Costel-Gabriel Bontea

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CLASSIFYING BICROSSED PRODUCTS OF TWO SWEEDLER'S HOPF ALGEBRAS

Costel-Gabriel Bontea, Bucureşti

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Abstract. We continue the study started recently by Agore, Bontea and Militaru in "Classifying bicrossed products of Hopf algebras" (2014), by describing and classifying all Hopf algebras E that factorize through two Sweedler's Hopf algebras. Equivalently, we classify all bicrossed products $H_4 \bowtie H_4$. There are three steps in our approach. First, we explicitly describe the set of all matched pairs $(H_4, H_4, \triangleright, \triangleleft)$ by proving that, with the exception of the trivial pair, this set is parameterized by the ground field k. Then, for any $\lambda \in k$, we describe by generators and relations the associated bicrossed product, $\mathcal{H}_{16,\lambda}$. This is a 16-dimensional, pointed, unimodular and non-semisimple Hopf algebra. A Hopf algebra E factorizes through H_4 and H_4 if and only if $E \cong H_4 \otimes H_4$ or $E \cong \mathcal{H}_{16,\lambda}$. In the last step we classify these quantum groups by proving that there are only three isomorphism classes represented by: $H_4 \otimes H_4$, $\mathcal{H}_{16,0}$ and $\mathcal{H}_{16,1} \cong D(H_4)$, the Drinfel'd double of H_4. The automorphism group of these objects is also computed: in particular, we prove that $\mathrm{Aut}_{\mathrm{Hopf}}(D(H_4))$ is isomorphic to a semidirect product of groups, $k^{\times} \rtimes \mathbb{Z}_2$.

Keywords: bicrossed product of Hopf algebras; Sweedler's Hopf algebra; Drinfel'd double

MSC 2010: 16T10, 16T05, 16S40

Introduction

Let A and H be two given Hopf algebras. The factorization problem for Hopf algebras consists in classifying up to an isomorphism all Hopf algebras that factorize through A and H, i.e., all Hopf algebras E containing A and H as Hopf subalgebras such that the multiplication map $A \otimes H \to E$, $a \otimes h \mapsto ah$ is bijective. The problem can be put in more general terms but we restrict ourselves to the case of Hopf algebras. For a detailed account on the subject the reader may consult [1].

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An important step in dealing with the factorization problem was made by Majid in [10], Proposition 3.12, who generalized to Hopf algebras the construction of the bicrossed product for groups introduced by Takeuchi in [16]. Although in [10] the construction is known under the name of double cross product, we will follow [9] and call it, just like in the case of groups, the bicrossed product construction. A bicrossed product of two Hopf algebras A and H is a new Hopf algebra $A \bowtie H$ associated with a matched pair $(A, H, \triangleright, \triangleleft)$ of Hopf algebras. It is proved in [10], Proposition 3.12, that a Hopf algebra E factorizes through A and H if and only if E is isomorphic to some bicrossed product of A and H. Thus, the factorization problem can be stated in a computational manner: for two Hopf algebras A and H describe the set of all matched pairs $(A, H, \triangleright, \triangleleft)$ and classify up to an isomorphism all the bicrossed products $A \bowtie H$. This way of approaching the problem was recently proposed in [1] with promising results regarding new examples of quantum groups. For example, in [1], Section 4, all bicrossed products $H_4 \bowtie k[C_n]$ are described by generators and relations and are classified. They are quantum groups at roots of unity $H_{4n,\omega}$ which are classified by the arithmetic of the ring $\mathbb{Z}_n$. In this paper we continue the study began in [1] by classifying all Hopf algebras that factorize through two Sweedler's Hopf algebras.

The paper is organized as follows. In Section 1 we set the notation and recall the bicrossed product construction of two Hopf algebras. In Section 2, the main section of this paper, we classify all Hopf algebras that factorize through two Sweedler's Hopf algebras. For this, we first compute all the matched pairs $(H_4, H_4, \triangleright, \triangleleft)$: except the trivial one, these are parameterized by the ground field k, which is an arbitrary field of characteristic $\neq 2$. We then describe by generators and relations the associated bicrossed products $H_4 \bowtie H_4$. These are: $H_4 \otimes H_4$ and $\mathcal{H}_{16,\lambda}$, where, for any $\lambda \in k$, $\mathcal{H}_{16,\lambda}$ is the 16-dimensional quantum group generated by g, x, G, X subject to the relations

$$g^2 = G^2 = 1, \quad x^2 = X^2 = 0, \quad gx = -xg, \quad GX = -XG,$$
$$gG = Gg, \quad gX = -Xg, \quad xG = -Gx, \quad xX + Xx = \lambda\,(1 - Gg)$$

with the coalgebra structure given such that g and G are group-likes, x is $(1, g)$-primitive and X is $(1, G)$-primitive. We then prove that there are only three isomorphism classes of Hopf algebras that factorize through two Sweedler's Hopf algebras: $H_4 \otimes H_4$, $\mathcal{H}_{16,0}$ and $\mathcal{H}_{16,1} \cong D(H_4)$, the Drinfel'd double of H_4. Finally, we prove that there exist the following isomorphisms of groups:

$$\mathrm{Aut}_{\mathrm{Hopf}}(D(H_4)) \cong k^\times \rtimes \mathbb{Z}_2, \quad \mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,0}) \cong (k^\times \times k^\times) \rtimes \mathbb{Z}_2 \cong \mathrm{Aut}_{\mathrm{Hopf}}(H_4 \otimes H_4).$$

Since the tensor product of two pointed coalgebras is pointed [14], Lemma 5.1.10, it follows that the bicrossed product of two pointed Hopf algebras is pointed. In particular, $H_4 \otimes H_4$ and $\mathcal{H}_{16,\lambda}$, for $\lambda \in k$ are pointed Hopf algebras of dimension 16. The classification of such Hopf algebras, over an algebraically closed field of characteristic zero, was considered in [6] and, without the pointedness assumption, in [8]. With the notation of [6], Theorem 5.2, we have $\mathbb{H}_4 \otimes H_4 = H_{(3)}$, $\mathcal{H}_{16,0} = H_{(4)}$ and $\mathcal{H}_{16,1} = H_{(5)}$. Thus, the Hopf algebras we obtain here have already appeared in literature.

The classification of pointed Hopf algebras has been the subject of intense study in the past years ([3], [6], [4], [5]) and many classification results are known, especially when the coradical is commutative. The most impressive result of this type was obtained by Andruskiewitsch and Schneider in [5] where the classification of all finite-dimensional pointed Hopf algebras with commutative coradical, whose dimension is not divisible by primes $\leqslant 7$, is given. Therefore, if one hopes to obtain really new examples of Hopf algebras by considering the factorization problem then one has better chances of succeeding if he considers pointed Hopf algebras with non-commutative coradical or non-pointed Hopf algebras.

Finally, we point out that the dual problem of classifying all the extensions of H_4 by H_4 was solved by García and Vay in [8], Lemma 2.8, where it is shown that all such extensions are isomorphic to the tensor product $H_4 \otimes H_4$. Their proof uses the cocycle bicrossproduct construction of [12] and [2] as a tool, but, instead of computing all the cocycle bicrossproducts of H_4 and H_4, they build their argument on the fact that an extension of a Hopf algebra A by another Hopf algebra B is a B-cleft extension of A, which allows them to use the description and the classification of the H_4-cleft extensions of an algebra A from [13] and [7]. The link between the factorization problem and the extension problem was observed by Majid, who shows in [11], Proposition 7.2.4, that the set of matched pairs $(A, H, \triangleright, \triangleleft)$ is in bijection with the set of bicrossproduct data (A^*, H, α, β) giving rise to cocycle bicrossproducts with trivial cocycles. Since $H_4^* \simeq H_4$ one sees that the above correspondence breaks down at the level of the isomorphism classes of the associated Hopf algebra products, which is not so surprising considering that the two kinds of products are different objects.

1. Preliminaries

We work over an arbitrary field k of characteristic $\neq 2$. All algebras, coalgebras, Hopf algebras are over k and $\otimes = \otimes_k$. We shall use the standard notation from Hopf algebras theory: in particular, for a coalgebra C, we use the Σ-notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for any $c \in C$ (summation understood). Let A and H be two Hopf algebras. A is called a left H-module coalgebra if there exists $\triangleright\colon H \otimes A \to A$, a morphism of coalgebras such that $(A, \triangleright)$ is also a left H-module. Similarly, H is called a right A-module coalgebra if there exists $\triangleleft\colon H \otimes A \to H$, a morphism of coalgebras such that $(H, \triangleleft)$ is a right A-module. The actions $\triangleright\colon H \otimes A \to A$ and

$\triangleleft\colon H \otimes A \to H$ are called *trivial* if $h \triangleright a = \varepsilon_H(h)a$ and $h \triangleleft a = \varepsilon_A(a)h$, respectively, for all $a \in A$ and $h \in H$.

A *matched pair* of Hopf algebras ([10], [9]) is a quadruple $(A, H, \triangleright, \triangleleft)$, where A and H are Hopf algebras, $\triangleright\colon H \otimes A \to A$ and $\triangleleft\colon H \otimes A \to H$ are coalgebra maps such that $(A, \triangleright)$ is a left H-module coalgebra, $(H, \triangleleft)$ is a right A-module coalgebra and the following compatibility conditions hold:

$$h \triangleright 1_A = \varepsilon_H(h)1_A,\ 1_H \triangleleft a = \varepsilon_A(a)1_H, \tag{1}$$

$$h \triangleright (ab) = (h_{(1)} \triangleright a_{(1)})((h_{(2)} \triangleleft a_{(2)}) \triangleright b), \tag{2}$$

$$(gh) \triangleleft a = (g \triangleleft (h_{(1)} \triangleright a_{(1)}))(h_{(2)} \triangleleft a_{(2)}), \tag{3}$$

$$h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleright a_{(2)} = h_{(2)} \triangleleft a_{(2)} \otimes h_{(1)} \triangleright a_{(1)} \tag{4}$$

for all a, $b \in A$, g, $h \in H$. If $(A, H, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras then the associated *bicrossed product* $A \bowtie H$ of A with H is the vector space $A \otimes H$ endowed with the tensor product coalgebra structure and the multiplication

$$(a \bowtie g) \cdot (b \bowtie h) := a(g_{(1)} \triangleright b_{(1)}) \bowtie (g_{(2)} \triangleleft b_{(2)})h \tag{5}$$

for all a, $b \in A$, g, $h \in H$, where we use $\bowtie$ for $\otimes$. $A \bowtie H$ is a Hopf algebra with the antipode given by the formula

$$S(a \bowtie h) := (1_A \bowtie S_H(h)) \cdot (S_A(a) \bowtie 1_H) \tag{6}$$

for all $a \in A$ and $h \in H$ [11], Theorem 7.2.2, [9], Theorem IX 2.3.

The basic example of a bicrossed product is the famous Drinfel'd double of a finite dimensional Hopf algebra H: $D(H) = (H^*)^{\mathrm{cop}} \bowtie H$, the bicrossed product associated with a given canonical matched pair [9], Theorem IX.3.5. For other examples of bicrossed products we refer to [1], [9], [11].

We recall that a Hopf algebra E *factorizes* through two Hopf algebras A and H if there exist injective Hopf algebra maps $i\colon A \to E$ and $j\colon H \to E$ such that the map

$$A \otimes H \to E, \quad a \otimes h \mapsto i(a)j(h)$$

is bijective. The next fundamental result is due to Majid [10], Proposition 3.12: A Hopf algebra E factorizes through two given Hopf algebras A and H if and only if there exists a matched pair of Hopf algebras $(A, H, \triangleright, \triangleleft)$ such that $E \cong A \bowtie H$. In light of this result, the factorization problem for Hopf algebras was restated [1] in a computational manner: for two given Hopf algebras, A and H, describe the set of all matched pairs $(A, H, \triangleright, \triangleleft)$ and classify up to isomorphisms all bicrossed products $A \bowtie H$.

2. The bicrossed products of two Sweedler's Hopf algebras

In this section we are going to classify all bicrossed products $H_4 \bowtie H_4$. Recall that Sweedler's 4-dimensional Hopf algebra, H_4, is generated by two elements, g and x, subject to the relations $g^2 = 1$, $x^2 = 0$ and $xg = -gx$. The coalgebra structure and the antipode are given by:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g,$$
$$\Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -gx.$$

In order to avoid confusion we will denote by $\mathbb{H}_4$ a copy of H_4, and by G and X the generators of $\mathbb{H}_4$. Thus, $G^2 = 1$, $X^2 = 0$, $GX = -XG$, G is a group-like element and X is an $(1, G)$-primitive element.

Recall that, for a Hopf algebra H, $\mathrm{G}(H) = \{g \in H;\ \Delta(g) = g \otimes g,\ \varepsilon(g) = 1\}$ is the set of group-like elements of H and, for g, $h \in \mathrm{G}(H)$, $\mathrm{P}_{g,h}(H) = \{x \in H;\ \Delta(x) = x \otimes g + h \otimes x\}$ is the set of (g, h)-primitive elements of H. For the Sweedler Hopf algebra we have

$$\mathrm{G}(H_4) = \{1, g\}, \quad \mathrm{P}_{1,1}(H_4) = \mathrm{P}_{g,g}(H_4) = \{0\}, \quad \mathrm{P}_{1,g}(H_4) = k(1 - g) \oplus kx$$

The next theorem describes the set of all matched pairs $(A = \mathbb{H}_4, H = H_4, \triangleright, \triangleleft)$.

Theorem 2.1. *Let k be a field of characteristic $\neq 2$. Then $(\mathbb{H}_4, H_4, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras if and only if both $(\triangleright, \triangleleft)$ are the trivial actions or the pair $(\triangleright, \triangleleft)$ is given by*

$\triangleright$	1	G	X	GX
1	1	G	X	GX
g	1	G	$-X$	$-GX$
x	0	0	$\lambda - \lambda G$	$\lambda - \lambda G$
gx	0	0	$\lambda - \lambda G$	$\lambda - \lambda G$

$\triangleleft$	1	G	X	GX
1	1	1	0	0
g	g	g	0	0
x	x	$-x$	$\lambda - \lambda g$	$-\lambda + \lambda g$
gx	gx	$-gx$	$-\lambda + \lambda g$	$\lambda - \lambda g$

for some $\lambda \in k$.

We prove this result in three steps. The first is Lemma 2.2 where we describe the set of all right $\mathbb{H}_4$-module coalgebra structures $\triangleleft$ on H_4 satisfying the normalizing condition $1 \triangleleft h = \varepsilon(h)1$ for all $h \in \mathbb{H}_4$. There will be four such families of actions, $\triangleleft^j$, $j = 1, 2, 3, 4$, parameterized by scalars a, b, c, $d \in k$.

The second step is Lemma 2.3 where we describe the set of all left H_4-module coalgebra structures $\triangleright$ on $\mathbb{H}_4$ satisfying the normalizing condition $h \triangleright 1 = \varepsilon(h)1$ for all $h \in H_4$. There will also be four families of such actions, $\triangleright^i$, $i = 1, 2, 3, 4$ parameterized by scalars $s, t, u, v \in k$.

The final step consists of a detailed analysis of the sixteen possibilities of choice for the pair of actions $(\triangleright^i, \triangleleft^j)$, for all i, $j = 1, 2, 3, 4$. This will show that the only ones that verify the axioms (2)–(4) of a matched pair are $(\triangleright^1, \triangleleft^1)$, i.e., the pair of trivial actions, and $(\triangleright^4, \triangleleft^4)$, in which case the actions take the form described in the statement.

We begin with

Lemma 2.2. *If $\triangleleft\colon H_4 \otimes \mathbb{H}_4 \to H_4$ is a right $\mathbb{H}_4$-module coalgebra structure such that $1 \triangleleft h = \varepsilon(h)1$ for all $h \in \mathbb{H}_4$, then $\triangleleft$ has one of the following forms:*

$\triangleleft^1$	1	G	X	GX
1	1	1	0	0
g	g	g	0	0
x	x	x	0	0
gx	gx	gx	0	0

$\triangleleft^2$	1	G	X	GX
1	1	1	0	0
g	g	g	0	0
x	x	x	0	0
gx	gx	$c-cg-gx$	$d-dg$	$-d+dg$

$\triangleleft^3$	1	G	X	GX
1	1	1	0	0
g	g	g	0	0
x	x	$a-ag-x$	$b-bg$	$-b+bg$
gx	gx	gx	0	0

$\triangleleft^4$	1	G	X	GX
1	1	1	0	0
g	g	g	0	0
x	x	$a-ag-x$	$b-bg$	$-b+bg$
gx	gx	$c-cg-gx$	$d-dg$	$-d+dg$

where a, b, c, $d \in k$.

Proof. Let $\triangleleft\colon H_4 \otimes \mathbb{H}_4 \to H_4$ be a right $\mathbb{H}_4$-module coalgebra structure such that $1 \triangleleft h = \varepsilon(h)1$ for all $h \in \mathbb{H}_4$. Then $1 \triangleleft G = 1$, $1 \triangleleft X = 0$, and $1 \triangleleft (GX) = 0$. Also, $g \triangleleft G \in \mathrm{G}(H_4)$. We have $g \triangleleft G \neq 1$, for otherwise $1 = 1 \triangleleft G = (g \triangleleft G) \triangleleft G = g \triangleleft 1 = g$. Therefore $g \triangleleft G = g$. Since $g \triangleleft X \in \mathrm{P}_{g,g}(H_4)$, we deduce that $g \triangleleft X = 0$. Similarly, $g \triangleleft (GX) = 0$. Observe that the actions of X and GX on g are compatible with the relations $X^2 = 0$ and $GX = -XG$.

We next show that

$\triangleleft$	1	G	X	GX
x	x	x	0	0

or

$\triangleleft$	1	G	X	GX
x	x	$a-ag-x$	$b-bg$	$-b+bg$

for some a, $b \in k$.

We have $x \triangleleft G \in \mathrm{P}_{1,g}(H_4)$, hence $x \triangleleft G = a - ag + bx$ for some a, $b \in k$. Since the action of G is compatible with $G^2 = 1$, we have

$$x = x \triangleleft 1 = (x \triangleleft G) \triangleleft G = (a - ag + bx) \triangleleft G = a + ba - (a + ba)g + b^2 x.$$

Thus, $b^2 = 1$ and $a(1 + b) = 0$. If $b = -1$ then there are no restrictions on a. Otherwise, $b = 1$ and $a = 0$. This shows that $x \triangleleft G = x$ or $x \triangleleft G = a - ag - x$, with $a \in k$.

We also have $x \triangleleft X \in \mathrm{P}_{1,g}(H_4)$, hence $x \triangleleft X = b - bg + cx$ for some b, $c \in k$. Using that $X^2 = 0$, we have

$$0 = x \triangleleft 0 = (x \triangleleft X) \triangleleft X = (b - bg + cx) \triangleleft X = cb - cbg + c^2 x$$

Thus, $c = 0$ and $x \triangleleft X = b - bg$.

If $x \triangleleft G = x$ then $x \triangleleft (GX) = x \triangleleft X$ and, if $x \triangleleft G = a - ag - x$ then $x \triangleleft (GX) = (a - ag - x) \triangleleft X = -x \triangleleft X$. Observe that, in both cases, $\varepsilon(x \triangleleft (GX)) = 0$, and $\Delta(x \triangleleft (GX)) = x_{(1)} \triangleleft (GX)_{(1)} \otimes x_{(2)} \triangleleft (GX)_{(2)}$.

It remains to see when $x \triangleleft (GX) = x \triangleleft (-XG)$. If $x \triangleleft G = x$, then

$$x \triangleleft (XG) = (b - bg) \triangleleft G = b - bg = x \triangleleft X = x \triangleleft (GX).$$

Thus, $x \triangleleft (GX) = x \triangleleft (-XG)$ implies $x \triangleleft X = x \triangleleft (GX) = 0$. If $x \triangleleft G = a - ag - x$, then

$$x \triangleleft (XG) = (b - bg) \triangleleft G = b - bg = x \triangleleft X = -x \triangleleft (GX).$$

In this case, the equality $x \triangleleft (GX) = x \triangleleft (-XG)$ is satisfied without further restrictions.

In a similar manner it can be shown that

$\triangleleft$	1	G	X	GX
gx	gx	gx	0	0

or

$\triangleleft$	1	G	X	GX
gx	gx	$c - cg - gx$	$d - dg$	$-d + dg$

for some c, $d \in k$. $\square$

Analogously to Lemma 2.2 we can prove

Lemma 2.3. *If $\triangleright\colon H_4 \otimes \mathbb{H}_4 \to \mathbb{H}_4$ is a left H_4-module coalgebra structure such that $h \triangleright 1 = \varepsilon(h)1$ for all $h \in H_4$, then $\triangleright$ has one of the following forms:*

$\triangleright^1$	1	G	X	GX
1	1	G	X	GX
g	1	G	X	GX
x	0	0	0	0
gx	0	0	0	0

$\triangleright^2$	1	G	X	GX
1	1	G	X	GX
g	1	G	X	$u - uG - GX$
x	0	0	0	$v - vG$
gx	0	0	0	$v - vG$

$\triangleright^3$	1	G	X	GX
1	1	G	X	GX
g	1	G	$s - sG - X$	GX
x	0	0	$t - tG$	0
gx	0	0	$t - tG$	0

$\triangleright^4$	1	G	X	GX
1	1	G	X	GX
g	1	G	$s - sG - X$	$u - uG - GX$
x	0	0	$t - tG$	$v - vG$
gx	0	0	$t - tG$	$v - vG$

where s, t, u, $v \in k$.

P r o o f. One can check the validity of the statement by employing the same arguments as those used in the proof of Lemma 2.2. A more elegant and shorter proof can be deduced from the following elementary remark: if $\triangleright\colon H \otimes C \to C$ is a left H-module coalgebra on C then $\triangleleft\colon C \otimes H^{\mathrm{cop}} \to C$, $c \triangleleft h := S(h) \triangleright c$ for all $c \in C$ and $h \in H$ is a right H^{cop}-module coalgebra on C, and the above correspondence is bijective if the antipode of H is bijective. We apply this observation to $H = H_4$ and $C = \mathbb{H}_4$, which is just a copy of H_4, taking into account that the antipode of H_4 is bijective and $H_4^{\mathrm{cop}} \cong H_4$. In this way, the proof of Lemma 2.3 follows from the one of Lemma 2.2. $\square$

We are now in a position to complete the proof of Theorem 2.1.

T h e p r o o f of Theorem 2.1. Let $(\mathbb{H}_4, H_4, \triangleright, \triangleleft)$ be a matched pair. Since $\triangleleft\colon H_4 \otimes \mathbb{H}_4 \to H_4$ is a right $\mathbb{H}_4$-module coalgebra structure satisfying $1 \triangleleft h = \varepsilon(h)1$ for all $h \in \mathbb{H}_4$, we deduce from Lemma 2.2 that $\triangleleft$ is one of the $\triangleleft^i$'s. Similarly, $\triangleright\colon H_4 \otimes \mathbb{H}_4 \to \mathbb{H}_4$ is a left H_4-module coalgebra structure satisfying $h \triangleright 1 = \varepsilon(h)1$ for all $h \in H_4$, hence, $\triangleright$ is one of the $\triangleright^j$'s, by Lemma 2.3. We next show that $(\mathbb{H}_4, H_4, \triangleright^j, \triangleleft^i)$ is a matched pair if and only if $(i,j) \in \{(1,1),(4,4)\}$ and, if $(i,j) = (4,4)$, then $\triangleright^i$ and $\triangleleft^j$ are defined as we have claimed.

First, if $i = 2,3$ or $j = 2,3$ then $(\mathbb{H}_4, H_4, \triangleright^i, \triangleleft^j)$ is not a matched pair. Indeed, if $i = 2,3$ then condition (2) is not satisfied for the triple (g, G, X), while if $j = 2,3$ then condition (3) is not satisfied for the triple (x, g, G).

Secondly, $(\mathbb{H}_4, H_4, \triangleright^4, \triangleleft^1)$ and $(\mathbb{H}_4, H_4, \triangleright^1, \triangleleft^4)$ are not matched pairs, since condition (4) fails to be fulfilled in the former case for the pair (x, GX) and in the latter case for the pair (gx, X).

We focus now our attention on when $(\mathbb{H}_4, H_4, \triangleright^4, \triangleleft^4)$ is a matched pair, and for this we look at the conditions (2)–(4). It is not hard to see that (4) is trivially fulfilled for all $(h,a) \in \{1, g, x, gx\} \times \{1, G, X, GX\} \setminus \{(x,X),(x,GX),(gx,X),(gx,GX)\}$. A straightforward computation shows that the same condition is satisfied by (x, X) if and only if $t = b$ and $a = s = 0$, by (gx, GX) if and only if $v = -d$ and $c = u = 0$, by (x, GX) if and only if $v = b$, and by (gx, X) if and only if $t = -d$. Thus, condition (4) is fulfilled if and only if $a = c = s = u = 0$, $t = v = b$ and $d = -b$. It remains to see that conditions (2) and (3) are compatible with the relations $G^2 = g^2 = 1$, $X^2 = x^2 = 0$, $gx = -xg$ and $GX = -XG$. Since this is straightforward, the proof is complete. $\square$

We are able to describe and classify all Hopf algebras that factorize through two Sweedler's Hopf algebras.

Theorem 2.4. *Let k be a field of characteristic $\neq 2$. Then:*

(1) *A Hopf algebra E factorizes through $\mathbb{H}_4$ and H_4 if and only if $E \cong \mathbb{H}_4 \otimes H_4$*

or $E \cong \mathcal{H}_{16,\lambda}$ for some $\lambda \in k$, where $\mathcal{H}_{16,\lambda}$ is the 16-dimensional Hopf algebra generated by g, x, G, X subject to the relations

$$g^2 = G^2 = 1, \quad x^2 = X^2 = 0, \quad gx = -xg, \quad GX = -XG,$$
$$gG = Gg, \quad gX = -Xg, \quad xG = -Gx, \quad xX + Xx = \lambda(1 - Gg)$$

with the coalgebra structure given by

$$\Delta(g) = g \otimes g, \ \ \Delta(x) = x \otimes 1 + g \otimes x, \ \ \Delta(G) = G \otimes G, \ \ \Delta(X) = X \otimes 1 + G \otimes X,$$
$$\varepsilon(g) = \varepsilon(G) = 1, \qquad \varepsilon(x) = \varepsilon(X) = 0.$$

(2) *$\mathcal{H}_{16,\lambda}$ is pointed, unimodular, and non-semisimple. Moreover,*

$$\mathrm{P}_{1,g}(\mathcal{H}_{16,\lambda}) = k(1-g) \oplus kx, \qquad \mathrm{P}_{1,G}(\mathcal{H}_{16,\lambda}) = k(1-G) \oplus kX,$$
$$\mathrm{P}_{1,gG}(\mathcal{H}_{16,\lambda}) = k(1-gG).$$

(3) *Up to an isomorphism of Hopf algebras, there are only three Hopf algebras that factorize through $\mathbb{H}_4$ and H_4, namely*

$$\mathbb{H}_4 \otimes H_4, \qquad \mathcal{H}_{16,0} \qquad \text{and} \qquad \mathcal{H}_{16,1} \cong D(H_4) \tag{7}$$

where $D(H_4)$ is the Drinfel'd double of H_4.

Proof. (1) The Hopf algebra $\mathcal{H}_{16,\lambda}$ is the explicit description of the bicrossed product $\mathbb{H}_4 \bowtie H_4$ associated with the non-trivial matched pair given in Theorem 2.1. In $\mathbb{H}_4 \bowtie H_4$ we make the canonical identifications: $G = G \bowtie 1$, $X = X \bowtie 1$, $g = 1 \bowtie g$, $x = 1 \bowtie x$. The defining relations of $\mathcal{H}_{16,\lambda}$ follow easily. For instance:

$$xX = (1 \bowtie x)(X \bowtie 1) = (\lambda - \lambda G) \bowtie 1 - X \bowtie x + G \bowtie (\lambda - \lambda g)$$
$$= \lambda 1 \bowtie 1 - X \bowtie x - \lambda G \bowtie g = \lambda 1 - Xx - \lambda Gg.$$

(2) $\mathcal{H}_{16,\lambda}$ is pointed because, as a coalgebra, it is the tensor product of two pointed coalgebras [14], Lemma 5.1.10. The coradical of $\mathcal{H}_{16,\lambda}$ is $k[\mathrm{G}(\mathbb{H}_4)] \otimes k[\mathrm{G}(H_4)]$, hence $\mathrm{G}(\mathcal{H}_{16,\lambda}) = \{1, g, G, gG\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Since H_4 is non-semisimple, so is $\mathcal{H}_{16,\lambda}$ [14], Corollary 3.2.3. $\mathcal{H}_{16,\lambda}$ is unimodular since $(X + GX)(x - gx)$ is simultaneously a non-zero left and right integral as can easily be verified. The last part follows by a routine check.

(3) $\mathcal{H}_{16,\lambda} \cong \mathcal{H}_{16,1}$ for $\lambda \in k^{\times}$, since the defining relations for $\mathcal{H}_{16,1}$ can be obtained from that of $\mathcal{H}_{16,\lambda}$ by replacing X by $\lambda^{-1}X$.

We next prove that $\mathbb{H}_4 \otimes H_4$, $\mathcal{H}_{16,0}$ and $\mathcal{H}_{16,1}$ are non-isomorphic Hopf algebras. Observe first that $\mathbb{H}_4 \otimes H_4$ is generated as an algebra by the two copies of Sweedler's

Hopf algebra, $\mathbb{H}_4$ and H_4, such that the generators of $\mathbb{H}_4$ commute with the generators of H_4. Moreover, the sets of skew-primitive elements of $\mathbb{H}_4 \otimes H_4$ have the same description as in (2). In order to distinguish between the generators of the Hopf algebras in question, we attach a prime sign, $'$, to the elements of $\mathcal{H}_{16,\,0}$ and two such signs to the elements of $\mathcal{H}_{16,\,1}$.

Assume $\varphi\colon \mathbb{H}_4 \otimes H_4 \to \mathcal{H}_{16,0}$ is a Hopf algebra isomorphism. Then $\varphi(g)$ is a group-like element of $\mathcal{H}_{16,0}$. Since the vector space of $(1,g)$-primitive elements must have the same dimension as the vector space of $(1,\varphi(g))$-elements, we have $\varphi(g) \in \{g', G'\}$. If $\varphi(g) = g'$ then $\varphi(G) = G'$ and $\varphi(X) \in \mathrm{P}_{1,G'}(\mathcal{H}_{16,\,0})$. Let $a,\, b \in k$ be such that $\varphi(X) = a(1-G') + bX'$. Taking into account that $\varphi(G)\varphi(X)\varphi(G) = -\varphi(X)$, we obtain that $a = 0$, hence $\varphi(X) = bX'$. Since g and X commute, we have $g'X' = \varphi(g)\varphi(X) = \varphi(X)\varphi(g) = bX'g' = -bg'X'$, hence $b = 0$. Thus $\varphi(X) = 0$, a contradiction with the fact that φ has a trivial kernel. A similar contradiction is obtained if $\varphi(g) = G'$, so we conclude that $\mathbb{H}_4 \otimes H_4$ and $\mathcal{H}_{16,0}$ are not isomorphic. Since the same argument works when we consider $\mathbb{H}_4 \otimes H_4$ and $\mathcal{H}_{16,1}$, we deduce that these Hopf algebras are not isomorphic as well.

Suppose now that $\varphi\colon \mathcal{H}_{16,0} \to \mathcal{H}_{16,1}$ is a Hopf algebra isomorphism. Then, as above, $\varphi(g') \in \{g'', G''\}$. If $\varphi(g') = g''$ then $\varphi(G') = G''$, $\varphi(x') \in \mathrm{P}_{1,g''}(\mathcal{H}_{16,1})$ and $\varphi(X') \in \mathrm{P}_{1,G''}(\mathcal{H}_{16,1})$. Let $a,\, b \in k$ be such that $\varphi(x') = a(1-g'') + bx''$. Since $\varphi(g)\varphi(x)\varphi(g) = -\varphi(x)$ it follows that $\varphi(x') = bx''$. A similar argument shows that $\varphi(X') = dX''$ for some $d \in k$. Using the fact that $x'X' + X'x' = 0$, we have

$$0 = \varphi(x'X' + X'x') = bd(x''X'' + X''x'') = bd(1 - g''G'').$$

Therefore $b = 0$ or $d = 0$, with either case leading to a contradiction. If $\varphi(g') = G''$ then we arrive at a similar contradiction so we conclude that $\mathcal{H}_{16,0} \ncong \mathcal{H}_{16,1}$.

Finally, we show that $\mathcal{H}_{16,1} \cong D(H_4)$. First, recall that $D(H_4)$ factorizes through $(H_4^*)^{\mathrm{cop}}$ and H_4. Also, if $\{1^*, g^*, x^*, (gx)^*\}$ denotes the dual basis of $\{1, g, x, gx\}$ then H_4^* is generated as an algebra by the group-like element $G = 1^* - g^*$ and by the $(G,1)$-primitive element $X = x^* + (gx)^*$, with the relations $G^2 = 1$, $X^2 = 0$, and $GX = -XG$. Therefore, $(H_4^*)^{\mathrm{cop}} \simeq \mathbb{H}_4$, so $D(H_4)$ factorizes through $\mathbb{H}_4$ and H_4. In order to see which of the three Hopf algebras from (7) $D(H_4)$ is, recall the Drinfel'd double as a matched pair. If H is a finite dimensional Hopf algebra, then $((H^*)^{\mathrm{cop}}, H, \triangleright, \triangleleft)$ is a matched pair, where $h \triangleright h^* = h^*\left(S^{-1}(h_{(2)})?h_{(1)}\right)$ and $h \triangleleft h^* = h^*\left(S^{-1}(h_{(3)})h_{(1)}\right)h_{(2)}$ for all $h \in H$, $h^* \in (H^*)^{\mathrm{cop}}$, and $D(H) \simeq (H^*)^{\mathrm{cop}} \bowtie H$. In our case, we have

$$\begin{aligned} x \triangleleft X &= X(S^{-1}(1)x) + X(S^{-1}(1)g)x + X(S^{-1}(x)g)g \\ &= X(x) + X(g)x + X(-x)g \\ &= 1 - g \end{aligned}$$

which shows that $D(H_4) \simeq \mathbb{H}_4 \bowtie_1 H_4$. $\square$

Remark 2.1. $\mathcal{H}_{16,0}$ is not the dual of $D(H_4)$ for otherwise $D(H_4)^*$ would be unimodular and so would H_4 ([15], Corollary 4).

Remark 2.2. The classification of pointed Hopf algebras of dimension 16 over an algebraically closed field of characteristic zero was done in [6]. With the notation of [6], Theorem 5.2, we have $\mathbb{H}_4 \otimes H_4 \simeq H_{(3)}$, $\mathcal{H}_{16,0} \simeq H_{(4)}$ and $\mathcal{H}_{16,1} \simeq H_{(5)}$.

As a consequence of Theorem 2.4 we are able to describe the group of Hopf algebra automorphisms of the three Hopf algebras from (7). We begin with the Drinfel'd double, $D(H_4)$.

Corollary 2.5. *Let k be a field of characteristic $\neq 2$. Then there exists an isomorphism of groups*

$$\mathrm{Aut}_{\mathrm{Hopf}}(D(H_4)) \cong k^* \rtimes_f \mathbb{Z}_2$$

where $k^* \rtimes_f \mathbb{Z}_2$ is the semidirect product associated with the action as automorphisms $f\colon \mathbb{Z}_2 \to \mathrm{Aut}(k^*)$, $f(1+2\mathbb{Z})(\alpha) = \alpha^{-1}$ for all $\alpha \in k^*$.

Proof. We use Theorem 2.4 and the description of $\mathcal{H}_{16,1}$ given in (1). Let φ be a Hopf algebra automorphism of $\mathcal{H}_{16,1}$. Then $\{\varphi(g), \varphi(G)\} = \{g, G\}$. If $(\varphi(g), \varphi(G)) = (g, G)$ then $\varphi(x) \in \mathrm{P}_{1,g}(\mathcal{H}_{16,1})$ and $\varphi(X) \in \mathrm{P}_{1,G}(\mathcal{H}_{16,1})$. Let a, b, c, $d \in k$ be such that $\varphi(x) = a(1-g) + bx$ and $\varphi(X) = c(1-G) + dX$. Taking into account that $\varphi(gxg) = -\varphi(x)$ and $\varphi(GXG) = -\varphi(X)$ we find that $a = c = 0$. Thus, $\varphi(x) = bx$ and $\varphi(X) = dX$. Considering now the relation $xX + Xx = 1 - gG$ and applying φ to both terms of the equation we find that $bd = 1$. Thus, $d = b^{-1}$. Since there are no further restrictions on b imposed by the fact that φ is a Hopf algebra isomorphism, we have obtained, for each $b \in k^\times$, an element $\varphi_b \in \mathrm{Aut}(\mathcal{H}_{16,1})$ given by

$$\varphi_b(g) = g, \quad \varphi_b(G) = G, \quad \varphi_b(x) = bx, \quad \varphi_b(X) = b^{-1}X.$$

If $(\varphi(g), \varphi(G)) = (G, g)$ then, by a reasoning similar to the one above, we have $\varphi(x) = dX$ and $\varphi(X) = d^{-1}x$ for some $d \in k^\times$, and each such φ is a Hopf algebra isomorphism of $\mathcal{H}_{16,1}$ that we denote by ψ_d.

Summarizing, we have obtained that the set of Hopf algebra automorphisms of $\mathcal{H}_{16,1}$ is $\mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,1}) = \{\varphi_b\,;\ b \in k^\times\} \cup \{\psi_d\,;\ d \in k^\times\}$, a disjoint union of two sets indexed by $k^\times$. The elements of $\mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,1})$ multiply according to the following rules:

$$\varphi_b\varphi_d = \varphi_{bd}, \quad \psi_b\psi_d = \varphi_{b^{-1}d}, \quad \psi_b\varphi_d = \psi_{bd}, \quad \varphi_b\psi_d = \psi_{b^{-1}d}$$

for all b, $d \in k^\times$. Let $k^\times \rtimes_f \mathbb{Z}_2$ be the semidirect product associated with $f\colon \mathbb{Z}_2 \to \mathrm{Aut}(k^\times)$, $f(1+2\mathbb{Z})(b) = b^{-1}$ for all $b \in k^\times$. Taking into account that multiplication in $k^\times \rtimes_f \mathbb{Z}_2$ is given by $(b, \hat{m}) \cdot (d, \hat{n}) = (bf(\hat{m})(d), \hat{m} + \hat{n})$ for all b, $d \in k^\times$ and $\hat{m}$,

$\hat{n} \in \mathbb{Z}_2$, it is easy to see that

$$\Gamma\colon\ k^\times \rtimes_f \mathbb{Z}_2 \to \mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,1}), \quad \Gamma(b,\hat{0}) = \varphi_b, \quad \Gamma(b,\hat{1}) = \psi_{b^{-1}}$$

for all $b \in k^\times$, is an isomorphisms of groups. $\square$

Corollary 2.6. *Let k be a field of characteristic $\neq 2$. Then there exist isomorphisms of groups*

$$\mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,0}) \cong (k^\times \times k^\times) \rtimes_g \mathbb{Z}_2 \cong \mathrm{Aut}_{\mathrm{Hopf}}(\mathbb{H}_4 \otimes H_4)$$

where $(k^\times \times k^\times) \rtimes_g \mathbb{Z}_2$ is the semidirect product associated with the action as automorphisms $g\colon\ \mathbb{Z}_2 \to \mathrm{Aut}(k^\times \times k^\times)$, $g(1+2\mathbb{Z})(b,d) = (d,b)$ for all $(b,d) \in k^\times \times k^\times$.

P r o o f. The method of proof in both cases is the same as the one used in proving Proposition 2.7. Moreover, both isomorphisms are obtained in exactly the same way, therefore we will limit ourselves to pointing out the one for $\mathcal{H}_{16,0}$.

The set of Hopf algebra isomorphisms of $\mathcal{H}_{16,0}$ is $\mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,0}) = \{\varphi_{b,d};\ (b,d) \in k^\times \times k^\times\} \cup \{\psi_{b,d};\ (b,d) \in k^\times \times k^\times\}$, where $\varphi_{b,d}$ and $\psi_{b,d}$ are defined by

$$\begin{aligned} &\varphi_{b,d}(g) = g, && \varphi_{b,d}(G) = G, && \varphi_{b,d}(x) = bx, && \varphi_{b,d}(X) = dX,\\ &\psi_{b,d}(g) = G, && \psi_{b,d}(G) = g, && \psi_{b,d}(x) = bX, && \psi_{b,d}(X) = dx. \end{aligned}$$

The elements of $\mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,0})$ multiply according to the following rules

$$\psi_{b,d}\psi_{c,e} = \varphi_{dc,be}, \quad \varphi_{b,d}\varphi_{c,e} = \varphi_{bc,de}, \quad \psi_{b,d}\varphi_{c,e} = \psi_{bc,de}, \quad \varphi_{b,d}\psi_{c,e} = \psi_{dc,be}$$

for all b, d, c, $e \in k^\times$. Considering the semi-direct product $(k^\times \times k^\times) \rtimes_g \mathbb{Z}_2$ associated with $g\colon\ \mathbb{Z}_2 \to \mathrm{Aut}(k^\times \times k^\times)$, $g(1+2\mathbb{Z})(b,d) = (d,b)$ for all $(b,d) \in k^\times \times k^\times$, we obtain that

$$\Gamma\colon\ (k^\times \times k^\times) \rtimes_g \mathbb{Z}_2 \to \mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{H}_{16,0}), \quad \Gamma((d,b),\hat{1}) = \psi_{b,d}, \quad \Gamma((b,d),\hat{0}) = \varphi_{b,d}$$

for all α, $\beta \in k^\times$ is an isomorphism of groups. $\square$

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References

[1] *A. L. Agore, C. G. Bontea, G. Militaru*: Classifying bicrossed products of Hopf algebras. Algebr. Represent. Theory *17* (2014), 227–264.

[2] *N. Andruskiewitsch, J. Devoto*: Extensions of Hopf algebras. St. Petersbg. Math. J. *7* (1996), 17–52; translation from Algebra Anal. *7* (1995), 22–61.

[3] *N. Andruskiewitsch, H.-J. Schneider*: Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. J. Algebra *209* (1998), 658–691.

[4] *N. Andruskiewitsch, H.-J. Schneider*: Finite quantum groups and Cartan matrices. Adv. Math. *154* (2000), 1–45.

[5] *N. Andruskiewitsch, H.-J. Schneider*: On the classification of finite-dimensional pointed Hopf algebras. Ann. Math. (2) *171* (2010), 375–417.

[6] *S. Caenepeel, S. Dăscălescu, Ş. Raianu*: Classifying pointed Hopf algebras of dimension 16. Commun. Algebra *28* (2000), 541–568.

[7] *Y. Doi, M. Takeuchi*: Quaternion algebras and Hopf crossed products. Commun. Algebra *23* (1995), 3291–3325.

[8] *G. A. García, C. Vay*: Hopf algebras of dimension 16. Algebr. Represent. Theory *13* (2010), 383–405.

[9] *C. Kassel*: Quantum Groups, Graduate Texts in Mathematics. Graduate Texts in Mathematics 155, Springer, New York, 1995.

[10] *S. Majid*: Physics for algebraists: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. J. Algebra *130* (1990), 17–64.

[11] *S. Majid*: Foundations of Quantum Groups Theory. Cambridge University Press, Cambridge, 1995.

[12] *S. Majid*: More examples of bicrossproduct and double cross product Hopf algebras. Isr. J. Math. *72* (1990), 133–148.

[13] *A. Masuoka*: Cleft extensions for a Hopf algebra generated by a nearly primitive element. Commun. Algebra *22* (1994), 4537–4559.

[14] *S. Montgomery*: Hopf Algebras and Their Actions on Rings. Expanded version of ten authors lectures given at the CBMS Conference on Hopf algebras and their actions on rings, DePaul University in Chicago, USA, 1992. Regional Conference Series in Mathematics 82, AMS, Providence, RI, 1993.

[15] *D. E. Radford*: Minimal quasitriangular Hopf algebras. J. Algebra *157* (1993), 285–315.

[16] *M. Takeuchi*: Matched pairs of groups and bismash products of Hopf algebras. Commun. Algebra *9* (1981), 841–882.

Author's addresses: Costel-Gabriel Bontea, Faculty of Mathematics and Computer-Science, University of Bucharest, str. Academiei 14, RO-010014, Bucharest, Romania; and Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium, e-mail: `costel.bontea@gmail.com`.