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SOME PROPERTIES OF THE DISTANCE LAPLACIAN EIGENVALUES OF A GRAPH

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Abstract. The distance Laplacian of a connected graph G is defined by $\mathcal{L} = \mathrm{Diag}(\mathrm{Tr}) - \mathcal{D}$, where $\mathcal{D}$ is the distance matrix of G, and $\mathrm{Diag}(\mathrm{Tr})$ is the diagonal matrix whose main entries are the vertex transmissions in G. The spectrum of $\mathcal{L}$ is called the distance Laplacian spectrum of G. In the present paper, we investigate some particular distance Laplacian eigenvalues. Among other results, we show that the complete graph is the unique graph with only two distinct distance Laplacian eigenvalues. We establish some properties of the distance Laplacian spectrum that enable us to derive the distance Laplacian characteristic polynomials for several classes of graphs.

Keywords: distance matrix; Laplacian; characteristic polynomial; eigenvalue

MSC 2010: 05C12, 05C31, 05C50, 05C76

1. Introduction

We begin by recalling some definitions. In this paper, we consider only connected simple, undirected and finite graphs, i.e., undirected graphs on a finite number of vertices without multiple edges or loops and in which any two vertices are connected by a sequence of edges. A graph is (usually) denoted by $G = G(V, E)$, where V is its vertex set and E its edge set. The *order* of G is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges.

As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by $K_{a,n-a}$, $1 \leqslant a \leqslant n-1$, the complete bipartite graph, and by K_n the complete graph, each on n vertices. A *kite* $\mathrm{Ki}_{n,\omega}$ is the graph obtained from a clique K_ω and a path $P_{n-\omega}$ by adding an edge between an endpoint of the path and a vertex from the clique. We denote by S_n^+ the graph obtained from a star S_n by adding an edge.

The *adjacency matrix* of G is a 0-1 $n \times n$-matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. The *adjacency spectrum* of G is the spectrum of its adjacency matrix. For more details about the adjacency spectrum of a graph see the books [5], [9], [10], [11].

The matrix $L = \mathrm{Diag}(\mathrm{Deg}) - A$, where $\mathrm{Diag}(\mathrm{Deg})$ is the diagonal matrix whose main entries are the degrees in G, is called the (*adjacency*) *Laplacian* of G. The adjacency Laplacian spectrum of G is the spectrum of L. More details about L and its spectrum can be found in the books [5], [11] and in the survey papers [21], [22].

Given two vertices u and v in a graph G, $d(u,v) = d_G(u,v)$ denotes the *distance* (the length of a shortest path) between u and v. The *Wiener index* $W(G)$ of a graph G is defined to be the sum of all distances in G, i.e.,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u,v).$$

The *transmission* $\mathrm{Tr}(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e.,

$$\mathrm{Tr}(v) = \sum_{u \in V} d(u,v).$$

A graph $G = (V,E)$ is said to be k-*transmission regular* if $\mathrm{Tr}(v) = k$ for every vertex $v \in V$.

The *distance matrix* $\mathcal{D}$ of a graph G is the matrix indexed by the vertices of G with $\mathcal{D}_{i,j} = d(v_i, v_j)$ and where $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j. Let $(\partial_1, \partial_2, \dots, \partial_n)$ denote the spectrum of $\mathcal{D}$. It is called the *distance spectrum* of the graph G. We assume that the distance eigenvalues are labeled such that $\partial_1 \geqslant \partial_2 \geqslant \dots \geqslant \partial_n$.

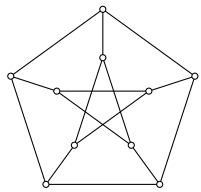
Following the way that the adjacency Laplacian matrix L is defined, we introduced in [3] the *distance Laplacian* $\mathcal{L}$ of a graph G as $\mathcal{L} = \mathrm{Diag}(\mathrm{Tr}) - \mathcal{D}$, where $\mathrm{Diag}(\mathrm{Tr})$ denotes the diagonal matrix of the vertex transmissions in G. The similarity is that in L the diagonal entries are the column (row) sums in the adjacency matrix and in $\mathcal{L}$ the diagonal entries are the column (row) sums in the distance matrix. Let $(\partial_1^{\mathcal{L}}, \partial_2^{\mathcal{L}}, \dots, \partial_n^{\mathcal{L}})$ denote the spectrum of $\mathcal{L}$. We call it the *distance Laplacian spectrum* of the graph G. We assume that the distance Laplacian eigenvalues are labeled such that $\partial_1^{\mathcal{L}} \geqslant \partial_2^{\mathcal{L}} \geqslant \dots \geqslant \partial_n^{\mathcal{L}}$.

To illustrate the definition, we present in Figure 1 the Petersen graph [17] with its different spectra.

For a graph G, let $P_{\mathcal{D}}^G(t)$ and $P_{\mathcal{L}}^G(t)$ denote the distance and the distance Laplacian characteristic polynomials respectively. For instance, the distance and the distance Laplacian spectra of the complete graph K_n are respectively its adjacency and Laplacian spectra, i.e.,

$$P_{\mathcal{D}}^{K_n}(t) = (t-n+1)(t+1)^{n-1};$$
$$P_{\mathcal{L}}^{K_n}(t) = t(t-n)^{n-1}.$$

Distance and distance Laplacian spectra of some common families of graphs can be found in [3] and below.



A-spectrum	$3^{(1)}$	$1^{(5)}$	$-2^{(4)}$
L-spectrum	$5^{(4)}$	$2^{(5)}$	$0^{(1)}$
$\mathcal{D}$-spectrum	$15^{(1)}$	$0^{(4)}$	$-3^{(5)}$
$\mathcal{L}$-spectrum	$18^{(5)}$	$15^{(4)}$	$0^{(1)}$

Figure 1. The Petersen graph and its different spectra.

Graphs with the same spectrum with respect to an associated matrix M are called *cospectral* graphs with respect to M, or *M-cospectral graphs.* Two M-cospectral non-isomorphic graphs G and H are called *M-cospectral mates* or *M-mates.* The question "which graph is defined by its A-spectrum" raised by Günthard and Primas [14] in 1956 in a paper relating spectral theory of graphs and Hückel's theory from chemistry. It was conjectured [14] that there are no A-cospectral mates. That conjecture was refuted in [7] for the class of trees, in [8] for the class of general graphs, and in [4] for the class of connected graphs (see Figure 2). The first infinite family of pairs of A-cospectral mates trees was constructed by Schwenk [23], who also proved that asymptotically every tree has a mate.

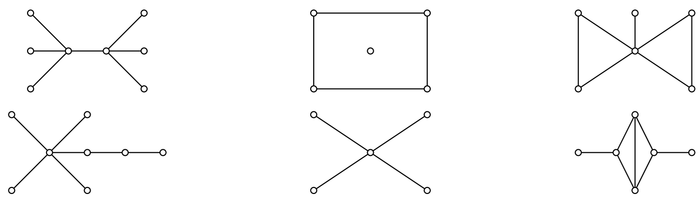


Figure 2. Two A-cospectral trees, graphs, and connected graphs.

The L-cospectrality is studied in [13], [15], [16], [20], [21], [24]. The smallest L-cospectral graphs, with respect to the order, contain 6 vertices, and are given in Figure 3.

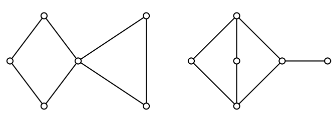


Figure 3. The smallest L-cospectral graphs.

Regarding the $\mathcal{D}$-cospectrality, the smallest (see Figure 4) $\mathcal{D}$-cospectral trees contain 17 vertices, and belong to an infinite family of pairs of $\mathcal{D}$-mates that can be constructed using *McKay's method*, described in [19]. In fact, these two trees are the only $\mathcal{D}$-cospectral trees on 17 vertices.

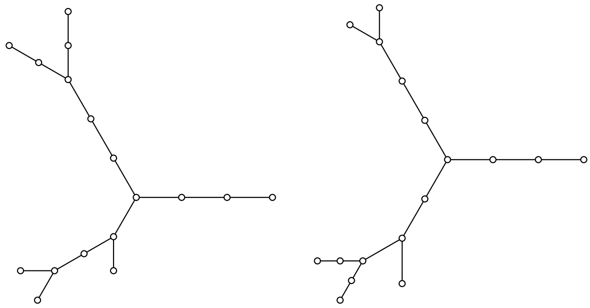


Figure 4. The smallest $\mathcal{D}$-cospectral trees.

Concerning the cospectrality with respect to the distance Laplacian matrix, the experiments done in [3], by enumerating all the 1346023 trees on at most 20 vertices, found no mates. Then, it was conjectured that every tree can be determined by its distance Laplacian spectrum. Over the class of graphs in general, there exist mates with respect to the distance Laplacian matrix. For instance, the graphs given in Figure 5 are not isomorphic (the graph on the left contains a triangle whose vertices have degree three, while the graph on the right does not contain such a triangle), but share the same distance Laplacian spectrum $(16.803542, 16, 16, 16, 14.624336, 14, 12, 10.572121, 0)$. Note that these graphs are not cospectral with respect to the distance matrix, but they are with respect to the adjacency Laplacian.

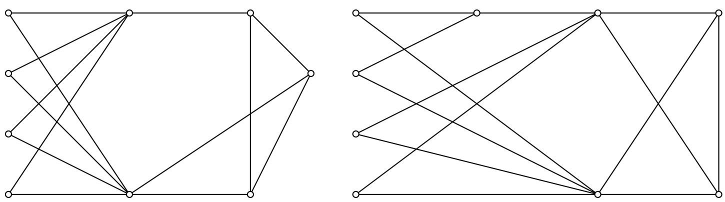


Figure 5. Two cospectral graphs on 9 vertices with respect to the distance Laplacian.

It is known [25] that the adjacency and (adjacency) Laplacian spectra are equivalent over the class of degree regular graphs. A similar result for the distance and the distance Laplacian spectra over the class of transmission regular graphs is proved in [3]. Also, equivalence between the Laplacian and the distance Laplacian spectra holds over the class of graphs with diameter 2 (see [3]).

Several other properties and results about the distance Laplacian spectra are discussed and proved in [3]. Among these result we recall the following theorem that will be used in the present paper.

Theorem 1.1 ([3]). *Let G be a graph on n vertices. Then $\partial^{\mathcal{L}}_{n-1} = n$ if and only if $\overline{G}$ is disconnected. Furthermore, the multiplicity of n as an eigenvalue of $\mathcal{L}$ is one less than the number of the connected components of $\overline{G}$.*

The above theorem establishes a connection between the second smallest distance Laplacian eigenvalue $\partial^{\mathcal{L}}_{n-1}$ of a graph G and the second smallest adjacency Laplacian eigenvalue (known as *algebraic connectivity* [12]) of its complement $\overline{G}$.

The rest of the paper is organized as follows. In Section 2, we study some particular eigenvalues. Among other results, we show that 0 is the smallest distance Laplacian eigenvalue, with multiplicity 1. We prove that the complete graph K_n, $n \geqslant 2$, is the only graph with exactly two distinct distance Laplacian eigenvalues. We show also how to compute some distance Laplacian eigenvalue and its multiplicity whenever the graph contains a clique or an independent set whose vertices share the same neighborhood. In Section 3, we list a series of open conjectures.

2. Some particular eigenvalues

In this section, we study some particular distance Laplacian eigenvalues. First, as for the Laplacian, 0 is also an eigenvalue of the distance Laplacian. Before proving this fact, recall the following well-known result from matrix theory.

Lemma 2.1 (Gershgorin Theorem, [18]). *Let $M = (m_{ij})$ be a complex $n \times n$-matrix and denote by $\lambda_1, \lambda_2, \ldots, \lambda_p$ its distinct eigenvalues. Then*

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subset \bigcup_{i=1}^{n} \Big\{ z\colon\ |z - m_{ii}| \leqslant \sum_{j \neq i} |m_{ij}| \Big\}.$$

Theorem 2.2. *For any graph G, we have $\partial^{\mathcal{L}}_{n} = 0$ with multiplicity* 1.

P r o o f. If $e = [1, 1, \ldots, 1]^t$ is the all ones n-vector, then $\mathcal{L}e = 0$. Thus $\partial = 0$ is an eigenvalue of $\mathcal{L}$. Since $\mathcal{L}$ is positive semi-definite, then $\partial^{\mathcal{L}}_{n} = 0$.

To prove that the multiplicity of $\partial^{\mathcal{L}}_{n} = 0$ is 1, it suffices to prove that the rank of $\mathcal{L}$ is $n-1$. Consider the matrix M obtained from $\mathcal{L}$ by the deletion of, say, the last row and the last column. Then M is strictly diagonally dominant. Using Lemma 2.1, 0 is not an eigenvalue of M. Thus $\det(M) \neq 0$ and therefore the rank of $\mathcal{L}$ is $n - 1$. $\square$

Some regularities in graphs are useful in calculating certain eigenvalues of the matrices related to these graphs. It is the case, for instance, for the largest eigenvalue of the adjacency matrix or the signless Laplacian whenever the graph is degree regular. The same is true for the largest eigenvalue of the distance Laplacian whenever the graph is transmission regular. Sometimes, a local regularity in a graph suffices to know some eigenvalue. We prove below that it is possible to know a distance Laplacian eigenvalue of a graph if it contains a clique or an independent set whose vertices share the same neighborhood.

Theorem 2.3. *Let G be a graph on n vertices. If $S = \{v_1, v_2, \dots, v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$, then $\partial = \mathrm{Tr}(v_i) = \mathrm{Tr}(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$ and $\partial + 2$ is an eigenvalue of $\mathcal{L}$ with multiplicity at least $p - 1$.*

Proof. Since the vertices in S share the same neighborhood, any vertex in $V - S$ is at the same distance from all vertices in S. Any vertex of S is at distance 2 from any other vertex in S. Thus all vertices in S have the same transmission, say ∂.

To show that $\partial + 2$ is a distance Laplacian eigenvalue with multiplicity $p - 1$, it suffices to observe that the matrix $(\partial + 2)I_n - \mathcal{L}$ contains p identical rows (columns). $\square$

Corollary 2.4.

(a) *The distance Laplacian characteristic polynomial of the star S_n is*

$$P_{\mathcal{L}}^{S_n}(t) = t \cdot (t - n) \cdot (t - 2n + 1)^{n-2}.$$

(b) *The distance Laplacian characteristic polynomial of the complete bipartite graph $K_{a,b}$ is*

$$P_{\mathcal{L}}^{K_{a,b}}(t) = t \cdot (t - n) \cdot (t - (2a + b))^{a-1} \cdot (t - (2b + a))^{b-1}.$$

(c) *Let $SK_{n,\alpha}$ denote the complete split graph, i.e., the complement of the disjoint union of a clique K_α and $n - \alpha$ isolated vertices. Then*

$$P_{\mathcal{L}}^{SK_{n,\alpha}}(t) = t \cdot (t - n)^{n-\alpha} \cdot (t - n - \alpha)^{\alpha-1}.$$

Proof. (a) The star S_n contains an independent set S of $n - 1$ vertices with a common neighborhood. Each vertex of S has a transmission of $2n - 1$. Thus by Theorem 2.3, $2n-1$ is a distance Laplacian eigenvalue with multiplicity at least $n-2$.

The complement of S_n contains exactly two components. Then, by Theorem 1.1, n is a simple eigenvalue of $\mathcal{L}^{S_n}$. Finally, using Theorem 2.2, we get the characteristic polynomial of $\mathcal{L}^{S_n}$.

(b) The complete bipartite graph $K_{a,b}$ contains two independent sets S_1 and S_2 with $|S_1| = a$ and $|S_2| = b$. The vertices of S_1 (resp. S_2) share the same neighborhood S_2 (resp. S_1). The transmission of each vertex of S_1 (resp. S_2) is $2a + b - 2$ (resp. $2b + a - 2$). Thus, by Theorem 2.3, $2a + b$ and $2b + a$ are eigenvalues of $\mathcal{L}^{K_{a,b}}$ with multiplicities at least $a - 1$ and $b - 1$ respectively. In addition, n and 0 are eigenvalues of $\mathcal{L}^{K_{a,b}}$, by Theorem 1.1 and Theorem 2.2, respectively.

(c) The independent set of $SK_{n,\alpha}$ contains α vertices sharing the same neighborhood and the same transmission $n + \alpha - 2$. Then, $n + \alpha$ is an $\mathcal{L}$-eigenvalue with multiplicity at least $\alpha - 1$. In addition, the complement of $SK_{n,\alpha}$ contains $n - \alpha + 1$ components. Thus n is an $\mathcal{L}$-eigenvalue with multiplicity $n - \alpha$. □

Theorem 2.5. *Let G be a graph on n vertices. If $K = \{v_1, v_2, \ldots, v_p\}$ is a clique of G such that $N(v_i) - K = N(v_j) - K$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = \mathrm{Tr}(v_i) = \mathrm{Tr}(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial + 1$ is an eigenvalue of $\mathcal{L}$ with multiplicity at least $p - 1$.*

The proof of this theorem is similar to that of the previous one and therefore omitted here.

Corollary 2.6.

(a) *The distance Laplacian characteristic polynomial of the graph S_n^+, obtained from the star S_n by adding an edge, is*

$$P_{\mathcal{L}}^{S_n^+}(t) = t \cdot (t - n) \cdot (t - 2n + 3) \cdot (t - 2n + 1)^{n-3}.$$

(b) *The distance Laplacian characteristic polynomial of the pineapple $PA_{n,p}$, obtained from a clique K_{n-p} by attaching $p > 0$ pending edges to a vertex from the clique, is*

$$P_{\mathcal{L}}^{PA_{n,p}}(t) = t \cdot (t - n) \cdot (t - n - p)^{n-p-2} \cdot (t - 2n + 1)^p.$$

Proof. (a) is a particular case of (b), with $p = n - 3$. Thus, it suffices to prove (b).

It is trivial that 0 is an eigenvalue of $\mathcal{L}^{PA_{n,p}}$. Since the complement of $PA_{n,p}$ contains two components, n is a simple eigenvalue of $\mathcal{L}^{PA_{n,p}}$. $PA_{n,p}$ contains an independent set of p (pending) vertices sharing the same neighborhood and the same transmission $2n-3$. Thus, by Theorem 2.3, $2n-1$ is an $\mathcal{L}$-eigenvalue with multiplicity at least $p-1$. $PA_{n,p}$ contains a clique on $n-p-1$ vertices sharing the same neighborhood (composed of the dominating vertex) and the same transmission $n+p-1$. By Theorem 2.5, $n+p$ is an $\mathcal{L}$-eigenvalue with multiplicity at least $n-p-2$. Now, exactly $n-1$ $\mathcal{L}$-eigenvalues are known. The remaining eigenvalue is equal to the difference between the sum of all transmissions and the sum of the $n-1$ known eigenvalues. It is easy to evaluate the remaining eigenvalue, which in fact equals $2n-1$. □

Theorem 2.7. *If G is a graph on $n \geqslant 2$ vertices then $m(\partial_1^{\mathcal{L}}) \leqslant n-1$ with equality if and only if G is the complete graph K_n.*

P r o o f. The inequality results immediately from Theorem 2.2. If the graph is complete, it is easy to see that equality holds. Now, let G be a graph such that $m(\partial_1^{\mathcal{L}}) = n-1$. Assume, without loss of generality that the vertices of G are labeled such that $\mathrm{Tr}_{\max} = \mathrm{Tr}(v_1) \geqslant \mathrm{Tr}(v_2) \geqslant \ldots \geqslant \mathrm{Tr}(v_n) = \mathrm{Tr}_{\min}$. Since $\mathcal{L}$ admits only two distinct eigenvalues, 0 and $\partial_1^{\mathcal{L}}$, and $e = [1,1,\ldots,1]^t$ is an eigenvector that belongs to 0, any vector $X = [x_1, x_2, \ldots, x_n]^t$, with $x_1 = 1$, $x_i = -1$ and $x_j = 0$ for $j \neq 1$ and $j \neq i$, is an eigenvector that belongs to $\partial_1^{\mathcal{L}}$. Using the characteristic relation $\mathcal{L} \cdot X = \partial_1 X$, we get $\mathrm{Tr}_{\max} + d(v_1, v_i) = \partial_1$ for every vertex v_i including the neighbors of v_1, i.e., all the vertices, but v_1, are neighbors of v_1. Therefore, $\mathrm{Tr}_{\max} = n-1$ which is true if and only if G is the complete graph. □

Theorem 2.8. *If G is a tree on $n \geqslant 3$ vertices, then $\partial_1^{\mathcal{L}} \geqslant 2n-1$ with equality if and only if G is the star S_n.*

P r o o f. It is easy to see that if G is the star S_n with $n \geqslant 3$ equality holds. If the tree G is not a star, then its diameter is at least 3. For $n = 3$, there is only one tree S_3. For $n = 4$, there are two trees, P_4 and S_4, and equality holds only for S_4. Assume that $n \geqslant 5$. Let the vertex set $\{v_1, v_2, \ldots, v_n\}$ of G be labeled such that $v_1v_2v_3v_4$ is a path. For $i \geqslant 5$, v_i is adjacent to v_1 or to v_2 and $d(v_i, v_4) \geqslant 3$, or v_i is adjacent to v_3 or to v_4 and $d(v_i, v_1) \geqslant 3$, or v_i is not adjacent to any of the four vertices, $d(v_i, v_1) \geqslant 3$ and $d(v_i, v_4) \geqslant 3$. Thus there are at least $n-3$ distances greater than or equal to 3. Then we have

$$\begin{aligned}\sum_{i=1}^{n-1} \partial_i^{\mathcal{L}} &= 2W \\ &\geqslant 2((n-1) + 2(n(n-1)/2 - (n-1) - (n-3)) + 3(n-3)) \\ &= 2n(n-1) - 4.\end{aligned}$$

Using Theorem 2.7, we get $m(\partial_1^{\mathcal{L}}) < n-1$ and therefore

$$\partial_1^{\mathcal{L}} > \frac{2W}{n-1} \geqslant 2n - \frac{4}{n-1} \geqslant 2n-1$$

for all $n \geqslant 5$. This completes the proof. □

3. Some conjectures

In this section, we list a series of conjectures about some particular distance Laplacian eigenvalues of a graph. These conjectures, as well as some of the results proved in this paper, were obtained using the AutoGraphiX system ([1], [2], [6]) devoted to conjecture-making in graph theory.

First, we conjecture about bounding the largest distance Laplacian eigenvalue.

Conjecture 3.1. *For any graph G on $n \geqslant 4$ vertices,*

- ▷ $\partial_1^{\mathcal{L}}(G) \leqslant \partial_1^{\mathcal{L}}(P_n)$ *with equality if and only if G is the path P_n;*
- ▷ *if G is unicyclic, then* $\partial_1^{\mathcal{L}}(G) \leqslant \partial_1^{\mathcal{L}}(\mathrm{Ki}_{n,3})$ *with equality if and only if G is the kite* $\mathrm{Ki}_{n,3}$*;*
- ▷ *if G is unicyclic and $n \geqslant 6$, then* $\partial_1^{\mathcal{L}}(G) \geqslant \partial_1^{\mathcal{L}}(S_n^+)$ *with equality if and only if G is the graph S_n^+, obtained from the star S_n by adding an edge.*

The next conjecture is about the multiplicity of the largest distance Laplacian eigenvalue. In Theorem 2.7, we proved that the complete graph K_n is the only graph with exactly two distinct distance Laplacian eigenvalues. Then, it becomes natural to consider the problem of characterizing the graphs with exactly three distance Laplacian eigenvalues. It is easy to check that the star S_n, $n \geqslant 3$, and the balanced complete bipartite graph $K_{p,p}$, $p \geqslant 2$, possess exactly three distance Laplacian eigenvalues. But, the problem is not yet solved, however, our experiments with AutoGraphiX led to the following conjecture.

Conjecture 3.2. *If G is a graph on $n \geqslant 2$ vertices and $G \not\cong K_n$, then $m(\partial_1^{\mathcal{L}}(G)) \leqslant n-2$ with equality if and only if G is the star S_n and if $n = 2p$ for the complete bipartite graph $K_{p,p}$.*

Finally, we give conjectures about the second largest distance Laplacian eigenvalue of a graph: lower and upper bounds over all graphs; a lower bound over all trees; and lower and upper bounds over unicyclic graphs.

Conjecture 3.3. *For any graph G on $n \geqslant 4$ vertices,*

- $\partial_2^{\mathcal{L}}(G) \geqslant n$ *with equality if and only if G is the complete graph K_n or K_n minus an edge;*
- *if $n \neq 7$, then $\partial_2^{\mathcal{L}}(G) \leqslant \partial_2^{\mathcal{L}}(P_n)$ with equality if and only if G is the path P_n;*
- *if G is a tree and $n \geqslant 5$, then $\partial_2^{\mathcal{L}}(G) \geqslant 2n-1$ with equality if and only if G is the star S_n;*
- *if G is unicyclic and $n \geqslant 10$, then $\partial_2^{\mathcal{L}}(G) \leqslant \partial_2^{\mathcal{L}}(\mathrm{Ki}_{n,3})$ with equality if and only if G is the kite $\mathrm{Ki}_{n,3}$;*
- *if G is unicyclic and $n \geqslant 6$, then $\partial_2^{\mathcal{L}}(G) \geqslant \partial_2^{\mathcal{L}}(S_n^+)$ with equality if and only if G is the graph S_n^+ obtained from the star S_n by adding an edge.*

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