Gaohua Tang; Yangjiang Wei; Yuanlin Li

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UNIT GROUPS OF GROUP ALGEBRAS OF SOME SMALL GROUPS

Gaohua Tang, Yangjiang Wei, Nanning, Yuanlin Li, St. Catharines

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Abstract. Let FG be a group algebra of a group G over a field F and $\mathscr{U}(FG)$ the unit group of FG. It is a classical question to determine the structure of the unit group of the group algebra of a finite group over a finite field. In this article, the structure of the unit group of the group algebra of the non-abelian group G with order 21 over any finite field of characteristic 3 is established. We also characterize the structure of the unit group of FA_4 over any finite field of characteristic 3 and the structure of the unit group of FQ_{12} over any finite field of characteristic 2, where $Q_{12} = \langle x, y; x^6 = 1, y^2 = x^3, x^y = x^{-1}\rangle$.

Keywords: group ring; unit group; augmentation ideal; Jacobson radical

MSC 2010: 16S34, 16U60, 20C05

1. Introduction and notations

Let FG be a group algebra of a group G over a field F and $\mathscr{U}(FG)$ the unit group of the group algebra FG. It is a classical question to determine the structure of the unit group of the group algebra of a finite group over a finite field. Recently there are quite a few papers which characterize the structures of unit groups of group algebras of certain small groups over finite fields (see for example [3], [5], [4], [6], [7], [9], [8], [10], [11], [15], [2], [16]). Most recently, in [2] Tang et al. determined the structures of unit groups of group algebras FG of any groups of order 21 over finite fields except for the case when G is the non-abelian group of order 21 and F is a field of characteristic 3. The first goal of this paper is to study this remaining case. We shall determine the structure of the Jacobson radical for this group algebra and then establish the structure of its unit group.

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There are three non-abelian groups of order 12: A_4, D_{12} and Q_{12}. In 2007, R. Sharma, J. Srivastava and M. Khan ([15]) characterized the unit group of FA_4 over a finite field. In the case when the characteristic of F is 2 or 3, they provided only a preliminary description of the unit group. In [8], J. Gildea established a complete characterization of the unit group of FA_4 over a finite field of characteristic 2. In [11], Gildea and Monaghan established the structure of unit groups of FD_{12} and FQ_{12} over a finite field of characteristic 3. Our second goal is to determine the structure of the group algebra FA_4 over a finite field of characteristic 3 and establish a complete characterization of the unit group of this group algebra. In 2011, Tang and Gao ([16]) described the structure of the unit group of the group algebra FQ_{12}. We shall determine the structure of the Jacobson radical of FQ_{12} over a finite field of characteristic 2 and provide a better characterization of the unit group of FQ_{12}. We note that other unit groups of group algebras of the groups of order 12 have been completely characterized (see [11], [15], [16] for details).

Throughout this paper, A_4 denotes the alternating group of degree 4, $Q_{12} = \langle x, y; x^6 = 1, y^2 = x^3, x^y = x^{-1}\rangle$, C_n denotes the cyclic group of order n, F is a finite field of characteristic p of order p^n, and F^* is the multiplicative group of F. We also denote by $M(n, F)$ and $\mathrm{GL}(n, F)$ the ring of all $n \times n$ matrices over a field F and the general linear group of degree n over a field F, respectively. Denote by $Z(FG)$ the center of FG.

Recall that the ring homomorphism $\varepsilon\colon FG \to F$ given by

$$\varepsilon\Bigl(\sum_{g\in G} a_g g\Bigr) = \sum_{g\in G} a_g$$

is called the augmentation mapping of FG and its kernel, denoted by $\Delta(G)$, is called the augmentation ideal of FG. For a subgroup H of G, we shall denote by $\Delta(G, H)$ the left ideal of FG generated by the set $\{h - 1;\ h \in H\}$. That is,

$$\Delta(G, H) = \Bigl\{\sum_{h\in H} \alpha_h(h - 1);\ \alpha_h \in FG\Bigr\}.$$

If H is a normal subgroup of G, then $\Delta(G, H)$ is a two-sided ideal. Note that the ideal $\Delta(G, G)$ coincides with the ideal $\Delta(G)$.

2. The unit group of $F_{3^n}G$ over the non-abelian group of order 21

In this section, we characterize the structure of the unit group of the group algebra FG of the non-abelian group G of order 21 over a finite field of characteristic 3, where $G = \langle x, y; x^7 = y^3 = 1, x^y = x^2\rangle$. We first state a useful definition and establish a lemma regarding the Jacobson radical of a group algebra.

Definition 2.1. Let p be a prime number and let S_p denote the set of all p-elements of a group G including the identity, i.e., $S_p = \{g \in G;\ g \text{ is a } p\text{-element}\}$. Define a map $T\colon G \to F$ by $T(g) = 1$ if $g \in S_p$ and $T(g) = 0$, otherwise. We now extend T linearly to a map from FG to F given by $T(\alpha) = \sum a_i T(g_i) = \sum a_i g_i \in FG$ for all α, which is the sum of all coefficients of p-elements including 1 in α. Define $\operatorname{Ker} T = \{\alpha \in FG;\ T(\alpha g) = 0\ \forall g \in G\}$.

Lemma 2.2. *Let F be a finite field of characteristic $p > 0$, G be a finite group, and T be the function defined above. Then*

(1) $J(FG) \subseteq \operatorname{Ker} T$.
(2) $\operatorname{Ker} T = \operatorname{Ann}(c)$, *where* $c = \sum\limits_{g\in S_p} g = \hat{S}_p$.
(3) $J(FG) \subseteq \operatorname{Ann}(c)$.

Proof. (1) Let $\alpha \in J(FG)$. Since $\alpha g \in J(FG)$ for all $g \in G$ and $J(FG)$ is nilpotent, αg is nilpotent. By Passman [13, Lemma 3.3], we have $T(\alpha g) = 0$ for all $g \in G$. Thus $\alpha \in \operatorname{Ker} T$ and so $J(FG) \subseteq \operatorname{Ker} T$, proving the first statement.

(2) This follows from the proof of [1, Lemma 3.2] that $\operatorname{Ker} T = \operatorname{Ann}(c)$.

(3) This follows immediately from (1) and (2) that $J(FG) \subseteq \operatorname{Ann}(c)$. □

We now state our main result in this section.

Theorem 2.3. *Let G be the group of order 21 described above and let F be a finite field of characteristic 3 of order 3^n. Then either $FG \cong M(3, F) \oplus M(3, F) \oplus FC_3$ when n is even, or $FG \cong M(3, F_2) \oplus FC_3$ when n is odd, where F_2 is the degree 2 extension field of F. Therefore, either $\mathscr{U}(FG) \cong \mathrm{GL}(3, F) \times \mathrm{GL}(3, F) \times F^* \times C_3^{2n}$ when n is even, or $\mathscr{U}(FG) \cong \mathrm{GL}(3, F_2) \times F^* \times C_3^{2n}$ when n is odd.*

Proof. Let $H = \langle x\rangle$ and $e = \frac{1}{|H|}\hat{H}$. Since H is a normal subgroup of G, $e = e^2$ is a central idempotent. Thus $FG = FG(1-e) \oplus FG(e) \cong \Delta(G, H) \oplus FC_3$ since $FG(1-e) = \Delta(G, H)$ and $FG(e) \cong F(G/H) \cong FC_3$ (see [14, Proposition 3.6.7] for the details). We next show that $\Delta(G, H) \cong \Delta(G)/J(FG)$ is semisimple and determine the structure of $\Delta(G, H)$. To do so, we compute the Jacobson radical $J(FG)$ of FG. By Lemma 2.2 (3), we know that $J(FG) \subseteq \operatorname{Ann}(c)$ where $c = \sum\limits_{g\in S_3} g = \hat{S}_3$, so we first compute $\operatorname{Ann}(c)$.

A direct computation shows that G has five conjugacy classes as follows:

$$b_0 = \{1\},\ b_1 = \{x, x^2, x^4\},\ b_2 = \{x^3, x^5, x^6\},$$
$$b_3 = \{y, xy, x^2y, x^3y, x^4y, x^5y, x^6y\} = \langle x\rangle y,$$

and

$$b_4 = \{y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, x^5y^2, x^6y^2\} = \langle x\rangle y^2.$$

Now we have $c = 1 + \hat{b}_3 + \hat{b}_4 = 1 + \hat{x}y + \hat{x}y^2$, which is the sum of all 3-elements including 1. Let $\alpha = \sum_{i=0}^{4} \alpha_i \in \mathrm{Ann}(c)$ where $\mathrm{supp}(\alpha_i) \subseteq b_i$ for $i = 0, 1, 2, 3, 4$. Then $\alpha c = 0$, and thus $\Big(\sum_{i=0}^{4} \alpha_i\Big)(1 + \hat{x}y + \hat{x}y^2) = 0$, so $0 = \sum_{i=0}^{4} \alpha_i + (\alpha_0 + \alpha_1 + \alpha_2)\hat{x}y + \alpha_3\hat{x}y + \alpha_4\hat{x}y + (\alpha_0 + \alpha_1 + \alpha_2)\hat{x}y^2 + \alpha_3\hat{x}y^2 + \alpha_4\hat{x}y^2 = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_4\hat{x}y + \alpha_3\hat{x}y^2) + (\alpha_3 + (\alpha_0 + \alpha_1 + \alpha_2)\hat{x}y + \alpha_4\hat{x}y^2) + (\alpha_4 + \alpha_3\hat{x}y + (\alpha_0 + \alpha_1 + \alpha_2)\hat{x}y^2) = (\alpha_0 + \alpha_1 + \alpha_2 + \varepsilon(\alpha_3 + \alpha_4)\hat{x}) + (\alpha_3 + \varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_4)\hat{x}y) + (\alpha_4 + \varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)\hat{x}y^2)$. This gives that $\alpha_0 + \alpha_1 + \alpha_2 = -\varepsilon(\alpha_3 + \alpha_4)\hat{x}, \alpha_3 = -\varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_4)\hat{x}y$ and $\alpha_4 = -\varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)\hat{x}y^2$. Set $a_1 = -\varepsilon(\alpha_3 + \alpha_4)$ and $a_2 = -\varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_4)$. Since $\varepsilon(\alpha) = \varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0$, we have $-\varepsilon(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) = \varepsilon(\alpha_4) = -(a_1 + a_2)$. Thus $\mathrm{Ann}(c) = \{a_1\hat{x} + a_2\hat{x}y - (a_1 + a_2)\hat{x}y^2;\ a_1, a_2 \in F\}$ and $\dim_F(\mathrm{Ann}(c)) = 2$.

We now show that $\mathrm{Ann}(c)^3 = 0$, so $\mathrm{Ann}(c) \subseteq J(FG)$. This together with Lemma 2.2 gives that $J(FG) = \mathrm{Ann}(c)$. Note that $(a_1\hat{x} + a_2\hat{x}y - (a_1 + a_2)\hat{x}y^2)(c_1\hat{x} + c_2\hat{x}y - (c_1 + c_2)\hat{x}y^2)(d_1\hat{x} + d_2\hat{x}y - (d_1 + d_2)\hat{x}y^2) = (a_1 - a_2)(c_1 - c_2)\hat{G}(d_1\hat{x} + d_2\hat{x}y - (d_1 + d_2)\hat{x}y^2) = (a_1 - a_2)(c_1 - c_2)(d_1 + d_2 - (d_1 + d_2))\hat{G} = 0$ (as $\mathrm{char}\, F = 3$). Thus $\mathrm{Ann}(c)^3 = 0$ and $J(FG) = \mathrm{Ann}(c)$.

Next we show that $\Delta(G) = \Delta(G, H) \oplus J(FG)$. Note that both $\Delta(G, H)$ and $J(FG)$ are contained in $\Delta(G)$, and $\dim_F \Delta(G, H) + \dim_F J(FG) = 18 + 2 = 20 = \dim_F \Delta(G)$. We now show that $\Delta(G, H) \cap J(FG) = 0$. Let $\beta \in \Delta(G, H) \cap J(FG)$. Then $\beta = \alpha(x - 1)$ where $\alpha \in FG$ (as $\beta \in \Delta(G, H)$ and $H = \langle x\rangle$), and also $\beta = a_1\hat{x} + a_2\hat{x}y - (a_1 + a_2)\hat{x}y^2$ for some $a_1, a_2 \in F$ (as $\beta \in J(FG)$). Thus $\beta\hat{x} = \alpha(x - 1)\hat{x} = 0$. On the other hand, $\beta\hat{x} = (a_1\hat{x} + a_2\hat{x}y - (a_1 + a_2)\hat{x}y^2)\hat{x} = \beta$ as $\hat{x}\hat{x} = \hat{x}$. So we have $\beta = \beta\hat{x} = 0$, showing that $\Delta(G, H) \cap J(FG) = 0$. Therefore, $\Delta(G) = \Delta(G, H) \oplus J(FG)$ and thus $\Delta(G, H) \cong \Delta(G)/J(FG)$. Since $FG/J(FG)$ is semisimple and $\Delta(G)/J(FG)$ is an ideal of $FG/J(FG)$, we conclude that $\Delta(G, H) \cong \Delta(G)/J(FG)$ is semisimple as desired.

Now we show that $Z(\Delta(G, H)) \subseteq Z(FG)$ and compute $Z(\Delta(G, H))$. Let $\alpha = \alpha_1(x - 1) \in Z(\Delta(G, H))$. Then $\alpha(x - 1) = (x - 1)\alpha$, so

$$(*) \qquad \alpha x = x\alpha.$$

We next show that α also commutes with y, and therefore, α is in the center of FG. Note that $\alpha y(x-1) = y(x-1)\alpha$ as $y(x-a) \in \Delta(G,H)$. By $(*)$, we have $(\alpha y - y\alpha)(x-1) = 0$. Similarly, we have $(x-1)(\alpha y - y\alpha) = 0$ as $(x-1)y \in \Delta(G,H)$. Thus $\alpha y - y\alpha \in \mathrm{Ann}(x-1) = \mathrm{Ann}(\Delta(G,H)) = (FG)\hat{x}$, so $\alpha y - y\alpha = \beta\hat{x}$ where $\beta \in FG$. Since $\alpha\hat{x} = \alpha_1(x-1)\hat{x} = 0$, we conclude that $(\alpha y - y\alpha)\hat{x} = \alpha\hat{x}y - y\alpha\hat{x} = 0$. On the other hand, $(\alpha y - y\alpha)\hat{x} = \beta\hat{x}\hat{x} = \beta\hat{x} = \alpha y - y\alpha$. This gives that $\alpha y - y\alpha = 0$, so $\alpha y = y\alpha$, implying $\alpha \in Z(FG)$. Thus $Z(\Delta(G,H)) \subseteq Z(FG)$. Let $\alpha \in Z(\Delta(G,H)) \subseteq Z(FG)$. Then $\alpha = a_0 + a_1\hat{b}_1 + a_2\hat{b}_2 + a_3\hat{b}_3 + a_4\hat{b}_4$. Since $\alpha\hat{x} = 0$, we obtain that $\alpha\hat{x} = (a_0 + 3a_1 + 3a_2)\hat{x} + 7a_3\hat{b}_3 + 7a_4\hat{b}_4 = a_0\hat{x} + a_3\hat{b}_3 + a_4\hat{b}_4 = 0$, so $a_0 = a_3 = a_4 = 0$. Thus $Z(\Delta(G,H)) = \{(a_1\hat{b}_1 + a_2\hat{b}_2) \mid a_1;\ a_2 \in F\}$ and $\dim_F Z(\Delta(G,H)) = 2$. Note also that for all $\alpha \in Z(\Delta(G,H))$, $\alpha^3 = (a_1\hat{b}_1 + a_2\hat{b}_2)^3 = a_1^3\hat{b}_2 + a_2^3\hat{b}_1$, so $\alpha^{3^n} = a_1\hat{b}_1^{3^n} + a_2\hat{b}_2^{3^n}$. If n is even, then $\alpha^{3^n} = \alpha$; if n is odd, then $\alpha^{3^{2n}} = \alpha$, but in general

$$(**) \qquad \alpha^{3^n} \neq \alpha.$$

Finally, we determine the structure of FG. Since $\Delta(G,H)$ is semisimple, $\dim_F(\Delta(G,H)) = 18$, and $\dim_F Z(\Delta(G,H)) = 2$, by $(**)$ we conclude that either $\Delta(G,H) \cong M(3,F) \oplus M(3,F)$ when n is even, or $\Delta(G,H) \cong M_3(F_2)$ when n is odd, where F_2 is the degree 2 extension field of F. This gives that either $FG \cong \Delta(G,H) \oplus FC_3 \cong M(3,F) \oplus M(3,F) \oplus FC_3$ when n is even, or $FG \cong M_3(F_2) \oplus FC_3$ when n is odd. Therefore, either $\mathscr{U}(FG) \cong \mathrm{GL}(3,F) \times \mathrm{GL}(3,F) \times F^* \times C_3^{2n}$ when n is even, or $\mathscr{U}(FG) \cong \mathrm{GL}(3,F_2) \times F^* \times C_3^{2n}$ when n is odd. This completes the proof. $\square$

3. Unit groups of $F_{3^n}A_4$ and $F_{2^n}Q_{12}$

As mentioned in the introduction, in 2007, R. Sharma, J. Srivastava and M. Khan provided a preliminary characterization of the unit group of FA_4 over a finite field of characteristic 3. In this section, we first determine the structure of the group algebra FA_4 over a finite field of characteristic 3. As a consequence, we establish a complete characterization of the unit group of this group algebra. Our first main result is as follows:

Theorem 3.1. *Let F be a finite field of characteristic 3 with order 3^n and A_4 the alternating group of degree 4. Then $FA_4 \cong \Delta(A_4,K_4) \oplus F(A_4/K_4) \cong M(3,F) \oplus FC_3$ where K_4 is the normal subgroup of order 4 in A_4. Moreover, $\mathscr{U}(FA_4) \cong \mathrm{GL}(3,F) \times F^* \times C_3^{2n}$.*

P r o o f. Note that $A_4 = K_4 \rtimes C_3$, where $K_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ and $C_3 = \langle y\rangle = \langle(123)\rangle$. A direct computation shows that A_4 has 4 conjugacy classes: $b_1 = \{1\}$, $b_2 = \{(12)(34), (13)(24), (14)(23)\}$, $b_3 = \{(123), (134), (142), (243)\} = K_4(123) = K_4y$, $b_4 = \{(132), (143), (124), (234)\} = K_4(123)^2 = K_4y^2$. Since K_4 is a normal subgroup of A_4, $e = (1/|K_4|)\hat{K}_4 = e^2$ is a central idempotent. Thus we have the decomposition $FA_4 = FA_4(1-e) \oplus FA_4(e) \cong \Delta(A_4, K_4) \oplus FC_3$. We next show that $\Delta(A_4, K_4)$ is semisimple and determine the structure of $\Delta(A_4, K_4)$. Our proof is similar to that of Theorem 2.3 and we shall include a complete proof for the convenience of the reader.

We first compute the Jacobson radical $J(FA_4)$ of this group algebra. As mentioned before, $J(FA_4) \subseteq \mathrm{Ann}(c)$ where $c = 1 + \hat{b}_3 + \hat{b}_4 = 1 + \hat{K}_4y + \hat{K}_4y^2$. Let $\alpha = \sum\limits_{i=0}^{2} \alpha_i \in \mathrm{Ann}(c)$ where $\mathrm{supp}(\alpha_i) \subseteq K_4y^i$, $i = 0, 1, 2$. Since $\alpha c = 0$, we obtain that $(\alpha_0 + \alpha_1 + \alpha_2)(1 + \hat{K}_4y + \hat{K}_4y^2 = (\alpha_0 + \varepsilon(\alpha_1 + \alpha_2)\hat{K}_4) + (\alpha_1 + \varepsilon(\alpha_0 + \alpha_2)\hat{K}_4y) + (\alpha_2 + \varepsilon(\alpha_0 + \alpha_1)\hat{K}_4y^2) = 0$. Thus $\alpha_0 = -\varepsilon(\alpha_1 + \alpha_2)\hat{K}_4, \alpha_1 = -\varepsilon(\alpha_0 + \alpha_2)\hat{K}_4y$ and $\alpha_2 = -\varepsilon(\alpha_0 + \alpha_1)\hat{K}_4y^2$. Let $a_1 = -\varepsilon(\alpha_1 + \alpha_2)$ and $a_2 = -\varepsilon(\alpha_0 + \alpha_2)$. Since $\varepsilon(\alpha) = \varepsilon(\alpha_0 + \alpha_1 + \alpha_2) = 0$, we conclude that $-\varepsilon(\alpha_0 + \alpha_1) = \varepsilon(\alpha_2) = -(a_1 + a_2)$. Therefore, $\mathrm{Ann}(c) = \{a_1\hat{K}_4 + a_2\hat{K}_4y - (a_1 + a_2)\hat{K}_4y^2;\ a_1, a_2 \in F\} = \{(a_1(1+y) + a_2y)(1-y)\hat{K}_4;\ a_1, a_2 \in F\}$. We now show that $\mathrm{Ann}(c)^3 = 0$. Indeed, for all $\alpha, \alpha', \alpha'' \in \mathrm{Ann}(c)$, $\alpha\alpha'\alpha'' = ((a_1(1+y) + a_2y)(1-y)\hat{K}_4)((a_1'(1+y) + a_2'y)(1-y)\hat{K}_4)((a_1''(1+y) + a_2''y) \times (1-y)\hat{K}_4) = (a_1(1+y) + a_2y)(a_1'(1+y) + a_2'y)(a_1''(1+y) + a_2''y)(1-y)^3(\hat{K}_4)^3 = 0$ as $(1-y)^3 = 1 - y^3 = 0$. As before, we obtain that $J(FA_4) = \mathrm{Ann}(c)$, $J(FA_4)^3 = 0$ and $\dim_F J(FA_4) = 2$.

Next we show that $\Delta(A_4) = \Delta(A_4, K_4) \oplus J(FA_4)$. First we show that $\Delta(A_4, K_4) \cap J(FA_4) = 0$. Let $\alpha \in \Delta(A_4, K_4) \cap J(FA_4)$. Since $\alpha \in \Delta(A_4, K_4)$, we have $\alpha\hat{K}_4 = 0$. On the other hand, since $\alpha \in J(FA_4)$, $\alpha = (a_1(1+y) + a_2y) \times (1-y)\hat{K}_4$, so $\alpha\hat{K}_4 = \alpha$. Thus $\alpha = \alpha\hat{K}_4 = 0$, proving $\Delta(A_4, K_4) \cap J(FA_4) = 0$. Since both $\Delta(A_4, K_4)$ and $J(FA_4)$ are contained in $\Delta(A_4)$, $\dim_F \Delta(A_4, K_4) + \dim_F J(FA_4) = 9 + 2 = \dim_F \Delta(A_4)$ and $\Delta(A_4, K_4) \cap J(FA_4) = 0$, we conclude $\Delta(A_4) = \Delta(A_4, K_4) \oplus J(FA_4)$, so $\Delta(A_4, K_4) \cong \Delta(A_4)/J(FA_4)$. Since $FA_4/J(FA_4)$ is semisimple and $\Delta(A_4)/J(FA_4)$ is an ideal of $FA_4/J(FA_4)$, we obtain that $\Delta(A_4, K_4) \cong \Delta(A_4)/J(FA_4)$ is semisimple as desired.

We now show that $Z(\Delta(A_4, K_4)) \subseteq Z(FA_4)$ and determine the structure of $Z(\Delta(A_4, K_4))$. Let $\alpha \in Z(\Delta(A_4, K_4))$. Then $\alpha(x-1) = (x-1)\alpha$, for all $x \in K_4$ (as $x - 1 \in \Delta(A_4, K_4)$), so $\alpha x = x\alpha$. Recall that $y = (123)$ and $y(x-1) \in \Delta(A_4, K_4)$, for all $x \in K_4$. Hence $\alpha y(x-1) - y(x-1)\alpha = (\alpha y - y\alpha)(1-x)$, so $\alpha y - y\alpha \in \mathrm{Ann}_l(x-1)$, for all $x \in K_4$. Similarly, since $(x-1)y \in \Delta(A_4, K_4)$, as before we can show that $\alpha y - y\alpha \in \mathrm{Ann}_r(x-1)$, for all $x \in K_4$. Thus $\alpha y - y\alpha \in \mathrm{Ann}(\Delta(A_4, K_4)) = (FA_4)\hat{K}_4$, so $\alpha y - y\alpha = \beta\hat{K}_4$, implying $\alpha y - y\alpha = (\alpha y - y\alpha)\hat{K}_4$. On the other hand,

$(\alpha y - y\alpha)\hat{K}_4 = \alpha\hat{K}_4 y - y\alpha\hat{K}_4 = 0$ (as $\alpha\hat{K}_4 = 0$ because $\alpha \in \Delta(A_4, K_4)$). Thus $\alpha y - y\alpha = 0$, implying $\alpha \in Z(FA_4)$ and $Z(\Delta(A_4, K_4)) \subseteq Z(FA_4)$. Let $\alpha \in Z(\Delta(A_4, K_4))$. Then $\alpha = a_1 + a_2\hat{b}_2 + a_3\hat{b}_3 + a_4\hat{b}_4 = a_1 + a_2\hat{b}_2 + a_3\hat{K}_4 y + a_4\hat{K}_4 y^2$ where $a_i \in F$. Since $\alpha\hat{K}_4 = 0$, we have $a_1\hat{K}_4 + a_3\hat{K}_4 y + a_4\hat{K}_4 y^2 = 0$, so $a_1 = a_3 = a_4 = 0$. Hence $Z(\Delta(A_4, K_4)) = \{a_2\hat{b}_2;\ a_2 \in F\}$. Finally, since $\Delta(A_4, K_4)$ is semisimple with dimension 9 and $\dim_F Z(\Delta(A_4, K_4)) = 1$, we conclude that $\Delta(A_4, K_4) \cong M(3, F)$. Therefore, $FA_4 \cong \Delta(A_4, K_4) \oplus F(A_4/K_4) \cong M(3, F) \oplus FC_3$. Consequently, we obtain that $\mathscr{U}(FA_4) \cong \mathrm{GL}(3, F) \times F^* \times C_3^{2n}$. This completes the proof. □

As mentioned earlier, in [16] Tang and Gao described the unit group of FQ_{12}, where F is a finite field of order 2^n and $Q_{12} = \langle x, y; x^6 = 1, y^2 = x^3, x^y = x^{-1}\rangle$. It was shown that the group $V_1 = 1 + J(FQ_{12})$ is nilpotent of class at most 2; however, the structure of the Jacobson radical $J(FQ_{12})$ was not established. In what follows, we shall establish the structure of the Jacobson radical $J(FQ_{12})$ and show that the group $V_1 = 1 + J(FQ_{12})$ is, in fact, abelian. As a consequence, we provide a better characterization of the the unit group of FQ_{12}.

Theorem 3.2. *Let F be a a finite field of order 2^n and $G = Q_{12}$ be the group of order 12 defined above. Then $FQ_{12} \cong \Delta(Q_{12}, Q'_{12}) \oplus FC_4$. Moreover,*

(1) *$\mathscr{U}(FQ_{12})/V_1 \cong \mathrm{GL}(2, F) \times F^*$, where $V_1 = 1 + J(FQ_{12}) \cong C_2^{5n} \times C_4^n$.*
(2) *$\mathscr{U}(FQ_{12}) \cong \mathscr{U}(\Delta(Q_{12}, Q'_{12})) \oplus \mathscr{U}(FC_4)$ where $\mathscr{U}(FC_4) \cong C_{2^n-1} \times C_2^n \times C_4^n$ and $\mathscr{U}(\Delta(Q_{12}, Q'_{12}))/(1 + J(\Delta(Q_{12}, Q'_{12}))) \cong \mathrm{GL}(2, F)$ where $1 + J(\Delta(Q_{12}, Q'_{12})) \cong C_2^{4n}$.*

Proof. Let $H = Q'_{12} = \{1, x^2, x^4\} = \langle x^2\rangle$ and $e = \hat{H}/|H| = \hat{H}$ which is a central idempotent. Then $FQ_{12} = FQ_{12}(1 - e) \oplus FQ_{12}(e) \cong \Delta(Q_{12}, H) \oplus FC_4$. We remark that unlike the case discussed before $\Delta(Q_{12}, H)$ is no longer semisimple. It is routine to check that Q_{12} has six conjugacy classes: $b_1 = \{1\}$, $b_2 = \{x^2, x^4\}$, $b_3 = \{x, x^5\}$, $b_4 = \{x^3\}$, $b_5 = \{y, x^2y, x^4y\} = Hy$, $b_6 = \{xy, x^3yx^5y\} = Hxy$.

We first compute the Jacobson radical $J(FQ_{12})$. As before, we can show that $J(FQ_{12}) = \mathrm{Ann}(c)$ where $c = 1 + \hat{b}_4 + \hat{b}_5 + \hat{b}_6$. Let $\alpha = \sum_{i=1}^{6} a_i x^{i-1} + \sum_{i=1}^{6} a_{i+6}x^{i-1}y \in \mathrm{Ann}(c)$. Rewrite $\alpha = \sum_{i=1}^{6} \alpha_i$ such that $\mathrm{supp}(\alpha_i) \subseteq b_i$ for $i = 1, 2, \ldots, 6$, i.e., $\alpha_1 = a_1$, $\alpha_2 = a_3x^2 + a_5x^4$, $\alpha_3 = a_2x + a_6x^5$, $\alpha_4 = a_4x^3$, $\alpha_5 = a_7y + a_9x^2y + a_{11}x^4y$ and $\alpha_6 = a_8xy + a_{10}x^3y + a_{12}x^5y$. Since $\alpha c = 0$, we obtain $\alpha c = \alpha + \alpha x^3 + \alpha\hat{b}_5 + \alpha\hat{b}_6 = ((\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(1 + x^3) + \varepsilon(\alpha_5 + \alpha_6)(1 + x)\hat{H}) + (\alpha_5 + \alpha_6x^3 + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\hat{b}_5) + (\alpha_6 + \alpha_5x^3 + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\hat{b}_6) = 0$. Thus

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(1 + x^3) + \varepsilon(\alpha_5 + \alpha_6)(1 + x)\hat{H} = 0,$$
$$\alpha_5 + \alpha_6x^3 + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\hat{b}_5 = 0$$

and

$$\alpha_6 + \alpha_5 x^3 + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\hat{b}_6 = 0.$$

Simplifying these equations gives

$$\begin{aligned}
&a_1 + a_4 + \varepsilon(\alpha_5 + \alpha_6) = 0,\\
&a_2 + a_5 + \varepsilon(\alpha_5 + \alpha_6) = 0,\\
&a_3 + a_6 + \varepsilon(\alpha_5 + \alpha_6) = 0,\\
&a_7 + a_{10} + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0,\\
&a_8 + a_{11} + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0,\\
&a_9 + a_{12} + \varepsilon(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0.
\end{aligned}$$

Further simplification gives $a_5 = a_1 + a_2 + a_4$, $a_6 = a_1 + a_3 + a_4$, $a_{10} = a_1 + a_4 + a_7$, $a_{11} = a_1 + a_4 + a_8$, and $a_{12} = a_1 + a_4 + a_9$. Thus, $J(FQ_{12}) = \mathrm{Ann}(c) = \{(a_1 + a_2x + a_3x^2 + a_7y + a_8xy + a_9x^2y)(1 + x^3) + (a_1 + a_4)(x^3 + x^4 + x^5)(1 + y);\ a_1, a_2, a_3, a_4, a_7, a_8, a_9 \in F\}$ and $\dim_F J(FQ_{12}) = 7$.

We next show that $J(FQ_{12})^4 = \mathrm{Ann}(c)^4 = 0$. Let $\alpha = (a_1 + a_2x + a_3x^2 + a_7y + a_8xy + a_9x^2y)(1+x^3) + (a_1+a_4)(x^3+x^4+x^5)(1+y)$ and $\beta = (d_1+d_2x+d_3x^2+d_7y+d_8xy+d_9x^2y)(1+x^3)+(d_1+d_4)(x^3+x^4+x^5)(1+y)$ be any two elements of $J(FQ_{12})$. Then $\alpha\beta = ((a_1+a_4)(d_1+d_2+d_3+d_7+d_8+d_9)+(d_1+d_4)(a_1+a_2+a_3+a_7+a_8+a_9))\hat{Q}_{12} + (a_1+a_4)(d_1+d_4)\hat{x}$ (as $(1+x^3)^2 = 0, (x^3+x^4+x^5)(1+y)(1+x^3) = \hat{Q}_{12}$, and $(x^3 + x^4 + x^5)(1 + y)(x^3 + x^4 + x^5)(1 + y) = \hat{x}$). Thus $\alpha\beta = \beta\alpha$, implying $J(FQ_{12})$ is commutative. Since $\hat{x}\hat{x} = 0$, $\hat{x}\hat{Q}_{12} = 0$, $\hat{Q}_{12}\hat{Q}_{12} = 0$, we have $\alpha\beta\gamma s = 0$, for all $\alpha, \beta, \gamma, s \in J(FQ_{12})$, so $J(FQ_{12})^4 = 0$.

Finally, we determine the structure of the unit group of FQ_{12}. Note that for all $\alpha \in J(FQ_{12})$, if $\alpha^2 = 0$, then $a_1+a_4 = 0$, so $\alpha = (a_1+a_2x+a_3x^2+a_7y+a_8xy+a_9x^2y) \times (1+x^3)$. Thus the number of elements in $J(FQ_{12})$ with this property is $|F|^6 = 2^{6n}$. So the number of elements α in $J(FQ_{12})$ for which $\alpha^4 = 0$, but $\alpha^2 \neq 0$, is $|F|^7 - |F|^6 = 2^{7n} - 2^{6n}$. This together with the fact that $J(FQ_{12})$ is commutative and $J(FQ_{12})^4 = 0$ gives that $1 + J(FQ_{12}) \cong C_2^{5n} \times C_4^n$. Since $FQ_{12}/J(FQ_{12}) \cong \Delta(Q_{12})/J(\Delta(Q_{12})) \oplus F$, we conclude that $\dim_F \Delta(Q_{12})/J(\Delta(Q_{12})) = 4$, so $\Delta(Q_{12})/J(\Delta(Q_{12})) \cong M(2, F)$ as it is non-commutative and semisimple (note that in $\Delta(Q_{12})/J(\Delta(Q_{12}))$, $(1 + x^2) + J(FQ_{12})$, $(1 + x^2)y + J(FQ_{12})$ do not commute). Note also that since $FQ_{12} \cong \Delta(Q_{12}, Q'_{12}) \oplus FC_4$, $FQ_{12}/J(FQ_{12}) \cong \Delta(Q_{12}, Q'_{12})/J(\Delta(Q_{12}, Q'_{12})) \oplus FC_4/J(FC_4) \cong \Delta(Q_{12}, Q'_{12})/J(\Delta(Q_{12}, Q'_{12}) \oplus F$, so $\Delta(Q_{12}, Q'_{12})/J(\Delta(Q_{12}, Q'_{12})) \cong \Delta(Q_{12})/J(\Delta(Q_{12})) \cong M(2, F)$. Thus $FQ_{12}/J(FQ_{12}) \cong M(2, F) \oplus F$. Therefore, $\mathscr{U}(FQ_{12})/V_1 \cong \mathscr{U}(FQ_{12}/J(FQ_{12})) \cong \mathrm{GL}(2, F) \times F^*$, where $V_1 = 1 + J(FQ_{12}) \cong C_2^{5n} \times C_4^n$, proving (1).

Since $FQ_{12} \cong \Delta(Q_{12}, Q'_{12}) \oplus FC_4$, $\mathscr{U}(FQ_{12}) \cong \mathscr{U}(\Delta(Q_{12}, Q'_{12})) \times \mathscr{U}(FC_4)$ and $J(FQ_{12}) \cong J(\Delta(Q_{12}, Q'_{12})) \oplus J(FC_4)$. It follows from [12, Theorem 3.3] that $\mathscr{U}(FC_4) \cong C_{2^n-1} \times C_2^n \times C_4^n$. As before, $\mathscr{U}(\Delta(Q_{12}, Q'_{12}))/(1 + J(\Delta(Q_{12}, Q'_{12}))) \cong \mathscr{U}(\Delta(Q_{12}, Q'_{12})/J(\Delta(Q_{12}, Q'_{12}))) \cong \mathrm{GL}(2, F)$. Since $J(FQ_{12}) \cong J(\Delta(Q_{12}, Q'_{12})) \oplus J(FC_4)$, $1 + J(FQ_{12}) \cong C_2^{5n} \times C_4^n$ and $1 + J(FC_4) \cong C_2^n \times C_4^n$, we conclude that $1 + J(\Delta(Q_{12}, Q'_{12})) \cong (1 + J(FQ_{12}))/(1 + J(FC_4)) \cong C_2^{4n}$, proving (2). $\square$

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Authors' addresses: Gaohua Tang, Yangjiang Wei, School of Mathematical Science, Guangxi Teachers Education University, Nanning, Guangxi, 530001, P. R. China, e-mail: `tanggaohua@163.com`, `gus02@163.com`; Yuanlin Li (corresponding author), Department of Mathematics, Brock University, St. Catharines, Ontario, Canada L2S 3A1, e-mail: `yli@brocku.ca`.