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AN A POSTERIORI ERROR ESTIMATE FOR THE STOKES-BRINKMAN PROBLEM IN A POLYGONAL DOMAIN

Pavel Burda[1], Martin Hasal[2]

[1] Czech University of Technology, Faculty of Mechanical Engineering,
Department of Mathematics
Karlovo nám. 13, CZ-121 35 Praha 2, Czech Republic
pavel.burda@fs.cvut.cz
[2] VŠB TU Ostrava
17. listopadu 15, 708 03 Ostrava-Poruba, Czech Republic
martin.hasal@vsb.cz

Abstract

We derive a residual based a posteriori error estimate for the Stokes-Brinkman problem on a two-dimensional polygonal domain. We use Taylor-Hood triangular elements. The link to the possible information on the regularity of the problem is discussed.

1. Introduction

In the paper we try to contribute to the technique of a posteriori error estimates for the finite element solution of linearized flow problems. In this respect we note that important results have already been obtained: concerning linear elliptic equations let us mention I. Babuška, W. C. Rheinboldt [2], I. Babuška, R. Durán, R. Rodríguez [3], concerning the Stokes problem e.g. M. Ainsworth, J. T. Oden [1], R. E. Bank, D. Welfert [5], C. Carstensen, S. Jansche [7], C. Johnson, R. Rannacher, M. Boman [12], R. Verfürth [15].

The goal of this paper is to link the problem of a posteriori error estimates as much as possible to the information on the regularity of the solution.

Let us illustrate it first on the Dirichlet problem for the Poisson equation

$$\begin{aligned} -\Delta u = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \qquad (1)$$

where Ω is a polygonal domain in R^2. Let u_h be the finite element solution of (1), with linear triangular elements. Let us denote

$$e = u - u_h,$$

the approximation error, and

$$R(u_h) = f + \Delta u_h,$$

the residual. Following the technique of K. Eriksson et al. [10], we first express the error by means of product of residual and solution of the dual problem, then use the Galerkin orthogonality and get the estimate of the error, in the L_2-norm:

$$\|e\|_0^2 \leq \sum_{K\in\mathcal{T}_h} \Big\{ \|R(u_h)\|_{0,K} \|\varphi - \pi_h\varphi\|_{0,K} + \sum_{l\in\partial K} \Big\| \frac{1}{2} \Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big]_l \Big\|_{0,l} \|\varphi - \pi_h\varphi\|_{0,l} \Big\}, \tag{2}$$

where φ is the solution of the dual problem

$$\begin{aligned} -\Delta\varphi &= e \quad \text{in} \quad \Omega, \\ \varphi &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3}$$

$\pi_h\varphi$ means the interpolant of φ. The sum in (2) is taken over all triangles in the triangulation $\mathcal{T}_h$, the symbol $[\![\frac{\partial u_h}{\partial \boldsymbol{n}}]\!]_l$ means the jump of the normal derivative $\frac{\partial u_h}{\partial \boldsymbol{n}}$ over the edge l of the triangle K.

Let us now distinguish 3 cases:

A) *General polygonal domain* Ω:

Let h_K be the largest side of the triangle K. The interpolation property together with the (low) regularity of the dual problem (3) yield

$$\|\varphi - \pi_h\varphi\|_{0,K} \leq C_I h_K \|\varphi\|_1 \leq C_I C_R h_K \|e\|_0.$$

Combining this with (2), we come to the a posteriori error estimate

$$\|e\|_0 \leq C_I C_R \sum_{K\in\mathcal{T}_h} h_K \Big\{ \|R(u_h)\|_{0,K} + h_K^{-\frac{1}{2}} \sum_{l\in\partial K} \Big\| \frac{1}{2} \Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big]_l \Big\|_{0,l} \Big\}. \tag{4}$$

B) *Convex polygon* Ω:

Now the regularity of the dual problem (3) is higher, cf. R. B. Kellogg, J. E. Osborn [13], and together with the interpolation property it gives

$$\|\varphi - \pi_h\varphi\|_{0,K} \leq C_I h_K^2 \|\varphi\|_2 \leq C_I C_R h_K^2 \|e\|_0.$$

Combining this with (2), we come to the more precise a posteriori estimate

$$\|e\|_0 \leq C_I C_R \sum_{K\in\mathcal{T}_h} h_K^2 \Big\{ \|R(u_h)\|_{0,K} + h_K^{-\frac{1}{2}} \sum_{l\in\partial K} \Big\| \frac{1}{2} \Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big]_l \Big\|_{0,l} \Big\}. \tag{5}$$

C) *Nonconvex polygon* Ω *with known singularity*:

It is well-known that the solution near the nonconvex corner, in the local spherical coordinates, has the form

$$u(r,\vartheta) = r^\gamma w(\vartheta),$$

where r is the distance from the corner, $\gamma \in (0,1)$. For instance, the case of the L-shaped domain with the interior angle $\omega = \frac{3}{2}\pi$ gives $\gamma = \frac{2}{3}$, cf. also [6]. Now the interpolation together with the above regularity gives

$$\|\varphi - \pi_h\varphi\|_{0,K} \le C_I h_K^{1+\gamma-\varepsilon}\|\varphi\|_{H^{1+\gamma-\varepsilon}} \le C_I C_R h_K^{1+\gamma-\varepsilon}\|e\|_0,\ \forall\varepsilon > 0,$$

which, combined with (2), finally leads to the a posteriori estimate

$$\|e\|_0 \le C_I C_R \sum_{K\in\mathcal{T}_h} h_K^{1+\gamma-\varepsilon}\left\{\|R(u_h)\|_{0,K} + h_K^{-\frac{1}{2}}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\left[\frac{\partial u_h}{\partial \boldsymbol{n}}\right]\!\right]_l\right\|_{0,l}\right\}, \tag{6}$$

valid $\forall\varepsilon > 0$. Of course, in (6) the parameter γ applies only in the nearest neighborhood of the corner.

Comparing the estimates (4), (5), (6) we see that the a posteriori error estimate depends significantly on the regularity of the problem. Having this in mind, we try to derive the a posteriori error estimate for the Stokes-Brinkman problem.

2. The Stokes-Brinkman model

Let Ω be a bounded Lipschitzian domain, $\Omega \subset R^2$, which consists of two parts: porous part Ω_p and fluid part Ω_f, $\bar\Omega = \bar\Omega_p \cup \bar\Omega_f$. The Stokes-Brinkman equation representing a mathematical model of a single phase flow in a porous/free flow media has the following form

$$\nu\mathbf{K}^{-1}\mathbf{v} + \nabla p - \nu^*\Delta\mathbf{v} = \mathbf{f} \quad \text{in } \Omega, \tag{7}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{8}$$

$$\mathbf{v} = \mathbf{w} \quad \text{on } \partial\Omega_D, \qquad \frac{\partial\mathbf{v}}{\partial n} - \mathbf{n}p = \mathbf{s} \quad \text{on } \partial\Omega_N, \tag{9}$$

where $\mathbf{v}$ is the vector of velocity, P is the pressure, $\mathbf{f}$ is the vector of external force, $\mathbf{n}$ is the outward-pointing normal to the boundary, ν^* is the effective viscosity and ν - the physical viscosity - is a uniform constant in the entire domain Ω. $\mathbf{K}$ is a symmetric permeability tensor, which in Ω_p is equal to the Darcy permeability of the porous media. Note that with the choice $\nu^* = 0$ in the vugular region Ω_p, the equation (7) reduces to the problem of Darcy's law. On the other hand by choosing $k_{ij} \to \infty$ (or very large) in fluid domain Ω_f, the equation (7) reduces to the problem of Stokes flow (here ν^* is taken equal to the physical fluid viscosity ν). Thus, the Stokes or Darcy's equations can be obtained by suitable choices of the parameters ν^* and $\mathbf{K}$ by defining them in vugular and rock matrix regions, respectively.

In the porous region ($\mathbf{K} < \infty$) it is known [14], that for moderately small permeabilities and pore fractions, the diffusive term $\nu^*\Delta\mathbf{v}$, where ν^* takes values close to the fluid viscosity ν, intoroduces only a small perturbation of the velocity and pressure fields in comparison with a pure Darcy law with $\nu^* = 0$. In [14] it is shown that Stokes-Brinkman equation with the choice $\nu^* = \nu$ in the porous region is very close to the solution of coupled Stokes and Darcy's equations.

The advantage of Stokes-Brinkman model is usage of uniform equations for porous and free flow domains. Boundary conditions between these two domains are represented by $\mathbf{K}$. This approach makes it possible to model heterogeneous material. Moreover, by a numerical point of view, it is easier to solve a monolithic system such as Stokes-Brinkman, in contrast to a coupled Darcy-Stokes system which requires an additional iterative scheme. Also, near the interface, Stokes-Brinkman equations allow us to avoid the typical grid refinement issues necessary for solving the interface between Darcy and Stokes region. On the other hand usage of Taylor-Hood elements for the whole domain requires big load of memory.

3. Weak formulation of Stokes-Brinkman equations

In what follows we denote $G = \mathbf{K}^{-1}$ and assume G is symmetric.

For the weak formulation we denote

$$\mathbf{H}^1_E := \{\mathbf{u} \in H^1(\Omega)^2 | \mathbf{u} = \mathbf{w} \text{ na } \partial\Omega_D\}, \tag{10}$$

$$\mathbf{H}^1_{E_0} := \{\mathbf{v} \in H^1(\Omega)^2 | \mathbf{v} = \mathbf{0} \text{ na } \partial\Omega_D\}. \tag{11}$$

Now the weak form of the Stokes-Brinkman problem reads:
Find $\boldsymbol{v} \in \mathbf{H}^1_{E_0}$ and $p \in L^2_0(\Omega)$ such that

$$\nu^* \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v^*} + \nu \int_\Omega \boldsymbol{v^T} G \boldsymbol{v^*} - \int_\Omega p \nabla \cdot \boldsymbol{v^*} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v^*} \qquad \forall \boldsymbol{v^*} \in \mathbf{H}^1_{E_0}, \tag{12}$$

$$\int_\Omega q \nabla \cdot \boldsymbol{v} = 0 \qquad \forall q \in L^2_0(\Omega). \tag{13}$$

Here $L^2_0(\Omega)$ is the space of L^2 functions having mean value zero.

On the space $V = \Big(H^1_0(\Omega)^2 \times L^2_0(\Omega) \Big)$ we define the bilinear form

$$\mathcal{A}\Big(\{\boldsymbol{v}, p\}, \{\boldsymbol{v}^*, p^*\}\Big) = \nu^* \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v}^* + \nu \int_\Omega \boldsymbol{v^T} G \boldsymbol{v}^* - \int_\Omega p \nabla \cdot \boldsymbol{v}^* - \int_\Omega p^* \nabla \cdot \boldsymbol{v} \tag{14}$$

where $(.,.)_0$ means the scalar product in L^2.

In what follows we assume $\mathbf{w} = 0$, i. e. only zero Dirichlet condition on the whole boundary $\partial\Omega$. Problem (12), (13) can be written as follows: find $\{\boldsymbol{v}, p\} \in V$, such that

$$\mathcal{A}\Big(\{\boldsymbol{v}, p\}, \{\boldsymbol{v}^*, p^*\}\Big) = (\boldsymbol{f}, \boldsymbol{v}^*)_0, \quad \forall \{\boldsymbol{v}^*, p^*\} \in V. \tag{15}$$

4. Finite element approximation

We suppose Ω to be a polygon, for simplicity. Let $\mathcal{T}_h$ be regular [11] triangulations of Ω. Let X^h, M^h be the finite element spaces of Taylor-Hood elements (cf. e.g. F. Brezzi, M. Fortin [4]), i.e.

$$X^h = \{\boldsymbol{v} \in H^1_0(\Omega)^2, \boldsymbol{v}/_T \in P^2(T)^2, T \in \mathcal{T}_h\},$$
$$M^h = \{p \in L^2_0(\Omega), p/_T \in P^1(T), T \in \mathcal{T}_h\}.$$

These satisfy the Babuška-Brezzi condition [4]. The finite element approximation of the Stokes-Brinkman problem consists in finding $\{\boldsymbol{v}_h, p_h\} \in X^h \times M^h$ such that

$$\mathcal{A}\Big(\{\boldsymbol{v}_h, p_h\}, \{\boldsymbol{v}_h^*, p_h^*\}\Big) = (\boldsymbol{f}, \boldsymbol{v}_h^*)_0 \ , \quad \forall \{\boldsymbol{v}_h^*, p_h^*\} \in X^h \times M^h. \tag{16}$$

5. A posteriori error estimate

We follow the idea of K. Eriksson et al. [10] who proved the a posteriori error estimate for the Poisson equation. We define the residual components by the relations

$$\boldsymbol{R_1}\{\boldsymbol{v}_h, p_h\} = \boldsymbol{f} + \nu^* \Delta \boldsymbol{v}_h - \nu G \boldsymbol{v}_h - \nabla p_h, \quad R_2\{\boldsymbol{v}_h, p_h\} = \text{div } \boldsymbol{v}_h. \tag{17}$$

Next we study the properties of the errors

$$\boldsymbol{e}_v = \boldsymbol{v} - \boldsymbol{v}_h \ , \ e_p = p - p_h \ ,$$

where $\{\boldsymbol{v}, p\}$ is the exact solution of (15), $\{\boldsymbol{v}_h, p_h\}$ is the approximate solution defined in (16). The V norm of $\{\boldsymbol{e}_v, e_p\}$ is

$$\|\{\boldsymbol{e}_v, e_p\}\|_V^2 = (\boldsymbol{e}_v, \boldsymbol{e}_v)_1 + (e_p, e_p)_0 = \int_\Omega (\boldsymbol{e}_v \cdot \boldsymbol{e}_v + \nabla \boldsymbol{e}_v : \nabla \boldsymbol{e}_v) + \int_\Omega e_p e_p.$$

By the Poincaré-Friedrichs inequality, cf. [9], as $\boldsymbol{e}_v \in H_0^1(\Omega)^2$

$$(\boldsymbol{e}_v, \boldsymbol{e}_v)_1 \leq C_P \int_\Omega \nabla \boldsymbol{e}_v : \nabla \boldsymbol{e}_v \tag{18}$$

5.1. Dual Stokes-Brinkman problem

To study the above norms we introduce the dual Brinkman-Stokes problem by

$$\begin{aligned} -\nu^* \Delta \boldsymbol{\varphi_v} + \nu G \boldsymbol{\varphi_v} + \nabla \varphi_p &= -\Delta \boldsymbol{e}_v \quad \text{in } \Omega, \ \text{here } \Delta \boldsymbol{e}_v \in H^{-1}(\Omega) \\ -\text{div } \boldsymbol{\varphi_v} &= e_p \quad \text{in } \Omega, \\ \boldsymbol{\varphi_v} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{19}$$

which in a weak form is: find $\boldsymbol{\varphi_v} \in H^1(\Omega)^2$ and $\varphi_p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (\nu^* \nabla \boldsymbol{\varphi_v}, \nabla \boldsymbol{v}^*)_0 + \nu((G\boldsymbol{\varphi_v}), \boldsymbol{v}^*) - (\varphi_p, \nabla \boldsymbol{v}^*)_0 &= (\nabla \boldsymbol{e}_v, \nabla \boldsymbol{v}^*)_0, \ \forall \boldsymbol{v}^* \in H_0^1(\Omega)^2, \\ (-\text{div } \boldsymbol{\varphi_v}, p^*)_0 &= (e_p, p^*)_0, \ \forall p^* \in L_0^2(\Omega), \end{aligned} \tag{20}$$

or, using the notation (14)

$$\mathcal{A}(\{\boldsymbol{\varphi_v}, \varphi_p\}, \{\boldsymbol{v}^*, p^*\}) = (\nabla \boldsymbol{e}_v, \nabla \boldsymbol{v}^*)_0 + (e_p, p^*)_0 \ , \ \forall \{\boldsymbol{v}^*, p^*\} \in V \ . \tag{21}$$

By (18) and (20) where we put $\boldsymbol{v}^* = \boldsymbol{e}_v, \ p^* = e_p$, we get

$$\begin{aligned} \frac{1}{C_P}(\boldsymbol{e}_v, \boldsymbol{e_v})_1 \leq \ (\nabla \boldsymbol{e_v}, \nabla \boldsymbol{e_v})_0 &= \nu^*(\nabla \boldsymbol{\varphi_v}, \nabla \boldsymbol{e_v})_0 + \nu((G\boldsymbol{\varphi_v}), \boldsymbol{e_v}) - (\varphi_p \nabla, \boldsymbol{e_v})_0 \\ &= \nu^*(\nabla \boldsymbol{\varphi_v}, \nabla \boldsymbol{v})_0 + \nu((G\boldsymbol{\varphi_v})\boldsymbol{v}) - (\varphi_p \nabla, \boldsymbol{v})_0 - \nu^*(\nabla \boldsymbol{\varphi_v}, \nabla \boldsymbol{v}_h)_0 \\ &\quad - \nu((G\boldsymbol{\varphi_v})\boldsymbol{v_h}) + (\varphi_p \nabla, \boldsymbol{v_h})_0, \end{aligned} \tag{22}$$

$$(e_p, e_p)_0 = (e_p, -\text{div } \boldsymbol{\varphi_v})_0 = -(p\nabla, \boldsymbol{\varphi_v})_0 + (p_h \nabla, \boldsymbol{\varphi_v})_0. \tag{23}$$

5.2. Estimation of the error by means of the residual and solution of the dual problem

Combining (22), (23), and (19) we get (as $C_P \geq 1$)

$$
\begin{aligned}
&\frac{1}{C_P}\Big\{(\boldsymbol{e}_v, \boldsymbol{e}_v)_1 + (e_p, e_p)_0\Big\} \\
&\leq \nu^*(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi_v})_0 + \nu((G\boldsymbol{v}\varphi_v)) - (p, \nabla \boldsymbol{\varphi_v})_0 - (\nabla \boldsymbol{v}, \varphi_p)_0 \\
&\quad + \sum_{K\in\mathcal{T}_h} \{-\nu^*(\nabla \boldsymbol{\varphi_v}, \nabla \boldsymbol{v}_h)_{0,K} - \nu((G\boldsymbol{v_h}\varphi_v)) + (p_h, \nabla \boldsymbol{\varphi_v})_{0,K} + (\varphi_p, \nabla \boldsymbol{v}_h)_{0,K}\} \\
&= (\boldsymbol{f}, \boldsymbol{\varphi_v})_0 + \sum_{K\in\mathcal{T}_h} \left\{(\nu^*\Delta \boldsymbol{v}_h, \boldsymbol{\varphi_v})_{0,K} - \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{\varphi_v} ds\right\} - \nu((G\boldsymbol{v_h}\varphi_v)) \qquad (24) \\
&\quad - \sum_{K\in\mathcal{T}_h} \left\{(\nabla p_h, \boldsymbol{\varphi_v})_{0,K} + \int_{\partial K} p_h \boldsymbol{\varphi_v} \cdot \boldsymbol{n} ds \ + (\text{div } \boldsymbol{v}_h, \varphi_p)_{0,K}\right\} \\
&= \sum_{K\in\mathcal{T}_h} (\boldsymbol{f} + \nu^*\Delta \boldsymbol{v}_h - \nu((G\boldsymbol{v_h}\varphi_v)) - \nabla p_h, \boldsymbol{\varphi_v})_{0,K} + \sum_{K\in\mathcal{T}_h} (\text{div } \boldsymbol{v}_h, \varphi_p)_{0,K} \\
&\quad - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{\varphi_v} ds + \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \boldsymbol{\varphi_v} \cdot \boldsymbol{n} ds
\end{aligned}
$$

In view of (16) we also have

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} (\boldsymbol{f} + \nu^*\Delta \boldsymbol{v}_h - \nu G\boldsymbol{v}_h - \nabla p_h, \boldsymbol{v}_h^*)_{0,K} + \ (\text{div } \boldsymbol{v}_h, p_h^*)_0 \\
&= (\boldsymbol{f}, \boldsymbol{v}_h^*)_0 + \sum_{K\in\mathcal{T}_h} \left\{(-\nu^*\nabla \boldsymbol{v}_h, \nabla \boldsymbol{v}_h^*)_{0,K} - \nu(G\boldsymbol{v}_h, \boldsymbol{v}_h^*) + \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{v}_h^* ds\right\} \\
&\quad + (\nabla p_h, \boldsymbol{v}_h^*)_0 - \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \boldsymbol{v}_h^* \cdot \boldsymbol{n} ds + (\text{div } \boldsymbol{v}_h, p_h^*)_0 \qquad (25) \\
&= 0 + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{v}_h^* ds - \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \boldsymbol{v}_h^* \cdot \boldsymbol{n} ds, \ \forall \{\boldsymbol{v}_h^*, p_h^*\} \in X^h \times M^h.
\end{aligned}
$$

This implies, taking $\boldsymbol{v}_h^* = \pi_h \boldsymbol{\varphi_v}$, $p_h^* = \pi_h \varphi_p$, the Clement interpolants, (cf. e.g. [8], p. 146) that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} (\boldsymbol{f} + \nu^*\Delta \boldsymbol{v}_h - \nu G\boldsymbol{v}_h - \nabla p_h, \pi_h \boldsymbol{\varphi_v}) + \ (\text{div } \boldsymbol{v}_h, \pi_h \varphi_p)_0 \\
&\qquad - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \pi_h \boldsymbol{\varphi_v} ds - \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \pi_h \boldsymbol{\varphi_v} \cdot \boldsymbol{n} ds = 0 \qquad (26)
\end{aligned}
$$

Now subtracting zero in (26) from (24) we get

$$
\begin{aligned}
&\frac{1}{C_P}\Big\{(\boldsymbol{e}_v,\boldsymbol{e}_v)_1+(e_p,e_p)_0\Big\}\\
&\le \sum_{K\in\mathcal{T}_h}(\boldsymbol{f}+\nu^*\Delta\boldsymbol{v}_h-\nu G\boldsymbol{v}_h-\nabla p_h,\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})_{0,K}+\ (\mathrm{div}\ \boldsymbol{v}_h,\varphi_p-\pi_h\varphi_p)_0\\
&\quad-\sum_{K\in\mathcal{T}_h}\int_{\partial K}\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}(\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})ds+\sum_{K\in\mathcal{T}_h}\int_{\partial K}p_h(\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})\cdot\boldsymbol{n}ds \qquad (27)\\
&=\sum_{K\in\mathcal{T}_h}(\boldsymbol{f}+\nu^*\Delta\boldsymbol{v}_h-\nu G\boldsymbol{v}_h-\nabla p_h,\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})_{0,K}+\ (\mathrm{div}\ \boldsymbol{v}_h,\varphi_p-\pi_h\varphi_p)_0\\
&\quad-\sum_{K\in\mathcal{T}_h}\sum_{l\in\partial K}\int_l\left(\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right)(\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})ds,
\end{aligned}
$$

where we denoted

$$
\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l=\left(\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right)\Big/_{l+}-\left(\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right)\Big/_{l_-}
$$

the jump along the common side l of two adjacent triangles. Then, using in turn the Schwarz inequality, the interpolation properties of $X^h,\ M^h$ (cf. e.g. [4]), and the estimate of the solution of the dual problem (19) (cf. [4]), we get the inequalities

$$
\begin{aligned}
&\|\boldsymbol{e}_v\|_1^2+\|e_p\|_0^2\\
&\le C_P\sum_{K\in\mathcal{T}_h}\Big\{\|\boldsymbol{R}_1\{\boldsymbol{v}_h,p_h\}\|_{0,K}\|\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v}\|_{0,K}+\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}\|\varphi_p-\pi_h\varphi_p\|_{0,K}\Big\}\\
&\quad+C_P\sum_{K\in\mathcal{T}_h}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\|\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v}\|_{0,l} \qquad (28)\\
&\le C_PC_I\sum_{K\in\mathcal{T}_h}\Big\{h_K\ \|\boldsymbol{R}_1\{\boldsymbol{v}_h,p_h\}\|_{0,K}\ \|\boldsymbol{\varphi_v}\|_1+\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}\ \|\varphi_p\|_0\Big\}\\
&\quad+C_PC_I\sum_{K\in\mathcal{T}_h}(h_K)^{\frac{1}{2}}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\|\boldsymbol{\varphi_v}\|_1\\
&\le C_PC_IC_R\sum_{K\in\mathcal{T}_h}\Bigg\{h_K\ \|\boldsymbol{R}_1\{\boldsymbol{v}_h,p_h\}\|_{0,K}\ +\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}\\
&\quad+\sum_{l\in\partial K}(h_K)^{\frac{1}{2}}\ \left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\Bigg\}\cdot\big\{\|\Delta\boldsymbol{e}_v\|_{-1}+\|e_p\|_0\big\}.
\end{aligned}
$$

Using then the relations

$$
\begin{aligned}
\|\Delta\boldsymbol{e}_v\|_{-1}&\equiv\sup_{\boldsymbol{v}^*\in H_0^1,\boldsymbol{v}^*\neq0}\frac{|(\Delta\boldsymbol{e}_v,\boldsymbol{v}^*)_0|}{\|\boldsymbol{v}^*\|_1}=\sup_{\boldsymbol{v}^*\in H_0^1,\boldsymbol{v}^*\neq0}\frac{|(\nabla\boldsymbol{e}_v,\nabla\boldsymbol{v}^*)_0|}{\|\boldsymbol{v}^*\|_1}\\
&\le\sup_{\boldsymbol{v}^*\in H_0^1,\boldsymbol{v}^*\neq0}\frac{\|\nabla\boldsymbol{e}_v\|_0\ \|\nabla\boldsymbol{v}^*\|_0}{\|\boldsymbol{v}^*\|_1}\le\|\nabla\boldsymbol{e}_v\|_0\le\|\boldsymbol{e}_v\|_1
\end{aligned}
$$

we get, by (28)

$$\left\{\|\boldsymbol{e}_v\|_1+\|e_p\|_0\right\}^2 \le 2\left\{\|\boldsymbol{e}_v\|_1^2+\|e_p\|_0^2\right\} \le 2C_PC_IC_R\sum_{K\in\mathcal{T}_h}\Bigg\{h_K\|\boldsymbol{R_1}\{\boldsymbol{v}_h,p_h\}\|_{0,K}$$
$$+\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}+h_K^{\frac{1}{2}}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\Bigg\}\cdot\left\{\|\boldsymbol{e}_v\|_1+\|e_p\|_0\right\}. \quad (29)$$

Upon cancelling $\left\{\|\boldsymbol{e}_v\|_1+\|e_p\|_0\right\}$ in (29) we finally get the following theorem:

Theorem 1. *Let Ω be a polygon in R^2. Let $\mathcal{T}_h$ be a family of regular triangulations of Ω. Let $\{\boldsymbol{v_h},p_h\}$ be the Taylor-Hood approximation of the solution $\{\boldsymbol{v},p\}$ of the Stokes-Brinkman problem. Then the error $\{\boldsymbol{e}_v,e_p\}$ satisfies the following a posteriori estimate*

$$\|\boldsymbol{e}_v\|_1+\|e_p\|_0 \le 2C_PC_IC_R\sum_{K\in\mathcal{T}_h}\Bigg\{h_K\|\boldsymbol{R}_1\{\boldsymbol{v}_h,p_h\}\|_{0,K}+\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}$$
$$+h_K^{\frac{1}{2}}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\Bigg\}. \quad (30)$$

where C_P, C_I, C_R are positive constants, residuals $\boldsymbol{R}_1$ and R_2 are defined in (17) .

Conclusions

The estimate in Theorem 1 applies to more general class of elements. Of course, for Taylor-Hood elements with continuous pressure the jumps of p_h along the common sides disappear.

Let us note that for convex domains stronger regularity applies to the Stokes problem, cf. [13], and better a posteriori error estimate may be expected.

For nonconvex domains with corners we do not obtain so strong regularity as in [13], cf. e.g. [6], but still the a posteriori error estimate should be better than in (30), as it was for the Poisson equation in (2).

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