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INJECTIVITY OF SECTIONS OF CONVEX HARMONIC MAPPINGS AND CONVOLUTION THEOREMS

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Abstract. We consider the class $\mathcal{H}_0$ of sense-preserving harmonic functions $f = h + \overline{g}$ defined in the unit disk $|z| < 1$ and normalized so that $h(0) = 0 = h'(0) - 1$ and $g(0) = 0 = g'(0)$, where h and g are analytic in the unit disk. In the first part of the article we present two classes $\mathcal{P}_H^0(\alpha)$ and $\mathcal{G}_H^0(\beta)$ of functions from $\mathcal{H}_0$ and show that if $f \in \mathcal{P}_H^0(\alpha)$ and $F \in \mathcal{G}_H^0(\beta)$, then the harmonic convolution is a univalent and close-to-convex harmonic function in the unit disk provided certain conditions for parameters α and β are satisfied. In the second part we study the harmonic sections (partial sums)

$$s_{n,n}(f)(z) = s_n(h)(z) + \overline{s_n(g)(z)},$$

where $f = h+\overline{g} \in \mathcal{H}_0$, $s_n(h)$ and $s_n(g)$ denote the n-th partial sums of h and g, respectively. We prove, among others, that if $f = h + \overline{g} \in \mathcal{H}_0$ is a univalent harmonic convex mapping, then $s_{n,n}(f)$ is univalent and close-to-convex in the disk $|z| < 1/4$ for $n \geqslant 2$, and $s_{n,n}(f)$ is also convex in the disk $|z| < 1/4$ for $n \geqslant 2$ and $n \neq 3$. Moreover, we show that the section $s_{3,3}(f)$ of $f \in \mathcal{C}_H^0$ is not convex in the disk $|z| < 1/4$ but it is convex in a smaller disk.

Keywords: harmonic mapping; partial sum; univalent mapping; convex mapping; starlike mapping; close-to-convex mapping; harmonic convolution; direction convexity preserving map

MSC 2010: 30C45

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1. Introduction and main results

One of the interesting features of a univalent harmonic mapping f is that if f is convex (starlike, convex in a direction α, respectively) in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$, then it does not hold in general that the function g defined by $g(z) = r^{-1} f(rz)$ is convex (starlike, convex in a direction α, respectively), for $r < 1$. The aim of this article is to discuss properties such as convolution results and sections of univalent harmonic mappings in the plane. Our theorems are generalizations of known results for univalent analytic mappings, which we now recall.

The class $\mathcal{S}$ of all univalent mappings h analytic in $\mathbb{D}$ normalized by $h(0) = h'(0) - 1 = 0$ is the central object in the study of univalent function theory, see [8], [24]. In 1928, Szegő [36] proved that if $h \in \mathcal{S}$ then all sections $s_n(h)(z) := \sum_{k=1}^{n} a_k z^k$ of $h = \sum_{k=1}^{n} a_k z^k$ are univalent in the disk $|z| < 1/4$ and the number $1/4$ cannot be replaced by a larger one. There exists a considerable amount of results in the literature concerning sections of mappings from $\mathcal{S}$ and some of its various geometric subclasses mentioned later in this section. We refer the reader to [8], Section 8.2, pages 243–246, for a general survey and to recent papers [20], [21], [22], [23], which stimulated further interest on this topic. Moreover, the theory of Hadamard convolution also plays a major role in dealing with such problems. See [9], [10], [32], [34]. However, corresponding questions for the class of univalent harmonic mappings seem to be difficult to handle as can be seen from the recent investigations of the authors [3], [4], [15], [16].

Let $\mathcal{H}$ be the class of all complex-valued harmonic functions $f = h + \overline{g}$ defined on $\mathbb{D}$, where h and g are analytic on $\mathbb{D}$ with the normalization $h(0) = 0 = h'(0) - 1$ and $g(0) = 0$. Set

$$\mathcal{H}_0 = \{f = h + \overline{g} \in \mathcal{H}\colon\ g'(0) = 0\}.$$

According to the work of Lewy [13], a function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense-preserving on $\mathbb{D}$ if and only if its Jacobian $J_f(z)$ is positive in $\mathbb{D}$, where

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2.$$

In view of this result, we observe that $J_f(z) > 0$ in $\mathbb{D}$ if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the (second complex) dilatation $\omega(z) = g'(z)/h'(z)$ of $f = h + \overline{g}$ is analytic in $\mathbb{D}$ and has the property that $|\omega(z)| < 1$ for $z \in \mathbb{D}$.

Following the pioneering work of Clunie and Sheil-Small [2], let $\mathcal{S}_H$ denote the subclass of $\mathcal{H}$ of functions that are sense-preserving and univalent in $\mathbb{D}$, and further let $\mathcal{S}_H^0 = \mathcal{S}_H \cap \mathcal{H}_0$. The class $\mathcal{S}_H^0$ reduces to $\mathcal{S}$ when $g(z)$ is identically zero. Note that each $f = h + \overline{g} \in \mathcal{H}_0$ has the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1.1}$$

For $p \geqslant 1$ and $q \geqslant 2$, we define the harmonic *sections* (*partial sums*) $s_{p,q}(f)$ of $f = h + \overline{g} \in \mathcal{H}_0$ as follows:

$$s_{p,q}(f)(z) = s_p(h)(z) + \overline{s_q(g)(z)}.$$

Also, denote by $\omega_{p,q}(f)$ the dilatation of the harmonic sections $s_{p,q}(f)(z)$.

Recall that a domain Ω is said to be close-to-convex if the complement of Ω can be written as a union of non-intersecting half-lines. A harmonic function $f \in \mathcal{H}$ is said to be convex (close-to-convex, starlike, respectively) in $|z| < r$ if it is univalent and the range $f(|z| < r)$ is convex (close-to-convex, starlike with respect to the origin, respectively). By $\mathcal{C}_H^0$ ($\mathcal{K}_H^0$, $\mathcal{S}_H^{0*}$, respectively) we denote the subclasses of functions in $\mathcal{S}_H^0$ which are convex (close-to-convex, starlike, respectively) in $|z| < 1$ just like $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{S}^*$ are the subclasses of functions in $\mathcal{S}$ mapping $\mathbb{D}$ onto these respective domains. The reader is referred to [2], [5], [6], [26] for many interesting results on planar univalent harmonic mappings.

Szegő [36] also proved that if $h \in \mathcal{C}$ ($\mathcal{S}^*$), then all sections $s_n(h)$ of h are convex (starlike) in the disk $|z| < 1/4$. Miki [19] showed that the same holds for close-to-convex functions in $\mathcal{S}$. We refer to [1], [12], [18], [20], [21], [27], [28], [32], [34], [35] for many interesting results and expositions on this topic for the case of conformal mappings. For the case of univalent harmonic mappings, almost nothing appears in the literature until recently, where for a given $\alpha < 1$, the authors in [15], [16] considered the class

$$\mathcal{P}_H^0(\alpha) = \{f = h + \overline{g} \in \mathcal{H}_0\colon\ \mathrm{Re}\,(h'(z) - \alpha) > |g'(z)| \text{ for } z \in \mathbb{D}\}$$

and discussed the properties of harmonic sections of functions from the class $\mathcal{P}_H^0 := \mathcal{P}_H^0(0)$ (see Lemmas E and F). We note that functions in $\mathcal{P}_H^0(\alpha)$ are univalent and close-to-convex in the unit disk $\mathbb{D}$ whenever $0 \leqslant \alpha < 1$. Moreover, $\mathcal{P}_H^0(\alpha) \subset \mathcal{P}_H^0$ for $0 \leqslant \alpha < 1$ and $\mathcal{P}_H^0 \subset \mathcal{K}_H^0$, so $\mathcal{P}_H^0 \subsetneq \mathcal{S}_H^0$. Also for $\beta < 1$, we define

$$\mathcal{G}_H^0(\beta) = \Big\{f = h + \overline{g} \in \mathcal{H}_0\colon\ \mathrm{Re}\Big(\frac{h(z)}{z}\Big) - \beta > \Big|\frac{g(z)}{z}\Big| \text{ for } z \in \mathbb{D}\Big\}$$

and observe that $\mathcal{G}_H^0(\beta) \subset \mathcal{G}_H^0(0) := \mathcal{G}_H^0$ for $0 \leqslant \beta < 1$. The classes $\mathcal{P}_H^0(\alpha)$ and $\mathcal{G}_H^0(\beta)$ will be considered to state and prove a new convolution result (see Theorem 1.1) along the lines of ideas of Ponnusamy [25] for analytic functions.

We define the harmonic *convolution* (*Hadamard product*) as follows: For $f = h + \overline{g} \in \mathcal{H}$ with the series expansions for h and g as in (1.1), and $F = H + \overline{G} \in \mathcal{H}$, where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} B_n z^n,$$

we define

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n z^n}.$$

Clearly, $f * F = F * f$. Then, for two subsets $\mathcal{P}, \mathcal{Q} \subset \mathcal{H}$, we define $\mathcal{P} * \mathcal{Q} = \{f * g\colon f \in \mathcal{P},\ g \in \mathcal{Q}\}$.

Theorem 1.1. *Let $\alpha, \beta \in [0, 1)$ and $\gamma = 1 - 2(1-\alpha)(1-\beta)$. Then hold $\mathcal{P}_H^0(\alpha) * \mathcal{G}_H^0(\beta) \subset \mathcal{K}_H^0$, whenever $\gamma \geqslant 0$. In particular, $\mathcal{P}_H^0 * \mathcal{G}_H^0(1/2) \subset \mathcal{K}_H^0$ and $\mathcal{P}_H^0(1/2) * \mathcal{G}_H^0 \subset \mathcal{K}_H^0$.*

The proof of Theorem 1.1 will be given in Section 2. We now present an example which shows that there are harmonic functions in $\mathcal{G}_H^0(\beta)$ that are not univalent in $\mathbb{D}$.

Example 1.1. Consider the harmonic function $f(z) = z + a(1-\beta)\overline{z}^2$, where $0 \leqslant \beta < 1$ and $a \in \mathbb{C}$. By the definition of $\mathcal{G}_H^0(\beta)$ it is clear that $f \in \mathcal{G}_H^0(\beta)$ if and only if $|a| \leqslant 1$. A direct calculation shows that f is univalent in $\mathbb{D}$ if and only if $|a| \leqslant (1-\beta)/2$. Thus if a is a complex number such that $|a| \in ((1-\beta)/2, 1]$ then $f \in \mathcal{G}_H^0(\beta)$, but is not necessarily univalent in $\mathbb{D}$.

Remark 1.1. Dorff in [3] (see also [4]) considered $\mathcal{S}_H^0$ mappings that are convex in one direction and these results have been extended by the present authors in [14], [17]. According to Theorem 1.1 and Example 1.1, it follows that the convolution of a non-univalent harmonic function with a certain class of harmonic functions could still be close-to-convex in $\mathbb{D}$. Note that $f(z) = z + \overline{z}^2/2$ belongs to $\mathcal{P}_H^0$ but is not convex in $\mathbb{D}$.

At this place it is worth recalling the well-known fact that the convolution of two convex functions in $\mathcal{C}_H^0$ is not necessarily univalent in $\mathbb{D}$ (see also [3]). To do this, we consider the harmonic convex mapping $f_0 = h_0 + \overline{g_0} \in \mathcal{C}_H^0$, where

$$h_0(z) = \frac{2z - z^2}{2(1-z)^2} \quad \text{and} \quad g_0(z) = \frac{-z^2}{2(1-z)^2}. \tag{1.2}$$

The function f_0 maps $\mathbb{D}$ harmonically onto the half-plane $\{w\colon \operatorname{Re} w > -1/2\}$ and can be obtained as the vertical shear (i.e. shear in the direction $\pi/2$) of the function

$l(z) = z/(1-z)$ with dilatation $\omega(z) = -z$. That is, h_0 and g_0 are obtained as the solution of the linear system

$$h_0(z) + g_0(z) = l(z) \quad \text{and} \quad g_0'(z)/h_0'(z) = -z$$

with the conditions $h_0(0) = g_0(0) = 0$ (see the shearing theorem due to Clunie and Sheil-Small [2], Theorem 5.3). The function f_0 plays the role of extreme for certain extremal problems for the class $\mathcal{C}_H^0$. Now, we see that the convolution $f_0 * f_1$ of the right-half plane mapping f_0 and the hexagon mapping (see [7]) defined by $f_1 = h_1 + \overline{g_1}$, where

$$h_1(z) = z + \sum_{n=2}^{\infty} \frac{z^{6n+1}}{6n+1} \quad \text{and} \quad g_1(z) = -\sum_{n=2}^{\infty} \frac{z^{6n-1}}{6n-1},$$

is not even locally univalent in $\mathbb{D}$. This is because the dilatation $\omega_{f_0 * f_1}$ of $f_0 * f_1$ has the property that

$$|\omega_{f_0*f_1}(z)| = \left|\frac{(g_0 * g_1)'(z)}{(h_0 * h_1)'(z)}\right| = \left|\frac{z^4(2+z^6)}{1+2z^6}\right| \not< 1 \quad \text{for every } z \in \mathbb{D}.$$

In order to state other results, we need to recall some standard notations and results on harmonic mappings.

A domain $D \subset \mathbb{C}$ is said to be convex in the direction α ($\alpha \in \mathbb{R}$) if for every $a \in \mathbb{C}$ the set $D \cap \{a + te^{\mathrm{i}\alpha} \colon t \in \mathbb{R}\}$ is either connected or empty. A univalent harmonic function f defined on $|z| < r$ is said to be *convex in the direction* α if $f(|z| < r)$ is convex in the direction α. We denote by $\mathcal{C}_H(\alpha)$ the family of normalized univalent harmonic functions which are convex in the direction α in $\mathbb{D}$. We may set $\mathcal{C}_H^0(\alpha) := \mathcal{C}_H(\alpha) \cap \mathcal{H}_0$.

Obviously, every function that is convex in the direction α ($0 \leqslant \alpha < \pi$) is necessarily close-to-convex, but the converse is not true. Clearly, a convex function is convex in every direction. The class of functions convex in one direction has been studied by many mathematicians (see, for example, [3], [11]) as a subclass of functions introduced by Robertson [29]. The case $\alpha = 0$ ($\alpha = \pi/2$) is called convex in real (vertical) direction.

Concerning the classical result of Szegő [36] for the class $\mathcal{C}$, it is natural to ask whether every section of $f \in \mathcal{C}_H^0$ is convex in some disk $|z| < r$. Thus, the first task is to derive properties of sections $s_{n,n}(f)$ of $f \in \mathcal{C}_H^0$. Moreover, in our theorems we see that $s_{2,2}(f)$ and $s_{4,4}(f)$ are (fully) convex in the disk $|z| < 1/4$. It is surprising to see that $s_{3,3}(f_0)$ is not convex in the disk $|z| < 1/4$ (see Theorem 4.2 and Figure 1), where f_0 is defined by (1.2).

This leads us to propose the following.

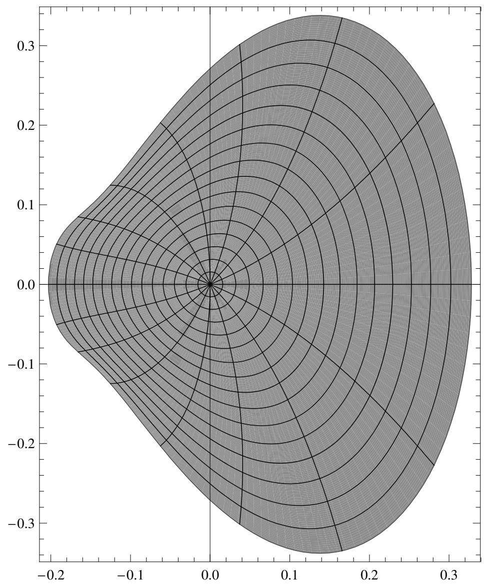

Figure 1. Images of $\mathbb{D}_{1/4}$ under $s_{3,3}(f_0)(z) = z + 3z^2/2 + 2z^3 - \overline{(z^2/2 + z^3)}$.

Problem 1.1. *Suppose that $f \in \mathcal{C}_H^0$. Is each section $s_{n,n}(f)$ convex in the disk $|z| < 1/4$ for $n \geqslant 2$ and $n \neq 3$?*

In this article, we solve this problem and our solution implies that for $n \geqslant 2$ and $n \neq 3$, each section $s_{n,n}(f)$ is convex in the direction of the real axis in the disk $|z| < 1/4$, in particular. On the other hand, Problem 1.1 remains open for the sections $s_{p,q}(f)$ of $f \in \mathcal{C}_H^0$ if $p \neq q$, $p \geqslant 1$ and $q \geqslant 2$. Thus, as in the case of conformal mappings, it is natural to raise the following question.

Problem 1.2. *Suppose that $f \in \mathcal{S}_H^0$ ($\mathcal{S}_H^{0*}$, $\mathcal{K}_H^0$, $\mathcal{C}_H^0$, $\mathcal{C}_H^0(\alpha)$). Determine $\varrho_{p,q}$ so that each section $s_{p,q}(f)$ belongs to the corresponding class in the disk $|z| < \varrho_{p,q}$ for $p \geqslant 1$ and $q \geqslant 2$.*

Solution to Problem 1.1 requires some ideas from the work of Ruscheweyh [31] and Ruscheweyh and Salinas [33].

In Section 3, we discuss the close-to-convexity of $s_{n,n}(f)$. In Section 4, we prove that $s_{2,2}(f)$ of $f \in \mathcal{C}_H^0$ is convex in the disk $|z| < 1/4$ while $s_{3,3}(f_0)$ is not convex in the disk $|z| < 1/4$. Finally, in Section 5, we prove that (see Theorem 5.2) for $n \geqslant 4$, each $s_{n,n}(f)$ is convex in the disk $|z| < 1/4$.

We end this section with the following conjecture.

Conjecture 1.1. *Suppose that $f \in \mathcal{C}_H^0$. Then $s_{3,3}(f)$ is convex in the direction of the real axis as well as the imaginary axis in the disk $|z| < 1/4$.*

2. Convolution theorem

We need the following well-known result which follows easily from the Herglotz representation for analytic functions with positive real part in the unit disk.

Lemma A. *If p is analytic in $\mathbb{D}$, $p(0) = 1$, and $\operatorname{Re} p(z) > 1/2$ in $\mathbb{D}$ then for any function F analytic in $\mathbb{D}$, the function $p * F$ takes values in the convex hull of the image of $\mathbb{D}$ under F.*

We next recall another important result due to Clunie and Sheil-Small [2], which relates the harmonic mapping $f = h + \overline{g}$ to the analytic functions $F_\lambda = h + \lambda g$.

Lemma B ([2]). *If a harmonic mapping $f = h + \overline{g}$ on $\mathbb{D}$ satisfies $|g'(0)| < |h'(0)|$ and the function $F_\lambda = h + \lambda g$ is close-to-convex for all $|\lambda| = 1$, then f is close-to-convex and univalent in $\mathbb{D}$.*

Proof of Theorem 1.1. Let $f_1 \in \mathcal{P}_H^0(\alpha)$ have the canonical decomposition $f_1 = h_1 + \overline{g_1}$ with

$$h_1(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g_1(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{2.1}$$

Let $f_2 \in \mathcal{G}_H^0(\beta)$ have the canonical decomposition $f_2 = h_2 + \overline{g_2}$ with

$$h_2(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad g_2(z) = \sum_{n=2}^{\infty} B_n z^n. \tag{2.2}$$

Now, we define $H = h_1 * h_2 + \overline{g_1 * g_2}$ and $H_\varepsilon = (h_1 * h_2) + \varepsilon(g_1 * g_2)$. Then $H(0) = 0 = H_\varepsilon(0)$ and $H_\varepsilon'(0) = 1$. We need to show that $H \in \mathcal{K}_H^0$. We remark that, as $(h_1 * h_2)'(0) = 1 > (g_1 * g_2)'(0) = 0$, by Lemma B, it is enough to prove that for all ε with $|\varepsilon| = 1$, the function H_ε is close-to-convex in $\mathbb{D}$.

By using the representations (2.1) and (2.2) we have

$$H_\varepsilon'(z) = 1 + \sum_{n=2}^{\infty} n a_n A_n z^{n-1} + \varepsilon \sum_{n=2}^{\infty} n b_n B_n z^{n-1}, \quad |\varepsilon| = 1.$$

Now we claim that $\operatorname{Re} H_\varepsilon'(z) > \gamma$, which will prove that H_ε is in $\mathcal{P}_H^0(\gamma)$.

Since $f_1 \in \mathcal{P}_H^0(\alpha)$, the function F_{ε_1} defined by

$$F_{\varepsilon_1}(z) = z + \frac{\sum_{n=2}^{\infty} a_n z^n + \varepsilon_1\bigl(\sum_{n=2}^{\infty} b_n z^n\bigr)}{1-\alpha}, \quad z \in \mathbb{D},$$

satisfies the condition $\operatorname{Re} F'_{\varepsilon_1}(z) > 0$, for all ε_1 with $|\varepsilon_1| = 1$. A simple calculation shows that the last inequality is equivalent to the inequality

$$\operatorname{Re}\left(1 + \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} n a_n z^{n-1} + \frac{\varepsilon_1}{2(1-\alpha)} \sum_{n=2}^{\infty} n b_n z^{n-1}\right) > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{2.3}$$

Similarly, as the function $f_2 \in \mathcal{G}_H^0(\beta)$, for $|\varepsilon_2| = 1$ we have the inequality

$$\operatorname{Re}\left(\frac{h_2(z)}{z} + \varepsilon_2 \frac{g_2(z)}{z}\right) > \beta, \quad z \in \mathbb{D},$$

which is equivalent to

$$\operatorname{Re}\left(1 + \frac{1}{2(1-\beta)} \sum_{n=2}^{\infty} A_n z^{n-1} + \frac{\varepsilon_2}{2(1-\beta)} \sum_{n=2}^{\infty} B_n z^{n-1}\right) > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{2.4}$$

Using Lemma A and the inequalities (2.3) and (2.4) we get

$$\operatorname{Re}\left(1 + \frac{1}{4(1-\alpha)(1-\beta)} \sum_{n=2}^{\infty} n a_n A_n z^{n-1} + \frac{\varepsilon_1\varepsilon_2}{4(1-\alpha)(1-\beta)} \sum_{n=2}^{\infty} n b_n B_n z^{n-1}\right) > \frac{1}{2}.$$

With $\gamma = 1 - 2(1-\alpha)(1-\beta)$, the above inequality becomes

$$\operatorname{Re}\left(1 + \sum_{n=2}^{\infty} n a_n A_n z^{n-1} + \varepsilon_1\varepsilon_2 \sum_{n=2}^{\infty} n b_n B_n z^{n-1}\right) > \gamma, \quad z \in \mathbb{D},$$

which shows that $\operatorname{Re} H'_{\varepsilon_1\varepsilon_2}(z) > \gamma$ for each $|\varepsilon_1| = 1$ and $|\varepsilon_2| = 1$. In particular, for $\gamma \geqslant 0$, $H_\varepsilon(z)$ is close-to-convex for all ε with $|\varepsilon| = 1$. The proof is complete. □

3. Close-to-convexity of sections $s_{n,n}(f)$ of convex functions f

By using Lemma B due to Clunie and Sheil-Small [2], we obtain the following result.

Theorem 3.1. *Suppose that $f = h + \overline{g} \in \mathcal{H}_0$ is sense-preserving in $\mathbb{D}$ and $F_\lambda = h + \lambda g$ is close-to-convex in $\mathbb{D}$ for every $|\lambda| = 1$. Then $s_{n,n}(f)$ is close-to-convex and univalent in the disk $|z| < 1/4$ for $n \geqslant 2$.*

P r o o f. Let $F_\lambda = h + \lambda g$ be close-to-convex. Then f is locally univalent in $\mathbb{D}$ and it follows that (see Miki [19]) $s_n(F_\lambda)$ is close-to-convex and univalent in the disk $|z| < 1/4$ for all $n \geqslant 2$. In other words, for each $n \geqslant 2$, the section $4s_n(F_\lambda)(z/4)$ is close-to-convex and univalent in the unit disk $|z| < 1$. We observe that

$$4s_n(F_\lambda)\Big(\frac{z}{4}\Big) = 4s_n(h)\Big(\frac{z}{4}\Big) + 4\lambda s_n(g)\Big(\frac{z}{4}\Big),$$

and so,

$$\big|(4s_n(h))'(0)\big| = 1 > 0 = \Big|\Big(4s_n(g)\Big(\frac{z}{4}\Big)\Big)'(0)\Big|.$$

By Lemma B, we find that

$$4s_n(h)\Big(\frac{z}{4}\Big) + \overline{4s_n(g)\Big(\frac{z}{4}\Big)} = 4s_{n,n}(f)\Big(\frac{z}{4}\Big)$$

is close-to-convex and univalent in the disk $|z| < 1$ for all $n \geqslant 2$. The desired conclusion follows. $\square$

Remark 3.1. We wish to emphasize that if $f = h + \overline{g} \in \mathcal{S}_H^{0*}$, then it is not necessarily true that the analytic functions $F_\lambda = h + \lambda g$ are univalent in $\mathbb{D}$ for all $|\lambda| = 1$. For example, for $|\lambda| = 1$, we consider

$$\varphi_\lambda(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \lambda\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} = h(z) + \lambda g(z) = z + \sum_{n=2}^{\infty}\varphi_{\lambda,n}z^n,$$

where

$$\varphi_{\lambda,n} = \frac{1}{6}(2n^2(1+\lambda) + 3n(1-\lambda) + (1+\lambda)) \quad \text{for all } n \geqslant 2.$$

When $\lambda = -1$, $\varphi_\lambda(z)$ reduces to the analytic Koebe function $k(z) = z/(1-z)^2$, which is univalent and starlike in $\mathbb{D}$. Moreover, $\varphi_\lambda(z)$ is easily seen to be univalent only for $\lambda = -1$. For φ_λ to be univalent in $\mathbb{D}$, it is necessary that $|\varphi_{\lambda,n}| \leqslant n$ for all $n \geqslant 2$. For $|\lambda| = 1$ ($\lambda \neq -1$), we see that $|\varphi_{\lambda,n}| > n$ for large values of n and hence, for these values of λ, $\varphi_\lambda(z)$ is not univalent in $\mathbb{D}$. Also, we observe that $K(z) = h + \overline{g}$ is the harmonic Koebe mapping which is indeed starlike in $\mathbb{D}$. This example shows that there is a limitation on the use of Lemma B. However, an analogue of Theorem 3.1 holds for the family $\mathcal{C}_H^0$ of univalent harmonic convex mappings.

Theorem 3.2. *Let $f = h + \overline{g} \in \mathcal{C}_H^0$. Then every section $s_{n,n}(f)$ is close-to-convex in the disk $|z| < 1/4$ for $n \geqslant 2$. In particular, $s_{n,n}(f)$ is univalent and sense-preserving in $|z| < 1/4$ for $n \geqslant 2$. The number $1/4$ cannot be replaced by a greater one.*

P r o o f. Let $f = h + \overline{g} \in \mathcal{C}_H^0$. Then the analytic functions $F_\lambda = h + \lambda g$ are close-to-convex in $\mathbb{D}$ (see [2], Theorem 5.7) for all $|\lambda| = 1$. According to the last observation and Theorem 3.1, we obtain that every section $s_{n,n}(f)$ is close-to-convex in the disk $|z| < 1/4$ for $n \geqslant 2$.

Next we prove the sharpness part. Consider the function $f_0 = h_0 + \overline{g_0} \in \mathcal{C}_H^0$ defined by (1.2). Then for $n = 2$, we see that $s_2'(h_0)(z) = 1+3z$ and $s_2'(g_0)(z) = -z$. Therefore, the dilatation $\omega_{2,2}(f_0)$ of f_0 is given by

$$\omega_{2,2}(f_0)(z) = \frac{s_2'(g_0)(z)}{s_2'(h_0)(z)} = \frac{-z}{1+3z}. \tag{3.1}$$

Since the Möbius transformation $w = M(z) = -z/(1+3z)$ maps the disk $|z| < 1/4$ onto the disk $|w - 3/7| < 4/7$, the relation (3.1) implies that $|\omega_{2,2}(f_0)(z)| < 1$ for $|z| < 1/4$. Moreover, at the boundary point $z = -1/4$, we have $\omega_{2,2}(f_0)(-1/4) = M(-1/4) = 1$, which shows that the radius 1/4 cannot be replaced by a larger one. The proof is complete. □

4. The sections $s_{2,2}(f)$ and $s_{3,3}(f_0)$

Let $\mathcal{A}_0$ denote the class of all functions $h(z) = \sum\limits_{k=1}^{\infty} a_k z^k$ analytic on the unit disk $\mathbb{D}$ and $\mathcal{A} = \{h \in \mathcal{A}_0\colon\ h'(0) = 1\}$.

A function $g \in \mathcal{A}_0$ is called *direction convexity preserving* ($g \in$ DCP) if and only if $g * h \in \mathcal{C}(\alpha)$ for all $h \in \mathcal{C}(\alpha)$ and all $\alpha \in \mathbb{R}$. Here $\mathcal{C}(\alpha)$ denotes the family of normalized univalent analytic functions in $\mathbb{D}$ which are convex in the direction α.

The class DCP is somewhat special in the following sense: for $g \in$ DCP, we do not necessarily have $g_r(z) := g(rz) \in$ DCP for $0 < r < 1$. We therefore define the DCP radius of an analytic function g to be $\max\{r\colon\ g_\varrho \in \text{DCP for } 0 < \varrho < r\}$.

From [31], we observe the following fact.

Lemma C. $s_2(z) = z + z^2 \in$ DCP *in the disk* $|z| < 1/4$.

We extend this lemma in Theorem 5.1 for an arbitrary section $s_n(z)$ of $z/(1-z)$. Let us now recall a convolution characterization for a function to be in the class DCP.

Lemma D ([33]). *Let* $p \in \mathcal{A}_0$. *Then* $p \,\widetilde{*}\, f := p * h + \overline{p * g} \in \mathcal{C}_H^0$ *for every* $f = h + \overline{g} \in \mathcal{C}_H^0$ *if and only if* $p \in$ DCP.

Before we proceed to state and prove our main results of this section, it is appropriate to include the definition of (fully) convex mappings and some known results on sections of functions from the class $\mathcal{P}_H^0$. For sense-preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}$, one has

$$\frac{\partial}{\partial\theta}\Big(\arg\Big(\frac{\partial}{\partial\theta} f(re^{i\theta})\Big)\Big) = \operatorname{Re}\frac{D^2 f(z)}{Df(z)} = \operatorname{Re}\frac{z(h'(z)+zh''(z)) + \overline{z(g'(z)+zg''(z))}}{zh'(z) - \overline{zg'(z)}},$$

where $z = re^{i\theta}$, $Df = zf_z - \overline{z}f_{\overline{z}}$ and $D^2 f = D(Df)$. Recall that if $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving, $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and the condition

$$\operatorname{Re}\frac{z(h'(z)+zh''(z)) + \overline{z(g'(z)+zg''(z))}}{zh'(z) - \overline{zg'(z)}} > 0 \quad \text{for all } z \in \mathbb{D}\setminus\{0\}$$

is satisfied, then f is univalent and *fully convex* in $\mathbb{D}$, i.e., the image of every subdisk $|z| < r < 1$ under f is convex.

It is appropriate to recall two recent results of the authors.

Lemma E ([16], Theorems 4, 5 and 6)**.** *Let $f \in \mathcal{P}_H^0$. Suppose that p and q satisfy one of the following conditions:*
(a) *$p = 1$ and $q \geqslant 2$,*
(b) *$3 \leqslant p < q$,*
(c) *$p = q \geqslant 2$,*
(d) *$p > q \geqslant 3$,*
(e) *$p = 3$ and $q = 2$.*
Then $s_{p,q}(f)$ is univalent and close-to-convex in $|z| < 1/2$. Moreover, we have
(f) *for $2 < q$, $s_{2,q}(f)$ is univalent and close-to-convex in $|z| < (3-\sqrt{5})/2 \approx 0.381966$,*
(g) *for $p \geqslant 4$, $s_{p,2}(f)$ is univalent and close-to-convex in $|z| < 0.433797$.*

Lemma F ([15], Theorems 2, 3 and 4)**.** *Let $f = h + \overline{g} \in \mathcal{P}_H^0$, and suppose that p and q satisfy one of the following conditions:*
(a) *$p = 1$ and $q \geqslant 2$,*
(b) *$3 \leqslant p < q$,*
(c) *$p = q \geqslant 2$,*
(d) *$p > q \geqslant 3$.*
Then $s_{p,q}(f)$ is convex in $|z| < 1/4$.
(e) *If $p = 2 < q$, then $s_{2,q}(f)$ is convex in $|z| < 0.210222$.*
(f) *If $q = 2 < p$, then $s_{p,2}(f)$ is convex in $|z| < 0.234906$.*

Now we explore the disk of convexity of $s_{n,n}(f)(z)$ when $f \in \mathcal{C}_H^0$. For $n = 2$, we obtain the following.

Theorem 4.1. *Let $f = h + \overline{g} \in \mathcal{C}_H^0$, where $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=2}^{\infty} b_n z^n$. Then the section $s_{2,2}(f) = z + a_2 z^2 + \overline{b_2 z^2}$ is convex in the disk $|z| < 1/4$. The number $1/4$ cannot be replaced by a greater one.*

Proof. Set $s_2(z) = z + z^2$. Then, by Lemmas C and D, we conclude that $r^{-1} s_2(rz) \widetilde{*} f(z)$ is convex in $\mathbb{D}$ for $0 < r \leqslant 1/4$. Since

$$r^{-1} s_2(rz) \widetilde{*} f(z) = z + r a_2 z^2 + \overline{r b_2 z^2} = r^{-1} s_{2,2}(f)(rz),$$

it follows that $r^{-1} s_{2,2}(f)(rz)$ is convex in $\mathbb{D}$ for $0 < r \leqslant 1/4$. This means that the section $s_{2,2}(f)$ is (fully) convex in the disk $|z| < 1/4$.

In order to prove the sharpness part, we consider the section $s_{2,2}(f_0)$ of $f_0 = h_0 + \overline{g_0} \in \mathcal{C}_H^0$, where h_0 and g_0 are given by (1.2). Note that

$$s_{2,2}(f_0)(z) = s_2(h_0)(z) + \overline{s_2(g_0)(z)} = z + \Big(\frac{3}{2}\Big) z^2 - \Big(\frac{1}{2}\Big) \overline{z^2}.$$

A computation gives

$$\operatorname{Re} \frac{z(z s_2'(h_0)(z))' + \overline{z(z s_2'(g_0)(z))'}}{z s_2'(h_0)(z) - \overline{z s_2'(g_0)(z)}} = \operatorname{Re} \frac{z + 6z^2 - 2\overline{z}^2}{z + 3z^2 + \overline{z}^2} = \operatorname{Re} \frac{1 + w(z)}{1 - w(z)},$$

where

$$w(z) = \frac{3z^2 - 3\overline{z}^2}{2z + 9z^2 - \overline{z}^2} \quad \text{and} \quad \lim_{z \to 0} \frac{1 + w(z)}{1 - w(z)} = 1.$$

Thus, for the convexity of $s_{2,2}(z)$ in the disk $|z| < 1/4$, it suffices to prove that $|w(z)| < 1$ for $0 < |z| < 1/4$, which is equivalent to

$$G(z) = |3z^2 - 3\overline{z}^2|^2 - |2z + 9z^2 - \overline{z}^2|^2 < 0 \quad \text{for } 0 < |z| < \frac{1}{4}.$$

Let $z = r\mathrm{e}^{\mathrm{i}\theta}$. Then a computation yields

$$\begin{aligned}
&G(r\mathrm{e}^{\mathrm{i}\theta}) \\
&\quad = 36r^4 \sin^2 2\theta - [(2r\cos\theta + 8r^2\cos 2\theta)^2 + (2r\sin\theta + 10r^2 \sin 2\theta)^2] \\
&\quad = 36r^4 \sin^2 2\theta - (4r^2 + 64r^4 + 36r^4 \sin^2 2\theta + 32r^3 \cos\theta\cos 2\theta + 40r^3 \sin\theta \sin 2\theta) \\
&\quad = -[4r^2 + 64r^4 + 32r^3\cos\theta(1 - 2\sin^2\theta) + 64r^3\sin^2\theta\cos\theta + 16r^3\sin^2\theta\cos\theta] \\
&\quad = -4r^2[1 + 16r^2 + 4r\cos\theta(2 + \sin^2\theta)] \\
&\quad = -4r^2[1 + 16r^2 + 4r\cos\theta(3 - \cos^2\theta)].
\end{aligned}$$

We observe that the function $B(x) = x(3 - x^2)$ is increasing on $[-1, 1]$ and therefore, from the last relation, we see that

$$G(re^{i\theta}) \leqslant -4r^2[1 + 16r^2 + 4rB(-1)] = -4r^2[1 + 16r^2 - 8r] = -4r^2(4r - 1)^2$$

for $r < 1/4$ and $-\pi < \theta \leqslant \pi$ with equality for $\theta = \pi$. Thus, $G(z) < 0$ for $0 < |z| < 1/4$ and hence, $|w(z)| < 1$ for $|z| < 1/4$. Finally, $s_{2,2}(f_0)$ of $f_0 = h_0 + \overline{g_0} \in \mathcal{C}_H^0$ is (fully) convex for $|z| < 1/4$ but not in a larger disk. The proof is complete. $\square$

For $n = 3$, we will show that $s_{3,3}(f_0)(z)$ is not convex in $|z| < 1/4$.

Theorem 4.2. *The harmonic section*

$$s_{3,3}(f_0)(z) = s_3(h_0)(z) + \overline{s_3(g_0)(z)} = z + \frac{3}{2}z^2 + 2z^3 - \frac{1}{2}\overline{z}^2 - \overline{z}^3$$

is not convex in the disk $|z| < 1/4$. Here $f_0 = h_0 + \overline{g_0} \in \mathcal{C}_H^0$, where h_0 and g_0 are given by (1.2).

P r o o f. By Theorem 3.2, $s_{3,3}(f_0)(z)$ is locally one-to-one and sense-preserving in $|z| < 1/4$. Now, by a computation, we have

$$F(z) = \operatorname{Re} \frac{z(zs_3'(h_0)(z))' + \overline{z(zs_3'(g_0)(z))'}}{zs_3'(h_0)(z) - \overline{zs_3'(g_0)(z)}} = \operatorname{Re} \frac{z + 6z^2 + 18z^3 - 2\overline{z}^2 - 9\overline{z}^3}{z + 3z^2 + 6z^3 + \overline{z}^2 + 3\overline{z}^3}. \tag{4.1}$$

Let $z_0 = \frac{1}{4}e^{2i\pi/3}$. Then, it follows that

$$F(z_0) = \operatorname{Re} \frac{\frac{1}{4}e^{2i\pi/3} + \frac{6}{16}e^{4i\pi/3} - \frac{2}{16}e^{2i\pi/3} + \frac{9}{64}}{\frac{1}{4}e^{2i\pi/3} + \frac{3}{16}e^{4i\pi/3} + \frac{1}{16}e^{2i\pi/3} + \frac{9}{64}}$$
$$= \operatorname{Re} \frac{\frac{1}{8}e^{2i\pi/3} + \frac{3}{8}e^{-2i\pi/3} + \frac{9}{64}}{\frac{5}{16}e^{2i\pi/3} + \frac{3}{16}e^{-2i\pi/3} + \frac{9}{64}} = \operatorname{Re} \frac{-\frac{7}{64} - \frac{\sqrt{3}}{8}i}{-\frac{7}{64} + \frac{\sqrt{3}}{16}i} = -\frac{47}{97} < 0.$$

This means that $s_{3,3}(f_0)(z)$ is not convex in the disk $|z| < 1/4$. $\square$

Remark 4.1. For the function $f_0 = h_0 + \overline{g_0} \in \mathcal{C}_H^0$ defined by (1.2), it can be easily seen that the function $F(z)$ defined by (4.1) satisfies the positivity condition $F(z) > 0$ for $|z| < 0.201254$ and thus, the disk of convexity of $s_{3,3}(f_0)$ is $|z| < r$, where r is close to the value 0.201254. Since the computation is lengthy, we do not wish to address it for the moment. However, in Theorem 5.3, we actually show that the section $s_{3,3}(f)(z)$ of every $f = h + \overline{g} \in \mathcal{C}_H^0$ is indeed convex in the disk $|z| < 0.201254$.

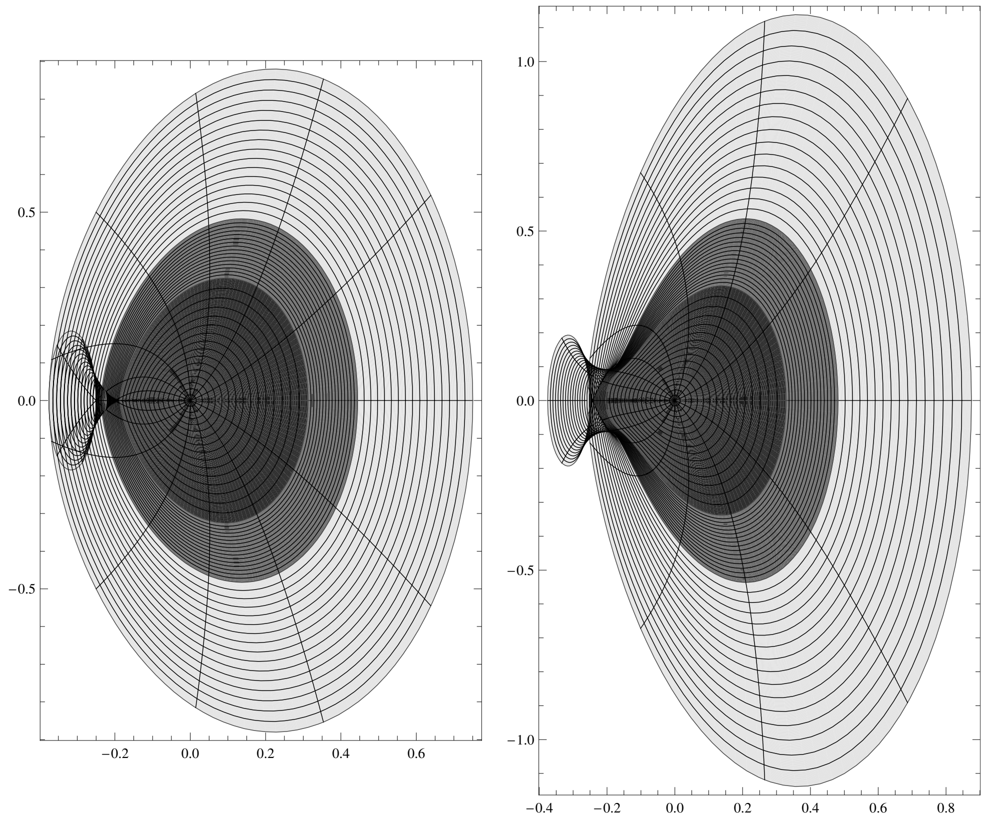

Figure 2. Images of $|z| < 1/4$, $1/4 < |z| < 1/3$, and $1/3 < |z| < 1/2$ under $s_{2,2}(f_0)(z)$ and $s_{3,3}(f_0)(z)$.

In Figure 2, images of $|z| < 1/4$, $1/4 < |z| < 1/3$, and $1/3 < |z| < 1/2$ under $s_{2,2}(f_0)(z)$ and $s_{3,3}(f_0)(z)$ are drawn in different shades. These pictures were drawn using Mathematica as plots of the images of equally spaced radial segments and concentric circles of the corresponding disk and of the two annuli.

5. Disk of convexity of $s_{n,n}(f)$

We need the following result for the proof of two remaining theorems.

Lemma G ([33], Theorem 2)**.** *Let g be analytic in $\mathbb{D}$. Then $g \in \mathrm{DCP}$ if and only if for each $t \in \mathbb{R}$, $g + \mathrm{i}tzg'$ is convex in the direction of the imaginary axis.*

For the proof of Theorem 5.1, we use a result of Royster and Ziegler [30] concerning analytic mappings convex in one direction.

Lemma H ([30], Theorem 1). *Let $\varphi(z)$ be a non-constant function analytic in $\mathbb{D}$. The function $\varphi(z)$ maps univalently $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis if and only if there are numbers μ and ν, $0 \leqslant \mu < 2\pi$ and $0 \leqslant \nu \leqslant \pi$, such that*

$$\operatorname{Re}\{F_{\mu,\nu}(z)\varphi'(z)\} \geqslant 0 \tag{5.1}$$

for all $z \in \mathbb{D}$, where $F_{\mu,\nu}(z) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\mu}(1 - 2z\mathrm{e}^{-\mathrm{i}\mu}\cos\nu + z^2\mathrm{e}^{-2\mathrm{i}\mu})$.

By using Lemmas G and H, we now prove the following theorem for $n \geqslant 4$ and in view of the technical details we present the proof of the case $n = 3$ separately in Theorem 5.3.

Theorem 5.1. *The section $s_n(z) := \sum\limits_{k=1}^{n} z^k = (z - z^{n+1})/(1 - z) \in \mathrm{DCP}$ in the disk $|z| < 1/4$ for $n \geqslant 4$.*

Proof. Let $\varphi(z) = s_n(z) + \mathrm{i}tzs_n'(z)$, where $t \in \mathbb{R}$. A computation yields that

$$\begin{aligned}\varphi'(z) &= \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} + \mathrm{i}t\frac{1-(n+1)^2z^n + n(n+2)z^{n+1}}{(1-z)^2} \\ &\quad - \mathrm{i}t\frac{2nz^{n+1}}{(1-z)^2} + \mathrm{i}t\frac{2\sum_{k=1}^n z^k}{(1-z)^2} \\ &= \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} + \mathrm{i}t\frac{1-(n+1)^2z^n + n^2z^{n+1} + 2\sum_{k=1}^n z^k}{(1-z)^2}.\end{aligned}$$

We now divide our proof into the following three cases.

Case 1: $t > 2/19$. Let $\mu = \nu = 0$. Then $F_{0,0}(z) = -\mathrm{i}(1-z)^2$. It follows that

$$F_{0,0}(z)\varphi'(z) = t\Bigg[1-(n+1)^2z^n + n^2z^{n+1} + 2\sum_{k=1}^n z^k\Bigg] - \mathrm{i}[1-(n+1)z^n + nz^{n+1}]$$

and

$$\operatorname{Re}\{F_{0,0}(z)\varphi'(z)\} \geqslant t - t\Bigg[(n+1)^2|z|^n + n^2|z|^{n+1} + 2\sum_{k=1}^n |z|^k\Bigg] - (n+1)|z|^n - n|z|^{n+1}.$$

It suffices to prove that the right hand side of the above inequality is larger than 0 for $|z| = 1/4$ and for all $n \geqslant 4$, since it is harmonic in $|z| < 1/4$. For $|z| = 1/4$, the above estimate takes the form

$$\begin{aligned}\operatorname{Re}\{F_{0,0}(z)\varphi'(z)\} &\geqslant t - t\Big[\frac{5n^2+8n+4}{4^{n+1}} + \frac{2-\frac{2}{4^n}}{3}\Big] - \frac{5n+4}{4^{n+1}} \\ &= \frac{t}{3} - \frac{5tn^2 + (8t+5)n + \frac{4t}{3} + 4}{4^{n+1}} := A(n).\end{aligned}$$

We see that $A(n)$ is monotonically increasing with respect to n for $n \geqslant 4$. It follows that

$$A(n) \geqslant A(4) = \frac{57t}{4^4} - \frac{6}{4^4} = \frac{3}{4^4}(19t-2) > 0$$

for $t > 2/19$, which implies that $\mathrm{Re}\{F_{0,0}(z)\varphi'(z)\} > 0$ for $n \geqslant 4$ and $|z| = 1/4$. Lemma H implies that $\varphi(z)$ is convex in the direction of the imaginary axis in the disk $|z| < 1/4$ if $t > 2/19$ and $n \geqslant 4$.

Case 2: $t < -2/19$. Let $\mu = \nu = \pi$. Then $F_{\pi,\pi}(z) = \mathrm{i}(1-z)^2$. It follows that

$$F_{\pi,\pi}(z)\varphi'(z) = -t\left[1-(n+1)^2z^n + n^2z^{n+1} + 2\sum_{k=1}^{n} z^k\right] + \mathrm{i}[1-(n+1)z^n + nz^{n+1}].$$

By a similar reasoning as in Case 1, we obtain that $\mathrm{Re}\{F_{\pi,\pi}(z)\varphi'(z)\} > 0$ for $n \geqslant 4$ and $|z| < 1/4$. By Lemma H, we thus see that $\varphi(z)$ is convex in the direction of the imaginary axis in the disk $|z| < 1/4$ if $t < -2/19$ and $n \geqslant 4$.

Case 3: $-2/19 \leqslant t \leqslant 2/19$. Let $\mu = \nu = \pi/2$. Then $F_{\pi/2,\pi/2}(z) = 1 - z^2 = (1-z)(1+z)$. It follows that

$$\begin{aligned} F_{\pi/2,\pi/2}(z)\varphi'(z) = {} & \frac{1+z}{1-z} + \frac{1+z}{1-z}(nz^{n+1} - (n+1)z^n) + \mathrm{i}t\frac{1+z}{1-z} \\ & + \mathrm{i}t\frac{1+z}{1-z}\left(-(n+1)^2z^n + n^2z^{n+1} + 2\sum_{k=1}^{n} z^k\right), \end{aligned}$$

and therefore,

$$\begin{aligned} \mathrm{Re}(F_{\pi/2,\pi/2}(z)\varphi'(z)) \geqslant {} & \frac{1-|z|}{1+|z|} - \frac{2\,|t|\,|z|}{(1-|z|)^2} - \frac{1+|z|}{1-|z|}(n|z|^{n+1} + (n+1)|z|^n) \\ & - |t|\frac{1+|z|}{1-|z|}\left((n+1)^2|z|^n + n^2|z|^{n+1} + 2\sum_{k=1}^{n}|z|^k\right). \end{aligned}$$

For $|z| = 1/4$, the above estimate takes the following form

$$\begin{aligned} \mathrm{Re}(F_{\pi/2,\pi/2}(z)\varphi'(z)) & \geqslant \frac{3}{5} - \frac{8|t|}{9} - \frac{5}{3}\frac{5n+4}{4^{n+1}} - \frac{5|t|}{3}\left(\frac{5n^2+8n+4}{4^{n+1}} + \frac{2}{3} - \frac{2}{3}\frac{1}{4^n}\right) \\ & = \frac{3}{5} - \frac{18|t|}{9} - \frac{5}{3}\frac{5|t|n^2 + (8|t|+5)n + 4 + \frac{4}{3}|t|}{4^{n+1}} := B(n). \end{aligned}$$

We observe that $B(n)$ is monotonically increasing with respect to n for $n \geqslant 4$. Hence,

$$B(n) \geqslant B(4) = \frac{1}{4^3}\left(\frac{359}{10} - \frac{5033|t|}{36}\right) > 0$$

for $-2/19 \leqslant t \leqslant 2/19$. Again, by Lemma H, we obtain that $\varphi(z)$ is convex in the direction of the imaginary axis in the disk $|z| < 1/4$ if $-2/19 \leqslant t \leqslant 2/19$ and for all $n \geqslant 4$.

The desired conclusion follows from Lemma G. □

Theorem 5.2. *Let $f = h + \overline{g} \in \mathcal{C}_H^0$. Then $s_{n,n}(f)$ is convex in the disk $|z| < 1/4$ for $n \geqslant 4$.*

P r o o f. By Theorem 5.1 and Lemma D, we conclude that $r^{-1}s_n(rz) \widetilde{*} f(z)$ is convex in $\mathbb{D}$ for $0 < r \leqslant 1/4$ and $n \geqslant 4$. Since

$$r^{-1}s_n(rz) \widetilde{*} f(z) = r^{-1}s_{n,n}(f)(rz),$$

it follows that $r^{-1}s_{n,n}(f)(rz)$ is convex in $\mathbb{D}$ for $0 < r \leqslant 1/4$ and $n \geqslant 4$. This means that the section $s_{n,n}(f)$ is (fully) convex in the disk $|z| < 1/4$ for $n \geqslant 4$. $\square$

Theorem 5.3. *Let $f = h + \overline{g} \in \mathcal{C}_H^0$. Then $s_{3,3}(f)$ is convex in the disk $|z| < 0.201254$.*

P r o o f. As in the proof of Theorem 5.2, it suffices to show that $s_3(z) := z + z^2 + z^3 \in \mathrm{DCP}$ in the disk $|z| < 0.201254$.

We only have to give the crucial steps and appropriate replacements in the proof of Theorem 5.1 for $n = 3$ and the rest of arguments follows from there. Thus, if $\varphi(z)$ is as in the proof of Theorem 5.1 with $n = 3$, then $\varphi'(z)$ takes the form

$$\varphi'(z) = \frac{1 - 4z^3 + 3z^4}{(1 - z)^2} + \mathrm{i}t\frac{1 - 16z^3 + 9z^4 + 2\sum_{k=1}^{3} z^k}{(1 - z)^2}.$$

Case 1: $t > 0.105712$. It follows from the proof of Theorem 5.1 that

$$\mathrm{Re}\{F_{0,\,0}(z)\varphi'(z)\} \geqslant t - t\left[16|z|^3 + 9|z|^4 + 2\sum_{k=1}^{3}|z|^k\right] - 4|z|^3 - 3|z|^4,$$

which for $|z| \leqslant 0.201254$ implies that

$$\begin{aligned}\mathrm{Re}\{F_{0,0}(z)\varphi'(z)\} \geqslant t\Big[1 &- 16(0.201254)^3 - 9(0.201254)^4 - 2\sum_{k=1}^{3}(0.201254)^k\Big] \\ &- 4(0.201254)^3 - 3(0.201254)^3 > 0\end{aligned}$$

for $t > t_0 \approx 0.10571184$. In particular, by Lemma H, we obtain that $\varphi(z)$ is convex in the direction of the imaginary axis in the disk $|z| < 0.201254$ if $t > 0.105712$.

Case 2: $t < -0.105712$. With $\mu = \nu = \pi$ we have $F_{\pi,\pi}(z) = \mathrm{i}(1 - z)^2$, and

$$F_{\pi,\pi}(z)\varphi'(z) = -t\left[1 - 16z^3 + 9z^4 + 2\sum_{k=1}^{3} z^k\right] + \mathrm{i}[1 - 4z^3 + 3z^4]$$

and by a similar reasoning as in Case 1, we obtain that

$$\operatorname{Re}\{F_{\pi,\pi}(z)\varphi'(z)\} > 0 \quad \text{for } |z| < 0.201254$$

and thus, $\varphi(z)$ is convex in the direction of the imaginary axis in the disk $|z| < 0.201254$ if $t < -0.105712$.

Case 3: $-0.105712 \leqslant t \leqslant 0.105712$. This case corresponds to $\mu = \nu = \pi/2$, so $F_{\pi/2,\pi/2}(z) = 1 - z^2$ and

$$\begin{aligned}\operatorname{Re}(F_{\pi/2,\pi/2}(z)\varphi'(z)) \geqslant \frac{1-|z|}{1+|z|} - \frac{2\,|t|\,|z|}{(1-|z|)^2} - \frac{1+|z|}{1-|z|}(3|z|^4 + 4|z|^3) \\ - |t|\frac{1+|z|}{1-|z|}\left(16|z|^3 + 9|z|^4 + 2\sum_{k=1}^{3}|z|^k\right).\end{aligned}$$

For $|z| = 0.201254$, the above estimate shows that

$$\operatorname{Re}(F_{\pi/2,\pi/2}(z)\varphi'(z)) \geqslant 0.608489 - 1.60093|t| > 0$$

for $|t| < 0.608489/1.60093$ (> 0.105712). Consequently, by Lemma H, we obtain that $\varphi(z)$ is convex in the direction of imaginary axis in the disk $|z| < 0.201254$ if $-0.105712 \leqslant t \leqslant 0.105712$ and for $n = 3$.

The cases 1 to 3 show that $s_3(z) := z + z^2 + z^3 \in \mathrm{DCP}$ in the disk $|z| < 0.201254$. □

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