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A SHARP UPPER BOUND FOR THE SPECTRAL RADIUS OF A NONNEGATIVE MATRIX AND APPLICATIONS

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Dedicated to the memory of Miroslav Fiedler

Abstract. We obtain a sharp upper bound for the spectral radius of a nonnegative matrix. This result is used to present upper bounds for the adjacency spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance Laplacian spectral radius, the distance signless Laplacian spectral radius of an undirected graph or a digraph. These results are new or generalize some known results.

Keywords: nonnegative matrix; spectral radius; graph; digraph

MSC 2010: 05C50, 15A18

1. Introduction

We begin by recalling some definitions. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all eigenvalues of an $n \times n$ matrix M. It is obvious that the eigenvalues may be complex numbers since M is not symmetric in general. We usually assume that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. The spectral radius of M is defined as $\varrho(M) = |\lambda_1|$, i.e., it is the largest modulus of the eigenvalues of M. If M is a nonnegative matrix, it follows from the Perron-Frobenius theorem that the spectral radius $\varrho(M)$ is an eigenvalue of M. If M is a nonnegative

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irreducible matrix, it follows from the Perron-Frobenius theorem that $\varrho(M) = \lambda_1$ is simple.

Let $G = (V, E)$ be a simple undirected graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The Laplacian matrix and the signless Laplacian matrix of G are defined as

$$L(G) = \mathrm{diag}(G) - A(G), \quad Q(G) = \mathrm{diag}(G) + A(G),$$

respectively, where $A(G) = (a_{ij})$ is the adjacency matrix of G, $\mathrm{diag}(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees of G and d_i is the degree of the vertex v_i. The spectral radii of $A(G)$, $L(G)$ and $Q(G)$, denoted by $\varrho(G)$, $\mu(G)$ and $q(G)$, are called the (adjacency) spectral radius of G, the Laplacian spectral radius of G, and the signless Laplacian spectral radius of G, respectively. In 1973, Fiedler in [10] studied the Laplacian spectra, in particular, the second smallest eigenvalue which is called algebra connectivity. Since then, the Laplacian matrix has been extensively investigated. Further, Fiedler in [9] gave an excellent survey for the Laplacian matrix.

Let $G = (V, E)$ be a connected undirected graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. For $u, v \in V(G)$, the distance between u and v, denoted by $d_G(u, v)$, is the length of the shortest path connecting them in G. The distance matrix of G is the $n \times n$ matrix $\mathcal{D}(G) = (d_{ij})$ where $d_{ij} = d_G(v_i, v_j)$. In fact, for $1 \leqslant i \leqslant n$, the transmission of vertex v_i, $\mathrm{Tr}_G(v_i)$ is just the i-th row sum of $\mathcal{D}(G)$. So for convenience, we also call $\mathrm{Tr}_G(v_i)$ the distance degree of vertex v_i in G, denoted by D_i, that is, $D_i = \sum\limits_{j=1}^{n} d_{ij} = \mathrm{Tr}_G(v_i)$.

Let $\mathrm{Tr}(G) = \mathrm{diag}(D_1, D_2, \ldots, D_n)$ be the diagonal matrix of vertex transmissions of G. The distance Laplacian matrix and the distance signless Laplacian matrix of G are the $n \times n$ matrices defined by Aouchiche and Hansen in [1] as

$$\mathcal{L}(G) = \mathrm{Tr}(G) - \mathcal{D}(G), \quad \mathcal{Q}(G) = \mathrm{Tr}(G) + \mathcal{D}(G).$$

The spectral radius of $\mathcal{D}(G)$, $\mathcal{L}(G)$ and $\mathcal{Q}(G)$, denoted by $\varrho^{\mathcal{D}}(G)$, $\mu^{\mathcal{D}}(G)$ and $q^{\mathcal{D}}(G)$, are called the distance spectral radius of G, the distance Laplacian spectral radius of G, and the distance signless Laplacian spectral radius of G, respectively.

Let $\vec{G} = (V, E)$ be a digraph, where $V = V(\vec{G}) = \{v_1, v_2, \ldots, v_n\}$ and $E = E(\vec{G})$ are the vertex set and arc set of $\vec{G}$, respectively. A digraph $\vec{G}$ is simple if it has no loops and multiple arcs. A digraph $\vec{G}$ is strongly connected if for every pair of vertices $v_i, v_j \in V(\vec{G})$, there are directed paths from v_i to v_j and from v_j to v_i. In this paper, we consider finite, simple digraphs.

Let $\vec{G}$ be a digraph. Denote by $N^+_{\vec{G}}(v_i) = \{v_j \in V(\vec{G})\colon\ (v_i, v_j) \in E(\vec{G})\}$ the set of the out-neighbors of v_i and by $d_i^+ = |N^+_{\vec{G}}(v_i)|$ the out-degree of the vertex v_i in $\vec{G}$.

For a digraph $\vec{G}$, let $A(\vec{G}) = (a_{ij})$ denote the adjacency matrix of $\vec{G}$, where a_{ij} is equal to the number of arcs (v_i, v_j). Let $\operatorname{diag}(\vec{G}) = \operatorname{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix of vertex out-degrees of $\vec{G}$ and

$$L(\vec{G}) = \operatorname{diag}(\vec{G}) - A(\vec{G}), \quad Q(\vec{G}) = \operatorname{diag}(\vec{G}) + A(\vec{G})$$

the Laplacian matrix of $\vec{G}$ and the signless Laplacian matrix of $\vec{G}$, respectively. The spectral radii of $A(\vec{G})$, $L(\vec{G})$ and $Q(\vec{G})$, denoted by $\varrho(\vec{G})$, $\mu(\vec{G})$ and $q(\vec{G})$, are called the (adjacency) spectral radius of $\vec{G}$, the Laplacian spectral radius of $\vec{G}$, and the signless Laplacian spectral radius of $\vec{G}$, respectively.

For $u, v \in V(\vec{G})$, the distance from u to v, denoted by $d_{\vec{G}}(u, v)$, is the length of the shortest directed path from u to v in $\vec{G}$. For $u \in V(\vec{G})$, the transmission of a vertex u in $\vec{G}$ is the sum of distances from u to all other vertices of $\vec{G}$, denoted by $\operatorname{Tr}_{\vec{G}}(u)$.

Let $\vec{G}$ be a strong connected digraph with vertex set $V(\vec{G}) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix of $\vec{G}$ is the $n \times n$ matrix $\mathcal{D}(\vec{G}) = (d_{ij})$, where $d_{ij} = d_{\vec{G}}(v_i, v_j)$. In fact, for $1 \leqslant i \leqslant n$, the transmission of the vertex v_i, $\operatorname{Tr}_{\vec{G}}(v_i)$ is just the i-th row sum of $\mathcal{D}(\vec{G})$. So for convenience, we also call $\operatorname{Tr}_{\vec{G}}(v_i)$ the distance degree of the vertex v_i in $\vec{G}$, denoted by D_i^+, that is, $D_i^+ = \sum\limits_{j=1}^{n} d_{ij} = \operatorname{Tr}_{\vec{G}}(v_i)$.

Let $\operatorname{Tr}(\vec{G}) = \operatorname{diag}(D_1^+, D_2^+, \ldots, D_n^+)$ be the diagonal matrix of vertex transmissions of $\vec{G}$. The distance Laplacian matrix and the distance signless Laplacian matrix of $\vec{G}$ are the $n \times n$ matrices defined similarly to the undirected graph by Aouchiche and Hansen in [1] as

$$\mathcal{L}(\vec{G}) = \operatorname{Tr}(\vec{G}) - \mathcal{D}(\vec{G}), \quad \mathcal{Q}(\vec{G}) = \operatorname{Tr}(\vec{G}) + \mathcal{D}(\vec{G}).$$

The spectral radii of $\mathcal{D}(\vec{G})$, $\mathcal{L}(\vec{G})$ and $\mathcal{Q}(\vec{G})$, denoted by $\varrho^{\mathcal{D}}(\vec{G})$, $\mu^{\mathcal{D}}(\vec{G})$ and $q^{\mathcal{D}}(\vec{G})$, are called the distance spectral radius of $\vec{G}$, the distance Laplacian spectral radius of $\vec{G}$ and the distance signless Laplacian spectral radius of $\vec{G}$, respectively.

Let $G = (V, E)$ be an undirected graph. For $v_i, v_j \in V$, if v_i is adjacent to v_j, we denote it by $i \sim j$. Moreover, we call $m_i = d_i^{-1} \sum\limits_{i \sim j} d_j$ the average degree of the neighbors of v_i. In addition, let $\vec{G} = (V, E)$ be a digraph. For $v_i, v_j \in V$, if arc $(v_i, v_j) \in E$, we denote it by $i \sim j$. Moreover, we call $m_i^+ = {d_i^+}^{-1} \sum\limits_{i \sim j} d_j^+$ the average out-degree of the out-neighbors of v_i, where d_i^+ is the out-degree of the vertex v_i in $\vec{G}$.

A regular graph is a graph where every vertex has the same degree. A *bipartite semi-regular* graph is a bipartite graph $G = (U, V, E)$ for which every two vertices on the same side of the given bipartition have the same degree as each other.

So far, there are many results on the bounds of the spectral radius of a matrix and a nonnegative matrix, the spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance Laplacian spectral radius and the distance signless Laplacian spectral radius of an undirected graph and a digraph, see [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [16], [18], [21].

The followings are some results on the above spectral radii of undirected graphs and digraphs in terms of degree, average degree, out-degree and so on.

$$\varrho(G) \leqslant \max_{1\leqslant i\leqslant n} \{\sqrt{d_i m_i}\}, \qquad \text{see [4]}, \tag{1.1}$$

$$\mu(G) \leqslant \max_{1\leqslant i\leqslant n} \{d_i + \sqrt{d_i m_i}\}, \qquad \text{see [19]}, \tag{1.2}$$

$$q(G) \leqslant \max_{1\leqslant i\leqslant n} \{d_i + \sqrt{d_i m_i}\}, \qquad \text{see [17]}, \tag{1.3}$$

$$q(\vec{G}) \leqslant \max_{1\leqslant i\leqslant n} \left\{d_i^+ + \sqrt{\sum_{j\sim i} d_j^+}\right\}, \qquad \text{see [3]}. \tag{1.4}$$

It may be noticed that there are few results about the distance Laplacian spectral radius of G, the Laplacian spectral radius of $\vec{G}$, the distance Laplacian spectral radius of $\vec{G}$ and the distance signless Laplacian spectral radius of $\vec{G}$. Maybe one reason is that the Laplacian matrix and the distance Laplacian matrix are not nonnegative matrices.

In this paper, we obtain sharp upper bounds for the spectral radius of a matrix or a nonnegative matrix in Section 2, and then we apply these bounds to various matrices associated with an undirected graph or a digraph. We obtain some new results or known results about various spectral radii, including the (adjacency) spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance Laplacian spectral radius, the distance signless Laplacian spectral radius and so on.

2. Main results

In this section, we will obtain the sharp upper bound for the spectral radius of a (nonnegative) matrix. The techniques used in this section is motivated by [17].

Theorem 2.1. *Let $B = (b_{ij})$ be an $n\times n$ nonnegative matrix, l_i the number of the nonzero entries except for the diagonal entry of the i-th row for any $i \in \{1, 2, \ldots, n\}$,* *that is,* $l_i = |\{b_{ij}\colon\ b_{ij} \neq 0,\ j \in \{1,2,\ldots,n\} \setminus \{i\}\}|$, *and let* $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ *be the eigenvector of* B *corresponding to the eigenvalue* $\varrho(B)$. *Then*

$$\varrho(B) \leqslant \max_{1\leqslant i\leqslant n} \left\{ b_{ii} + \sqrt{\sum_{k=1,\, k\neq i}^{n} l_k b_{ki}^2} \right\}. \tag{2.1}$$

Moreover, if the equality in (2.1) *holds, then*

$$b_{ii} + \sqrt{\sum_{k=1,\, k\neq i}^{n} l_k b_{ki}^2} = b_{jj} + \sqrt{\sum_{k=1,\, k\neq j}^{n} l_k b_{kj}^2}$$

for any $i, j \in \{s\colon\ x_s \neq 0,\ 1 \leqslant s \leqslant n\}$. *Furthermore, if* B *is irreducible, and the equality in* (2.1) *holds, then*

$$b_{ii} + \sqrt{\sum_{k=1,\, k\neq i}^{n} l_k b_{ki}^2} = b_{jj} + \sqrt{\sum_{k=1,\, k\neq j}^{n} l_k b_{kj}^2}$$

for any $i, j \in \{1, 2, \ldots, n\}$.

P r o o f. For each $i \in \{1, 2, \ldots, n\}$, by $BX = \varrho(B)X$, we have $\varrho(B)x_i = \sum\limits_{j=1}^{n} b_{ij}x_j$, then

$$(\varrho(B) - b_{ii})x_i = \sum_{j\neq i, b_{ij}\neq 0} b_{ij}x_j,$$

and thus by the Cauchy inequality, we have

$$(\varrho(B) - b_{ii})^2 x_i^2 = \biggl(\sum_{j\neq i, b_{ij}\neq 0} b_{ij}x_j\biggr)^2 \leqslant l_i \sum_{j\neq i, b_{ij}\neq 0} (b_{ij}x_j)^2.$$

Then

$$\begin{aligned}
\sum_{i=1}^{n} [(\varrho(B) - b_{ii})x_i]^2 &\leqslant \sum_{i=1}^{n} \biggl(l_i \sum_{j\neq i, b_{ij}\neq 0} (b_{ij}x_j)^2\biggr) \\
&= \sum_{i=1}^{n} \biggl(\sum_{j\neq i, b_{ij}\neq 0} l_i b_{ij}^2 x_j^2\biggr) \\
&= \sum_{i=1}^{n} \biggl[\biggl(\sum_{k=1, k\neq i}^{n} l_k b_{ki}^2\biggr) x_i^2\biggr],
\end{aligned}$$

thus we have

$$(2.2)\qquad \sum_{i=1}^{n}\Big((\varrho(B)-b_{ii})^2-\sum_{k=1,k\neq i}^{n} l_k b_{ki}^2\Big)x_i^2\leqslant 0.$$

Therefore there must exist some $j\in\{1,2,\dots,n\}$ such that

$$(\varrho(B)-b_{jj})^2-\sum_{k\neq j} l_k b_{kj}^2\leqslant 0,$$

so

$$\varrho(B)\leqslant b_{jj}+\sqrt{\sum_{k\neq j} l_k b_{kj}^2}\leqslant \max_{1\leqslant i\leqslant n}\Big\{b_{ii}+\sqrt{\sum_{k\neq i} l_k b_{ki}^2}\Big\}.$$

If $\varrho(B)=\max\limits_{1\leqslant i\leqslant n}\Big\{b_{ii}+\sqrt{\sum\limits_{k\neq i} l_k b_{ki}^2}\Big\}$, then for any $j\in\{1,2,\dots,n\}$ we have $\varrho(B)\geqslant b_{jj}+\sqrt{\sum\limits_{k\neq j} l_k b_{kj}^2}$, then

$$(2.3)\qquad (\varrho(B)-b_{jj})^2-\sum_{k\neq j} l_k b_{kj}^2\geqslant 0,$$

and thus

$$(2.4)\qquad \sum_{j=1}^{n}\Big((\varrho(B)-b_{jj})^2-\sum_{k\neq j} l_k b_{kj}^2\Big)x_j^2\geqslant 0.$$

Combining (2.2) and (2.4) implies that

$$\sum_{i=1}^{n}\Big((\varrho(B)-b_{ii})^2-\sum_{k\neq i} l_k b_{ki}^2\Big)x_i^2=0.$$

Noting that (2.3) holds for any $j\in\{1,2,\dots,n\}$, we have $(\varrho(B)-b_{ii})^2-\sum\limits_{k\neq i} l_k b_{ki}^2=0$ for any $i\in\{s\colon\ x_s\neq 0,\ 1\leqslant s\leqslant n\}$, and thus $b_{ii}+\sqrt{\sum\limits_{k\neq i} l_k b_{ki}^2}=b_{jj}+\sqrt{\sum\limits_{k\neq j} l_k b_{kj}^2}$ for any $i,j\in\{s\colon\ x_s\neq 0,\ 1\leqslant s\leqslant n\}$.

Furthermore, if B is irreducible, then $x_i>0$ for each $i\in\{1,2,\dots,n\}$ by the Perron-Frobenius theorem, and thus $b_{ii}+\sqrt{\sum\limits_{k=1,k\neq i}^{n} l_k b_{ki}^2}=b_{jj}+\sqrt{\sum\limits_{k=1,k\neq j}^{n} l_k b_{kj}^2}$ for any $i,j\in\{1,2,\dots,n\}$ if the equality in (2.1) holds. $\square$

It is natural that we want to know under what conditions the equality in (2.1) holds.

Question 2.2. Look for the necessity and sufficiency conditions for the equality in (2.1) to hold.

Lemma 2.3 ([13]). *Let $B = (b_{ij})$ be an $n \times n$ nonnegative irreducible matrix and $A = (a_{ij})$ a complex matrix. Let $|A| = (|a_{ij}|)$. If $b_{ij} \geqslant |a_{ij}|$ for any $i, j \in \{1, 2, \ldots, n\}$, which we denote by $B \geqslant |A|$, then $\varrho(B) \geqslant \varrho(A)$.*

By Lemma 2.3 we know that for any connected graph G and any strong connected digraph $\vec{G}$, $\mu(G) \leqslant q(G)$ and $\mu(\vec{G}) \leqslant q(\vec{G})$. In fact, we have

Lemma 2.4 ([2]). *Let $G = (V, E)$ be a connected undirected graph on n vertices. Then $\mu(G) \leqslant q(G)$, with equality holding if and only if G is a bipartite graph.*

Corollary 2.5. *Let $A = (a_{ij})$ be an $n \times n$ complex irreducible matrix, l_i the number of the nonzero entries except for the diagonal entry of the ith row for any $i \in \{1, 2, \ldots, n\}$, that is, $l_i = |\{a_{ij} \colon a_{ij} \neq 0, \ j \in \{1, 2, \ldots, n\} \setminus \{i\}\}|$. Then*

$$\varrho(A) \leqslant \max_{1 \leqslant i \leqslant n} \left\{ |a_{ii}| + \sqrt{\sum_{k=1, k \neq i}^{n} l_k |a_{ki}|^2} \right\}. \tag{2.5}$$

If the equality holds, then $|a_{ii}| + \sqrt{\sum_{k=1, k\neq i}^{n} l_k |a_{ki}|^2} = |a_{jj}| + \sqrt{\sum_{k=1, k\neq j}^{n} l_k |a_{kj}|^2}$ for any $i, j \in \{1, 2, \ldots, n\}$.

Proof. Let $B = (|a_{ij}|)$, then B is a nonnegative irreducible matrix. Thus $\varrho(A) \leqslant \varrho(B)$ by Lemma 2.3, and (2.5) holds by Theorem 2.1. $\square$

3. Various spectral radii of an undirected graph

Let G be an undirected graph. The adjacency matrix $A(G)$, the Laplacian matrix $L(G)$, the signless Laplacian matrix $Q(G)$, and the (adjacency) spectral radius $\varrho(G)$, the Laplacian spectral radius $\mu(G)$, the signless Laplacian spectral radius $q(G)$ are defined as in Introduction. Let G be a connected undirected graph. The distance matrix $\mathcal{D}(G)$, the distance Laplacian matrix $\mathcal{L}(G)$, the distance signless Laplacian matrix $\mathcal{Q}(G)$, and the distance spectral radius $\varrho^{\mathcal{D}}(G)$, the distance Laplacian spectral radius $\mu^{\mathcal{D}}(G)$, the distance signless Laplacian spectral radius $q^{\mathcal{D}}(G)$ are defined as in Introduction. In this section, we will apply Theorem 2.1 to $A(G)$, $Q(G)$, $\mathcal{D}(G)$ and $\mathcal{Q}(G)$, and apply Corollary 2.5 to $L(G)$ and $\mathcal{L}(G)$, to obtain some new results or known results on the spectral radius.

3.1. Adjacency spectral radius of an undirected graph.

Lemma 3.1. *Let $G = (V, E)$ be a simple connected undirected graph with vertex set $V = \{v_1, v_2 \ldots, v_n\}$. For any $v_i \in V$, the degree of v_i and the average degree of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. Then $d_1m_1 = d_2m_2 = \ldots = d_nm_n$ holds if and only if G is a regular undirected graph or a bipartite semi-regular undirected graph.*

P r o o f. If G is a regular undirected graph or a bipartite semi-regular undirected graph, we can check that $d_1m_1 = d_2m_2 = \ldots = d_nm_n$ holds immediately.

Conversely, let $d_1m_1 = d_2m_2 = \ldots = d_nm_n$ hold. Now we show that G is a regular or a bipartite semi-regular undirected graph.

Let a vertex v_n be the lowest degree vertex in G, that is, $d_n = \min\{d_i \colon 1 \leqslant i \leqslant n\}$. Let $d_n = r$, and let the neighbors of v_n be $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$. Let $d_{i_1} = \max\{d_{i_j} \colon 1 \leqslant j \leqslant r\}$, denoted by $s = d_{i_1}$. Then $m_n \leqslant s$ by $m_i = d_i^{-1} \sum\limits_{i\sim j} d_j$ and thus $d_nm_n = rm_n \leqslant rs$.

On the other hand, for a vertex v_{i_1} we have $d_{i_1}m_{i_1} = sm_{i_1} \geqslant rs$, then $rs \leqslant d_{i_1}m_{i_1} = d_nm_n \leqslant rs$, thus $d_{i_1}m_{i_1} = d_nm_n = rs$, which implies $m_n = s$ and $m_{i_1} = r$. Therefore by the definitions of s and r, we know $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ must have the same degree, say, $s = d_{i_1} = d_{i_2} = \ldots = d_{i_r}$, and all the neighbors of v_{i_1} must have the same degree r.

Similarly to the above arguments, we can show that the vertices with degree r are adjacent to the vertices with degree s, and the vertices with degree s are adjacent to the vertices with degree r in G.

Now we assume that G is not bipartite. Then G has at least an odd cycle. Let $C = v_{j_1}v_{j_2}\ldots v_{j_{2k-1}}v_{j_{2k}}v_{j_{2k+1}}v_{j_1}$ be an odd cycle of length $2k+1$ in G. Clearly, the degree of the vertex v_{j_1}, say d_{j_1}, is either r or s. Without loss of generality, we assume that $d_{j_1} = r$. Since the vertices with degree r are adjacent to the vertices with degree s, and the vertices with degree s are adjacent to the vertices with degree r, hence $d_{j_2} = s$, $d_{j_3} = r$, $\ldots$, $d_{j_{2k-1}} = r$, $d_{j_{2k}} = s$, $d_{j_{2k+1}} = r$ and $d_{j_1} = s$ by $v_{j_{2k+1}}$ and v_{j_1} being adjacent, thus $r = s$, which implies that G is regular.

Hence the undirected graph G is a regular undirected graph or a bipartite semi-regular undirected graph. $\square$

Theorem 3.2 ([4], Theorem 1)**.** *Let $G = (V, E)$ be a simple undirected graph on n vertices. Then $\varrho(G) \leqslant \max\limits_{1\leqslant i\leqslant n} \sqrt{d_im_i}$. Moreover, if G is connected, then the equality holds if and only if G is a regular or bipartite semi-regular undirected graph.*

P r o o f. We apply Theorem 2.1 to $A(G)$.

Since $b_{ii} = 0$,

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

and $l_i = d_i$ for $i = 1, 2, \ldots, n$, hence $b_{ii} + \sqrt{\sum\limits_{k \neq i} l_k b_{ki}^2} = \sqrt{d_i m_i}$ for $i = 1, 2, \ldots, n$, and thus $\varrho(G) \leqslant \max\limits_{1 \leqslant i \leqslant n} \sqrt{d_i m_i}$ by (2.1).

If G is connected, now we show the equality holds if and only if G is a regular or bipartite semi-regular undirected graph.

If G is connected and $\varrho(G) = \max\limits_{1 \leqslant i \leqslant n} \sqrt{d_i m_i}$, then $\sqrt{d_1 m_1} = \sqrt{d_2 m_2} = \ldots = \sqrt{d_n m_n}$ by Theorem 2.1, and thus $d_1 m_1 = d_2 m_2 = \ldots = d_n m_n$. Therefore, G is a regular or bipartite semi-regular undirected graph by Lemma 3.1.

On the other hand, if G is connected and G is a regular or bipartite semi-regular undirected graph, then $d_1 m_1 = d_2 m_2 = \ldots = d_n m_n$ by Lemma 3.1, thus $\sqrt{d_1 m_1} = \sqrt{d_2 m_2} = \ldots = \sqrt{d_n m_n}$, and $\varrho(G) \leqslant \max\limits_{1 \leqslant i \leqslant n} \sqrt{d_i m_i} = \sqrt{d_1 m_1}$. Then we complete the proof in the following two cases.

Case 1: G is a regular undirected graph with degree r. It is well known that $r = \sqrt{d_1 m_1}$ is an eigenvalue of G, so $\sqrt{d_1 m_1} \leqslant \varrho(G)$. Thus $\varrho(G) = \max\limits_{1 \leqslant i \leqslant n} \sqrt{d_i m_i} = r$ by $\varrho(G) \leqslant \sqrt{d_1 m_1} = r$.

Case 2: G is a bipartite semi-regular undirected graph.

We assume that the two bipartitions of G have degree r and s, respectively. It is easy to check that $\sqrt{rs} = \sqrt{d_1 m_1}$ is an eigenvalue of G, so $\sqrt{rs} \leqslant \varrho(G)$. Thus $\varrho(G) = \max\limits_{1 \leqslant i \leqslant n} \sqrt{d_i m_i} = \sqrt{rs}$ by $\varrho(G) \leqslant \sqrt{d_1 m_1} = \sqrt{rs}$. □

3.2. (Signless) Laplacian spectral radius of an undirected graph.

Lemma 3.3 ([13]). *Let A be a nonnegative matrix with the spectral radius $\varrho(A)$ and the row sums $r_1, r_2, \ldots, r_n$. Then $\min\limits_{1 \leqslant i \leqslant n} r_i \leqslant \varrho(A) \leqslant \max\limits_{1 \leqslant i \leqslant n} r_i$. Moreover, if A is an irreducible matrix, then one of the equalities holds if and only if the row sums of A are all equal.*

Lemma 3.4. *Let $G = (V, E)$ be a simple connected undirected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For any $v_i \in V$, the degree of v_i and the average degree of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. Then $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \ldots = d_n + \sqrt{d_n m_n}$ holds if and only if G is regular.*

P r o o f. If G is regular, we can check that $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \ldots = d_n + \sqrt{d_n m_n}$ holds immediately.

Conversely, let $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \ldots = d_n + \sqrt{d_n m_n}$ hold. Now we show G is regular.

Let a vertex v_n be the lowest degree vertex in G, that is, $d_n = \min\{d_i \colon 1 \leqslant i \leqslant n\}$. Let $d_n = r$, and let the neighbors of v_n be $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$. Let $d_{i_1} = \max\{d_{i_j} \colon 1 \leqslant j \leqslant r\}$, denoted by $s = d_{i_1}$. It is obvious that $r \leqslant s$, $m_n \leqslant s$ by $m_i = d_i^{-1} \sum_{i \sim j} d_j$, and thus $d_n + \sqrt{d_n m_n} \leqslant r + \sqrt{rs}$.

On the other hand, for the vertex v_{i_1}, we have $d_{i_1} m_{i_1} = s m_{i_1} \geqslant rs$, then $d_{i_1} + \sqrt{d_{i_1} m_{i_1}} \geqslant s + \sqrt{rs}$. Thus $s + \sqrt{rs} \leqslant d_{i_1} + \sqrt{d_{i_1} m_{i_1}} = d_n + \sqrt{d_n m_n} \leqslant r + \sqrt{rs} \leqslant s + \sqrt{rs}$, which implies $m_n = s$, $m_{i_1} = r$ and $r = s$. Therefore by the definitions of s and r, we know $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ must have the same degree, that is, $s = d_{i_1} = d_{i_2} = \ldots = d_{i_r}$, and all the neighbors of v_{i_1} must have the same degree r $(= s)$.

Similarly to the above arguments, we can show that the vertices with degree r are adjacent to the vertices with degree s, and the vertices with degree s are adjacent to the vertices with degree r in G. Then G is regular by $r = s$. $\square$

Theorem 3.5. *Let $G = (V, E)$ be a simple undirected graph on n vertices. Then*

(i) (see [17]) $q(G) \leqslant \max\limits_{1 \leqslant i \leqslant n} \{d_i + \sqrt{d_i m_i}\}$. *Moreover, if G is connected, the equality holds if and only if G is a regular undirected graph.*

(ii) (See [19].) *If G is connected, then $\mu(G) \leqslant \max\limits_{1 \leqslant i \leqslant n} \{d_i + \sqrt{d_i m_i}\}$, and the equality holds if and only if G is a bipartite regular undirected graph.*

Proof. First, we show (i) holds. We apply Theorem 2.1 to $Q(G)$.

Since $b_{ii} = d_i$,

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise,} \end{cases}$$

$l_i = d_i$ for $i = 1, 2, \ldots, n$, we have $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = d_i + \sqrt{d_i m_i}$ for $i = 1, 2, \ldots, n$, and thus $q(G) \leqslant \max\limits_{1 \leqslant i \leqslant n} \{d_i + \sqrt{d_i m_i}\}$ by (2.1).

Now we show that if G is connected, then the equality holds if and only if G is regular.

If G is a connected undirected graph and $q(G) = \max\limits_{1 \leqslant i \leqslant n} \{d_i + \sqrt{d_i m_i}\}$, then $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \ldots = d_n + \sqrt{d_n m_n}$ by Theorem 2.1, and thus G is regular by Lemma 3.4.

On the other hand, if G is connected and G is regular with degree r, then $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \ldots = d_n + \sqrt{d_n m_n} = 2r$ by Lemma 3.4 and $\max\limits_{1 \leqslant i \leqslant n} \{d_i + \sqrt{d_i m_i}\} = 2r$. It is well known that $q(G) = 2r$ by Lemma 3.3, so $q(G) = \max\limits_{1 \leqslant i \leqslant n} \{d_i + \sqrt{d_i m_i}\}$.

Similarly to the proof of (i), by Corollary 2.5, Lemma 2.4 and the result of (i), we can show (ii) immediately, so we omit it. □

3.3. Distance spectral radius of an undirected graph.

Theorem 3.6. *Let $G=(V,E)$ be a connected undirected graph on n vertices. Then*

$$\varrho^D(G) \leqslant \max_{1\leqslant i\leqslant n} \sqrt{(n-1)\sum_{k=1}^{n} d_{ki}^2}. \tag{3.1}$$

If the equality holds, then $\sum\limits_{k=1}^{n} d_{ki}^2 = \sum\limits_{k=1}^{n} d_{kj}^2$ for any $i,j \in \{1,2,\ldots,n\}$.

P r o o f. We apply Theorem 2.1 to $\mathcal{D}(G)$. Since $b_{ii}=d_{ii}=0$, $b_{ij}=d_{ij}\neq 0$ when $i\neq j$ and $l_i=n-1$ for $i=1,2,\ldots,n$, we have $b_{ii}+\sqrt{\sum\limits_{k\neq i} l_k b_{ki}^2} = \sqrt{(n-1)\sum\limits_{k=1}^{n} d_{ki}^2}$ for $i=1,2,\ldots,n$, and thus (3.1) holds by (2.1).

It is obvious that if the equality holds, then $\sum\limits_{k=1}^{n} d_{ki}^2 = \sum\limits_{k=1}^{n} d_{kj}^2$ for any $i,j=1,2,\ldots,n$ by Theorem 2.1. □

3.4. Distance (signless) Laplacian spectral radius of an undirected graph.

Theorem 3.7. *Let $G=(V,E)$ be a connected undirected graph on n vertices. Then*

$$q^D(G) \leqslant \max_{1\leqslant i\leqslant n} \left\{ D_i + \sqrt{(n-1)\sum_{k=1}^{n} d_{ki}^2} \right\}, \tag{3.2}$$

and

$$\mu^D(G) \leqslant \max_{1\leqslant i\leqslant n} \left\{ D_i + \sqrt{(n-1)\sum_{k=1}^{n} d_{ki}^2} \right\}. \tag{3.3}$$

Moreover, if the equality in (3.2) *(or* (3.3)*) holds, then $D_i + \sqrt{(n-1)\sum\limits_{k=1}^{n} d_{ki}^2} = D_j + \sqrt{(n-1)\sum\limits_{k=1}^{n} d_{kj}^2}$ for any $i,j\in\{1,2,\ldots,n\}$.*

P r o o f. We apply Theorem 2.1 to $\mathcal{Q}(G)$. Since $b_{ii} = D_i$, $b_{ij} = d_{ij}$ where $i \neq j$, and $l_i = n-1$ for $i = 1, 2, \ldots, n$, hence $b_{ii} + \sqrt{\sum\limits_{k \neq i} l_k b_{ki}^2} = D_i + \sqrt{(n-1) \sum\limits_{k=1}^{n} d_{ki}^2}$ for $i = 1, 2, \ldots, n$, and thus (3.2) holds.

Similarly, we apply Corollary 2.5 to $\mathcal{L}(G)$ and (3.3) holds. □

4. Various spectral radii of a digraph

Let $\vec{G}$ be a connected digraph. The adjacency matrix $A(\vec{G})$, the Laplacian matrix $L(\vec{G})$, the signless Laplacian matrix $Q(\vec{G})$, and the (adjacency) spectral radius $\varrho(\vec{G})$, the Laplacian spectral radius $\mu(\vec{G})$, the signless Laplacian spectral radius $q(\vec{G})$ are defined as in Introduction. Let $\vec{G}$ be a connected digraph. The distance matrix $\mathcal{D}(\vec{G})$, the distance Laplacian matrix $\mathcal{L}(\vec{G})$, the distance signless Laplacian matrix $\mathcal{Q}(\vec{G})$, and the distance spectral radius $\varrho^{\mathcal{D}}(\vec{G})$, the distance Laplacian spectral radius $\mu^{\mathcal{D}}(\vec{G})$, the distance signless Laplacian spectral radius $q^{\mathcal{D}}(\vec{G})$ are defined as in Introduction. In this section, we will apply Theorem 2.1 to $A(\vec{G})$, $Q(\vec{G})$, $\mathcal{D}(\vec{G})$ and $\mathcal{Q}(\vec{G})$, and apply Corollary 2.5 to $L(\vec{G})$ and $\mathcal{L}(\vec{G})$, to obtain some new results or known results on the spectral radius.

4.1. Adjacency spectral radius of a digraph.

Theorem 4.1 ([20], Corollary 3.2). *Let $\vec{G} = (V, E)$ be a digraph on n vertices. Then*

$$\varrho(\vec{G}) \leqslant \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{k \sim i} d_k^+}.$$

If $\vec{G}$ is connected and the equality holds, then $\sum\limits_{k \sim 1} d_k^+ = \sum\limits_{k \sim 2} d_k^+ = \ldots = \sum\limits_{k \sim n} d_k^+$.

P r o o f. We apply Theorem 2.1 to $A(\vec{G})$. Since $b_{ii} = 0$, $b_{ij} = \begin{cases} 1 \text{ if } (v_i, v_j) \in E; \\ 0 \text{ otherwise,} \end{cases}$ and $l_i = d_i^+$ for $i = 1, 2, \ldots, n$, we have $b_{ii} + \sqrt{\sum\limits_{k \neq i} l_k b_{ki}^2} = \sqrt{\sum\limits_{k \sim i} d_k^+}$, and thus $\varrho(\vec{G}) \leqslant \max\limits_{1 \leqslant i \leqslant n} \sqrt{\sum\limits_{k \sim i} d_k^+}$ by (2.1).

It is obvious that if $\vec{G}$ is connected and the equality holds, then $\sum\limits_{k \sim 1} d_k^+ = \sum\limits_{k \sim 2} d_k^+ = \ldots = \sum\limits_{k \sim n} d_k^+$ by Theorem 2.1. □

4.2. (Signless) Laplacian spectral radius of a digraph.

Theorem 4.2. *Let $\vec{G}=(V,E)$ be a digraph on n vertices. Then*

(i) (See [3], Theorem 3.3.) $q(\vec{G}) \leqslant \max\limits_{1\leqslant i\leqslant n}\Big\{d_i^+ + \sqrt{\sum\limits_{j\sim i} d_j^+}\Big\}$. *Moreover, if $\vec{G}$ is connected and the equality holds, then $d_1^+ + \sqrt{\sum\limits_{j\sim 1} d_j^+} = d_2^+ + \sqrt{\sum\limits_{j\sim 2} d_j^+} = \ldots = d_n^+ + \sqrt{\sum\limits_{j\sim n} d_j^+}$.*

(ii) *If $\vec{G}$ is connected, then $\mu(\vec{G}) \leqslant \max\limits_{1\leqslant i\leqslant n}\Big\{d_i^+ + \sqrt{\sum\limits_{j\sim i} d_j^+}\Big\}$, and if the equality holds, then $d_1^+ + \sqrt{\sum\limits_{j\sim 1} d_j^+} = d_2^+ + \sqrt{\sum\limits_{j\sim 2} d_j^+} = \ldots = d_n^+ + \sqrt{\sum\limits_{j\sim n} d_j^+}$.*

P r o o f. We apply Theorem 2.1 to $Q(\vec{G})$. Since $b_{ii}=d_i^+$, $b_{ij}=\begin{cases}1 \text{ if } (v_i,v_j)\in E;\\ 0 \text{ otherwise,}\end{cases}$ and $l_i=d_i^+$ for $i=1,2,\ldots,n$, then $b_{ii}+\sqrt{\sum\limits_{k\neq i} l_k b_{ki}^2} = d_i^+ + \sqrt{\sum\limits_{j\sim i} d_j^+}$ for $i=1,2,\ldots,n$, and thus $q(\vec{G}) \leqslant \max\limits_{1\leqslant i\leqslant n}\Big\{d_i^+ + \sqrt{\sum\limits_{j\sim i} d_j^+}\Big\}$ by (2.1).

It is obvious that if $\vec{G}$ is connected and the equality holds then $d_1^+ + \sqrt{\sum\limits_{j\sim 1} d_j^+} = d_2^+ + \sqrt{\sum\limits_{j\sim 2} d_j^+} = \ldots = d_n^+ + \sqrt{\sum\limits_{j\sim n} d_j^+}$ by Theorem 2.1.

Similarly to the proof of (i), we can show (ii) immediately by Corollary 2.5, so we omit it. $\square$

4.3. Distance spectral radius of a digraph.

Theorem 4.3. *Let $\vec{G}=(V,E)$ be a strongly connected digraph on n vertices. Then*

$$\varrho^D(\vec{G}) \leqslant \max_{1\leqslant i\leqslant n}\sqrt{(n-1)\sum_{k=1}^{n} d_{ki}^2}. \tag{4.1}$$

If the equality holds, then $\sum\limits_{k=1}^{n} d_{ki}^2 = \sum\limits_{k=1}^{n} d_{kj}^2$ for any $i,j\in\{1,2,\ldots,n\}$.

P r o o f. We apply Theorem 2.1 to $\mathcal{D}(\vec{G})$. Since $b_{ii}=d_{ii}=0$, $b_{ij}=d_{ij}\neq 0$, and $l_i=n-1$ for any $i=1,2,\ldots,n$, hence $b_{ii}+\sqrt{\sum\limits_{k\neq i} l_k b_{ki}^2} = \sqrt{(n-1)\sum\limits_{k=1}^{n} d_{ki}^2}$ for $i=1,2,\ldots,n$, and thus (4.1) holds by (2.1).

It is easy to see that if the equality holds, then $\sum_{k=1}^{n} d_{ki}^2 = \sum_{k=1}^{n} d_{kj}^2$ for any $i, j \in \{1, 2, \ldots, n\}$. $\square$

4.4. Distance (signless) Laplacian spectral radius of a digraph.

Theorem 4.4. *Let $\vec{G} = (V, E)$ be a strongly connected digraph on n vertices. Then*

$$q^D(\vec{G}) \leqslant \max_{1 \leqslant i \leqslant n} \left\{ D_i^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{ki}^2} \right\}, \tag{4.2}$$

and

$$\mu^D(\vec{G}) \leqslant \max_{1 \leqslant i \leqslant n} \left\{ D_i^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{ki}^2} \right\}. \tag{4.3}$$

Moreover, if the equality in (4.2) *(or* (4.3)*) holds then* $D_i^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{ki}^2} = D_j^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{kj}^2}$ *for any* $i, j \in \{1, 2, \ldots, n\}$.

P r o o f. We apply Theorem 2.1 to $\mathcal{Q}(\vec{G})$. Since $b_{ii} = D_i^+$, $b_{ij} = d_{ij} \neq 0$ for all $i \neq j$, $b_{ii} = d_{ii} = 0$, and $l_i = n - 1$ for $i = 1, 2, \ldots, n$, hence $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = D_i^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{ki}^2}$ for $i = 1, 2, \ldots, n$, and thus (4.2) holds by (2.1). By Corollary 2.5 and (i), (4.3) holds.

It is easy to see that if the equality in (4.2) (or (4.3)) holds then $D_i^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{ki}^2} = D_j^+ + \sqrt{(n-1) \sum_{k=1}^{n} d_{kj}^2}$ for any $i, j \in \{1, 2, \ldots, n\}$ by Theorem 2.1. $\square$

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