}_n$ give different improvements. We will finally in examples compare our bounds with one another (Section 8) and with a bound of Hong and Loewy in [4] (Section 9). Concluding remarks (Section 10) complete our paper.

2. Asymptotics of λ_n and l_n

It is well-known, see [8], equation (25), that

$$s_n = \frac{6}{\pi^2} n^2 \log n + O(n^2),$$

so

$$l_n = \frac{6}{\pi^2} n \log n + O(n).$$

Experiments make us conjecture that

$$\lambda_n > \frac{6}{\pi^2} n \log n =: v_n. \tag{3}$$

It is also well-known, see [2], Theorem, that

$$\lambda_n = O(n^{1+\varepsilon}) \tag{4}$$

for all $\varepsilon > 0$ but

$$\lambda_n \neq O(n(\log n)^k) \tag{5}$$

for all $k \geqslant 1$. Therefore (3) is true if n is large enough. In fact, v_n is then a very poor bound, because

$$\lim_{n\to\infty} \frac{v_n}{\lambda_n} = 0$$

by (4) and (5).

3. First attempt: $e_{ij} = 1$ if $i \neq j$ and $a_{ij} \geqslant 2$

We obtained the bound u_n by replacing all off-diagonal entries of $\mathbf{A}_n$ by one. To improve it, we replace by two all of them that are at least two. In other words, we define $\mathbf{E}_n$ by setting $e_{ij} = 1$ if $i \neq j$ and $a_{ij} \geqslant 2$, and $e_{ij} = 0$ otherwise. The number of ones before the diagonal is $i - 1 - \varphi(i)$, where $i > 1$ and φ is the Euler totient function. Hence

$$\tau_n = 2\sum_{i=2}^{n}(i - 1 - \varphi(i)) = n^2 - n + 2(1 - \Phi(n)),$$

where

$$\Phi(n) = \sum_{i=1}^{n} \varphi(i).$$

By (2),

$$\lambda_n > \frac{3n-1}{2} + n - 1 + 2\frac{1-\Phi(n)}{n} = \frac{5n-3}{2} + 2\frac{1-\Phi(n)}{n} =: w_n.$$

Asymptotically, see [6], Section I.21,

$$\Phi(n) = \frac{3}{\pi^2} n^2 + O(n^\delta)$$

for some δ with $1 < \delta < 2$; hence

$$w_n = \left(\frac{5}{2} - \frac{6}{\pi^2}\right)n + O(n^\delta)$$

for some δ with $0 < \delta < 1$.

4. Second attempt: Restrict i and j even

To find a (weaker) bound without $\Phi(n)$, we restrict i and j to be even. So we set $e_{ij} = 1$ if i and j are different and even, and $e_{ij} = 0$ otherwise. Then

$$\tau_n = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right).$$

By (2),

$$\lambda_n > \frac{3n-1}{2} + \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) =: x_n.$$

If n is even, then

$$x_n = \frac{3n-1}{2} + \frac{1}{2}\left(\frac{n}{2} - 1\right) = \frac{7n}{4} - 1.$$

If n is odd, then

$$x_n = \frac{3n-1}{2} + \frac{n-1}{2n}\left(\frac{n-1}{2} - 1\right) = \frac{7n}{4} - \frac{3}{2} + \frac{3}{4n}.$$

Asymptotically

$$x_n = \frac{7n}{4} + O(1).$$

5. Third attempt: Change $e_{ij} = 2$ if $i \neq j$ and $3 \mid i, j$

If i and j are multiples of three and $i \neq j$, then $a_{ij} \geqslant 3$ but $b_{ij} = 1$. The number of such pairs (i, j) is

$$\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) =: \alpha_n.$$

"Old $\mathbf{E}_n$" (i.e., $\mathbf{E}_n$ constructed in the previous section) has then either $e_{ij} = 0$ or $e_{ij} = 1$. We change all these entries into two. Call "new $\mathbf{E}_n$" the matrix effecting so.

If $i \neq j$ and $6 \mid i, j$, then old $e_{ij} = 1$. The number of such pairs (i, j) is

$$\left\lfloor \frac{n}{6} \right\rfloor \left(\left\lfloor \frac{n}{6} \right\rfloor - 1 \right) =: \beta_n.$$

If $i \neq j$ and $3 \mid i,j$ but not $6 \mid i,j$, then old $e_{ij} = 0$. The number of such pairs is $\alpha_n - \beta_n$. Therefore we obtain "new τ_n" by adding

$$2(\alpha_n - \beta_n) + \beta_n = 2\alpha_n - \beta_n$$

to "old τ_n". Hence, by (2),

$$\lambda_n > \frac{3n-1}{2} + \frac{1}{n}\Big[\Big\lfloor\frac{n}{2}\Big\rfloor\Big(\Big\lfloor\frac{n}{2}\Big\rfloor - 1\Big) + 2\Big\lfloor\frac{n}{3}\Big\rfloor\Big(\Big\lfloor\frac{n}{3}\Big\rfloor - 1\Big) - \Big\lfloor\frac{n}{6}\Big\rfloor\Big(\Big\lfloor\frac{n}{6}\Big\rfloor - 1\Big)\Big] =: x'_n.$$

The polynomial expression of x'_n depends on the remainder

$$r = n - 6\Big\lfloor\frac{n}{6}\Big\rfloor.$$

If $r = 0$, then $\lfloor\frac{1}{2}n\rfloor = \frac{1}{2}n$, $\lfloor\frac{1}{3}n\rfloor = \frac{1}{3}n$, $\lfloor\frac{1}{6}n\rfloor = \frac{1}{6}n$; so

$$x'_n = \frac{3n-1}{2} + \frac{1}{n}\Big[\frac{n}{2}\Big(\frac{n}{2} - 1\Big) + 2\frac{n}{3}\Big(\frac{n}{3} - 1\Big) - \frac{n}{6}\Big(\frac{n}{6} - 1\Big)\Big] = \frac{35n}{18} - \frac{3}{2}.$$

If $r = 1$, then $\lfloor\frac{1}{2}n\rfloor = \frac{1}{2}(n-1)$, $\lfloor\frac{1}{3}n\rfloor = \frac{1}{3}(n-1)$, $\lfloor\frac{1}{6}n\rfloor = \frac{1}{6}(n-1)$; so

$$\begin{aligned} x'_n &= \frac{3n-1}{2} + \frac{1}{n}\Big[\frac{n-1}{2}\Big(\frac{n-1}{2} - 1\Big) + 2\frac{n-1}{3}\Big(\frac{n-1}{3} - 1\Big) - \frac{n-1}{6}\Big(\frac{n-1}{6} - 1\Big)\Big] \\ &= \frac{35n}{18} - \frac{43}{18} + \frac{13}{9n}. \end{aligned}$$

We continue similarly. If $r = 2$, then

$$x'_n = \frac{35n}{18} - \frac{41}{18} + \frac{16}{9n}.$$

If $r = 3$, then

$$x'_n = \frac{35n}{18} - \frac{11}{6}.$$

If $r = 4$, then

$$x'_n = \frac{35n}{18} - \frac{31}{18} - \frac{2}{9n}.$$

If $r = 5$, then

$$x'_n = \frac{35n}{18} - \frac{47}{18} + \frac{13}{9n}.$$

This procedure can be pursued further. The next step is to change $e_{ij} = 3$ if i and j are multiples of four and $i \neq j$. But we stop here, because the calculations become complicated.

6. Fourth attempt: $e_{i,ki} = e_{ki,i} = i - 1$

Denote $n_i = \lfloor n/i \rfloor$. The entries

$$a_{i,2i} = a_{i,3i} = \ldots = a_{i,n_i i} = i,$$
$$a_{2i,i} = a_{3i,i} = \ldots = a_{n_i i,i} = i, \quad i = 2, 3, \ldots, n_2,$$

are greater than one, but the corresponding entries are $b_{ij} = 1$. In order to give them their original values, we define $\mathbf{E}_n$ by

$$e_{i,2i} = e_{i,3i} = \ldots = e_{i,n_i i} = i - 1,$$
$$e_{2i,i} = e_{3i,i} = \ldots = e_{n_i i,i} = i - 1, \quad i = 2, 3, \ldots, n_2,$$

and $e_{ij} = 0$ otherwise. Then

$$\begin{aligned}\tau_n &= \sum_{i=2}^{n_2} 2 \sum_{k=2}^{n_i} e_{i,ki} = \sum_{i=2}^{n_2} 2(n_i - 1)(i - 1) \\ &= 2[(n_2 - 1) + (n_3 - 1) \cdot 2 + (n_4 - 1) \cdot 3 + \ldots + (n_{n_2 - 1} - 1)(n_2 - 2) + 1 \cdot (n_2 - 1)] \\ &= 2\{[1 + \ldots + (n_2 - 1)] + [1 + \ldots + (n_3 - 1)] + \ldots + [1 + \ldots + (n_{n_2-1} - 1)] + 1\} \\ &= 2 \sum_{k=2}^{n_2} [1 + 2 + \ldots + (n_k - 1)] = \sum_{k=2}^{n_2} n_k(n_k - 1),\end{aligned}$$

which is tedious to compute. So we underestimate it.

Because

$$n_k > \frac{n}{k} - 1,$$

we have

$$\begin{aligned}\tau_n > \sum_{k=2}^{n_2} \Big(\frac{n}{k} - 1\Big)\Big(\frac{n}{k} - 2\Big) &= \sum_{k=2}^{n_2} \Big(\frac{n^2}{k^2} - 3\frac{n}{k} + 2\Big) \\ &= n^2 \sum_{k=2}^{n_2} \frac{1}{k^2} - 3n \sum_{k=2}^{n_2} \frac{1}{k} + 2(n_2 - 1).\end{aligned}$$

Hence, by (2),

$$\lambda_n > \frac{3n - 1}{2} + n \sum_{k=2}^{n_2} \frac{1}{k^2} - 3 \sum_{k=2}^{n_2} \frac{1}{k} + \frac{2(n_2 - 1)}{n} =: y_n. \tag{6}$$

If n is even, then

$$y_n = \frac{3n-1}{2} + n\sum_{k=2}^{n/2}\frac{1}{k^2} - 3\sum_{k=2}^{n/2}\frac{1}{k} + \frac{2(\frac{1}{2}n-1)}{n}$$
$$= \frac{3n}{2} + n\sum_{k=2}^{n/2}\frac{1}{k^2} - 3\sum_{k=2}^{n/2}\frac{1}{k} + \frac{1}{2} - \frac{2}{n}.$$

If n is odd, then

$$y_n = \frac{3n-1}{2} + n\sum_{k=2}^{(n-1)/2}\frac{1}{k^2} - 3\sum_{k=2}^{(n-1)/2}\frac{1}{k} + \frac{2(\frac{1}{2}(n-1)-1)}{n}$$
$$= \frac{3n}{2} + n\sum_{k=2}^{(n-1)/2}\frac{1}{k^2} - 3\sum_{k=2}^{(n-1)/2}\frac{1}{k} + \frac{1}{2} - \frac{3}{n}.$$

Since

$$\sum_{k=1}^{n}\frac{1}{k} = O(\log n)$$

and

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6} + O\Big(\frac{1}{n}\Big), \tag{7}$$

we have asymptotically

$$y_n = \frac{3n}{2} + n\Big(\frac{\pi^2}{6} - 1 + O\Big(\frac{1}{n}\Big)\Big) + O(\log n) = \Big(\frac{\pi^2}{6} + \frac{1}{2}\Big)n + O(\log n).$$

7. Fifth attempt: Underestimate y_n

We underestimate y_n in order to find a polynomial expression. We apply the inequalities

$$\sum_{k=1}^{n}\frac{1}{k} < \log n, \qquad \sum_{k=1}^{n}\frac{1}{k^2} > \frac{2n(2n-1)}{(2n+1)^2}\frac{\pi^2}{6}.$$

The first inequality is easy to show. The second is from Wikipedia, where it is shown in order to prove (7). A reference to Yaglom and Yaglom [9] is given there. Now

$$n\sum_{k=2}^{n_2}\frac{1}{k^2} - 3\sum_{k=2}^{n_2}\frac{1}{k} > n\Big[\frac{2n_2(2n_2-1)}{(2n_2+1)^2}\frac{\pi^2}{6} - 1\Big] - 3\log n_2,$$

which implies, by (6),

$$\lambda_n > \frac{3n-1}{2} + \Big[\frac{2n_2(2n_2-1)}{(2n_2+1)^2}\frac{\pi^2}{6} - 1\Big]n - 3\log n_2 + \frac{2(n_2-1)}{n}$$
$$= \Big[\frac{2n_2(2n_2-1)}{(2n_2+1)^2}\frac{\pi^2}{6} + \frac{1}{2}\Big]n - \frac{1}{2} - 3\log n_2 + \frac{2(n_2-1)}{n} =: y'_n.$$

If n is even, then

$$y'_n = \Big[\frac{2\cdot\frac{1}{2}n(2\cdot\frac{1}{2}n-1)}{(2\cdot\frac{1}{2}n+1)^2}\frac{\pi^2}{6} + \frac{1}{2}\Big]n - \frac{1}{2} - 3\log\frac{n}{2} + \frac{2(\frac{1}{2}n-1)}{n}$$
$$= \frac{n^2(n-1)}{(n+1)^2}\frac{\pi^2}{6} + \frac{n+1}{2} - 3\log\frac{n}{2} - \frac{2}{n}.$$

If n is odd, then

$$y'_n = \Big[\frac{2\cdot\frac{1}{2}(n-1)(2\cdot\frac{1}{2}(n-1)-1)}{(2\cdot\frac{1}{2}(n-1)+1)^2}\frac{\pi^2}{6} + \frac{1}{2}\Big]n - \frac{1}{2} - 3\log\frac{n-1}{2} + \frac{2(\frac{1}{2}(n-1)-1)}{n}$$
$$= \frac{(n-1)(n-2)}{n}\frac{\pi^2}{6} + \frac{n+1}{2} - 3\log\frac{n-1}{2} - \frac{3}{n}.$$

Asymptotically

$$y'_n = \Big(\frac{\pi^2}{6} + \frac{1}{2}\Big)n + O(\log n).$$

8. Examples

In the asymptotic expression of all our bounds (excluding the conjectured bound v_n), the main term is of the form cn. The coefficient c (with four digits precision) is

for u_n: $c = \frac{3}{2} = 1.5$,
for x_n: $c = \frac{7}{4} = 1.75$,
for w_n: $c = \frac{5}{2} - 6/\pi^2 = 1.892$,
for x'_n: $c = \frac{35}{18} = 1.944$,
for y'_n, y_n: $c = \frac{1}{6}\pi^2 + \frac{1}{2} = 2.145$.

Therefore, and since $v_n = O(n\log n)$ by definition, we have

$$u_n < x_n < w_n < x'_n < y'_n < y_n < v_n \tag{8}$$

when n is large.

Example 1. $n = 3$, $\lambda_3 = 4.214$, $l_3 = u_3 = 4$. Since $\mathbf{B}_3 = \mathbf{A}_3$, there is nothing to be improved.

Example 2. $n = 4$, $\lambda_4 = 6.421$, $l_4 = 6$, $u_4 = 5.5$. In all our procedures, $\mathbf{B}_4 + \mathbf{E}_4 = \mathbf{A}_4$. So $w_4 = x_4 = x'_4 = 6 = l_4$, but $y_4 = 5.5 = u_4$. The benefit obtained in changing $\mathbf{B}_4$ is then lost in computing y_4. The bound $y'_4 = 3.079$. The conjectured bound $v_4 = 3.371$.

Example 3. $n = 5$, $\lambda_5 = 7.770$, $l_5 = 7.4$, $u_5 = 7$. Again all procedures work completely; so $w_5 = x_5 = x'_5 = 7.4 = l_5$. The bound $y_5 = 7.15$ is better than u_5. The gain in changing $\mathbf{B}_5$ is thus larger than the loss in computing y_5. The bound $y'_5 = 4.268$. The conjectured bound $v_4 = 4.892$.

Example 4. $n = 6$, $\lambda_6 = 11.05$, $l_6 = 10.17$, $u_6 = 8.5$. The bound $w_6 = 9.833$. The procedure of Section 5 yields $\mathbf{B}_6 + \mathbf{E}_6 = \mathbf{A}_6$, but that in Section 4 does not. We have $x_6 = 9.5$ and $x'_6 = 10.17 = l_6$. The bound $y_6 = 8.833$ is better than u_6. The bound $y'_6 = 5.913$. The conjectured bound $v_6 = 6.536$.

Example 5. $n = 20$, $\lambda_{20} = 49.62$, $l_{20} = 44$, $u_{20} = 29.5$. In the previous examples, the bound y'_n and the conjectured bound v_n are the poorest, but they improve when n increases. The bound $y_{20} = 35.61$ is better than $x_{20} = 34$ but worse than $x'_{20} = 36.71$. The bound $y'_{20} = 31.84$ is better than u_{20} but worse than x_{20}. The bound $w_{20} = 35.8$. The conjectured bound $v_{20} = 36.42$.

Example 6. $n = 50$, $\lambda_{50} = 156.73$, $l_{50} = 134.5$, $u_{50} = 74.5$. We have $x_{50} = 86.5$ and $x'_{50} = 94.98$. The bound $y_{50} = 97.30$ is better than x'_{50}. The bound $y'_{50} = 93.28$ is better than x_{50} but worse than x'_{50}. The bound $w_{50} = 92.58$. The conjectured bound $v_{50} = 118.91$. The ordering

$$u_{50} < x_{50} < w_{50} < y'_{50} < x'_{50} < y_{50} < v_{50}$$

is almost the same as the asymptotic ordering (8). Only y'_{50} and x'_{50} are reversed.

Example 7. $n = 150$, $\lambda_{150} = 617.0$, $l_{150} = 498.3$, $u_{150} = 224.5$. Now $x_{150} = 261.5$, $w_{150} = 282.1$, $x'_{150} = 290.2$, $y'_{150} = 304.4$, $y_{150} = 308.5$, $v_{150} = 456.9$ are in the asymptotic ordering.

9. Comparison with a bound of Hong and Loewy

Hong and Loewy proved as a special case of [4], Theorem 4.7 (ii), that

$$\lambda_n \geqslant \frac{n\mathrm{e}^{-\gamma}}{\log n}\Bigl(1 - \frac{c}{\log n}\Bigr),$$

where γ is Euler's constant and c is a certain positive number. Since c is unknown and cannot easily be overestimated, this bound is useless in comparison.

These authors actually studied power gcd matrices. So let $\mathbf{A}_n^{(p)}$ denote the entrywise p'th power of $\mathbf{A}_n$ with largest eigenvalue μ_n. A special case of [4], Theorem 4.7 (i), states that if $p > 1$, then

$$\mu_n \geqslant \frac{n^p}{\zeta(p)} =: h_n,$$

where ζ is the Riemann zeta function. We use this bound in comparison in two ways.

First, because $\mathbf{A}_n^{(p)} \geqslant \mathbf{A}_n$ (entrywise), we have

$$\mu_n \geqslant \lambda_n,$$

see [5], Theorem 8.1.18. Hence our bounds apply also to μ_n but are poor unless p is near to one. On the other hand, if $p \to 1$, then $\zeta(p) \to \infty$ and so $h_n \to 0$. Therefore h_n is poor if p is near to 1, which favors our bounds unless n is very large.

Second, applying to $\mathbf{A}^{(p)}$ the procedures described in Sections 1 and 3, we obtain

$$\mu_n > \frac{1}{n}\sum_{k=1}^{n} k^p + n - 1 =: \widetilde{u}_n,$$

$$\mu_n > \frac{1}{n}\sum_{k=1}^{n} k^p + n - 1 + (2^p - 1)\Bigl(n - 1 + 2\frac{1 - \Phi(n)}{n}\Bigr) =: \widetilde{w}_n.$$

If p is an integer, the power sum can be expressed polynomially by using Faulhaber's formula in [10].

We compare our bounds with h_n for $p = 2, 1.5, 1.1$. If p is not an integer and n is not small, the bounds $\widetilde{u}_n$ and $\widetilde{w}_n$ are tedious to compute with a non-programmable calculator. Therefore we consider these bounds only in case of $p = 2$. We denote by f_n and g_n the best and, respectively, the worst of the bounds presented in Sections 1 and 3–7.

Example 8. $p = 2$, $\mu_4 = 17.514$, $\mu_5 = 25.37$, $\mu_6 = 40.30$. The bound $\widetilde{u}_4 = 10.5$ is better than $h_4 = 9.727$, but $h_5 = 15.20$ is better than $\widetilde{u}_5 = 15$. The bound $\widetilde{w}_5 = 15.4$ is better than h_5, but $h_6 = 21.89$ is better than $\widetilde{w}_6 = 21.50$. The bound h_n is better than our bounds if $n \geqslant 6$, and remarkably better if n is large.

Example 9. $p = 1.5$, $\mu_6 = 19.36$, $\mu_{20} = 125.65$, $\mu_{150} = 3050.2$. Again our bounds are better for small n. For example, $g_6 = y'_6 = 5.913$ is better than $h_6 = 5.626$. As n increases, h_n begins to do better, but the range of n where our bounds succeed is wider than in Example 8. The bound $h_{20} = 34.24$, for example, beats $g_{20} = u_{20} = 29.50$ but loses to $f_{20} = x'_{20} = 36.71$. Again h_n is remarkably better if n is large.

Example 10. $p = 1.1$, $\mu_4 = 6.918$, $\mu_{20} = 58.09$, $\mu_{150} = 810.63$. Now our bounds are better for all matrices of reasonable size. For example, $g_4 = y'_4 = 3.079$, $h_4 = 0.434$, $g_{150} = u_{150} = 224.5$, $h_{150} = 23.39$. Even for $n = 1.01 \cdot 10^{12}$ the bound $g_n = u_n = 1.5150 \cdot 10^{12}$ is better than $h_n = 1.5139 \cdot 10^{12}$, but for $n = 1.02 \cdot 10^{12}$ the ordering changes: $g_n = u_n = 1.5300 \cdot 10^{12}$, $h_n = 1.5304 \cdot 10^{12}$.

10. Conclusions and remarks

We expected that $l_n = s_n/n$ is a quite good lower bound for λ_n. By underestimating s_n, we found several easily computable bounds. We compared them with one another and studied their asymptotical behavior. We also noted that $\lambda_n > v_n$ if n is large, and conjectured this for all n. The examples suggest a stronger conjecture that actually $l_n > v_n$. We also compared our bounds with a bound of Hong and Loewy. For this purpose, we extended u_n and w_n to concern the largest eigenvalue of $\mathbf{A}_n^{(p)}$, $p > 1$.

By using the vector $\mathbf{A}_n\mathbf{e}_n$ instead of $\mathbf{e}_n$ in the Rayleigh quotient, we obtain

$$\lambda_n > \frac{(\mathbf{A}_n\mathbf{e})^T\mathbf{A}_n(\mathbf{A}_n\mathbf{e}_n)}{\mathbf{e}_n^T\mathbf{e}_n} = \frac{\operatorname{su}\mathbf{A}_n^3}{\operatorname{su}\mathbf{A}_n^2},$$

where su denotes the sum of entries. This bound is better than l_n but seems difficult to be underestimated for our purpose.

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