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CENTRAL LIMIT THEOREM FOR GIBBSIAN U-STATISTICS OF FACET PROCESSES

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Abstract. A special case of a Gibbsian facet process on a fixed window with a discrete orientation distribution and with increasing intensity of the underlying Poisson process is studied. All asymptotic joint moments for interaction U-statistics are calculated and the central limit theorem is derived using the method of moments.

Keywords: central limit theorem; facet process; U-statistics

MSC 2010: 60D05, 60G55

1. Introduction

In the present paper we use the methods developed in [9] to calculate all moments of Gibbsian U-statistics of facets in a bounded window of arbitrary Euclidean dimension. These moments are used to derive the central limit theorem for such statistics. Central limit theorems for U-statistics of Poisson processes were derived based on the Malliavin calculus and the Stein method in [7]. The effort to extend developments of this type to functionals of a wider class of spatial processes, e.g. Gibbs processes, was initiated in [8].

Our calculations are based on the achievements in [1], where functionals of spatial point processes given by a density with respect to the Poisson process were investigated using the Fock space representation from [4]. This formula is applied to the product of a functional and the density and using a special class of functionals called U-statistics closed formulas for mixed moments of functionals are obtained. In processes with densities the key characteristic is the correlation function (see [3]) of arbitrary order.

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As in [9] we call facets some compact subsets of hyperplanes with a given shape, size and orientation. Natural geometrical characteristics of the union of facets, based on Hausdorff measure of the intersections of pairs, triplets, etc., of facets form U-statistics. Building a parametric density from exponential family, the limitations for the space of parameters have to be given, the so-called submodels are investigated. In application of the moment formulas we are interested in the limit behaviour as the intensity of the reference Poisson process tends to infinity.

We restrict ourselves to the facet model with finitely many orientations corresponding to canonical vectors. In [9] basic asymptotic properties of the studied U-statistics are derived. When the order of the submodel is not greater than the order of the observed U-statistic, then asymptotically the mean value of the U-statistic vanishes. This leads to a degeneracy in the sense that some orientations are missing. On the other hand, when the order of the submodel is greater than the order of the observed U-statistic, then the limit of the correlation function is finite and nonzero and under selected standardization the U-statistic tends almost surely to its nonzero expectation. By changing the standardization, however, we achieve a finite nonzero asymptotic variance. In the present paper we simplify the calculation of moments so that we are able to calculate any asymptotic moment.

2. Central limit theorem

We call facets some compact subsets of hyperplane in $\mathbb{R}^d$. Let

$$\widetilde{Y} = \{B \subset \mathbb{R}^d \colon \exists\, l \in [d],\ (x_1, \ldots, x_d) \in B,\ x_l = z_l;\ |x_i - z_i| \leqslant b,\ i \in [d] \setminus \{l\}\},$$

where

$$(z_1, \ldots, z_d) \in [0, b]^d, \quad [d] = \{1, \ldots, d\},$$

be a space of facets with fixed size b, orientations restricted only to elementary vectors $e_1, \ldots, e_d$ and centres in a cube $[0, b]^d$ (facets are $(d-1)$-dimensional cubes). The space $\widetilde{Y}$ is isomorphic to $Y = [0, b]^d \times \{2b\} \times \{e_1, \ldots, e_d\}$, where the three parts are the set of facet centres, possible sizes of the facet and possible orientations of the hyperplane containing the facet. Therefore we will call Y also the space of facets and use it instead of $\widetilde{Y}$. We also define the isomorphism $\iota\colon Y \to \widetilde{Y}$ by

$$\iota((z_1, \ldots, z_d), 2b, e_l) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \colon x_l = z_l;\ |x_i - z_i| \leqslant b,\ i \in [d] \setminus \{l\}\},$$

where

$$(z_1, \ldots, z_d) \in [0, b]^d, \quad l \in [d].$$

Moreover, let $(\mathbf{N}, \mathcal{N})$ be a measurable space of integer-valued finite measures on Y, where $\mathcal{N}$ is the smallest σ-algebra which makes the mappings $\mathbf{x} \mapsto \mathbf{x}(A)$, $\mathbf{x} \in \mathbf{N}$, measurable for all Borel sets $A \subset Y$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\eta_a\colon (\Omega, \mathcal{A}, P) \to (\mathbf{N}, \mathcal{N})$ a finite Poisson process of facets with intensity measure $a\lambda$ on Y, $a \geqslant 1$, in the form

$$a\lambda(\mathrm{d}(z, r, \varphi)) = a\chi(z)\,\mathrm{d}z\delta_{2b}(\mathrm{d}r)\frac{1}{d}\sum_{i=1}^{d}\delta_{e_i}(\mathrm{d}\varphi),$$

where we have fixed the facet size, uniform distribution of the facet orientation and $\chi\colon [0, b]^d \to \mathbb{R}_+$. We also define interaction U-statistics (using Hausdorff measure $\mathbb{H}^{d-j}$ of order $d-j$)

$$G_j(\mathbf{x}) = \frac{1}{j!}\sum_{(x_1,\ldots,x_j)\in\mathbf{x}^j_{\neq}} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j}\iota(x_i)\right), \quad \mathbf{x} \in \mathbf{N},$$

where $\mathbf{x}^j_{\neq}$ is set of all j-tuples of distinct facets from the support of $\mathbf{x}$. Furthermore, we define the process μ_a with density

$$p(\mathbf{x}) = c_a \exp\left(\sum_{i=1}^{d}\nu_i G_i(\mathbf{x})\right)$$

with respect to P_{η_a}, the probability distribution of η_a on space $(\mathbf{N}, \mathcal{N})$, where $a \geqslant 1$, ν_i is a real parameter and

$$c_a = \frac{1}{\mathbb{E}\exp\left(\sum_{i=1}^{d}\nu_i G_i(\eta_a)\right)}.$$

Fulfilling of the condition $\nu_i \leqslant 0$, $i = 2, \ldots, d$ ensures that $p(\mathbf{x}) \in L^2(P_{\eta_a})$. Necessity of these conditions will be discussed later. We also use the notion of a submodel $\mu_a^{(l)}$ of order l, where $\nu_j = 0$, $j \neq l$, and $\nu_l < 0$. We will explore properties of such submodels of order higher than 1, because in the case of $\mu_a^{(1)}$ we deal with a Poisson process (see [9]). We will also use a correlation function of order k, see [3], formula (2.2), which is defined as

$$\varrho_k(x_1, \ldots, x_k, \mu_a^{(c)}) = \frac{\mathbb{E}\exp(\nu_c G_c(\eta_a \cup \{x_1, \ldots, x_k\}))}{\mathbb{E}\exp(\nu_c G_c(\eta_a))},$$

where $x_1, \ldots, x_k \in Y$.

We can use a short expression for moment formulas using diagrams and partitions, see [6], [4], Theorem 1.1. Let $\widetilde{\Pi}_k$ be the set of all partitions $\{J_i\}$ of $[k]$, where J_i are disjoint blocks and $\bigcup J_i = [k]$. For $k = k_1 + \ldots + k_m$ and blocks

$$J_i = \{j\colon\ k_1 + \ldots + k_{i-1} < j \leqslant k_1 + \ldots + k_i\}, \quad i = 1, \ldots, m,$$

consider the partition $\pi = \{J_i,\ 1 \leqslant i \leqslant m\}$ and let $\Pi_{k_1,\ldots,k_m} \subset \widetilde{\Pi}_k$ be the set of all partitions $\sigma \in \widetilde{\Pi}_k$ such that $|J \cap J'| \leqslant 1$ for all $J \in \pi$ and all $J' \in \sigma$. Here $|J|$ is the cardinality of a block $J \in \sigma$. We will refer to blocks of π as rows and we denote by $S(\sigma) = |\{J \in \pi\colon\ \forall J' \in \sigma,\ |J \cap J'| = 1 \Rightarrow |J'| = 1\}|$ the number of singleton rows of partition σ.

For a partition $\sigma \in \Pi_{k_1\ldots k_m}$ and measurable functions $f_j\colon\ B \to \mathbb{R}$, $j = 1, \ldots, m$, we define the function $\left(\bigotimes_{j=1}^m f_j\right)_\sigma\colon\ B^{|\sigma|} \to \mathbb{R}$ by replacing all variables of the tensor product $\bigotimes_{j=1}^m f_j$ that belong to the same block of σ by a new common variable, $|\sigma|$ is the number of blocks in σ. We define $\Pi_{1,\ldots,s}^{(m_1,\ldots,m_s)} = \Pi_{1,\ldots,1,\ldots,s,\ldots,s}$, where i repeats m_i times for $i = 1, \ldots, s$. We put $\binom{p}{q} = 0$ for $q > p$. Now we state the main theorem of the paper.

Theorem 1. *Denote*

$$\widetilde{G}_j(\mu_a^{(c)}) = \frac{G_j(\mu_a^{(c)}) - \mathbb{E}G_j(\mu_a^{(c)})}{a^{j-1/2}}, \quad 1 \leqslant j \leqslant d,\ 2 \leqslant c \leqslant d.$$

Then

$$(\widetilde{G}_1(\mu_a^{(c)}), \ldots, \widetilde{G}_d(\mu_a^{(c)})) \xrightarrow{\mathcal{D}} \mathbf{Z},\ c = 2, \ldots, d, \tag{1}$$

as a tends to infinity, where $\mathbf{Z} \sim N(0, \Sigma)$, $\Sigma = \{\theta_{ij}\}_{i,j=1}^d$,

$$\theta_{kl} = \frac{c-1}{d^{k+l-1}}\binom{c-2}{k-1}\binom{c-2}{l-1} I_{kl},$$

$$I_{kl} = \int_{([0,b]^d)^{k+l-1}} \mathbb{H}^{d-k}\Bigl(\bigcap_{i=1}^k \iota(s_i, 2b, e_i)\Bigr)\mathbb{H}^{d-l}\Bigl(\bigcap_{i=2}^l \iota(s_{i+k-1}, 2b, e_i) \cap \iota(s_1, 2b, e_1)\Bigr) \times \chi(s_1)\,\mathrm{d}s_1 \ldots \chi(s_{k+l-1})\,\mathrm{d}s_{k+l-1}.$$

Moreover,

$$G_j(\mu_a^{(c)}) \xrightarrow{L^1} 0, \quad c \in \{2, \ldots, d\},\ j \geqslant c,$$

$$\frac{G_j(\mu_a^{(c)})}{a^j} \xrightarrow{L^2} \frac{I_j}{d^j}\binom{c-1}{j}, \quad c \in \{2, \ldots, d\},\ j < c,$$

where

$$I_j = \int_{([0,b]^d)^j} \mathbb{H}^{d-j}\Big(\bigcap_{i=1}^{j} \iota(s_i, 2b, e_i)\Big)\chi(s_1)\,\mathrm{d}s_1 \ldots \chi(s_j)\,\mathrm{d}s_j.$$

R e m a r k 1. Random variables $\widetilde{G}_c(\mu_a^{(c)}), \widetilde{G}_{c+1}(\mu_a^{(c)}), \ldots, \widetilde{G}_d(\mu_a^{(c)})$ are asymptotically degenerate, i.e., their expectations tend to zero and asymptotic covariances of these variables are $\theta_{kl} = 0$, $k \geqslant c$, $l \in [d]$.

R e m a r k 2. For the random vector $(\widetilde{G}_1(\eta_a), \ldots, \widetilde{G}_d(\eta_a))$ we have similar results, see [5], Theorem 4.1, with $\theta_{kl} = \frac{d}{d^{k+l-1}}\binom{d-1}{k-1}\binom{d-1}{l-1} I_{kl}$.

Corollary 1. *It holds that*

$$\frac{G_j(\mu_a^{(c)}) - \mathbb{E}G_j(\mu_a^{(c)})}{a^{j-\frac{1}{2}}} \xrightarrow{\mathcal{D}} Z, \quad c = 2, \ldots, d,\ j < c, \tag{2}$$

as a tends to infinity, where $Z \sim N(0, \theta_{jj})$.

Now we state three auxiliary lemmas, whose proofs are in Section 4.

Lemma 1. *It holds that*

$$\varrho_p(x_1, \ldots, x_p, \mu_a^{(c)}) = \frac{\mathbb{E}\exp(\nu_c G_c(\eta_a \cup \{x_1, \ldots, x_p\}))}{\mathbb{E}\exp(\nu_c G_c(\eta_a))} \to \frac{\binom{d-k}{d-c+1}}{\binom{d}{d-c+1}}, \tag{3}$$

as a tends to infinity, where $x_i \in Y$ and k is the number of distinct facet orientations among $\{x_1, \ldots, x_p\}$ and $c \geqslant 2$. Moreover, there exist $a_0 \geqslant 1$, $R > 0$, $S > 0$, which do not depend on $x_1, \ldots, x_p$, such that

$$\left|\varrho_p(x_1, \ldots, x_p, \mu_a^{(c)}) - \frac{\binom{d-k}{d-c+1}}{\binom{d}{d-c+1}}\right| < R\mathrm{e}^{-Sa} \quad \forall\, a \geqslant a_0.$$

R e m a r k 3. Consider matrix $A = \{a_{ij}\}_{i,j=1}^{d}$, where $\forall\, i\ \exists\, j_i\colon\ a_{ij_i} = 1$; $a_{ij} = 0$, $j \neq j_i$ and $\forall\, j\, \exists\, i_j\colon\ a_{i_j j} = 1$; $a_{ij} = 0$, $i \neq i_j$. Then it can be shown that for rotation of facets $\widetilde{A}\colon Y \to Y$, $\widetilde{A}((z, r, \varphi)) = (z, r, A\varphi)$ around their centers given by A the following relation holds:

$$\varrho_p(x_1, \ldots, x_p, \mu_a^{(c)}) = \varrho_p(\widetilde{A}(x_1), \ldots, \widetilde{A}(x_p), \mu_a^{(c)}).$$

For some $\mathbf{n}^{(d)} = (n_1, \ldots, n_d)$ we define

$$R^{c,p}(q, d, \mathbf{n}^{(d)}) = \sum_{\substack{F \subset [d] \\ c-p \leqslant |F| \leqslant c \\ |F \cup [q]| + p - q \geqslant c}} \prod_{j \in F} n_j.$$

If n_i is the number of facets among $u_1, \dots, u_n$ with orientation e_i, then specially $R^{c,0}(0, d, \mathbf{n}^{(d)})$ is the total number of intersections of all c-tuples among $u_1, \dots, u_n$ and $R^{c,p}(p, d, \mathbf{n}^{(d)})$ is the total number of intersections of all c-tuples among facets $u_1, \dots, u_n, (\mathbf{z}^1, 2b, e_1), \dots, (\mathbf{z}^p, 2b, e_p)$.

Lemma 2. *For any $\nu < 0$, $p < c \leqslant d$, $c \geqslant 2$ there exist $R > 0$, $S > 0$, such that for $a \geqslant 1$,*

$$\left|\sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \frac{a^{n_1+\ldots+n_d}}{n_1! \dots n_d!} \exp(\nu R^{c,p}(p, d, \mathbf{n}^{(d)}) - a(c-1)) - \frac{(d-p)!}{(c-1-p)!}\right| < R\mathrm{e}^{-Sa}.$$

For any $\nu < 0$, $p = c \leqslant d$, $c \geqslant 2$ there exist $R > 0$, $S > 0$, such that for $a \geqslant 1$,

$$\left|\sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \frac{a^{n_1+\ldots+n_d}}{n_1! \dots n_d!} \exp(\nu R^{c,p}(p, d, \mathbf{n}^{(d)}) - a(c-1))\right| < R\mathrm{e}^{-Sa}.$$

Lemma 3. *For any $c \leqslant d$, $m_1, \dots, m_d \in \mathbb{N}^0$ and $\sigma \in \Pi_{1,\dots,d}^{(m_1,\dots,m_d)}$, there exist $R > 0$, $S > 0$, $a_0 \geqslant 1$, such that for $a \geqslant a_0$,*

$$\begin{aligned}&\left|\int_{Y^{|\sigma|}} \left(\bigotimes_{j=1}^{d}((\overline{\mathbb{H}}^{d-j})^{\otimes m_j})\right)_{\sigma}(u_1, \dots, u_{|\sigma|})\varrho_{|\sigma|}(u_1, \dots, u_{|\sigma|}, \mu_a^{(c)})\lambda^{|\sigma|}(\mathrm{d}(u_1, \dots, u_{|\sigma|}))\right.\\ &\left.-\int_{Y_{c-1}^{|\sigma|}} \left(\bigotimes_{j=1}^{d}((\overline{\mathbb{H}}^{d-j})^{\otimes m_j})\right)_{\sigma}(u_1, \dots, u_{|\sigma|})\lambda^{|\sigma|}(\mathrm{d}(u_1, \dots, u_{|\sigma|}))\right| < R\mathrm{e}^{-Sa},\end{aligned}$$

where $Y_{c-1} = [0, b]^d \times \{2b\} \times \{e_1, \dots, e_{c-1}\}$ is the space of facets with $d - c + 1$ orientations missing (which can be selected arbitrarily) and $\overline{\mathbb{H}}^{d-j}(u_1, \dots, u_j) = \mathbb{H}^{d-j}\Big(\bigcap\limits_{i=1}^{j} \iota(u_i)\Big)$.

Remark 4. The expression

$$\int_{Y^{|\sigma|}} \left(\bigotimes_{j=1}^{d}((\overline{\mathbb{H}}^{d-j})^{\otimes m_j})\right)_{\sigma}(u_1, \dots, u_{|\sigma|})\varrho_{|\sigma|}(u_1, \dots, u_{|\sigma|}, \mu_a^{(c)})\lambda^{|\sigma|}(\mathrm{d}(u_1, \dots, u_{|\sigma|}))$$

in the statement of Lemma 3 is used in the calculation of the product of moments and this lemma shows that for large a the correlation function can be removed from the integral, if we consider only $d - c + 1$ orientations instead of d.

This will be used in the proof of Theorem 1.

R e m a r k 5. Consider the process μ_a with density in more general form $p(\mathbf{x}) = c_a \exp\Bigl(\sum\limits_{i=1}^{d} \nu_i G_i(\mathbf{x})\Bigr)$. Assume that there is $c \geqslant 2$, $\nu_c > 0$, and select minimal such c. Then

$$
\begin{aligned}
(4)\qquad & \mathbb{E} \exp\Bigl(\sum_{j=1}^{d} \nu_j G_j(\eta_a)\Bigr) \\
&= \sum_{n=0}^{\infty} \frac{a^n \mathrm{e}^{-aT}}{n!} \int_{Y^n} \exp\Bigl(\sum_{j=1}^{d} \nu_j G_j(\{u_1,\dots,u_n\})\Bigr) \lambda^n(\mathrm{d}(u_1,\dots,u_n)) \\
&\geqslant \mathrm{e}^{-aT} \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \frac{(aT/d)^{n_1+\dots+n_d}}{n_1!\dots n_d!} \exp\Bigl(\sum_{j=1}^{d} \nu_j' \sum_{\{i_1,\dots,i_j\}\subset[d]} \prod_{l=1}^{j} n_{i_l}\Bigr) \\
&\geqslant \mathrm{e}^{-aT} \sum_{n_1=0}^{\infty} \dots \sum_{n_c=0}^{\infty} \frac{(aT/d)^{n_1+\dots+n_c}}{n_1!\dots n_c!} \exp\Bigl(\sum_{j=1}^{c} \nu_j' \sum_{\{i_1,\dots,i_j\}\subset[c]} \prod_{l=1}^{j} n_{i_l}\Bigr) \\
&\geqslant \mathrm{e}^{-aT} \sum_{n=0}^{\infty} \frac{(aT/d)^{nc}}{(n!)^c} \exp\Bigl(\sum_{j=1}^{c} \nu_j' \binom{c}{j} n^j\Bigr),
\end{aligned}
$$

where

$$
\nu_j' = \begin{cases} \nu_j \inf\Bigl\{\mathbb{H}^{d-j}\Bigl(\bigcap\limits_{i=1}^{j} \iota(u_i)\Bigr),\ u_1,\dots,u_j \in Y,\ \mathbb{H}^{d-j}\Bigl(\bigcap\limits_{i=1}^{j} \iota(u_i)\Bigr) > 0\Bigr\}, & \nu_j \geqslant 0, \\ \nu_j \sup\Bigl\{\mathbb{H}^{d-j}\Bigl(\bigcap\limits_{i=1}^{j} \iota(u_i)\Bigr),\ u_1,\dots,u_j \in Y,\ \mathbb{H}^{d-j}\Bigl(\bigcap\limits_{i=1}^{j} \iota(u_i)\Bigr) > 0\Bigr\}, & \nu_j < 0, \end{cases}
$$

and $T = \int_{[0,b]^d} \chi(z)\,\mathrm{d}z$. We first set the last $d-c$ summing variables to zero and then sum only over the summands where all of the summing variables have the same value. It can be proven (e.g. by using ratio test), that the sum in (4) is divergent, because $\nu_j' \binom{c}{j} > 0$ at the highest power in the exponential. Therefore $p \notin L^1(P_{\eta_a})$ in this case. On the other hand, the non-positivity of parameter ν implies $p \in L^2(P_{\eta_a})$, which finally leads to $\nu_l \leqslant 0,\ l \geqslant 2 \iff p \in L^2(P_{\eta_a})$.

R e m a r k 6. Consider the process μ_a with density $p(\mathbf{x}) = c_a \exp\Bigl(\sum\limits_{i=1}^{d} \nu_i G_i(\mathbf{x})\Bigr)$, $\nu_l \leqslant 0$, $l \geqslant 2$. Assume there is $c \geqslant 2$, $\nu_c < 0$ and select minimal such c. Then using similar techniques as in the proof of Lemma 1 and Lemma 2 we can show that there exist $R > 0$, $S > 0$, $a_0 \geqslant 1$ such that

$$
\Bigl|\varrho_p(x_1,\dots,x_p,\mu_a) - \lim_{a\to\infty} \varrho_p(x_1,\dots,x_p,\mu_a^{(c)})\Bigr| < R\mathrm{e}^{-Sa}, \quad a \geqslant a_0,
$$

which leads to the same asymptotic distribution of statistics $(\widetilde{G}_1(\mu_a),\dots,\widetilde{G}_d(\mu_a))$ as $(\widetilde{G}_1(\mu_a^{(c)}),\dots,\widetilde{G}_d(\mu_a^{(c)}))$.

3. Proof of the central limit theorem

Proof of Theorem 1. It holds (see [1], Theorem 3) that

$$(5)\qquad \mathbb{E}G_j(\mu_a^{(c)}) = \frac{a^j}{j!}\int_{Y^j} \mathbb{H}^{d-j}\Big(\bigcap_{i=1}^{j}\iota(u_i)\Big)\varrho_j(u_1,\dots,u_j,\mu_a^{(c)})\lambda^j(\mathrm{d}(u_1,\dots,u_j)),$$

$$(6)\qquad \mathbb{E}\prod_{j=1}^{c-1} G_j^{m_j}(\mu_a^{(c)}) = \sum_{\sigma\in\Pi_{1,\dots,c-1}^{(m_1,\dots,m_{c-1})}}\prod_{j=1}^{c-1}\frac{1}{j!^{m_j}}a^{|\sigma|}\int_{Y^{|\sigma|}}\Big(\bigotimes_{j=1}^{c-1}(\overline{\mathbb{H}}^{d-j})^{\otimes t_j}\Big)_\sigma$$
$$\times (u_1,\dots,u_{|\sigma|})\varrho_{|\sigma|}(u_1,\dots,u_{|\sigma|},\mu_a^{(c)})\lambda^{|\sigma|}(\mathrm{d}(u_1,\dots,u_{|\sigma|})).$$

We can also get a relation for joint moments of centered random variables

$$(7)\qquad \mathbb{E}\prod_{j=1}^{c-1}\widetilde{G}_j^{m_j}(\mu_a^{(c)}) = \frac{1}{a^M}\mathbb{E}\prod_{j=1}^{c-1}(G_j(\mu_a^{(c)}) - \mathbb{E}G_j(\mu_a^{(c)}))^{m_j}$$
$$= \frac{1}{a^M}\sum_{i_1=0}^{m_1}\dots\sum_{i_{c-1}=0}^{m_{c-1}}\binom{m_1}{i_1}\dots\binom{m_{c-1}}{i_{c-1}}(-1)^{\sum_{j=1}^{c-1}i_j}$$
$$\times\,\mathbb{E}\Big(\prod_{j=1}^{c-1}G_j^{m_j-i_j}(\mu_a^{(c)})\Big)\prod_{j=1}^{c-1}(\mathbb{E}G_j(\mu_a^{(c)}))^{i_j},$$

where $\sum\limits_{j=1}^{c-1}(j-\frac{1}{2})m_j = M$.

Firstly we calculate expectations of the U-statistics. Using Lemma 3, we obtain

$$\frac{\mathbb{E}G_j(\mu_a^{(c)})}{a^j} \to \frac{1}{j!}\int_{Y_{c-1}^j}\mathbb{H}^{d-j}\Big(\bigcap_{i=1}^{j}\iota(u_i)\Big)\lambda^j(\mathrm{d}(u_1,\dots,u_j)) = \frac{1}{d^j}I_j\binom{c-1}{j},$$

where $\binom{c-1}{j}$ is the number of combinations how to select distinct j orientations from $c-1$ and $j!$ is the number of their allocations into the j positions, d^j is the number of all j-selections of d orientations. The value I_j is an integral of the Hausdorff measure of the intersection of j facets with distinct orientations. It does not depend on the currently selected orientations, they only need to be distinct, otherwise I_j would be 0, because only non-parallel facets intersect.

Using Lemma 1 for $j \geqslant c$, we have that $\varrho_j(u_1,\dots,u_j,\mu_a^{(c)})$ tends to zero at exponential rate, and therefore $\lim\limits_{a\to\infty} a^j\varrho_j(u_1,\dots,u_j,\mu_a^{(c)}) = 0$. Moreover, the limit and the integral can be interchanged by using Lebesgue's dominated convergence theorem and we obtain

$$G_j(\mu_a^{(c)}) \xrightarrow{L^1} 0,\quad c\in\{2,\dots,d\},\ j\geqslant c.$$

Therefore, we only need to investigate the U-statistics of order lower than c.

Secondly we calculate all joint moments. To do this we need to first use formula (7) and Lemma 3—we use the limit values of the correlation function, which we justify later. To describe the relation between the original formula and the formula with the correlation function replaced by its limit value we use the symbol $\simeq$:

$$\left(\prod_{j=1}^{c-1} j!^{m_j}\right)\mathbb{E}\left(\prod_{j=1}^{c-1} G_j^{m_j-i_j}(\mu_a^{(c)})\right)\prod_{j=1}^{c-1}(\mathbb{E}G_j(\mu_a^{(c)}))^{i_j}$$
$$\simeq \prod_{j=1}^{c-1}\left(\int_{Y_{c-1}^j}\mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j}\iota(u_i)\right)\lambda^j(\mathrm{d}(u_1,\dots,u_j))\right)^{i_j}\sum_{\sigma\in\Pi_{1,\dots,c-1}^{(m_1-i_1,\dots,m_{c-1}-i_{c-1})}} a^{|\sigma|+\sum_{j=1}^{c-1} j i_j}$$
$$\times \int_{Y_{c-1}^{|\sigma|}}\left(\bigotimes_{j=1}^{c-1}((\overline{\mathbb{H}}^{d-j})^{\otimes(m_j-i_j)})\right)_\sigma (u_1,\dots,u_{|\sigma|})\lambda^{|\sigma|}(\mathrm{d}(u_1,\dots,u_{|\sigma|})).$$

We are interested only in terms with power higher than or equal to M, because the other terms will tend to zero with increasing a, i.e., partitions fulfilling condition $|\sigma| \geqslant M - \sum_{j=1}^{c-1} i_j j$. Also, we do not have to examine odd moments, i.e., those with $\sum_{j=1}^{c-1} m_j$ odd, because there is no summand with the power of a matching M in the denominator, thus asymptotically they can only be zero or infinite. Therefore, if we prove that all even moments tend to some finite value, then all odd moments tend to zero.

Select $\mathbf{s} = (s_1,\dots,s_{c-1})$, so that $m_i \geqslant s_i \geqslant 0$, $i \in [c-1]$, $\exists\, j \in [c-1]$, $m_j > s_j$, choose any partition $\sigma_{\mathbf{s}} \in \Pi_{1,\dots,c-1}^{\mathbf{s}}$ fulfilling conditions $|\sigma_{\mathbf{s}}| \geqslant M - \sum_{j=1}^{c-1} i_j j$ and $S(\sigma_{\mathbf{s}}) = 0$, i.e., each block of π is connected to any other block of π by some block of $\mathbf{s}$. Then for $\mathbf{t} = (t_1,\dots,t_{c-1})$, $m_i \geqslant t_i \geqslant s_i$, $i \in [c-1]$, $\exists\, j \in [c-1]$, $t_j > s_j$ there are partitions $\sigma_{\mathbf{t}} \in \Pi_{1,\dots,c-1}^{\mathbf{t}}$, which have only additional singleton rows compared to $\sigma_{\mathbf{s}}$, $S(\sigma_{\mathbf{t}}) = \sum_{i=1}^{c-1}(t_i - s_i)$, $|\sigma_{\mathbf{t}}| - |\sigma_{\mathbf{s}}| = \sum_{i=1}^{c-1}(t_i - s_i) i$ and it holds that

$$a^{|\sigma_{\mathbf{t}}|}\int_{Y_{c-1}^{|\sigma_{\mathbf{t}}|}}\left(\bigotimes_{j=1}^{c-1}(\overline{\mathbb{H}}^{d-j})^{\otimes t_j}\right)_{\sigma_{\mathbf{t}}}(u_1,\dots,u_{|\sigma_{\mathbf{t}}|})\lambda^{|\sigma_{\mathbf{t}}|}(\mathrm{d}(u_1,\dots,u_{|\sigma_{\mathbf{t}}|}))$$
$$= a^{|\sigma_{\mathbf{s}}|}\int_{Y_{c-1}^{|\sigma_{\mathbf{s}}|}}\left(\bigotimes_{j=1}^{c-1}(\overline{\mathbb{H}}^{d-j})^{\otimes s_j}\right)_{\sigma_{\mathbf{s}}}(u_1,\dots,u_{|\sigma_{\mathbf{s}}|})\lambda^{|\sigma_{\mathbf{s}}|}(\mathrm{d}(u_1,\dots,u_{|\sigma_{\mathbf{s}}|}))$$
$$\times \prod_{j=1}^{c-1}\left(a^j\int_{Y_{c-1}^j}\mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j}\iota(u_i)\right)\lambda^j(\mathrm{d}(u_1,\dots,u_j))\right)^{t_j-s_j},$$

because we can separate the singleton rows corresponding to the functions $\overline{\mathbb{H}}^k$ in the tensor product, which can be integrated separately, because they do not have any common variables with the other functions in the tensor product and the integral is equal to the expectation of U-statistic. We can see that all summands corresponding to any of the partitions $\sigma_{\mathbf{t}}$ in the evaluation of (7) contain a common term

$$\Theta = a^{|\sigma_{\mathbf{s}}|} \int_{Y_{c-1}^{|\sigma_{\mathbf{s}}|}} \left(\bigotimes_{j=1}^{c-1} \left(\overline{\mathbb{H}}^{d-j}\right)^{\otimes s_j} \right)_{\sigma_{\mathbf{s}}} (u_1, \ldots, u_{|\sigma_{\mathbf{s}}|}) \lambda^{|\sigma_{\mathbf{s}}|}(\mathrm{d}(u_1, \ldots, u_{|\sigma_{\mathbf{s}}|}))$$
$$\times \prod_{j=1}^{c-1} \left(a^j \int_{Y_{c-1}^j} \mathbb{H}^{d-j} \left(\bigcap_{i=1}^{j} \iota(u_i) \right) \lambda^j(\mathrm{d}(u_1, \ldots, u_j)) \right)^{m_j - s_j}$$

and then we sum over all such partitions $\sigma_{\mathbf{t}}$:

$$\Theta \sum_{i_1=s_1}^{m_1} \cdots \sum_{i_{c-1}=s_{c-1}}^{m_{c-1}} \binom{m_1}{i_1} \cdots \binom{m_{c-1}}{i_{c-1}} \binom{i_1}{s_1} \cdots \binom{i_{c-1}}{s_{c-1}} (-1)^{\sum_{j=1}^{c-1} i_j}$$
$$= \Theta(-1)^{\sum_{j=1}^{c-1} s_j} \binom{m_1}{s_1} \cdots \binom{m_{c-1}}{s_{c-1}} \sum_{i_1=0}^{m_1-s_1} \cdots \sum_{i_{c-1}=0}^{m_{c-1}-s_{c-1}} \binom{m_1 - s_1}{i_1} \cdots$$
$$\times \binom{m_{c-1} - s_{c-1}}{i_{c-1}} (-1)^{\sum_{j=1}^{c-1} i_j} = 0,$$

where we use the binomial theorem for summing with necessary condition $\sum_{j=1}^{c-1} s_j < \sum_{j=1}^{c-1} m_j$ and $\binom{m_j}{i_j}$ are original coefficients from formula (7) and $\binom{i_j}{s_j}$ is the number of options how to select additional singleton rows.

Therefore all partitions with any singleton rows or containted within $\Pi_{1,\ldots,c-1}^{\mathbf{s}}$, $\mathbf{s} < \mathbf{m} = (m_1, \ldots, m_{c-1})$ cancel each other out. But we have to take into account that we have carried out the calculation with the limit values of correlation functions and all the integrals are multiplied by a in polynomial, thus we have to deal with the speed of convergence. Consider

$$\sum_{i=1}^{N} a^p \Upsilon_i \int_{Y^p} \left(\bigotimes_{j=1}^{c-1} \left(\overline{\mathbb{H}}^{d-j}\right)^{\otimes t_j} \right)_p (u_1, \ldots, u_p) \varrho_{i,p}(u_1, \ldots, u_p, \mu_a^{(c)}) \lambda^p(\mathrm{d}(u_1, \ldots, u_p)),$$

where

$$\Upsilon_i, \ t_j \in \mathbb{N}^0, \ \varrho_{i,p}(u_1, \ldots, u_p, \mu_a^{(c)}) = \left(\bigotimes_{J \in \sigma_i} \varrho_{|J|}(\cdot, \mu_a^{(c)}) \right)(u_1, \ldots, u_p), \ \sigma_i \in \widetilde{\Pi}_p,$$

and $\varrho_{i,p}$ is such that $|\varrho_{i,p}(u_1,\dots,u_p,\mu_a^{(c)}) - \varrho_{i,p}| < R\mathrm{e}^{-Sa}$ for some $R,S>0$ and

$$\sum_{i=1}^{N} a^p \Upsilon_i \int_{Y^p} \left(\bigotimes_{j=1}^{c-1} \left(\overline{\mathbb{H}}^{d-j}\right)^{\otimes t_j}\right)_p (u_1,\dots,u_p)\varrho_{i,p}\lambda^p(\mathrm{d}(u_1,\dots,u_p)) = 0.$$

For practical purposes we omit variables $u_1,\dots,u_p$ from the following formulas and it holds that

$$\begin{aligned} a^p &\left|\sum_{i=1}^{N} \Upsilon_i \int_{Y^p} \left(\bigotimes_{j=1}^{c-1} \left(\overline{\mathbb{H}}^{d-j}\right)^{\otimes t_j}\right)_p (\varrho_{i,p}(\cdot,\mu_a^{(c)}) - \varrho_{i,p})\,\mathrm{d}\lambda^p\right| \\ &\leqslant a^p\left|\sum_{i=1}^{N} \Upsilon_i \int_{Y^p} \left(\bigotimes_{j=1}^{c-1} \left(\overline{\mathbb{H}}^{d-j}\right)^{\otimes t_j}\right)_p |\varrho_{i,p}(\cdot,\mu_a^{(c)}) - \varrho_{i,p}|\,\mathrm{d}\lambda^p\right| \\ &\leqslant a^p R\mathrm{e}^{-Sa}\left|\sum_{i=1}^{N} \Upsilon_i \int_{Y^p} \left(\bigotimes_{j=1}^{c-1} \left(\overline{\mathbb{H}}^{d-j}\right)^{\otimes t_j}\right)_p \mathrm{d}\lambda^p\right| \to 0. \end{aligned}$$

Now we are left only with partitions σ which do not contain any pure singleton rows. All of these partitions are contained in $\Pi^{\mathbf{m}}_{1,\dots,c-1}$. These partitions have each row connected exactly to one other row by one block of two elements in σ ($|\sigma| = M$) and therefore, if we omit all the mentioned partitions, then

$$\begin{aligned} &\prod_{j=1}^{c-1} j!^{m_j}\ \mathbb{E}\left(\prod_{j=1}^{c-1} G_j^{m_j-i_j}(\mu_a^{(c)})\right)\prod_{j=1}^{c-1}(\mathbb{E}G_j(\mu_a^{(c)}))^{i_j} \\ &\simeq \sum_{k_1^{(2)},\dots,k_{m_2}^{(2)}=1}^{2} \cdots \sum_{k_1^{(c-1)},\dots,k_{m_{c-1}}^{(c-1)}=1}^{c-1} \sum_{\sigma\in\widetilde{\Pi}_K,\, J\in\sigma:\ |J|=2} \prod_{J=\{b_1,b_2\}\in\sigma} a^{\tau(b_1)+\tau(b_2)-1} \\ &\qquad \times \int_{Y_{c-1}^{\tau(b_1)+\tau(b_2)-1}} \mathbb{H}^{d-\tau(b_1)}\left(\bigcap_{i=1}^{\tau(b_1)} \iota(x_i)\right)\mathbb{H}^{d-\tau(b_2)}\left(\bigcap_{i=1}^{\tau(b_2)-1} \iota(x_{\tau(b_1)+i})\cap\iota(x_1)\right) \\ &\qquad \times \lambda^{\tau(b_1)+\tau(b_2)-1}(\mathrm{d}(x_1,\dots,x_{\tau(b_1)+\tau(b_2)-1})), \\ &\qquad\qquad \tau(s) = \max_{j\in[c-1]}\left\{\sum_{i=1}^{j-1} m_i < s\right\}, \quad K = \sum_{j=1}^{c-1} m_j, \end{aligned}$$

where we sum first over all possible selections of common elements among the partitions and then over all possible pairings of partition rows, we also divide the integral into several parts, where each part consists only of elements which are in the same block of a partition. Function τ connects each row of a partition to its length. We have

$$(8)\quad \int_{Y_{c-1}^{\tau(b_1)+\tau(b_2)-1}} \mathbb{H}^{d-\tau(b_1)}\left(\bigcap_{i=1}^{\tau(b_1)} \iota(x_i)\right)$$

$$\times \mathbb{H}^{d-\tau(b_2)}\left(\bigcap_{i=1}^{\tau(b_2)-1} \iota(x_{\tau(b_1)+i}) \cap \iota(x_1)\right)\lambda^{\tau(b_1)+\tau(b_2)-1}(\mathrm{d}(x_1,\ldots,x_{\tau(b_1)+\tau(b_2)-1}))$$

$$= \frac{(c-1)(\tau(b_1)-1)!(\tau(b_2)-1)! I_{\tau(b_1)\tau(b_2)}\binom{c-2}{\tau(b_1)-1}\binom{c-2}{\tau(b_2)-1}}{d^{\tau(b_1)+\tau(b_2)-1}},$$

$$(9)\quad \sum_{k_1^{(2)},\ldots,k_{m_2}^{(2)}=1}^{2} \cdots \sum_{k_1^{(c-1)},\ldots,k_{m_{c-1}}^{(c-1)}=1}^{c-1} 1 = \prod_{j=1}^{c-1} j^{m_j},$$

where $c-1$ is the number of choices of the one common facet orientation, $\binom{c-2}{\tau(b_1)-1}$, $\binom{c-2}{\tau(b_2)-1}$ are the numbers of combinations how to select the remaining distinct orientations of the rest of the facet orientations in the first and the second function in the integrand and $(\tau(b_1)-1)!$, $(\tau(b_2)-1)!$ are the numbers of their allocations into $\tau(b_1)-1$ and $\tau(b_2)-1$ positions, $d^{\tau(b_1)+\tau(b_2)-1}$ is the $(\tau(b_1)+\tau(b_2)-1)$-selection of d orientations (even non-distinct ones) and $I_{\tau(b_1)\tau(b_2)}$ is the integral over facets with fixed orientations over the space of the facet centres. Then using (8) and (9)

$$\mathbb{E}\left(\prod_{j=1}^{c-1} G_j^{m_j-i_j}(\mu_a^{(c)})\right)\prod_{j=1}^{c-1}(\mathbb{E}G_j(\mu_a^{(c)}))^{i_j}$$

$$\simeq \left(\frac{a}{d}\right)^M \sum_{\sigma\in\widetilde{\Pi}_K, J\in\sigma:\ |J|=2} \sum_{k_1^{(2)},\ldots,k_{m_2}^{(2)}=1}^{2} \cdots \sum_{k_1^{(c-1)},\ldots,k_{m_{c-1}}^{(c-1)}=1}^{c-1} \prod_{J=\{b_1,b_2\}\in\sigma} \prod_{j=1}^{c-1} \frac{1}{j!^{m_j}}$$

$$\times (\tau(b_1)-1)!(\tau(b_2)-1)!(c-1) I_{\tau(b_1)\tau(b_2)}\binom{c-2}{\tau(b_1)-1}\binom{c-2}{\tau(b_2)-1}$$

$$= \left(\frac{a}{d}\right)^M \sum_{\sigma\in\widetilde{\Pi}_K,\ J\in\sigma:\ |J|=2} \prod_{J=\{b_1,b_2\}\in\sigma} (c-1) I_{\tau(b_1)\tau(b_2)}\binom{c-2}{\tau(b_1)-1}\binom{c-2}{\tau(b_2)-1},$$

because the factorial terms $(\tau(b_1)-1)!(\tau(b_2)-1)!$ and the sums cancel out with the term $\prod 1/j!^{m_j}$. Therefore, we have

$$\mathbb{E}\prod_{j=1}^{c-1} \widetilde{G}_j^{m_j}(\mu_a^{(c)}) \simeq \sum_{\sigma\in\widetilde{\Pi}_K, J\in\sigma:\ |J|=2} \prod_{J=\{b_1,b_2\}\in\sigma}$$

$$\times \frac{(c-1) I_{\tau(b_1)\tau(b_2)}}{d^{\tau(b_1)+\tau(b_1)-1}}\binom{c-2}{\tau(b_1)-1}\binom{c-2}{\tau(b_2)-1}$$

and as a special case we get

$$\mathbb{E}\widetilde{G}_i(\mu_a^{(c)})\widetilde{G}_j(\mu_a^{(c)}) \simeq \frac{(c-1)I_{ij}}{d^{i+j-1}}\binom{c-2}{i-1}\binom{c-2}{j-1}.$$

Now consider the vector of multivariate normal distribution $(X_1, \ldots, X_d) \sim N(0, \Sigma)$. Then for any joint moment we have

$$\mathbb{E}\prod_{j=1}^{d} X_j^{m_j} = \sum_{\sigma \in \widetilde{\Pi}_K, J \in \sigma\colon |J|=2} \prod_{J=\{b_1,b_2\}\in\sigma} \mathbb{E}X_{\tau(b_1)}X_{\tau(b_2)}.$$

We can see that asymptotically the distribution of statistics has the property of normal distribution, i.e., joint moments of centered variables are equal to the sum over all pairs of unordered random variables and this implies the central limit theorem, because normal distribution is defined by its moments, see [2], Theorem 30.2.

There is only one remaining statement to prove:

$$\frac{G_j(\mu_a^{(c)})}{a^j} \xrightarrow{L^2} \frac{I_j}{d^j}\binom{c-1}{j}, \quad c \in \{2, \ldots, d\},\ j < c.$$

The first moment of the random variable on the left-hand side tends to the right-hand side and the variance tends to zero as can be seen from the proof of the central limit theorem. □

4. Proofs of lemmas

Proof of Lemma 1. First consider the submodel $\mu_a^{(c)}$ and facets $x_1, \ldots, x_p$ with $k = p \leqslant c$ distinct orientations. Moreover, without loss of generality we consider orientations $e_1, \ldots, e_p$, because if we apply rotations from Remark 3, then the value of the correlation function does not change. It holds that

$$\varrho_p(x_1, \ldots, x_p, \mu_a^{(c)}) = \frac{\sum\limits_{n=0}^{\infty} \frac{a^n}{n!} \int_{Y^n} \exp(\nu_c G_c\{u_1, \ldots, u_n, x_1, \ldots, x_p\})\lambda^n(\mathrm{d}(u_1, \ldots, u_n))}{\sum\limits_{n=0}^{\infty} \frac{a^n}{n!} \int_{Y^n} \exp(\nu_c G_c\{u_1, \ldots, u_n\})\lambda^n(\mathrm{d}(u_1, \ldots, u_n))}.$$

We can obtain bounds for this expression by using the bounds for the volumes of intersection of facets

$$b^{d-c} \leqslant \mathbb{H}^{d-c}\Bigl(\bigcap_{i=1}^{c} y_i\Bigr) \leqslant (2b)^{d-c}$$

as follows:

$$
\begin{aligned}
(10)\qquad & \frac{\sum\limits_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum\limits_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c(2b)^{d-c}R^{c,p}(p,d,\mathbf{n}^{(d)}))}{\sum\limits_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum\limits_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c b^{d-c}R^{c,0}(0,d,\mathbf{n}^{(d)}))}\\
&\leqslant \varrho_p(x_1,\ldots,x_p,\mu_a^{(c)})\\
&\leqslant \frac{\sum\limits_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum\limits_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c b^{d-c}R^{c,p}(p,d,\mathbf{n}^{(d)}))}{\sum\limits_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum\limits_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c(2b)^{d-c}R^{c,0}(0,d,\mathbf{n}^{(d)}))},
\end{aligned}
$$

where $T=\int_{[0,b]^d}\chi(z)\,\mathrm{d}z$, n_i are the numbers of facets among $u_1,\ldots,u_n$ with orientations e_i, $i=1,\ldots,d$ and $\mathbf{n}^{(d)}=(n_1,\ldots,n_d)$. Furthermore, we will make use of the definition of $R^{c,p}(q,d,\mathbf{n}^{(d)})$, because specially $R^{c,0}(0,d,\mathbf{n}^{(d)})$ is the total number of intersections of all c-tuples of the facets among $u_1,\ldots,u_n$ and $R^{c,p}(p,d,\mathbf{n}^{(d)})$ is the total number of intersections of all c-tuples of the facets $u_1,\ldots,u_n$, $x_1,\ldots,x_p$.

Then we substitute aT/d for α, extend both fractions by $\mathrm{e}^{-\alpha(c-1)}$ and we get in the case of the lower bound of (10)

$$
\frac{\sum\limits_{n_1=0}^{\infty}\cdots\sum\limits_{n_d=0}^{\infty}\frac{\alpha^{n_1+\ldots+n_d}}{n_1!\ldots n_d!}\exp(\nu_c(2b)^{d-c}R^{c,p}(p,d,\mathbf{n}^{(d)})-\alpha(c-1))}{\sum\limits_{n_1=0}^{\infty}\cdots\sum\limits_{n_d=0}^{\infty}\frac{\alpha^{n_1+\ldots+n_d}}{n_1!\ldots n_d!}\exp(\nu_c b^{d-c}R^{c,0}(0,d,\mathbf{n}^{(d)})-\alpha(c-1))}
$$

and in the case of the upper bound of (10)

$$
\frac{\sum\limits_{n_1=0}^{\infty}\cdots\sum\limits_{n_d=0}^{\infty}\frac{\alpha^{n_1+\ldots+n_d}}{n_1!\ldots n_d!}\exp(\nu_c b^{d-c}R^{c,p}(p,d,\mathbf{n}^{(d)})-\alpha(c-1))}{\sum\limits_{n_1=0}^{\infty}\cdots\sum\limits_{n_d=0}^{\infty}\frac{\alpha^{n_1+\ldots+n_d}}{n_1!\ldots n_d!}\exp(\nu_c(2b)^{d-c}R^{c,0}(0,d,\mathbf{n}^{(d)})-\alpha(c-1))}.
$$

Using Lemma 2 we get the limit of the lower and upper bound in the same form $\binom{d-p}{d-c+1}/\binom{d}{d-c+1}=\binom{d-k}{d-c+1}/\binom{d}{d-c+1}$. For $d\geqslant p=k>c$ we can get an upper bound in (10) by using $p=c$. This upper bound tends to zero.

Now consider more than one facet with the same orientation among $x_1,\ldots x_p$ and with $k<c$ distinct orientations, which are without loss of generality set to $e_1,\ldots,e_k$ and the maximum number of facets with the same orientation is κ. Then we can

bound the correlation function in the following way:

$$\frac{\sum_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c\kappa^d(2b)^{d-c}R^{c,k}(k,d,\mathbf{n}^{(d)}))}{\sum_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c b^{d-c}R^{c,0}(0,d,\mathbf{n}^{(d)}))}$$
$$\leqslant \varrho_p(x_1,\ldots,x_p,\mu_a^{(c)})$$
$$\leqslant \frac{\sum_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c b^{d-c}R^{c,k}(k,d,\mathbf{n}^{(d)}))}{\sum_{n=0}^{\infty}\frac{(aT/d)^n}{n!}\sum_{n_1+\ldots+n_d=n}\binom{n}{n_1,\ldots,n_d}\exp(\nu_c\kappa^d(2b)^{d-c}R^{c,0}(0,d,\mathbf{n}^{(d)}))}.$$

These bounds lead to expressions in the same form as in the case of different orientations and therefore we proceed in the same way and get the value of the limit $\binom{d-k}{d-c+1}/\binom{d}{d-c+1}$. For $d\geqslant k\geqslant c$ we need only the lower bound for the number of intersections in the form $R^{c,k}(k,d,\mathbf{n}^{(d)})$, which forms an upper bound for the correlation function. This upper bound tends to zero.

Bounds for the numerator and denominator of the correlation function converge to their limits with at least exponential rate and we can also see that the upper bounds can be selected to depend only on ν, s and c, therefore they do not depend on currently selected facets $x_1,\ldots,x_p$ in the argument of the correlation function.

Rate of convergence can be extended to the whole fraction, if we denote by $A(a)$ the value of the numerator and by $B(a)$ the value of the denominator on the left-hand side in (3), and by A and B the limits of the numerator and denominator on the right-hand side in (3), respectively, then there exist $R_1,R_2,S_1,S_2>0$, such that for every $a\geqslant 1$ we have

$$|A(a)-A|<R_1A\mathrm{e}^{-S_1a},\quad |B(a)-B|<R_2B\mathrm{e}^{-S_2a}.$$

If we choose $a_0=\max\{-\frac{1}{S_2}\log\frac{1}{2R_2},1\}$, $R=4\max\{R_1,R_2\}$, and $S=\min\{S_1,S_2\}$, then for $a\geqslant a_0$ we get the bounds

$$\frac{A(a)}{B(a)}-\frac{A}{B}\leqslant\frac{A}{B}\frac{R_1\mathrm{e}^{-S_1a}+R_2\mathrm{e}^{-S_2a}}{1-R_2\mathrm{e}^{-S_2a}}\leqslant\frac{A}{B}R\mathrm{e}^{-Sa},$$
$$\frac{A(a)}{B(a)}-\frac{A}{B}\geqslant-\frac{A}{B}\frac{R_1\mathrm{e}^{-S_1a}+R_2\mathrm{e}^{-S_2a}}{1+R_2\mathrm{e}^{-S_2a}}\geqslant-\frac{A}{B}R\mathrm{e}^{-Sa}.$$

□

P r o o f of Lemma 2. We set

$$I^p(a,c,t,s)=\sum_{n_1=0}^{\infty}\ldots\sum_{n_s=0}^{\infty}\frac{a^{n_1+\ldots+n_s}}{n_1!\ldots n_s!}\exp(\nu R^{c,p}(t,s,\mathbf{n}^{(s)})-a(c-1)).$$

First we calculate the values of the limit by calculating the sum over $(n_1 > 0 \wedge \ldots \wedge n_d > 0)$ to show that this value tends to zero as a tends to infinity. We show this only for $p = 0$ because for $p > 0$, we get the upper bound using $p = 0$, because $R^{c,p} \geqslant R^{c,0}$ and the sum is non-negative. In the following we use the Chernoff bound for tail probabilities of the Poisson distribution

$$\sum_{l=0}^{m} \frac{s^l}{l!} \leqslant \frac{(\mathrm{e}s)^m}{m^m}, \quad m < s.$$

1. First we assume that all the summing variables are between 0 and $a^{2/3}$, i.e., $a^{2/3} > n_1 > 0 \wedge \ldots \wedge a^{2/3} > n_d > 0$. Then

$$\sum_{n_1>0}^{a^{2/3}} \cdots \sum_{n_d>0}^{a^{2/3}} \frac{a^{n_1+\ldots+n_d}}{n_1! \ldots n_d!} \exp(\nu R^{c,0}(0, d, \mathbf{n}^{(d)}) - a(c-1))$$
$$\leqslant \sum_{n_1>0}^{a^{2/3}} \cdots \sum_{n_d>0}^{a^{2/3}} \frac{a^{n_1+\ldots+n_d}}{n_1! \ldots n_d!} \exp(-a(c-1)) \leqslant \left(\frac{(\mathrm{e}a)^{da^{2/3}}}{(a^{2/3})^{da^{2/3}}}\right) \mathrm{e}^{-a(c-1)} \to 0,$$

   where we used d times the Chernoff bound.

2. Now we assume that one of the summing variables is greater than $a^{2/3}$, without loss of generality we select n_d, i.e., $a^{2/3} > n_1 > 0 \wedge \ldots \wedge a^{2/3} > n_{d-1} > 0 \wedge n_d \geqslant a^{2/3}$. Then

$$\sum_{n_1>0}^{a^{2/3}} \cdots \sum_{n_{d-1}>0}^{a^{2/3}} \sum_{n_d \geqslant a^{2/3}}^{\infty} \frac{a^{n_1+\ldots+n_d}}{n_1! \ldots n_d!} \exp(\nu R^{c,0}(0, d, \mathbf{n}^{(d)}) - a(c-1))$$
$$\leqslant \sum_{n_1>0}^{a^{2/3}} \cdots \sum_{n_{d-1}>0}^{a^{2/3}} \sum_{n_d=0}^{\infty} \frac{a^{n_1+\ldots+n_d}}{n_1! \ldots n_d!} \exp(\nu n_d - a(c-1))$$
$$= \sum_{n_1>0}^{a^{2/3}} \cdots \sum_{n_{d-1}>0}^{a^{2/3}} \frac{a^{n_1+\ldots+n_{d-1}}}{n_1! \ldots n_{d-1}!} \exp(a\mathrm{e}^{\nu} - a(c-1))$$
$$\leqslant \frac{(\mathrm{e}a)^{(d-1)a^{2/3}}}{(a^{2/3})^{(d-1)a^{2/3}}} \exp(a\mathrm{e}^{\nu} - a(c-1)) \to 0,$$

   because $\mathrm{e}^{\nu} - (c-1) < 0$.

3. When at least two of the summing variables are greater than $a^{2/3}$, without loss of generality we select n_{d-1} and n_d, then we have

$$\sum_{n_1>0}^{a^{2/3}} \dots \sum_{n_{d-2}>0}^{a^{2/3}} \sum_{n_{d-1}\geqslant a^{2/3}}^{\infty} \sum_{n_d\geqslant a^{2/3}}^{\infty} \frac{a^{n_1+\ldots+n_d}}{n_1!\ldots n_d!} \exp(\nu R^{c,0}(0,d,\mathbf{n}^{(d)}) - a(c-1))$$
$$\leqslant \sum_{n_1>0}^{a^{2/3}} \dots \sum_{n_{d-2}>0}^{a^{2/3}} \sum_{n_{d-1}\geqslant a^{2/3}}^{\infty} \sum_{n_d\geqslant a^{2/3}}^{\infty} \frac{a^{n_1+\ldots+n_d}}{n_1!\ldots n_d!} \exp(\nu a^{4/3} - a(c-1))$$
$$\leqslant \exp(\nu a^{4/3} + a(d+1-c)) \to 0.$$

4. The same applies to the case where more than two variables are greater than $a^{2/3}$, because we are able to find terms with higher power of a in the exponential.

Therefore, we need only to examine the remaining terms where at least one of the variables is equal to zero, thus we replace $I^p(a,c,p,d)$ by d sums, where one variable is set to zero

$$I^p(a,c,p,d) \approx pI^p(a,c,p-1,d-1) + (d-p)I^p(a,c,p,d-1), \tag{11}$$

where $\approx$ is the equality after omitting the summands which tend to zero on the left-hand side, $I^p(a,c,p-1,d-1)$ is the sum after setting to zero one of the variables $n_1,\dots,n_p$, $I^p(a,c,p,d-1)$ is the sum after setting to zero one of the variables $n_{p+1},\dots,n_d$ and the coefficients are the counts of possible selections of these variables. It can be shown that

$$R^{c,p}(t,c-1,\mathbf{n}^{c-1}) \begin{cases} = 0, & t = p, \\ \geqslant n_{c-1}, & t < p, \end{cases}$$

and therefore, we have

$$\lim_{a\to\infty} I^p(a,c,t,c-1) = \begin{cases} 0, & t < p, \\ 1, & t = p. \end{cases} \tag{12}$$

Because the series on the right-hand side of (11) is in the same form as the original one and we can again sum only over the indices where at least one is equal to zero, thus we repeat $(d-c+1)$ times the step in (11) and we get

$$I^p(a,c,p,d) \approx \sum_{j=0}^{d-c+1} c_j I^p(a,c,p-j,c-1), \tag{13}$$

where $c_j \in \mathbb{N}$. All summands tend to zero with one exception of $c_0 I^p(a,c,p,c-1)$ with

$$c_0 = \begin{cases} \dfrac{(d-p)!}{(c-1-p)!}, & c > p, \\ 0, & c = p, \end{cases} \tag{14}$$

which is the number of all selections of variables set to zero from $n_{p+1}, \ldots, n_d$ in $d-c+1$ steps. The overall speed of convergence is implied by the convergence speed of every part of the sum, which converges to its limit at least at exponential rate. □

Proof of Lemma 3. The limit of the correlation function depends only on the number k of distinct orientations among the facets $(u_1, \ldots, u_{|\sigma|})$, then the correlation function tends to $\binom{d-k}{d-c+1}/\binom{d}{d-c+1}$ and thus we can write

$$\begin{aligned}
&\int_{Y^{|\sigma|}} \Big(\bigotimes_{j=1}^{d} (\overline{\mathbb{H}}^{d-j})^{\otimes t_j}\Big)_\sigma (u_1, \ldots, u_{|\sigma|}) \varrho_{|\sigma|}(u_1, \ldots, u_{|\sigma|}, \mu_a^{(c)}) \lambda^{|\sigma|}(\mathrm{d}(u_1, \ldots, u_{|\sigma|})) \\
&= \sum_{k=1}^{d} \binom{d}{k} \int_{(Y^{|\sigma|})_{[k]}} \Big(\bigotimes_{j=1}^{d} ((\overline{\mathbb{H}}^{d-j})^{\otimes m_j})\Big)_\sigma (u_1, \ldots, u_{|\sigma|}) \\
&\qquad\qquad \times \varrho_{|\sigma|}(u_1, \ldots, u_{|\sigma|}, \mu_a^{(c)}) \lambda^{|\sigma|}(\mathrm{d}(u_1, \ldots, u_{|\sigma|})) \\
&\leqslant \sum_{k=1}^{d} \binom{d}{k} \int_{(Y^{|\sigma|})_{[k]}} \Big(\bigotimes_{j=1}^{d} (\overline{\mathbb{H}}^{d-j})^{\otimes t_j}\Big)_\sigma (u_1, \ldots, u_{|\sigma|}) \frac{\binom{d-k}{d-c+1}}{\binom{d}{d-c+1}} \lambda^{|\sigma|}(\mathrm{d}(u_1, \ldots, u_{|\sigma|})) \\
&\quad + \sum_{k=1}^{d} \binom{d}{k} \int_{(Y^{|\sigma|})_{[k]}} \Big(\bigotimes_{j=1}^{d} ((\overline{\mathbb{H}}^{d-j})^{\otimes m_j})\Big)_\sigma (u_1, \ldots, u_{|\sigma|}) \\
&\qquad\qquad \times \left| \frac{\binom{d-k}{d-c+1}}{\binom{d}{d-c+1}} - \varrho_{|\sigma|}(u_1, \ldots, u_{|\sigma|}, \mu_a^{(c)}) \right| \lambda^{|\sigma|}(\mathrm{d}(u_1, \ldots, u_{|\sigma|})) \\
&\leqslant \sum_{k=1}^{c-1} \binom{c-1}{k} \int_{(Y^{|\sigma|})_{[k]}} \Big(\bigotimes_{j=1}^{d} (\overline{\mathbb{H}}^{d-j})^{\otimes t_j}\Big)_\sigma (u_1, \ldots, u_{|\sigma|}) \lambda^{|\sigma|}(\mathrm{d}(u_1, \ldots, u_{|\sigma|})) \\
&\quad + \sum_{k=1}^{d} \binom{d}{k} \int_{(Y^{|\sigma|})_{[k]}} \Big(\bigotimes_{j=1}^{d} (\overline{\mathbb{H}}^{d-j})^{\otimes t_j}\Big)_\sigma (u_1, \ldots, u_{|\sigma|}) \lambda^{|\sigma|}(\mathrm{d}(u_1, \ldots, u_{|\sigma|})) R\mathrm{e}^{-Sa},
\end{aligned}$$

where $(Y^{|\sigma|})_{[k]}$ is a subspace of $Y^{|\sigma|}$, where facets $u_1, \ldots, u_{|\sigma|}$ use orientations $e_1, \ldots, e_k$ (each orientation is used at least by one of the facets), $\binom{d}{k}$ is the number of possible selections of orientations used. We have an upper bound for the expression in the absolute value and we can get a lower bound in the same way. □

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