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ISOGEOMETRIC ANALYSIS FOR FLUID FLOW PROBLEMS

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Abstract

The article is devoted to the simulation of viscous incompressible fluid flow based on solving the Navier-Stokes equations. As a numerical model we chose isogeometrical approach. Primary goal of using isogemetric analysis is to be always geometrically exact, independently of the discretization, and to avoid a time-consuming generation of meshes of computational domains. For higher Reynolds numbers, we use stabilization techniques SUPG and PSPG. All methods mentioned in the paper are demonstrated on a standard test example – flow in a lid-driven cavity.

1. Introduction

Typically in engineering practice, design is done in CAD systems and meshes, needed for the finite element analysis, are generated from CAD data. Primary goal of using isogeometric analysis is to be geometrically exact, independently of the discretization. Then we do not need to create any other mesh - the mesh of the so-called "NURBS elements" is acquired directly from CAD representation. Further refinement of the mesh or increasing the order of basis functions are very simple, efficient and robust.

2. NURBS Surfaces

NURBS surface of degree p, q is determined by a control net $\mathbf{P}$ (of control points $P_{i,j}$, $i = 0, \ldots, n$, $j = 0, \ldots, m$), weights $w_{i,j}$ of these control points and two knot vectors $U = (u_0, \ldots, u_{n+p+1})$, $V = (v_0, \ldots, v_{m+q+1})$ and is given by a parametrization

$$S(u, v) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{m} w_{i,j} P_{i,j} N_{i,p}(u) M_{j,q}(v)}{\sum_{i=0}^{n} \sum_{j=0}^{m} w_{i,j} N_{i,p}(u) M_{j,q}(v)} = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} R_{i,j}(u, v). \tag{1}$$

B-spline basis functions $N_{i,p}(u)$ and $M_{j,q}(v)$ are determined by knot vectors U and V and degrees p and q, respectively, by a formula (for $N_{i,p}(u)$, $M_{j,q}(v)$ is constructed by the similar way)

$$
\begin{aligned}
N_{i,0}(u) &= \begin{cases} 1 & u_i \leq t < u_{i+1} \\ 0 & \text{otherwise} \end{cases} \\
N_{i,p}(u) &= \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u).
\end{aligned} \tag{2}
$$

Knot vector is a non-decreasing sequence of real numbers which determines the distribution of a parameter on the corresponding curve/surface. B-spline basis functions (see Figure 1) of degree p are C^{p-1}-continuous in general. Knot repeated k times in the knot vector decreases the continuity of B-spline basis functions by $k-1$. Support of B-spline basis functions is local – it is nonzero only on the interval $[t_i, t_{i+p+1}]$ in the parameter space and each B-spline basis function is non-negative, i.e., $N_{i,p}(t) \geq 0, \forall t$. See [7] for more information.

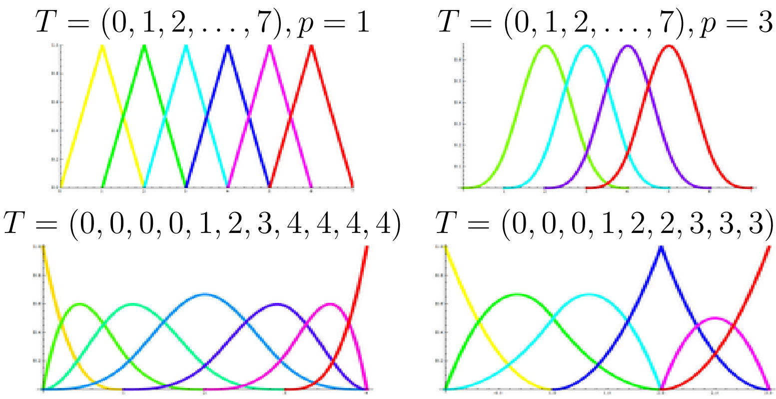

Figure 1: B-spline basis functions

3. Stationary Navier-Stokes equations

The model of viscous flow of an incompressible Newtonian fluid can be described by the Navier-Stokes equations in the common form

$$
\begin{aligned}
\nabla p + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} &= f, \\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned} \tag{3}
$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ is the vector function describing flow velocity, $p = p(\boldsymbol{x})$ is the pressure normalized by density function, ν describes kinematic viscosity and f additional body forces acting on the fluid. The boundary value problem is considered as the system (3) together with boundary conditions

$$
\begin{aligned}
\boldsymbol{u} &= \boldsymbol{w} \quad \text{on } \partial\Omega_D \quad \text{(Dirichlet condition)} \\
\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - \boldsymbol{n} p &= \boldsymbol{0} \quad \text{on } \partial\Omega_N \quad \text{(Neumann condition).}
\end{aligned} \tag{4}
$$

If the velocity is specified everywhere on the boundary, then the pressure solution is only unique up to a (hydrostatic) constant.

Let V be a velocity solution space and V_0 be the corresponding space of test functions, i.e.,

$$\begin{aligned} V &= \{\boldsymbol{u} \in H^1(\Omega)^d | \boldsymbol{u} = \boldsymbol{w} \text{ on } \partial\Omega_D\} \\ V_0 &= \{\boldsymbol{v} \in H^1(\Omega)^d | \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega_D\}. \end{aligned} \tag{5}$$

Then a weak formulation of the boundary value problem is: find $\boldsymbol{u} \in V$ and $p \in L_2(\Omega)$ such that

$$\begin{aligned} \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + \int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \boldsymbol{v} - \int_\Omega p \nabla \cdot \boldsymbol{v} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in V_0 \\ \int_\Omega q \nabla \cdot \boldsymbol{u} &= 0 \qquad \forall q \in L_2(\Omega) \end{aligned}$$

3.1. Approximation using isogeometric analysis

We define the finite-dimensional spaces $V^h \subset V$, $V_0^h \subset V_0$, $W^h \subset L_2(\Omega)$ and their basis functions. We want to find $\boldsymbol{u}_h^{k+1} \in V^h$ and $p_h^{k+1} \in W^h$ such that for all $\boldsymbol{v}_h \in V_0^h$ and $q_h \in W^h$ it holds

$$\nu \int_\Omega \nabla \boldsymbol{u}_h^{k+1} : \nabla \boldsymbol{v}_h + \int_\Omega (\boldsymbol{u}_h^k \cdot \nabla \boldsymbol{u}_h^{k+1}) \boldsymbol{v}_h - \int_\Omega p_h^{k+1} \nabla \cdot \boldsymbol{v}_h = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h, \tag{6}$$

$$\int_\Omega q_h \nabla \cdot \boldsymbol{u}_h^{k+1} = 0. \tag{7}$$

This approach is based on the Picard's method (fixed point iteration). For isogeometric analysis, basis functions of V_0^h and W^h are NURBS basis functions obtained from the NURBS description of the computational domain (for velocity and pressure). We can express $\boldsymbol{u}_h^k$ and p_h^k as a linear combination of the basis functions (2) (we use the values $p = 3, q = 3$ for the velocity and $p = 2, q = 2$ for the pressure in the follow-up examples). These linear combinations are substituted to (6) and (7). Linearization is done with help of Picard's iteration and we obtain a sequence of solutions $(\boldsymbol{u}_h^k, p_h^k) \in V^h \times W^h$, which converges to the weak solution. We obtain a matrix formulation of the problem in the form

$$\begin{bmatrix} \boldsymbol{A} + \boldsymbol{N}(\boldsymbol{u}^k) & \boldsymbol{0} & -\boldsymbol{B}_1^\top \\ \boldsymbol{0} & \boldsymbol{A} + \boldsymbol{N}(\boldsymbol{u}^k) & -\boldsymbol{B}_2^\top \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{k+1} \\ \boldsymbol{u}_2^{k+1} \\ \boldsymbol{p}^{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 - (\boldsymbol{A}^* + \boldsymbol{N}^*(\boldsymbol{u}^k)) \cdot \boldsymbol{u}_1^* \\ \boldsymbol{f}_2 - (\boldsymbol{A}^* + \boldsymbol{N}^*(\boldsymbol{u}^k)) \cdot \boldsymbol{u}_2^* \\ -\boldsymbol{B}_1^* \cdot \boldsymbol{u}_1^* - \boldsymbol{B}_2^* \cdot \boldsymbol{u}_2^* \end{bmatrix}, \tag{8}$$

where

$$\begin{aligned} \boldsymbol{A} &= \left[A_{ij}\right]_{1 \le i \le n_d^u, 1 \le j \le n_d^u}, & \boldsymbol{A}^* &= \left[A_{ij}\right]_{1 \le i \le n_d^u, n_d^u+1 \le j \le n_v^u}, \\ \boldsymbol{N}(\boldsymbol{u}) &= \left[N_{ij}(\boldsymbol{u})\right]_{1 \le i \le n_d^u, 1 \le j \le n_d^u}, & \boldsymbol{N}^*(\boldsymbol{u}), &= \left[N_{ij}(\boldsymbol{u})\right]_{1 \le i \le n_d^u, n_d^u+1 \le j \le n_v^u}, \\ \boldsymbol{B}_k &= \left[B_{kij}\right]_{1 \le i \le n^p, 1 \le j \le n_d^u}, & \boldsymbol{B}_k^* &= \left[B_{kij}\right]_{1 \le i \le n^p, n_d^u+1 \le j \le n_v^u}, \end{aligned} \tag{9}$$

$$
\begin{aligned}
A_{ij} &= \nu \int_\Omega (\nabla R_i^u \cdot J^{-1}) \cdot (\nabla R_j^u \cdot J^{-1}) |\det J|, \\
N_{ij}(\boldsymbol{u}) &= \int_\Omega R_i^u \left[\left(\sum_{l=1}^{n_v^u} (u_{1l}, u_{2l}) R_l^u \right) \cdot (\nabla R_j^u \cdot J^{-1}) \right] |\det J|, \\
B_{kij} &= \int_\Omega R_i^p \left[(\nabla R_j^u \cdot J^{-1}) \cdot \boldsymbol{e}_k \right] |\det J|.
\end{aligned}
\tag{10}
$$

Here n_d^u is the number of points where the Dirichlet boundary condition is not defined and $\boldsymbol{u}_1^*, \boldsymbol{u}_2^*$ are fixed coefficient so that the Dirichlet boundary condition is satisfied. J is the Jacobi matrix of a mapping from parametric domain to the computational domain. The initial nonlinear Navier-Stokes problem was transformed to the sequential solving of linear systems.

In the follow-up examples, we use strong imposition of Dirichlet boundary conditions. If the given function $\boldsymbol{w}$ belongs to V^h, Dirichlet boundary condition is prescribed directly on control points describing $\partial\Omega_D$. Otherwise, we have to find an approximation $\boldsymbol{w}_h$ of $\boldsymbol{w}$ in V^h and again prescribe this condition directly on control points.

3.2. LBB (Ladyženskaja-Babuška-Brezzi) condition

In general, it is not possible to use an arbitrary combination of discretizations for pressure and velocity for solving Stokes problem in order for given discretizations to be stable, it needs to satisfy the so-called LBB condition (or inf-sup condition). It can be shown that one of such suitable choices of discretizations is represented by spaces with basis function of degree p (for pressure) and degree $p+1$ (for velocity) obtained with the help of p-refinement (see [1] for more details).

4. Stabilization methods

The solving of Navier-Stokes equations leads to numerical nonstability for high Reynolds numbers. We review two methods to reduce nonphysical oscillations based on the construction of test functions in special forms (see for example [6]).

4.1. SUPG - Streamline Upwind/Petrov-Galerkin

The first equation (3) is multiplied by test function $\overline{\boldsymbol{v}}$ in the form

$$
\overline{\boldsymbol{v}} = \boldsymbol{v} + \tau_S \boldsymbol{u} \cdot \nabla \boldsymbol{v}, \tag{11}
$$

where

$$
\tau_S = \frac{h}{2\deg(\boldsymbol{u})\|\boldsymbol{u}\|} \left(\coth P - \frac{1}{P} \right), \tag{12}
$$

h is the element diameter in the direction of the $\boldsymbol{u}$ and $P = \frac{\|\boldsymbol{u}\|h}{2\nu}$ is the local Péclet number which determines whether the problem is locally convection dominated or diffusion dominated. Then we integrate over Ω and use Picard's linearization method.

The first equation has the form

$$\underbrace{\nu\int\limits_{\Omega}\nabla\boldsymbol{u}^{k+1}:\nabla\boldsymbol{v}}_{\boldsymbol{A}}+\underbrace{\int\limits_{\Omega}\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{u}^{k+1}}_{\boldsymbol{N}(u)}-\underbrace{\int\limits_{\Omega}p^{k+1}\nabla\cdot\boldsymbol{v}}_{\boldsymbol{B}}-\underbrace{\nu\int\limits_{\Omega}\Delta\boldsymbol{u}^{k+1}\tau_S\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{v}}_{SUPG}+$$
$$+\underbrace{\int\limits_{\Omega}(\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{u}^{k+1})\tau_S\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{v}}_{SUPG}+\underbrace{\int\limits_{\Omega}\nabla p^{k+1}\tau_S\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{v}}_{SUPG}=\int\limits_{\Omega}\boldsymbol{f}\cdot\overline{\boldsymbol{v}}. \tag{13}$$

4.2. PSPG (Pressure Stabilized/Petrov-Galerkin)

In this case we multiply the first equation (3) by the test function in the form

$$\overline{\boldsymbol{v}}=\boldsymbol{v}+\tau_S\boldsymbol{u}\cdot\nabla\boldsymbol{v}+\tau_P\nabla q, \tag{14}$$

where $0\leq\tau_P\leq\tau_S$ and integrate over Ω. By application of Picard's method we have

$$\underbrace{\nu\int\limits_{\Omega}\nabla\boldsymbol{u}^{k+1}:\nabla\boldsymbol{v}}_{\boldsymbol{A}}+\underbrace{\int\limits_{\Omega}\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{u}^{k+1}}_{\boldsymbol{N}(u)}-\underbrace{\int\limits_{\Omega}p^{k+1}\nabla\cdot\boldsymbol{v}}_{\boldsymbol{B}}- \tag{15}$$
$$-\underbrace{\nu\int\limits_{\Omega}\Delta\boldsymbol{u}^{k+1}\tau_S\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{v}}_{SUPG}+\underbrace{\int\limits_{\Omega}(\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{u}^{k+1})\tau_S\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{v}}_{SUPG}+\underbrace{\int\limits_{\Omega}\nabla p^{k+1}\tau_S\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{v}}_{SUPG}+$$
$$+\underbrace{\int\limits_{\Omega}\tau_P\nabla p^{k+1}\nabla q}_{PSPG}-\underbrace{\nu\int\limits_{\Omega}\tau_P\Delta\boldsymbol{u}^{k+1}\nabla q}_{PSPG}+\underbrace{\int\limits_{\Omega}(\boldsymbol{u}^{k}\cdot\nabla\boldsymbol{u}^{k+1})\tau_P\nabla q}_{PSPG}=\int\limits_{\Omega}\boldsymbol{f}\cdot\overline{\boldsymbol{v}}$$

If we use these stabilization techniques, the LBB condition need not be satisfied.

5. Non-stationary Navier-Stokes problem

For the simplicity we solve the homogeneous problem

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t}+\nabla p+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{u}&=0\quad\text{in }\Omega\times(0,T)\\ \nabla\cdot\boldsymbol{u}&=0\quad\text{v }\Omega\end{aligned} \tag{16}$$

with the initial and boundary conditions

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x},t)&=\boldsymbol{w}(\boldsymbol{x},t)\quad\text{on }\partial\Omega\times[0,T],\\ \boldsymbol{u}(\boldsymbol{x},0)&=\boldsymbol{u}_0(\boldsymbol{x})\quad\text{in }\Omega.\end{aligned} \tag{17}$$

A method described in [4] is used. It is given $\boldsymbol{u}^0$, $\theta\in(0,\frac{1}{2})$, $\alpha\in(0,1)$, $\beta\in(0,1)$ and we search for $\boldsymbol{u}^1$,$\boldsymbol{u}^2$, …, $\boldsymbol{u}^n$ by the following three steps:

1. step

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+\theta}-\boldsymbol{u}^{n}}{\theta\Delta t}+\nabla p^{n+\theta}-\alpha\nu\Delta\boldsymbol{u}^{n+\theta} &= \beta\nu\Delta\boldsymbol{u}^{n}-\boldsymbol{u}^{n}\cdot\nabla\boldsymbol{u}^{n}\\
\nabla\cdot\boldsymbol{u}^{n+\theta} &= \mathbf{0}\\
\boldsymbol{u}^{n+\theta} &= \boldsymbol{g}^{n+\theta} \text{ on } \partial\Omega
\end{aligned}
\tag{18}
$$

2. step

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1-\theta}-\boldsymbol{u}^{n+\theta}}{(1-2\theta)\Delta t}-\beta\nu\Delta\boldsymbol{u}^{n+1-\theta}+\boldsymbol{u}^{*}\cdot\nabla\boldsymbol{u}^{n+1-\theta} &= \alpha\nu\Delta\boldsymbol{u}^{n+\theta}-\nabla p^{n+\theta}\\
\boldsymbol{u}^{n+1-\theta} &= \boldsymbol{g}^{n+1-\theta} \text{ on } \partial\Omega
\end{aligned}
\tag{19}
$$

3. step

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n+1-\theta}}{\theta\Delta t}+\nabla p^{n+1}-\alpha\nu\Delta\boldsymbol{u}^{n+1} &= \beta\nu\Delta\boldsymbol{u}^{n+1-\theta}-\boldsymbol{u}^{*}\cdot\nabla\boldsymbol{u}^{n+1-\theta}\\
\nabla\cdot\boldsymbol{u}^{n+1} &= \mathbf{0}\\
\boldsymbol{u}^{n+1} &= \boldsymbol{g}^{n+1} \text{ on } \partial\Omega
\end{aligned}
\tag{20}
$$

This is a self-starting scheme. Choosing $\alpha = \beta = \frac{1}{2}$ or setting $\theta = 1 - \frac{1}{\sqrt{2}}$ with $\alpha + \beta = 1$ gives second-order accuracy as $\Delta t \to 0$. In particular, setting $\theta = 1 - \frac{1}{\sqrt{2}}$ and $\alpha = \frac{1-2\theta}{1-\theta}, \beta = \frac{\theta}{1-\theta}$ gives a method which is second-order accurate in time, unconditionally stable and has good asymptotic properties.

6. Examples

We present test example, which is symmetric to the well-known test problem, the so-called lid-driven cavity flow in 2D. The only difference is that the moving wall is situated at the bottom of the cavity. This change has no compelling reason, the test problem is sufficient for testing the solver and comparing the results with benchmark ones.

It should be noted that the presented solver uses both presented stabilization techniques, it means that the degree of basis functions for pressure is one less than the degree of velocity basis functions and the PSPG stabilization technique is also used. Using only one technique is sufficient for the stable solution and we tested both of them as well as their combination.

6.1. Stationary flow

The first experiment is devoted to the stationary flow. So we solve stationary Navier-Stokes equations (3) with the bottom boundary moving from left to right ($\boldsymbol{u} = (u_x, 0)$) and no-slip boundary condition on the other walls. Figure 2 shows the solutions with the three different Reynolds numbers and instability for higher ones. The solution of the same problem where the stabilization methods are used is illustrated on Figure 3. It is known (see for example [5]), that the solution of

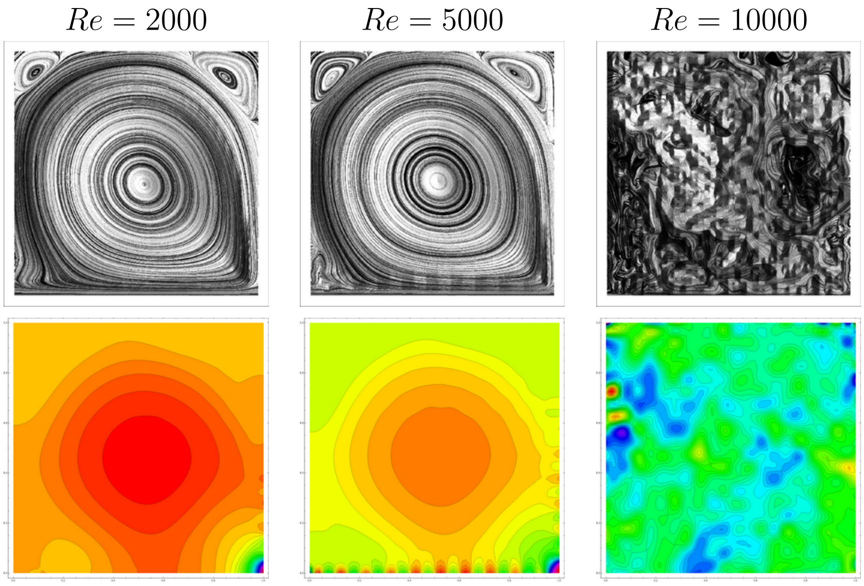

Figure 2: Stationary Navier-Stokes problem. Solution without stabilization techniques. Velocity is illustrated at the upper figures, pressure is illustrated at lower figures.

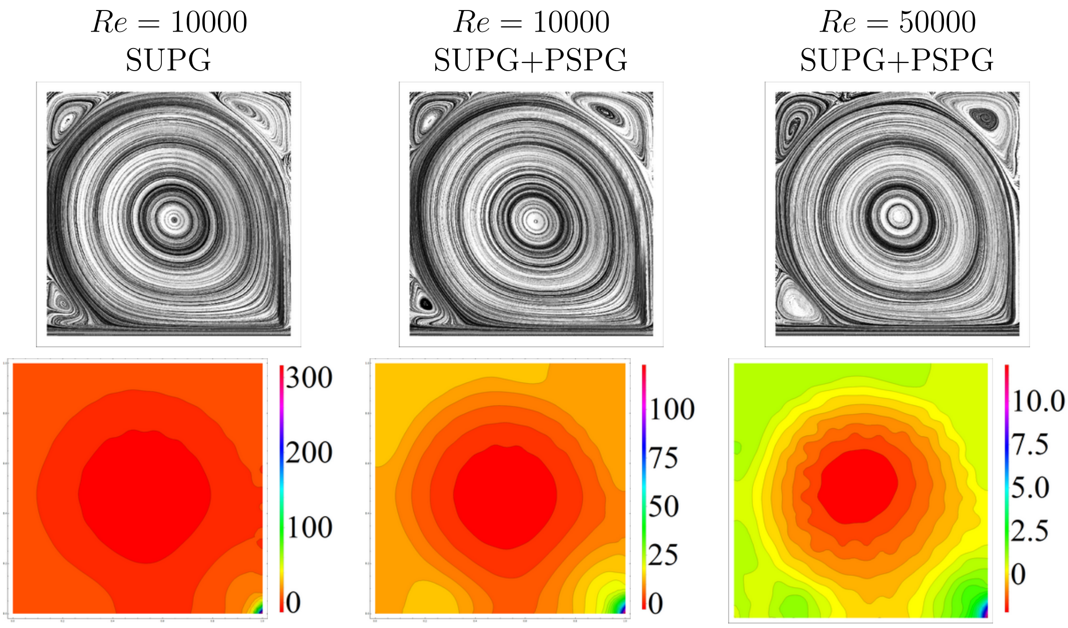

Figure 3: Stationary Navier-Stokes problem. Solution with stabilization techniques. Velocity is illustrated at the upper figures, pressure is illustrated at lower figures.

this test example has a stable solution only for much smaller Reynolds numbers than presented $Re = 50000$. So the result for the $Re = 50000$ is not very physically meaningful, it is rather an example of the used stabilization techniques. It should be also noted, that the NURBS discretization uses fewer elements than the finite element discretization in general. This coarse discretization causes more artificial viscosity.

6.2. Non-stationary flow

The second example is devoted to the non-stationary flow. We solve non-stationary Navier-Stokes equations (16) with the same boundary conditions as in the first example. Initial condition is described by the zero velocity inside the cavity ($\boldsymbol{u} = \mathbf{0}$). Solution with Reynolds number Re $= 1000$ is illustrated at Figure 4.

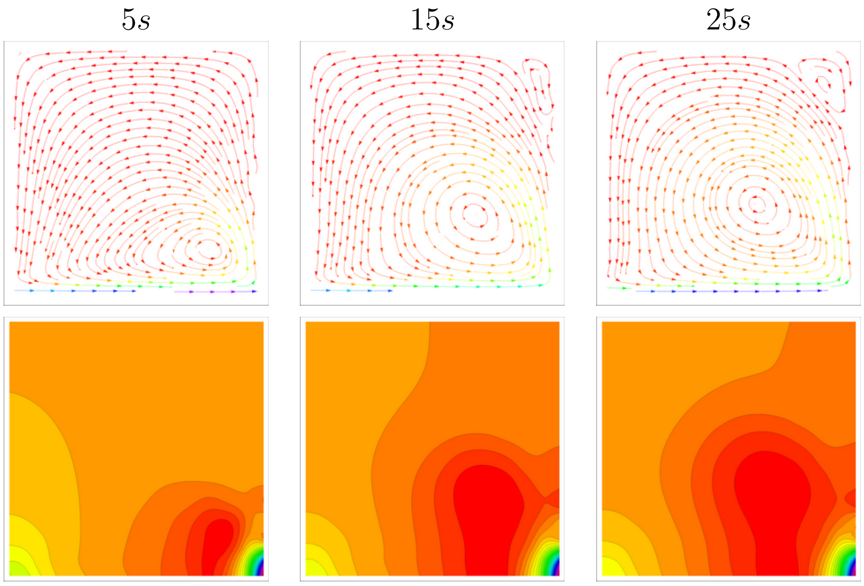

Figure 4: Non-stationary Navier-Stokes problem. Solution with stabilization techniques. Velocity is illustrated at the upper figures, pressure is illustrated at lower figures.

7. Conclusion

We developed and tested an isogeometric analysis based solver for solving stationary and nonstationary flow based on Navier-Stokes equations. The presented results show that the isogeometric analysis is a suitable tool for solving such complex problems. Iterative solution of stationary Navier-Stokes equations converges only for relatively low Reynolds numbers. Therefore, it is necessary to use stabilization methods (e.g. SUPG, PSPG, see [2]). The problems with oscillations can be solved by the SOLD methods [6]. The presented scheme for solving non-stationary

Navier-Stokes equations is currently enlarged by turbulence model. The turbulence is included by the RANS equations using $k - \omega$ model [3].

Acknowledgements

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