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AF-ALGEBRAS AND TOPOLOGY OF MAPPING TORI

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Abstract. The paper studies applications of C^*-algebras in geometric topology. Namely, a covariant functor from the category of mapping tori to a category of AF-algebras is constructed; the functor takes continuous maps between such manifolds to stable homomorphisms between the corresponding AF-algebras. We use this functor to develop an obstruction theory for the torus bundles of dimension 2, 3 and 4. In conclusion, we consider two numerical examples illustrating our main results.

Keywords: Anosov diffeomorphism; AF-algebra

MSC 2010: 46L85, 55S35

1. Introduction

This paper studies applications of operator algebras in topology; the operator algebras in question are the so-called *AF-algebras* and the topological spaces are certain *mapping tori*, i.e., circle bundles with a fiber M and monodromy $\varphi\colon M \to M$. Recall that a very fruitful approach to topology consists in construction of maps (functors) from topological spaces to certain algebraic objects, so that continuous maps between the spaces become homomorphisms of the corresponding algebraic entities. The functors usually take value in the finitely generated groups (abelian or not) and, therefore, reduce topology to a simpler algebraic problem.

The rings of operators on a Hilbert space are neither finitely generated nor commutative and, at the first glance, if ever such a reduction exists, it will not simplify the problem. Yet it is not so: we define an operator algebra, the so-called fundamental AF-algebra, which yields a set of simple obstructions (invariants) to the existence

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of continuous maps in a class of manifolds fibering over the circle. One obstruction turns out to be the Galois group of the fundamental AF-algebra; this invariant dramatically simplifies for a class of the so-called tight torus bundles, so that topology boils down to a division test for a finite set of natural numbers.

1.1. AF**-algebras** [4]**.** The C^*-algebra A is an algebra over the complex numbers $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$, $a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leqslant \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for every $a, b \in A$. Any commutative C^*-algebra A is isomorphic to the C^*-algebra $C_0(X)$ of continuous complex-valued functions on a locally compact Hausdorff space X vanishing at infinity; the algebras which are not commutative are deemed as non-commutative topological spaces. A *stable homomorphism* $A \to A'$ is defined as the (usual) homomorphism $A \otimes \mathcal{K} \to A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a Hilbert space; such a homomorphism corresponds to a continuous map between the non-commutative spaces A and A'.

The matrix algebra $M_n(\mathbb{C})$ is an example of non-commutative finite-dimensional C^*-algebra; a natural generalization are approximately finite-dimensional (AF-)algebras, which are given by an ascending sequence $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots$ of finite-dimensional semi-simple C^*-algebras $M_i = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$ and homomorphisms φ_i arranged into an infinite graph as follows. Two sets of vertices $V_{i_1}, \ldots, V_{i_k}$ and $V_{i'_1}, \ldots, V_{i'_k}$ are joined by the b_{rs} edges, whenever the summand M_{i_r} contains b_{rs} copies of the summand $M_{i'_s}$ under the embedding φ_i; as $i \to \infty$, one gets a *Bratteli diagram* of the AF-algebra. Such a diagram is defined by an infinite sequence of *incidence matrices* $B_i = (b_{rs}^{(i)})$. If the homomorphisms $\varphi_1 = \varphi_2 = \ldots = \mathrm{const}$, the AF-algebra is called *stationary*; its Bratteli diagram looks like an infinite graph with the incidence matrix $B = (b_{rs})$ repeated over and over again.

1.2. AF**-algebra of measured foliation.** Let M be a compact manifold of dimension m and $\mathcal{F}$ a codimension k measured foliation of M; it is known that $\mathcal{F}$ is tangent to the hyperplane $\omega(p) = 0$ at each point $p \in M$, where $\omega \in H^k(M; \mathbb{R})$ is a closed k-form, see e.g. [8]. Denote by $\lambda_i > 0$ the periods of ω against a basis in the homology group $H_k(M)$ and consider the vector $\theta = (\theta_1, \ldots, \theta_{n-1})$, where $\theta_i = \lambda_{i+1}/\lambda_1$ and $n = \operatorname{rank} H_k(M)$. Let $\lim_{i\to\infty} B_i$ be the Jacobi-Perron continued fraction convergent to the vector $(1, \theta)$; here $B_i \in GL_n(\mathbb{Z})$ are the nonnegative matrices with $\det(B_i) = 1$, see [3].

An AF-algebra $\mathbb{A}_{\mathcal{F}}$ is called *associated* to $\mathcal{F}$, if its Bratteli diagram is given by the matrices B_i; the Bratteli diagram defines an isomorphism class of $\mathbb{A}_{\mathcal{F}}$, see [4]. The algebra $\mathbb{A}_{\mathcal{F}}$ has a spate of remarkable properties, e.g., the topologically conjugate (or, induced) foliations have stably isomorphic (or, stably homomorphic) AF-algebras

(Lemma 1); the dimension group of $\mathbb{A}_{\mathcal{F}}$, see [5], coincides with the Plante group $P(\mathcal{F})$ of foliation $\mathcal{F}$, see [8].

1.3. Fundamental AF-algebras and main result. Let $\varphi\colon M \to M$ be an Anosov diffeomorphism of M, see [1]; if p is a fixed point of φ, then φ defines an invariant measured foliation $\mathcal{F}$ of M given by the stable manifold $W^s(p)$ of φ at the point p, see [9], page 760. The associated AF-algebra $\mathbb{A}_{\mathcal{F}}$ is stationary (Lemma 2); we call the latter a *fundamental AF-algebra* and denote it by $\mathbb{A}_{\varphi} := \mathbb{A}_{\mathcal{F}}$. Consider the mapping torus of φ, i.e., a manifold $M_{\varphi} := M \times [0,1]/\sim$, where $(x,0) \sim (\varphi(x),1)$, for all $x \in M$. Let $\mathcal{M}$ be a category of the mapping tori of all Anosov diffeomorphisms; the arrows of $\mathcal{M}$ are continuous maps between the mapping tori.

Likewise, let $\mathcal{A}$ be a category of all fundamental AF-algebras; the arrows of $\mathcal{A}$ are stable homomorphisms between the fundamental AF-algebras. By $F\colon \mathcal{M} \to \mathcal{A}$ we understand a map given by the formula $M_{\varphi} \mapsto \mathbb{A}_{\varphi}$, where $M_{\varphi} \in \mathcal{M}$ and $\mathbb{A}_{\varphi} \in \mathcal{A}$. Our main result can be stated as follows.

Theorem 1. *The map F is a functor which sends each continuous map $N_{\psi} \to M_{\varphi}$ to a stable homomorphism $\mathbb{A}_{\psi} \to \mathbb{A}_{\varphi}$ of the corresponding fundamental AF-algebras.*

1.4. Applications. Theorem 1 has a natural application, since stable homomorphisms of the fundamental AF-algebras are easier to detect than continuous maps between manifolds N_{ψ} and M_{φ}; such homomorphisms are in bijection with the inclusions of certain $\mathbb{Z}$-modules lying in a (real) algebraic number field. Often it is possible to prove that no inclusion is possible and, thus, draw a topological conclusion about the maps (an obstruction theory).

Namely, since $\mathbb{A}_{\psi}$ is stationary, it has a constant incidence matrix B; we denote the splitting field of the polynomial $\det(B - xI)$ by K_{ψ} and call $\mathrm{Gal}(K_{\psi};\mathbb{Q})$ the *Galois group* of the algebra $\mathbb{A}_{\psi}$. Suppose that $h\colon \mathbb{A}_{\psi} \to \mathbb{A}_{\varphi}$ is a stable homomorphism; since the corresponding invariant foliations $\mathcal{F}_{\psi}$ and $\mathcal{F}_{\varphi}$ are induced, their Plante groups are included $P(\mathcal{F}_{\varphi}) \subseteq P(\mathcal{F}_{\psi})$ and, therefore, $\mathbb{Q}(\lambda_{B'}) \subseteq K_{\psi}$, where $\lambda_{B'}$ is the Perron-Frobenius eigenvalue of the matrix B' attached to $\mathbb{A}_{\varphi}$. Thus, stable homomorphisms are in bijection with subfields of the algebraic number field K_{ψ}; their classification achieves perfection in terms of the Galois theory, since the subfields are in a one-to-one correspondence with the subgroups of $\mathrm{Gal}(\mathbb{A}_{\psi})$, see [7].

In particular, when $\mathrm{Gal}(\mathbb{A}_{\psi})$ is simple, there are only trivial stable homomorphisms; thus, the structure of $\mathrm{Gal}(\mathbb{A}_{\psi})$ is an obstruction (an invariant) to the existence of a continuous map between the manifolds N_{ψ} and M_{φ}. Is our invariant effective? The answer is positive for a class of the so-called tight torus bundles; in this case N_ψ is given by a monodromy matrix, which is similar to the matrix B. The obstruction theory for the tight torus bundles of any dimension can be completely determined; it reduces to a divisibility test for a finite set of natural numbers. For the sake of clarity, the test is done in dimension $m = 2$, 3 and 4 and followed by numerical examples.

Remark 1. Notice that for the tight torus bundles (see Section 3.2) our results can be proved using the theory of hyperbolic diffeomorphism $\psi\colon T^m \to T^m$ alone; however, our approach seems to be more general, leading to essentially new topological invariants.

2. Preliminaries

2.1. Measured foliations. By a q-dimensional, class C^r foliation of an m-dimensional manifold M we understand a decomposition of M into a union of disjoint connected subsets $\{\mathcal{L}_\alpha\}_{\alpha\in A}$, called *leaves*, see [6]. They must satisfy the following property: each point in M has a neighborhood U and a system of local class C^r coordinates $x = (x^1, \dots, x^m)\colon U \to \mathbb{R}^m$ such that for each leaf $\mathcal{L}_\alpha$, the components of $U \cap \mathcal{L}_\alpha$ are described by the equations $x^{q+1} = \text{const}, \dots, x^m = \text{const}$. Such a foliation is denoted by $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha\in A}$. The number $k = m - q$ is called the codimension of the foliation.

An example of a codimension k foliation $\mathcal{F}$ is given by a closed k-form ω on M: the leaves of $\mathcal{F}$ are tangent to the hyperplane $\omega(p) = 0$ at each point p of M. The C^r-foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of codimension k are said to be C^s-conjugate ($0 \leqslant s \leqslant r$), if there exists an (orientation-preserving) diffeomorphism of M, of class C^s, which maps the leaves of $\mathcal{F}_0$ onto the leaves of $\mathcal{F}_1$; when $s = 0$, $\mathcal{F}_0$ and $\mathcal{F}_1$ are *topologically conjugate*. Denote by $f\colon N \to M$ a map of class C^s ($1 \leqslant s \leqslant r$) of a manifold N into M; the map f is said to be *transverse* to $\mathcal{F}$, if for all $x \in N$ it holds that $T_y(M) = \tau_y(\mathcal{F}) + f_* T_x(N)$, where $\tau_y(\mathcal{F})$ are the vectors of $T_y(M)$ tangent to $\mathcal{F}$ and $f_*\colon T_x(N) \to T_y(M)$ is the linear map on tangent vectors induced by f, where $y = f(x)$.

If the map $f\colon N \to M$ is transverse to a foliation $\mathcal{F}' = \{\mathcal{L}\}_{\alpha\in A}$ on M, then f induces a class C^s foliation $\mathcal{F}$ on N, where the leaves are defined as $f^{-1}(\mathcal{L}_\alpha)$ for all $\alpha \in A$; it is immediate that $\operatorname{codim}(\mathcal{F}) = \operatorname{codim}(\mathcal{F}')$. We shall call $\mathcal{F}$ an *induced foliation*. When f is a submersion, it is transverse to any foliation of M; in this case, the induced foliation $\mathcal{F}$ is correctly defined for all $\mathcal{F}'$ on M, see [6], page 373. Notice that for $M = N$ the above definition corresponds to topologically conjugate foliations $\mathcal{F}$ and $\mathcal{F}'$. To introduce measured foliations, denote by P and Q two

k-dimensional submanifolds of M, which are everywhere transverse to a foliation $\mathcal{F}$ of codimension k.

Consider a collection of C^r homeomorphisms between subsets of P and Q induced by a return map along the leaves of $\mathcal{F}$. The collection of all such homeomorphisms between subsets of all possible pairs of transverse manifolds generates a *holonomy pseudogroup* of $\mathcal{F}$ under composition of the homeomorphisms, see [8], page 329. A foliation $\mathcal{F}$ is said to have measure preserving holonomy, if its holonomy pseudogroup has a nontrivial invariant measure, which is finite on compact sets; for brevity, we call $\mathcal{F}$ a *measured foliation*.

An example of measured foliation is a foliation determined by the closed k-form ω; the restriction of ω to a transverse k-dimensional manifold determines a volume element, which gives a positive invariant measure on open sets. Each measured foliation $\mathcal{F}$ defines an element of the cohomology group $H^k(M;\mathbb{R})$, see [8]; in the case of $\mathcal{F}$ given by a closed k-form ω, such an element coincides with the de Rham cohomology class of ω, ibid.

In view of the isomorphism $H^k(M;\mathbb{R}) \cong \mathrm{Hom}(H_k(M),\mathbb{R})$, foliation $\mathcal{F}$ defines a linear map h from the k-th homology group $H_k(M)$ to $\mathbb{R}$; by the *Plante group* $P(\mathcal{F})$ we shall understand a finitely generated abelian subgroup $h(H_k(M)/\,\mathrm{Tors})$ of the real line $\mathbb{R}$. If $\{\gamma_i\}$ is a basis of the homology group $H_k(M)$, then the periods $\lambda_i = \int_{\gamma_i}\omega$ are generators of the group $P(\mathcal{F})$, see [8].

2.2. AF-algebra of measured foliation. Let $\lambda = (\lambda_1,\dots,\lambda_n)$ be a basis of the Plante group $P(\mathcal{F})$ of a measured foliation $\mathcal{F}$ such that $\lambda_i > 0$. Take a vector $\theta = (\theta_1,\dots,\theta_{n-1})$ with $\theta_i = \lambda_{i+1}/\lambda_1$; the Jacobi-Perron continued fraction of vector $(1,\theta)$ (or, projective class of vector λ) is given by the formula ([3], page 13):

$$\begin{pmatrix}1\\ \theta\end{pmatrix} = \lim_{i\to\infty}\begin{pmatrix}0 & 1\\ I & b_1\end{pmatrix}\cdots\begin{pmatrix}0 & 1\\ I & b_i\end{pmatrix}\begin{pmatrix}0\\ \mathbb{I}\end{pmatrix} = \lim_{i\to\infty} B_i\begin{pmatrix}0\\ \mathbb{I}\end{pmatrix}, \tag{2.1}$$

where $b_i = (b_1^{(i)},\dots,b_{n-1}^{(i)})^{\mathrm{T}}$ is a vector of nonnegative integers, I the unit matrix and $\mathbb{I} = (0,\dots,0,1)^{\mathrm{T}}$; the b_i are obtained from θ by the Euclidean algorithm, see [3], pages 2–3, for details. An AF-algebra given by the Bratteli diagram with the incidence matrices B_i (and unital homomorphisms $M_i \to M_{i+1}$) will be called *associated* to the foliation $\mathcal{F}$; we shall denote such an algebra by $\mathbb{A}_{\mathcal{F}}$. Taking another basis of the Plante group $P(\mathcal{F})$ gives an AF-algebra which is stably isomorphic to $\mathbb{A}_{\mathcal{F}}$; this is an algebraic recast of the main property of the Jacobi-Perron fractions.

It is known that the Bratteli diagram defines the AF-algebra up to an isomorphism, see [4]; by $\mathbb{A}_{\mathcal{F}}$ we mean the isomorphism class. Note that if $\mathcal{F}'$ is a measured foliation on a manifold M and $f\colon N \to M$ is a submersion, the induced foliation $\mathcal{F}$ on N is a measured foliation. We shall denote by $\mathcal{MF}ol$ the category of all manifolds with measured foliations (of fixed codimension), whose arrows are submersions of the manifolds; by $\mathcal{M}_0\mathcal{F}ol$ we understand a subcategory of $\mathcal{MF}ol$ consisting of manifolds whose foliations have a unique transverse measure.

Let $\mathcal{R}ng$ be the category of the (isomorphism classes of) AF-algebras given by *convergent* Jacobi-Perron fractions (2.1), so that the arrows of $\mathcal{R}ng$ are the stable homomorphisms of the AF-algebras. By F we denote a map between $\mathcal{M}_0\mathcal{F}ol$ and $\mathcal{R}ng$ given by the formula $\mathcal{F} \mapsto \mathbb{A}_{\mathcal{F}}$. Notice that F is correctly defined, since foliations with a unique measure have convergent Jacobi-Perron fractions; this assertion follows from, see [2].

Lemma 1. *The map $F\colon \mathcal{M}_0\mathcal{F}ol \to \mathcal{R}ng$ is a functor which sends any pair of induced foliations to a pair of stably homomorphic AF-algebras.*

2.3. Fundamental AF-algebras. Let M be an m-dimensional manifold and $\varphi\colon M \to M$ a diffeomorphism of M; recall that an orbit of point $x \in M$ is the subset $\{\varphi^n(x);\ n \in \mathbb{Z}\}$ of M. Finite orbits $\varphi^m(x) = x$ are called periodic; when $m = 1$, x is a *fixed point* of diffeomorphism φ. A fixed point p is *hyperbolic* if the eigenvalues λ_i of the linear map $D\varphi(p)\colon T_p(M) \to T_p(M)$ do not lie on the unit circle. If $p \in M$ is a hyperbolic fixed point of a diffeomorphism $\varphi\colon M \to M$, denote by $T_p(M) = V^s + V^u$ the corresponding decomposition of the tangent space under the linear map $D\varphi(p)$, where V^s (V^u) is the eigenspace of $D\varphi(p)$ corresponding to $|\lambda_i| > 1$ ($|\lambda_i| < 1$).

For a submanifold $W^s(p)$ there exists a contraction $g\colon W^s(p) \to W^s(p)$ with fixed point p_0 and an injective equivariant immersion $J\colon W^s(p) \to M$, such that $J(p_0) = p$ and $DJ(p_0)\colon T_{p_0}(W^s(p)) \to T_p(M)$ is an isomorphism; the image of J defines an immersed submanifold $W^s(p) \subset M$ called a *stable manifold* of φ at p. Clearly, $\dim(W^s(p)) = \dim(V^s)$. We say that $\varphi\colon M \to M$ is an *Anosov diffeomorphism* (see [1]) if the following condition is satisfied: there exists a splitting of the tangent bundle $T(M)$ into a continuous Whitney sum $T(M) = E^s + E^u$, invariant under $D\varphi\colon T(M) \to T(M)$, so that $D\varphi\colon E^s \to E^s$ is contracting and $D\varphi\colon E^u \to E^u$ is expanding.[1]

[1] It follows from definition that the Anosov diffeomorphism imposes a restriction on the topology of manifold M, in the sense that not each manifold can support such a diffeomorphism; however, if one Anosov diffeomorphism exists on M, there are infinitely many (conjugacy classes of) such diffeomorphisms on M. It is an open problem by Stephen Smale, which M can carry an Anosov diffeomorphism; so far, it has been proved that the hyperbolic diffeomorphisms of m-dimensional tori and certain automorphisms of the nilmanifolds are Anosov, see [9]. It is worth mentioning that on each surface of genus $g \geqslant 1$ there exists a rich family of the so-called pseudo-Anosov diffeomorphisms, see [10], to which our theory fully applies.

Let p be a fixed point of the Anosov diffeomorphism $\varphi\colon M \to M$ and $W^s(p)$ its stable manifold. Since $W^s(p)$ cannot have self-intersections or limit compacta, $W^s(p) \to M$ is a dense immersion, i.e., the closure of $W^s(p)$ is the entire M. Moreover, if q is a periodic point of φ of period n, then $W^s(q)$ is a translate of $W^s(p)$, i.e., locally they look like two parallel lines.

Consider a foliation $\mathcal{F}$ of M whose leaves are the translates of $W^s(p)$; then $\mathcal{F}$ is a continuous foliation [9], page 760, which is invariant under the action of diffeomorphism φ on its leaves, i.e., φ moves leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}$. The holonomy of $\mathcal{F}$ preserves the Lebesgue measure and, therefore, $\mathcal{F}$ is a measured foliation; we shall call it an *invariant measured foliation* and denote by $\mathcal{F}_\varphi$. The AF-algebra of foliation $\mathcal{F}$ is called *fundamental*, when $\mathcal{F} = \mathcal{F}_\varphi$, where φ is an Anosov diffeomorphism; the following is a basic property of such algebras.

Lemma 2. *Any fundamental AF-algebra is stationary.*

3. Proofs

P r o o f of Lemma 1. Let $\mathcal{F}'$ be a measured foliation on M, given by a closed form $\omega' \in H^k(M;\mathbb{R})$; let $\mathcal{F}$ be the measured foliation on N, induced by a submersion $f\colon N \to M$. Roughly speaking, we have to prove that the diagram in Figure 1 is commutative; the proof amounts to the fact that the periods of form ω' are contained among the periods of form $\omega \in H^k(N;\mathbb{R})$ corresponding to the foliation $\mathcal{F}$.

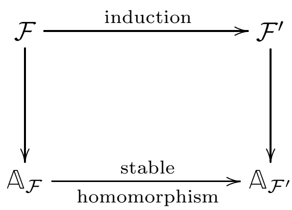

Figure 1. Functor $F\colon \mathcal{M}_0\mathcal{F}ol \to \mathcal{R}ng$.

The map f defines a homomorphism $f_*\colon H_k(N) \to H_k(M)$ of the k-th homology groups; let $\{e_i\}$ and $\{e_i'\}$ be a basis in $H_k(N)$ and $H_k(M)$, respectively. Since $H_k(M) = H_k(N)/\ker(f_*)$, we shall denote by $[e_i] := e_i + \ker(f_*)$ a coset representative of e_i; these can be identified with the elements $e_i \notin \ker(f_*)$. The integral $\int_{e_i} \omega$ defines a scalar product $H_k(N) \times H^k(N;\mathbb{R}) \to \mathbb{R}$, so that f_* is a linear self-adjoint operator; thus, we can write:

$$\lambda_i' = \int_{e_i'} \omega' = \int_{e_i'} f^*(\omega) = \int_{f_*^{-1}(e_i')} \omega = \int_{[e_i]} \omega \in P(\mathcal{F}), \tag{3.1}$$

where $P(\mathcal{F})$ is the Plante group (the group of periods) of foliation $\mathcal{F}$. Since λ_i' are generators of $P(\mathcal{F}')$, we conclude that $P(\mathcal{F}') \subseteq P(\mathcal{F})$. Note that $P(\mathcal{F}') = P(\mathcal{F})$ if and only if f_* is an isomorphism.

One can apply a criterion of the stable homomorphism of AF-algebras; namely, $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are stably homomorphic, if and only if there exists a positive homomorphism $h\colon G \to H$ between their dimension groups G and H, see [5], page 15. But $G \cong P(\mathcal{F})$ and $H \cong P(\mathcal{F}')$, while $h = f_*$. Thus, $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are stably homomorphic.

The functor F is compatible with the composition; indeed, let $f\colon N \to M$ and $f'\colon L \to N$ be submersions. If $\mathcal{F}$ is a measured foliation of M, one gets the induced foliations $\mathcal{F}'$ and $\mathcal{F}''$ on N and L, respectively; these foliations fit the diagram $(L, \mathcal{F}'') \overset{f'}{\to} (N, \mathcal{F}') \overset{f}{\to} (M, \mathcal{F})$ and the corresponding Plante groups are included: $P(\mathcal{F}'') \supseteq P(\mathcal{F}') \supseteq P(\mathcal{F})$. Thus, $F(f' \circ f) = F(f') \circ F(f)$, since the inclusion of the Plante groups corresponds to the composition of homomorphisms; Lemma 1 is proved. $\square$

P r o o f of Lemma 2. Let $\varphi\colon M \to M$ be an Anosov diffeomorphism; we proceed by showing that the invariant foliation $\mathcal{F}_\varphi$ is given by the form $\omega \in H^k(M; \mathbb{R})$, which is an eigenvector of the linear map $[\varphi]\colon H^k(M; \mathbb{R}) \to H^k(M; \mathbb{R})$ induced by φ. Indeed, let $0 < c < 1$ be the contracting constant of the stable sub-bundle E^s of diffeomorphism φ and Ω the corresponding volume element; by definition, $\varphi(\Omega) = c\Omega$.

Note that Ω is given by restriction of the form ω to a k-dimensional manifold, transverse to the leaves of $\mathcal{F}_\varphi$. The leaves of $\mathcal{F}_\varphi$ are fixed by φ and, therefore, $\varphi(\Omega)$ is given by a multiple $c\omega$ of form ω. Since $\omega \in H^k(M; \mathbb{R})$ is a vector whose coordinates define $\mathcal{F}_\varphi$ up to a scalar, we conclude that $[\varphi](\omega) = c\omega$, i.e., ω is an eigenvector of the linear map $[\varphi]$. Let $(\lambda_1, \ldots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F}_\varphi)$ such that $\lambda_i > 0$. Notice that φ acts on λ_i as multiplication by a constant c; indeed, since $\lambda_i = \int_{\gamma_i} \omega$, we have:

$$\lambda_i' = \int_{\gamma_i} [\varphi](\omega) = \int_{\gamma_i} c\omega = c \int_{\gamma_i} \omega = c\lambda_i, \tag{3.2}$$

where $\{\gamma_i\}$ is a basis in $H_k(M)$. Since φ preserves the leaves of $\mathcal{F}_\varphi$, one concludes that $\lambda_i' \in P(\mathcal{F}_\varphi)$; therefore, $\lambda_j' = \sum b_{ij}\lambda_i$ for a nonnegative integer matrix $B = (b_{ij})$. According to Bauer [2], the matrix B can be written as a finite product:

$$B = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ I & b_p \end{pmatrix} := B_1 \ldots B_p, \tag{3.3}$$

where $b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^{\mathrm{T}}$ is a vector of nonnegative integers and I the unit matrix. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Consider a purely periodic Jacobi-Perron continued fraction:

$$\lim_{i \to \infty} \overline{B_1 \ldots B_p} \begin{pmatrix} 0 \\ \square \end{pmatrix}, \tag{3.4}$$

where $\mathbb{I} = (0, \dots, 0, 1)^{\mathrm{T}}$; by a basic property of such fractions, it converges to an eigenvector $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ of matrix $B_1 \dots B_p$, see [3], Chapter 3. But $B_1 \dots B_p = B$ and λ is an eigenvector of matrix B; therefore, vectors λ and λ' are collinear. Collinear vectors are known to have the same continued fractions; thus, we have

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i\to\infty} \overline{B_1 \dots B_p} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \tag{3.5}$$

where $\theta = (\theta_1, \dots, \theta_{n-1})$ and $\theta_i = \lambda_{i+1}/\lambda_1$. Since vector $(1, \theta)$ unfolds into a periodic Jacobi-Perron continued fraction, we conclude that the AF-algebra $\mathbb{A}_\varphi$ is stationary. Lemma 2 is proved. $\square$

P r o o f of Theorem 1. Let $\psi\colon N \to N$ and $\varphi\colon M \to M$ be a pair of Anosov diffeomorphisms; denote by $(N, \mathcal{F}_\psi)$ and $(M, \mathcal{F}_\varphi)$ the corresponding invariant foliations of manifolds N and M, respectively. In view of Lemma 1, it is sufficient to prove that the diagram in Figure 2 is commutative. We shall split the proof in a series of lemmas.

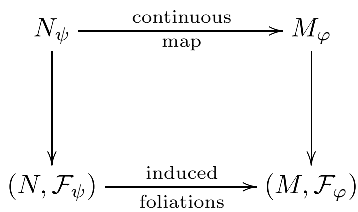

Figure 2. Mapping tori and invariant foliations.

Lemma 3. *There exists a continuous map $N_\psi \to M_\varphi$, whenever $f \circ \varphi = \psi \circ f$ for a submersion $f\colon N \to M$.*

P r o o f. (i) Suppose that $h\colon N_\psi \to M_\varphi$ is a continuous map; let us show that there exists a submersion $f\colon N \to M$ such that $f \circ \varphi = \psi \circ f$. Both N_ψ and M_φ fiber over the circle S^1 with the projection map p_ψ and p_φ, respectively; therefore, the diagram in Figure 3 is commutative. Let $x \in S^1$; since $p_\psi^{-1} = N$ and $p_\varphi^{-1} = M$,

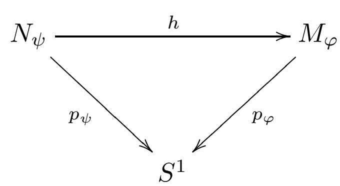

Figure 3. The fiber bundles N_ψ and M_φ over S^1.

the restriction of h to x defines a submersion $f\colon N \to M$, i.e., $f = h_x$. Moreover, since ψ and φ are monodromy maps of the bundle, it holds that

$$p_{\psi}^{-1}(x+2\pi) = \psi(N), \quad p_{\varphi}^{-1}(x+2\pi) = \varphi(M). \tag{3.6}$$

From the diagram in Figure 3, we get: $\psi(N) = p_{\psi}^{-1}(x+2\pi) = f^{-1}(p_{\varphi}^{-1}(x+2\pi)) = f^{-1}(\varphi(M)) = f^{-1}(\varphi(f(N)))$; thus, $f\circ\psi = \varphi\circ f$. The necessary condition of Lemma 3 follows.

(ii) Suppose that $f\colon N \to M$ is a submersion such that $f\circ\varphi = \psi\circ f$; we have to construct a continuous map $h\colon N_{\psi} \to M_{\varphi}$. Recall that

$$\begin{aligned} N_{\psi} &= \{N \times [0,1];\ (x,0) \sim (\psi(x),1)\}, \\ M_{\varphi} &= \{M \times [0,1];\ (y,0) \sim (\varphi(y),1)\}. \end{aligned} \tag{3.7}$$

We shall identify the points of N_{ψ} and M_{φ} using the substitution $y = f(x)$; it remains to verify that such an identification will satisfy the gluing condition $y \sim \varphi(y)$. In view of the condition $f\circ\varphi = \psi\circ f$, we have

$$y = f(x) \sim f(\psi(x)) = \varphi(f(x)) = \varphi(y). \tag{3.8}$$

Thus, $y \sim \varphi(y)$ and, therefore, the map $h\colon N_{\psi} \to M_{\varphi}$ is continuous. The sufficient condition of Lemma 3 is proved. □

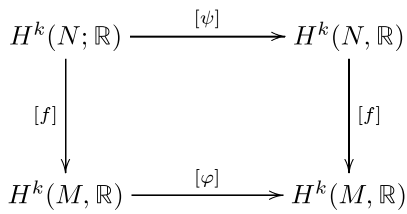

Figure 4. The linear maps $[\psi]$, $[\varphi]$ and $[f]$.

Lemma 4. *If a submersion $f\colon N \to M$ satisfies the condition $f\circ\varphi = \psi\circ f$ for the Anosov diffeomorphisms $\psi\colon N \to N$ and $\varphi\colon M \to M$, then the invariant foliations $(N, \mathcal{F}_{\psi})$ and $(M, \mathcal{F}_{\varphi})$ are induced by f.*

P r o o f. The invariant foliations $\mathcal{F}_{\psi}$ and $\mathcal{F}_{\varphi}$ are measured; we shall denote by $\omega_{\psi} \in H^k(N;\mathbb{R})$ and $\omega_{\varphi} \in H^k(M;\mathbb{R})$ the corresponding cohomology class, respectively. We shall denote by $[\psi]$ and $[\varphi]$ the linear maps on $H^k(N;\mathbb{R})$ and $H^k(M;\mathbb{R})$ induced by ψ and φ; we write as $[f]$ the linear map between $H^k(N;\mathbb{R})$ and $H^k(M;\mathbb{R})$

induced by f. Notice that $[\psi]$ and $[\varphi]$ are isomorphisms, while $[f]$ is generally a homomorphism. It was shown earlier that ω_ψ and ω_φ are eigenvectors of linear maps $[\psi]$ and $[\varphi]$, respectively; in other words, we have

$$[\psi]\omega_\psi = c_1\omega_\psi, \quad [\varphi]\omega_\varphi = c_2\omega_\varphi, \tag{3.9}$$

where $0 < c_1 < 1$ and $0 < c_2 < 1$. Consider the diagram in Figure 4, which involves the linear maps $[\psi]$, $[\varphi]$ and $[f]$; the diagram is commutative, since the condition $f \circ \varphi = \psi \circ f$ implies that $[\varphi] \circ [f] = [f] \circ [\psi]$. Take the eigenvector ω_ψ and consider its image under the linear map $[\varphi] \circ [f]$:

$$[\varphi] \circ [f](\omega_\psi) = [f] \circ [\psi](\omega_\psi) = [f](c_1\omega_\psi) = c_1([f](\omega_\psi)). \tag{3.10}$$

Therefore, the vector $[f](\omega_\psi)$ is an eigenvector of the linear map $[\varphi]$; let us compare it with the eigenvector ω_φ:

$$[\varphi]([f](\omega_\psi)) = c_1([f](\omega_\psi)), \quad [\varphi]\omega_\varphi = c_2\omega_\varphi. \tag{3.11}$$

We conclude, therefore, that ω_φ and $[f](\omega_\psi)$ are collinear vectors, such that $c_1^m = c_2^n$ for some integers $m, n > 0$; a scaling gives us $[f](\omega_\psi) = \omega_\varphi$. The latter is an analytic formula which says that the submersion $f\colon N \to M$ induces the foliation $(N, \mathcal{F}_\psi)$ from the foliation $(M, \mathcal{F}_\varphi)$. Lemma 4 is proved. □

To finish our proof of Theorem 1, let $N_\psi \to M_\varphi$ be a continuous map; by Lemma 3, there exists a submersion $f\colon N \to M$ such that $f \circ \varphi = \psi \circ f$. Lemma 4 says that in this case the invariant measured foliations $(N, \mathcal{F}_\psi)$ and $(M, \mathcal{F}_\varphi)$ are induced. On the other hand, from Lemma 2 we know that the Jacobi-Perron continued fraction connected to foliations $\mathcal{F}_\psi$ and $\mathcal{F}_\varphi$ is periodic and, hence, convergent, see [3]; therefore, one can apply Lemma 1, which says that the AF-algebra $\mathbb{A}_\psi$ is stably homomorphic to the AF-algebra $\mathbb{A}_\varphi$. The latter are, by definition, the fundamental AF-algebras of the Anosov diffeomorphisms ψ and φ, respectively. Theorem 1 is proved. □

4. Applications of Theorem 1

4.1. Galois group of the fundamental AF-algebra. Let $\mathbb{A}_\psi$ be a fundamental AF-algebra and B its primitive incidence matrix, i.e., B is not a power of some positive integer matrix. Suppose that the characteristic polynomial of B is irreducible and let K_ψ be its splitting field; then K_ψ is a Galois extension of $\mathbb{Q}$. We call $\mathrm{Gal}(\mathbb{A}_\psi) := \mathrm{Gal}(K_\psi; \mathbb{Q})$ the *Galois group* of the fundamental AF-algebra $\mathbb{A}_\psi$; such a group is determined by the AF-algebra $\mathbb{A}_\psi$. The second algebraic field is connected to the Perron-Frobenius eigenvalue λ_B of the matrix B; we shall denote this field $\mathbb{Q}(\lambda_B)$. Note that $\mathbb{Q}(\lambda_B) \subseteq K_\psi$ and $\mathbb{Q}(\lambda_B)$ is not, in general, a Galois extension of $\mathbb{Q}$; the obstacle are complex roots of the polynomial $\mathrm{char}(B)$ and if there are no such roots then $\mathbb{Q}(\lambda_B) = K_\psi$, see e.g. [7]. There is still a group $\mathrm{Aut}(\mathbb{Q}(\lambda_B))$ of automorphisms of $\mathbb{Q}(\lambda_B)$ fixing the field $\mathbb{Q}$ and $\mathrm{Aut}(\mathbb{Q}(\lambda_B)) \subseteq \mathrm{Gal}(K_\psi)$ is a subgroup inclusion.

Lemma 5. *If $h\colon \mathbb{A}_\psi \to \mathbb{A}_\varphi$ is a stable homomorphism, then $\mathbb{Q}(\lambda_{B'}) \subseteq K_\psi$ is a field inclusion, where B' is the matrix associated to φ.*

P r o o f. Notice that the nonnegative matrix B becomes strictly positive, when a proper power of it is taken; we always assume B positive. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a basis of the Plante group $P(\mathcal{F}_\psi)$. Following the proof of Lemma 2, one concludes that $\lambda_i \in K_\psi$; indeed, $\lambda_B \in K_\psi$ is the Perron-Frobenius eigenvalue of B, while λ is the corresponding eigenvector. The latter can be scaled so that $\lambda_i \in K_\psi$. Any stable homomorphism $h\colon \mathbb{A}_\psi \to \mathbb{A}_\varphi$ induces a positive homomorphism of their dimension groups $[h]\colon G \to H$; but $G \cong P(\mathcal{F}_\psi)$ and $H \cong P(\mathcal{F}_\varphi)$. From the inclusion $P(\mathcal{F}_\varphi) \subseteq P(\mathcal{F}_\psi)$, one gets $\mathbb{Q}(\lambda_{B'}) \cong P(\mathcal{F}_\varphi) \otimes \mathbb{Q} \subseteq P(\mathcal{F}_\psi) \otimes \mathbb{Q} \cong \mathbb{Q}(\lambda_B) \subseteq K_\psi$ and, therefore, $\mathbb{Q}(\lambda_{B'}) \subseteq K_\psi$. Lemma 5 follows. $\square$

Corollary 1. *If $h\colon \mathbb{A}_\psi \to \mathbb{A}_\varphi$ is a stable homomorphism, then* $\mathrm{Aut}(\mathbb{Q}(\lambda_{B'}))$ (*or,* $\mathrm{Gal}(\mathbb{A}_\varphi)$) *is a subgroup* (*or, a normal subgroup*) *of* $\mathrm{Gal}(\mathbb{A}_\psi)$.

P r o o f. The (Galois) subfields of the Galois field K_ψ are bijective with the (normal) subgroups of the group $\mathrm{Gal}(K_\psi)$, see [7]. $\square$

4.2. Tight torus bundles. Let $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$ be an m-dimensional torus; let ψ_0 be a $m \times m$ integer matrix with $\det(\psi_0) = 1$, such that it is similar to a positive matrix. The matrix ψ_0 defines a linear transformation of $\mathbb{R}^m$ which preserves the lattice $L \cong \mathbb{Z}^m$ of points with integer coordinates. There is an induced diffeomorphism ψ of the quotient $T^m \cong \mathbb{R}^m/\mathbb{Z}^m$ onto itself; this diffeomorphism $\psi\colon T^m \to T^m$ has a fixed point p corresponding to the origin of $\mathbb{R}^m$. Suppose that ψ_0 is hyperbolic, i.e., there are no eigenvalues of ψ_0 at the unit circle; then p is a hyperbolic fixed point of ψ and the stable manifold $W^s(p)$ is the image of the corresponding eigenspace of ψ_0 under the projection $\mathbb{R}^m \to T^m$. If $\mathrm{codim}\, W^s(p) = 1$, the hyperbolic linear transformation ψ_0 (and the diffeomorphism ψ) will be called *tight*.

Lemma 6. *The tight hyperbolic matrix ψ_0 is similar to the matrix B of the fundamental AF-algebra $\mathbb{A}_\psi$.*

P r o o f. Since $H_k(T^m;\mathbb{R}) \cong \mathbb{R}^{m!/k!(m-k)!}$, one gets $H_{m-1}(T^m;\mathbb{R}) \cong \mathbb{R}^m$; in view of the Poincaré duality, $H^1(T^m;\mathbb{R}) = H_{m-1}(T^m;\mathbb{R}) \cong \mathbb{R}^m$. Since $\operatorname{codim} W^s(p) = 1$, the measured foliation $\mathcal{F}_\psi$ is given by a closed form $\omega_\psi \in H^1(T^m;\mathbb{R})$ such that $[\psi]\omega_\psi = \lambda_\psi\omega_\psi$, where λ_ψ is the eigenvalue of the linear transformation $[\psi]\colon H^1(T^m;\mathbb{R}) \to H^1(T^m;\mathbb{R})$. It is easy to see that $[\psi] = \psi_0$, because $H^1(T^m;\mathbb{R}) \cong \mathbb{R}^m$ is the universal cover for T^m, where the eigenspace $W^u(p)$ of ψ_0 is the span of the eigenform ω_ψ. On the other hand, from the proof of Lemma 2 we know that the Plante group $P(\mathcal{F}_\psi)$ is generated by the coordinates of vector ω_ψ; the matrix B is nothing but the matrix ψ_0 written in a new basis of $P(\mathcal{F}_\psi)$. Each change of basis in the $\mathbb{Z}$-module $P(\mathcal{F}_\psi)$ is given by an integer invertible matrix S; therefore, $B = S^{-1}\psi_0 S$. Lemma 6 follows. □

Let $\psi\colon T^m \to T^m$ be a hyperbolic diffeomorphism; the mapping torus T^m_ψ will be called a (hyperbolic) *torus bundle* of dimension m. Let $k = |\mathrm{Gal}(\mathbb{A}_\psi)|$; it follows from the Galois theory that $1 < k \leqslant m!$. Denote by t_i the cardinality of a subgroup $G_i \subseteq \mathrm{Gal}(\mathbb{A}_\psi)$.

Corollary 2. *There are no (nontrivial) continuous maps $T^m_\psi \to T^{m'}_\varphi$, whenever $t'_i \nmid k$ for all $G'_i \subseteq \mathrm{Gal}(\mathbb{A}_\varphi)$.*

P r o o f. If $h\colon T^m_\psi \to T^{m'}_\varphi$ was a continuous map to a torus bundle of dimension $m' < m$, then, by Theorem 1 and Corollary 1, $\mathrm{Aut}(\mathbb{Q}(\lambda_{B'}))$ (or, $\mathrm{Gal}(\mathbb{A}_\varphi)$) would be a nontrivial subgroup (or, normal subgroup) of the group $\mathrm{Gal}(\mathbb{A}_\psi)$; since $k = |\mathrm{Gal}(\mathbb{A}_\psi)|$, one concludes that one of t'_i divides k. This contradicts our assumption. □

Definition 1. The torus bundle T^m_ψ is called robust, if there exists $m' < m$ such that no continuous map $T^m_\psi \to T^{m'}_\varphi$ exists.

Are there robust bundles? It is shown in this section that for $m = 2$, 3 and 4 there are infinitely many robust torus bundles; a reasonable guess is that it is true in any dimension.

Case 1: $m = 2$. This case is trivial; ψ_0 is a hyperbolic matrix and always tight. The polynomial $\mathrm{char}(\psi_0) = \mathrm{char}(B)$ is an irreducible quadratic polynomial with two real roots; $\mathrm{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_2$ and, therefore, $|\mathrm{Gal}(\mathbb{A}_\psi)| = 2$. Formally, T^2_ψ is robust, since no torus bundle of a smaller dimension is defined.

Case 2: $m = 3$. The matrix ψ_0 is hyperbolic; it is always tight, since one root of $\mathrm{char}(\psi_0)$ is real and isolated inside or outside the unit circle.

Corollary 3. *Let*

$$\psi_0(b,c) = \begin{pmatrix} -b & 1 & 0 \\ -c & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{4.1}$$

be such that $\operatorname{char}(\psi_0(b,c)) = x^3 + bx^2 + cx + 1$ *is irreducible and* $-4b^3 + b^2c^2 + 18bc - 4c^3 - 27$ *is the square of an integer; then* T^3_ψ *admits no continuous map to any* T^2_φ.

Proof. The polynomial $\operatorname{char}(\psi_0(b,c)) = x^3 + bx^2 + cx + 1$ and the discriminant $D = -4b^3 + b^2c^2 + 18bc - 4c^3 - 27$. By Theorem 13.1 in [7], $\operatorname{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_3$ and, therefore, $k = |\operatorname{Gal}(\mathbb{A}_\psi)| = 3$. For $m' = 2$, it was shown that $\operatorname{Gal}(\mathbb{A}_\varphi) \cong \mathbb{Z}_2$ and, therefore, $t'_1 = 2$. Since $2 \nmid 3$, Corollary 2 says that no continuous map $T^3_\psi \to T^2_\varphi$ can be constructed. □

Example 1. There are infinitely many matrices $\psi_0(b,c)$ satisfying the assumptions of Corollary 3; below are a few numerical examples of robust bundles:

$$\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Notice that the above matrices are not pairwise similar; it can be gleaned from their traces. Thus, they represent topologically distinct torus bundles.

Case 3: $m = 4$. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a quartic polynomial. Consider the associated cubic polynomial $r(x) = x^3 - bx^2 + (ac - 4d)x + 4bd - a^2d - c^2$; denote by L the splitting field of $r(x)$.

Corollary 4. *Let*

$$\psi_0(a,b,c) = \begin{pmatrix} -a & 1 & 0 & 0 \\ -b & 0 & 1 & 0 \\ -c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

be tight and such that $\operatorname{char}(\psi_0(a,b,c)) = x^4 + ax^3 + bx^2 + cx + 1$ *is irreducible and one of the following holds:*

(i) $L = \mathbb{Q}$;

(ii) $r(x)$ *has a unique root* $t \in \mathbb{Q}$ *and* $h(x) = (x^2 - tx + 1)(x^2 + ax + (b - t))$ *splits over* L;

(iii) $r(x)$ *has a unique root* $t \in \mathbb{Q}$ *and* $h(x)$ *does not split over* L.

Then T^4_ψ *admits no continuous map to any* T^3_φ *with* $D > 0$.

P r o o f. According to Theorem 13.4 in [7], $\mathrm{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in case (i); $\mathrm{Gal}(\mathbb{A}_\psi) \cong \mathbb{Z}_4$ in case (ii); and $\mathrm{Gal}(\mathbb{A}_\psi) \cong D_4$ (the dihedral group) in case (iii). Therefore, $k = |\mathbb{Z}_2 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4| = 4$ or $k = |D_4| = 8$. On the other hand, for $m' = 3$ with $D > 0$ (all roots are real), we have $t_1' = |\mathbb{Z}_3| = 3$ and $t_2' = |S_3| = 6$. Since $3; 6 \nmid 4; 8$, Corollary 2 says that a continuous map $T^4_\psi \to T^3_\varphi$ is impossible. $\square$

Example 2. There are infinitely many matrices ψ_0 which satisfy the assumption of Corollary 4; indeed, consider a family

$$\psi_0(a,c) = \begin{pmatrix} -2a & 1 & 0 & 0 \\ -a^2-c^2 & 0 & 1 & 0 \\ -2c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3}$$

where $a, c \in \mathbb{Z}$. The associated cubic polynomial becomes $r(x) = x(x^2 - (a^2+c^2)x + 4(ac-1))$, so that $t = 0$ is a rational root; then $h(x) = (x^2+1)(x^2+2ax+a^2+c^2)$. The matrix $\psi_0(a,c)$ satisfies one of the conditions (i)–(iii) of Corollary 4 for each $a, c \in \mathbb{Z}$; it remains to eliminate the (non-generic) matrices which are not tight or irreducible. Thus, $\psi_0(a,c)$ defines a family of topologically distinct robust bundles.

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