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TWO-PHASE CONVECTION IN ICY SATELLITES

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Praha

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Space missions to Jupiter and Saturn provided first estimates of the inner structure of their icy moons and, for some of them (Titan, Europa, Enceladus) suggested the presence of subsurface liquid water reservoirs. Similar subsurface oceans have probably been present in the interior of each differentiated icy satellite, at least at some instant during its evolution. Due to the presence of both water phases and possibly other tracers (methane, ammonium etc.), the proper physical description of these satellites requires the traditional thermal convection approach to be generalized to capture the multicomponent interaction. We present a set of equations derived from the principles of continuum thermodynamics which govern the flow of a two-phase material. Employing the dimensional analysis we identify the leading order terms and arrive at a reduced system of equations that can be compared with other formalisms, such as *McKenzie* [1984] and *Šrámek et al.* [2007]. To gain some insight into this highly complex system we first focus on a 1D Cartesian case where most of the interaction phenomena characteristic of a two-phase system (porosity shock waves, material singularities, etc.) are already present. The numerical treatment of this basic case already requires advanced numerical techniques (TVD schemes) and serves as a good starting point for further computations involving more realistic scenarios.

1. Introduction

The problem of thermal convection in partially molten ice layers on icy satellites is a particular example of the multiphase flow phenomena involved in many geophysical applications, such as planetary differentiation, Earth core compositional convection and flow beneath mid-ocean ridges to name just a few examples [e.g., *McKenzie*, 1984, *Spiegelman*, 1993, *Bercovici et al.*, 2001, *Šrámek et al.*, 2007]. The usual approach to this problem, common to all the above mentioned studies, is based on a

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physical description of thermomechanical interactions of the phases at some representative microscale by methods of single continuum thermodynamics. In the present paper, we adopt a different approach treating the medium as a mixture where both components are omnipresent. The detailed description of the microstructure (often missing or uncertain) is here replaced by formal interaction terms and couplings following from the principles of rational mixture thermodynamics [e.g., *Truesdell*, 1969, *Samohýl*, 1987]. This methodical framework provides the maximal class of thermodynamically consistent material models for the considered process and it easily allows further extensions, such as inclusion of tidal heating, methane tracer effects, etc. A particular shape of the interaction and constitutive functionals must be, however, determined independently from experiments or from the material theory.

2. Two-phase problem in rational mixture thermodynamics

Following *Samohýl* [1987], the general balance laws in a two-component reacting mixture involve balances of mass and linear momentum for both components and a single balance of internal energy for the mixture as a whole (under assumption of a common temperature) and read as[1]

$$\frac{D_f\overline{\varrho}_f}{Dt} + \overline{\varrho}_f \nabla \cdot \boldsymbol{v}_f = r_f \,, \tag{1}$$

$$\frac{D_m\overline{\varrho}_m}{Dt} + \overline{\varrho}_m \nabla \cdot \boldsymbol{v}_m = r_m = -r_f \,, \tag{2}$$

$$\overline{\varrho}_f \frac{D_f\boldsymbol{v}_f}{Dt} = \nabla \cdot \boldsymbol{T}_f + \overline{\varrho}_f \boldsymbol{b}_f + \boldsymbol{k}_f \,, \tag{3}$$

$$\overline{\varrho}_m \frac{D_m\boldsymbol{v}_m}{Dt} = \nabla \cdot \boldsymbol{T}_m + \overline{\varrho}_m \boldsymbol{b}_m + \boldsymbol{k}_m \,, \tag{4}$$

$$\begin{aligned} \overline{\varrho}_f \frac{D_f\varepsilon_f}{Dt} + \overline{\varrho}_m \frac{D_m\varepsilon_m}{Dt} &= \boldsymbol{T}_f : \boldsymbol{D}_f + \boldsymbol{T}_m : \boldsymbol{D}_m + Q - \nabla \cdot \boldsymbol{q} - \\ &- r_f\varepsilon_f - r_m\varepsilon_m - \boldsymbol{v}_f \cdot \boldsymbol{k}_f - \boldsymbol{v}_m \cdot \boldsymbol{k}_m - \frac{1}{2} r_f v_f^2 - \frac{1}{2} r_m v_m^2 \,. \end{aligned} \tag{5}$$

With $i \in \{f, m\}$, the following notation holds: $\frac{D_i}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_i \cdot \nabla$ is the material time derivative, $\boldsymbol{v}_i$ the velocity, $r_f = -r_m$ the volume rate of mass-change of the fluid and matrix, respectively, $\overline{\varrho}_i$ the mass density, that can be written in terms of porosity ϕ and material (volume) densities ϱ_f and ϱ_m as $\overline{\varrho}_f = \varrho_f\phi$ and $\overline{\varrho}_m = \varrho_m(1-\phi)$, $\boldsymbol{T}_i$ the partial Cauchy stress tensor, $\boldsymbol{k}_i$ the interaction volume force between the i-th component and the rest of the mixture, $\boldsymbol{b}_i = -g\boldsymbol{e}_z$ denotes the outer volume force (gravity), ε_i the internal energy, $\boldsymbol{D}_i = \frac{1}{2}[\nabla\boldsymbol{v}_i + (\nabla\boldsymbol{v}_i)^T]$ the strain-rate tensor (symmetric part of the velocity gradient), Q the total internal heating (not considered further in the text) and

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[1] Subscripts: f (fluid), m (ice matrix).

$\boldsymbol{q}$ the total heat flux. We have not mentioned angular momentum balances because for non-polar mixtures (considered here) they only imply symmetry of the partial Cauchy stress tensors.

Linearizing these equations around the thermodynamic equilibrium with respect to the water content (or, equivalently, porosity) and following the approach of *Souček* [2005], we obtain

$$\boldsymbol{T}_f = -P\phi\boldsymbol{I}\,, \tag{6}$$

where P is the pressure (we consider the same pressure in both phases),

$$\boldsymbol{k}_f = -\boldsymbol{k}_m - r_f\boldsymbol{v}_{dif} = -\phi\nu(T,P)\boldsymbol{v}_{dif} - \omega(T,P)\nabla\phi\,, \tag{7}$$

where $\boldsymbol{v}_{dif} = \boldsymbol{v}_f - \boldsymbol{v}_m$, and $\nu(T,P)$, $\omega(T,P)$ are yet undetermined functions of pressure and temperature whose interpretation requires comparison with material theory (see eq. (14)), and

$$\boldsymbol{T}_m = -P(1-\phi)\boldsymbol{I} - \Pi_m\boldsymbol{I} + (1-\phi)\boldsymbol{\sigma}_m\,, \tag{8}$$

where

$$\boldsymbol{\sigma}_m = \mu_m(1+\gamma\phi)\Big[\nabla\boldsymbol{v}_m + (\nabla\boldsymbol{v}_m)^T - \frac{2}{3}(\nabla\cdot\boldsymbol{v}_m)\boldsymbol{I}\Big], \tag{9}$$

and Π_m is the dynamic pressure in the matrix, its identification can be done by comparison with the material theory (see eq. (15)); μ_m and γ are the reference shear matrix viscosity and corresponding porosity dependence factor, respectively [see *Greve*, 1997]. The material time derivative of internal energies can be written in the following form

$$\bar{\varrho}_f\frac{D_f\varepsilon_f}{Dt} = \phi\Big(c_p^f\varrho_f - \alpha_f P\Big)\frac{D_fT}{Dt} + \phi\Big(\beta_f P - \alpha_f T\Big)\frac{D_fP}{Dt}, \tag{10}$$

and

$$\bar{\varrho}_m\frac{D_m\varepsilon_m}{Dt} = (1-\phi)\Big(c_p^m\varrho_m - \alpha_m P\Big)\frac{D_mT}{Dt} + (1-\phi)\Big(\beta_m P - \alpha_m T\Big)\frac{D_mP}{Dt}. \tag{11}$$

Here T denotes the absolute temperature, which due to the assumption of thermodynamic equilibrium is the same in both phases and determined by the local pressure (see eq. (22)). For the meaning of material parameters (c_p^i, α_i and β_i) see Table 1. The internal energy difference can be expressed for a reversible quasi-static phase-change process using the first law of thermodynamics as follows

$$[\varepsilon] = \varepsilon_f - \varepsilon_m = T[s] - P\left[\frac{1}{\varrho}\right] = L - P\left(\frac{\varrho_m - \varrho_f}{\varrho_f\varrho_m}\right), \tag{12}$$

where $[s]$ is the specific entropy difference between the phases and $L = T[s]$ is the specific latent heat of melting of ice. Employing also the following identity [e.g., *Samohýl*, 1987]

$$\boldsymbol{v}_f\cdot\boldsymbol{k}_f + \boldsymbol{v}_m\cdot\boldsymbol{k}_m + \frac{1}{2}r_f\boldsymbol{v}_f^2 + \frac{1}{2}r_m\boldsymbol{v}_m^2 = \boldsymbol{v}_{dif}\cdot\boldsymbol{k}_f + \frac{1}{2}r_f\boldsymbol{v}_{dif}^2\,, \tag{13}$$

the previous system can be formally compared with the approach of *McKenzie* [1984] and *Šrámek et al.* [2007]. The comparison reveals that the appropriate functional form of interaction force (7) and dynamic pressure Π_m in (8) reads

$$\boldsymbol{k}_f = -\frac{\mu_f \phi^2}{k(\phi)} \boldsymbol{v}_{dif} + P \nabla \phi \ , \tag{14}$$

$$\Pi_m = -\frac{1-\phi}{\phi} \mu_m (\nabla \cdot \boldsymbol{v}_m) \ , \tag{15}$$

where μ_f is the shear viscosity of fluid component. With these identifications and using the "ansatz" for permeability $k(\phi) = k_0 \phi^2$ [e.g., *Spiegelman*, 1993, *Choblet and Parmentier*, 2001, *Šrámek et al.*, 2007] the resultant equations can be written as follows

$$\varrho_f \frac{\partial \phi}{\partial t} + \nabla \cdot [\varrho_f \phi \boldsymbol{v}_f] = r_f \ , \tag{16}$$

$$-\varrho_m \frac{\partial \phi}{\partial t} + \nabla \cdot [\varrho_m (1-\phi) \boldsymbol{v}_m] = -r_f \ , \tag{17}$$

$$\varrho_f \phi \frac{D_f \boldsymbol{v}_f}{Dt} = -\phi \nabla P + \varrho_f \phi \boldsymbol{g}_0 - \frac{\mu_f}{k_0} \boldsymbol{v}_{dif} \ , \tag{18}$$

$$\begin{aligned} \varrho_m (1-\phi) \frac{D_m \boldsymbol{v}_m}{Dt} = -(1-\phi) \nabla P + \varrho_m (1-\phi) \boldsymbol{g}_0 + \\ + \mu_m \nabla \left[\frac{1-\phi}{\phi} (\nabla \cdot \boldsymbol{v}_m) \right] + \\ + \mu_m \nabla \cdot \left((1-\phi)(1+\gamma\phi) \left[\nabla \boldsymbol{v}_m + (\nabla \boldsymbol{v}_m)^T - \frac{2}{3} (\nabla \cdot \boldsymbol{v}_m) \boldsymbol{I} \right] \right) + \\ + \frac{\mu_f}{k_0} \boldsymbol{v}_{dif} - r_f \boldsymbol{v}_{dif} \ , \end{aligned} \tag{19}$$

and

$$\begin{aligned} \left[(\varrho_f c_p^f - \alpha_f P)\phi + (\varrho_m c_p^m - \alpha_m P)(1-\phi) \right] \frac{\partial T}{\partial t} + \\ + \left[(\beta_f P - \alpha_f T)\phi + (\beta_m P - \alpha_m T)(1-\phi) \right] \frac{\partial P}{\partial t} + \\ + \phi \boldsymbol{v}_f \cdot \left[(\varrho_f c_p^f - \alpha_f P) \nabla T + (\beta_f P - \alpha_m T) \nabla P \right] + \\ + (1-\phi) \boldsymbol{v}_m \cdot \left[(\varrho_m c_p^m - \alpha_m P) \nabla T + (\beta_m P - \alpha_m T) \nabla P \right] = \\ = \nabla \cdot [k_T(\phi) \nabla T] + \frac{1-\phi}{\phi} \mu_m (\nabla \cdot \boldsymbol{v}_m)^2 + \\ + \mu_m (1-\phi)(1+\gamma\phi) \left[\nabla \boldsymbol{v}_m + (\nabla \boldsymbol{v}_m)^T - \frac{2}{3} (\nabla \cdot \boldsymbol{v}_m) \boldsymbol{I} \right] : \nabla \boldsymbol{v}_m - \\ - r_f L + \frac{\mu_f}{k_0} \boldsymbol{v}_{dif}^2 - \frac{1}{2} r_f \boldsymbol{v}_{dif}^2 \ , \end{aligned} \tag{20}$$

where

$$k_T(\phi) = k_T^m \left[1 - \phi \frac{k_T^m - k_T^f}{\frac{2k_T^m + k_T^f}{3}} \right], \tag{21}$$

[*McKenzie*, 1984]. The first two equations describe the conservation of mass of fluid, eq. (16), and matrix, eq. (17), phases, respectively. The second two equations express the conservation of linear momentum of the fluid (the so-called *Darcy law*, eq. (18)) and the matrix, eq. (19). Finally, eq. (20) represents the conservation of energy of the whole mixture. As mentioned above, we assume that the mixture is close to the thermodynamic equilibrium which implies that it can be characterized by only one value of temperature, namely the melting temperature. The temperature is thus no longer an independent variable but it can be computed from the pressure field using the Clausius-Clapeyron equation [*Simon and Glatzel*, 1929]

$$T(P) = T_0 \left(\frac{P - P_0}{a} + 1 \right)^{\frac{1}{b}}, \tag{22}$$

where T_0, P_0, a and b are given in Table 1.

2.1 Dimensional analysis

We introduce the following scaling [2]

$$\begin{aligned} \phi &= \boldsymbol{\phi}_0 \phi' & t &= \frac{D_0^2 \varrho_m^0 c_p^m}{k_T^m} t' & \boldsymbol{r} &= D_0 \boldsymbol{r}' \\ \boldsymbol{v}_m &= \frac{k_T^m}{D_0 \varrho_m^0 c_p^m} \boldsymbol{v}'_m & \boldsymbol{v}_{dif} &= \frac{k_0 \Delta\varrho^0 g_0}{\mu_f} \boldsymbol{\phi}_0 \boldsymbol{v}'_{dif} & r_f &= \frac{k_T^m}{D_0^2 c_p^m} r'_f \end{aligned} \tag{23.1}$$

$$\begin{aligned} \Pi = P - P_H &= \frac{\mu_m k_0 \Delta\varrho^0 g_0}{D_0 \mu_f} \boldsymbol{\phi}_0 \Pi' & \Delta\varrho^0 &= \varrho_f^0 - \varrho_m^0 & \boldsymbol{g}_0 &= -g_0 \boldsymbol{e}_z \\ P_H &= \varrho_m^0 g_0 D_0 P'_0 & \varrho_f &= \varrho_f^0 \varrho'_f & \varrho_m &= \varrho_m^0 \varrho'_m, \end{aligned} \tag{23.2}$$

where the primes denote the dimensionless quantities, P_H is the hydrostatic pressure and Π is the difference from the hydrostatic pressure. The scaling for time, length and matrix velocity coincide with the standard scaling used for mantle convection [e.g., *Schubert*, 2001], while the scalings for the other quantities are specific for the two-phase system, cf. *Spiegelman* [1993]. We end up with 22 parameters (a, α_f, α_m, b, β_f, β_m, c_p^f, c_p^m, γ, D_0, $\boldsymbol{\phi}_0$, g_0, k_0, k_T^f, k_T^m, L, μ_f, μ_m, P_0, ϱ_f^0, ϱ_m^0, T_0) and the problem involves 4 physical units (m, s, kg, K). It follows from Buckingham's Π-Theorem [*Hutter and Jöhnk*, 2004] that we can find $22 - 4 = 18$ dimensionless numbers which

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[2] The meaning of parameters is given in Table 1.

fully characterize the behavior of the system. These numbers are given below together with their values corresponding to the parameters of icy satellites listed in Table 1:

$$\begin{array}{lll}
\widetilde{\mathscr{R}}_a = \dfrac{g_0 D_0^3 c_p^m (\varrho_m^0)^2}{k_T^m \mu_m} \sim 10^{10} & \mathscr{P}_r = \dfrac{c_p^m \mu_m}{k_T^m} \sim 9 \cdot 10^{16} & \mathscr{D}_i = \dfrac{\alpha_m g_0 D_0}{c_p^m} \sim 10^{-2} \\[2ex]
\Gamma = \beta_m \varrho_m^0 g_0 D_0 \sim 10^{-2} & \tau_\alpha = \alpha_m T_0 \sim 5 \cdot 10^{-2} & \chi = \dfrac{k_0}{D_0^2} \sim 10^{-19} \\[2ex]
\Lambda = \dfrac{L}{D_0 g_0} \sim 2.1 & \mathscr{K} = \dfrac{k_T^m - k_T^f}{\frac{2k_T^m + k_T^f}{3}} \sim 1.0 & \varrho = \dfrac{\varrho_f^0}{\varrho_m^0} \sim 1.1 \\[2ex]
\mathscr{C} = \dfrac{c_p^f}{c_p^m} \sim 2.0 & \mathscr{A} = \dfrac{\alpha_f}{\alpha_m} \sim 1.8 & \mathscr{M} = \dfrac{\mu_f}{\mu_m} \sim 10^{-17} \\[2ex]
\mathscr{B} = \dfrac{\beta_f}{\beta_m} \sim 5 & \mathscr{P}_H = \dfrac{\varrho_m^0 g_0 D_0}{a} \sim -0.3 & \mathscr{P}_0 = \dfrac{P_0}{a} \sim 0 \\[2ex]
\boldsymbol{\phi}_0 \sim 5 \cdot 10^{-2} & \boldsymbol{\gamma} \sim 184 & \boldsymbol{b} \sim 9 \, ,
\end{array} \tag{24}$$

and, for convenience, we also introduce (already dependent) dimensionless parameter $\Delta\varrho = \frac{\Delta\varrho^0}{\varrho_m^0} = \varrho - 1 \sim 0.1$. Some of these numbers are used in the classical framework of thermal convection (Prandtl number $\mathscr{P}_r$, Dissipation number $\mathscr{D}_i$) or are little modified from the classical thermal convection form (modified Rayleigh number $\widetilde{\mathscr{R}}_a$). Some are specific to the two-phase system and related to the following features—melting of the material(Λ), diffusive motion of fluid through the matrix (χ), porosity dependence of viscosity ($\boldsymbol{\gamma}$) and porosity itself ($\boldsymbol{\phi}_0$). Several dimensionless numbers relate the material parameters of both phases ($\mathscr{K}$, ϱ, $\mathscr{C}$, $\mathscr{A}$, $\mathscr{M}$, $\mathscr{B}$). Two numbers measure the effect of temperature and pressure variations on the melting rate (Γ, τ_α) and, finally, three numbers come from Clausius-Clapeyron equation ($\mathscr{P}_H$, $\mathscr{P}_0$, $\boldsymbol{b}$).

In accordance with the approach in a single-component convection, we expand the full equations around the hydrostatic equilibrium state and apply the above scalings. Using the reference values of physical parameters from the Table 1, we evaluate the magnitude of each term in (16–20) and neglect terms of small importance (e.g., the inertia forces, the viscous dissipation, the kinetic energy). The reduced system of equations reads as follows (omitting the primes for dimensionless variables): [3]

$$A_1 \left[\varrho_f \frac{\partial \phi}{\partial t} + \nabla \cdot [\varrho_f \phi \boldsymbol{v}_m] \right] + A_2 \nabla \cdot [\varrho_f \phi \boldsymbol{v}_{dif}] = r_f \, , \tag{25}$$

$$-B_1 \left[\varrho_m \frac{\partial \phi}{\partial t} + \nabla \cdot [\varrho_m \phi \boldsymbol{v}_m] \right] + \nabla \cdot [\varrho_m \boldsymbol{v}_m] = -r_f \, , \tag{26}$$

$$C_1 \boldsymbol{v}_{dif} = -\left[\varrho \varrho_f - \varrho_m \right] C_2 \phi \boldsymbol{e}_z \, , \tag{27}$$

$$
\begin{aligned}
0 = -D_1\nabla\Pi + \nabla\left[\frac{1-\boldsymbol{\phi}_0\phi}{\boldsymbol{\phi}_0\phi}(\nabla\cdot\boldsymbol{v}_m)\right] - \left[\varrho\varrho_f - \varrho_m\right]D_2\phi\boldsymbol{e}_z + \\
+\nabla\cdot\left[(1+\boldsymbol{\gamma\phi}_0\phi)(1-\boldsymbol{\phi}_0\phi)\left(\nabla\boldsymbol{v}_m + (\nabla\boldsymbol{v}_m)^T - \frac{2}{3}(\nabla\cdot\boldsymbol{v}_m)\boldsymbol{I}\right)\right],
\end{aligned}
\tag{28}
$$

$$
\begin{gathered}
E_1 r_f = \nabla\cdot\left[c_3(\phi)\frac{\partial f(P_H)}{\partial P}\nabla P_H\right] - \\
-\phi(\boldsymbol{v}_m + E_2\boldsymbol{v}_{dif})\cdot\nabla P_H\left[E_3\varrho_f\frac{\partial f(P_H)}{\partial P} - E_4P_H\frac{\partial f(P_H)}{\partial P} + E_5P_H - E_4f(P_H)\right] - \\
-c_4(\phi)\boldsymbol{v}_m\cdot\nabla P_H\left[\varrho_m'\frac{\partial f(P_H)}{\partial P} - E_6P_H\frac{\partial f(P_H)}{\partial P} + +E_7P_H - E_6f(P_H)\right] + \\
+E_8P_H\left[\frac{\phi}{\varrho_f}\boldsymbol{\phi}_0(\boldsymbol{v}_m + E_2\boldsymbol{v}_{dif})\cdot\nabla\varrho_f + \frac{c_4(\phi)}{\varrho_m}\boldsymbol{v}_m\cdot\nabla\varrho_m\right] + \\
+E_9\boldsymbol{v}_{dif}^2 .
\end{gathered}
\tag{29}
$$

2.2 Solution of equations for a 1D Cartesian case

As already pointed out, investigating first a simple 1D Cartesian case will provide a good starting point for better understanding of the processes involved in the problem and will allow us to choose appropriate numerical tools applicable to more realistic cases. Besides the very simple geometry, we will consider constant material densities and a matrix viscosity that depends on porosity only. Also, the energy balance equation will not be considered first, and the melt production r_f will be taken as a constant parameter, that is, we will only focus on the mechanical interaction. As boundary conditions, we consider no-slip for matrix velocity at the base of the domain

$$v_m(0,t) = 0\,, \tag{30}$$

and free-slip and purely solid ice at the upper surface:

$$\boldsymbol{T}_m\cdot\boldsymbol{n}(1,t) = 0\,, \tag{31}$$

$$\phi(1,t) = 0\,. \tag{32}$$

We consider the initial condition in the following form:

$$\phi(z,0) = \frac{-10\,(z-1)}{1+(10\,(z-1))^4}\,, \tag{33}$$

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[3] $A_1 = \varrho\boldsymbol{\phi}_0$, $A_2 = \varrho\boldsymbol{\Delta\varrho}\mathscr{M}^{-1}\boldsymbol{\chi}\widetilde{\mathscr{R}}_a\boldsymbol{\phi}_0^2$, $B_1 = \boldsymbol{\phi}_0$, $C_1 = \boldsymbol{\Delta\varrho}\mathscr{R}_a\boldsymbol{\phi}_0$, $C_2 = \widetilde{\mathscr{R}}_a\boldsymbol{\phi}_0$, $D_1 = \widetilde{\mathscr{R}}_a\boldsymbol{\chi\Delta\varrho}\mathscr{M}^{-1}\boldsymbol{\phi}_0$, $D_2 = \widetilde{\mathscr{R}}_a\boldsymbol{\phi}_0$, $E_1 = \mathscr{D}_i\boldsymbol{\Lambda\tau}_\alpha^{-1}$, $E_2 = \widetilde{\mathscr{R}}_a\boldsymbol{\Delta\varrho\chi\phi}_0\mathscr{M}^{-1}$, $E_3 = \mathscr{C}\boldsymbol{\varrho\phi}_0$, $E_4 = \mathscr{A}\mathscr{D}_i\boldsymbol{\phi}_0$, $E_5 = \mathscr{B}\mathscr{D}_i\boldsymbol{\Gamma\phi}_0\boldsymbol{\tau}_\alpha$, $E_6 = \mathscr{D}_i$, $E_7 = \mathscr{D}_i\boldsymbol{\Gamma\tau}_\alpha^{-1}$, $E_8 = \mathscr{D}_i\boldsymbol{\tau}_\alpha^{-1}$, $E_9 = \widetilde{\mathscr{R}}_a\mathscr{D}_i\boldsymbol{\chi}\left(\boldsymbol{\Delta\varrho\phi}_0\right)^2(\mathscr{M}\tau_\alpha)^{-1}$, $c_3(\phi) = 1 - \mathscr{K}\boldsymbol{\phi}_0\phi$, $c_4(\phi) - 1 - \boldsymbol{\phi}_0\phi$, $f(P) = = (\mathscr{P}_HP - \mathscr{P}_0 + 1)^{\frac{1}{\boldsymbol{b}}}$

Table 1. Notation and reference values considered.

Variable	Meaning	Reference value used	Physical unit
a [2]	*constant in Clausius-Clapeyron eq.*	$-395.2 \cdot 10^6$	Pa
α_f [1]	*thermal expansivity of water*	$3 \cdot 10^{-4}$	K^{-1}
α_m [1]	*thermal expansivity of ice*	$1.7 \cdot 10^{-4}$	K^{-1}
b [2]	*constant in Clausius-Clapeyron eq.*	9	—
β_f [1]	*isothermal compressibility of water*	$5 \cdot 10^{-10}$	Pa^{-1}
β_m [1]	*isothermal compressibility of ice*	10^{-10}	Pa^{-1}
c_p^f [1]	*isobaric heat capacity of water*	4180	$\mathrm{J\,kg^{-1}\,K^{-1}}$
c_p^m [1]	*isobaric heat capacity of ice*	2100	$\mathrm{J\,kg^{-1}\,K^{-1}}$
γ [3]	*constant in porosity dependence of μ_m*	184	—
$\Delta\varrho^0 = \varrho_f^0 - \varrho_m^0$ [1]	*density difference*	80	$\mathrm{kg\,m^{-3}}$
D_0 [4]	*length scale of the problem*	10^5	m
ϕ [3]	*porosity (volume fraction of fluid phase)*	$5 \cdot 10^{-2}$	—
g_0 [5]	*acceleration due to gravity*	1.35	$\mathrm{m\,s^{-2}}$
k_0 [6]	*constant in permeability relationship*	10^{-9}	$\mathrm{m^2}$
k_T^f [5]	*water thermal conductivity*	0.56	$\mathrm{W\,m^{-1}K^{-1}}$
k_T^m [1]	*ice thermal conductivity*	2.3	$\mathrm{W\,m^{-1}K^{-1}}$
L [1]	*latent heat of melting of ice*	$284 \cdot 10^3$	$\mathrm{J\,kg^{-1}}$
μ_f	*shear viscosity of water*	10^{-3}	Pa s
μ_m [1]	*shear viscosity of ice*	10^{14}	Pa s
P_0 [2]	*reference pressure*	0	Pa
ϱ_f^0 [1]	*density of water*	10^3	$\mathrm{kg\,m^{-3}}$
ϱ_m^0 [1]	*density of ice*	920	$\mathrm{kg\,m^{-3}}$
T_0	*melting temperature at zero pressure*	273	K

[1] *Tobie et al.* [2005] and references therein

[2] *Chizhov* [1993]

[3] *Greve* [1997]

[4] typical thickness of ice shells on Europa, Titan and Enceladus

[5] *Grasset and Sotin* [1996] and references therein

[6] compiled from *Scott and Stevenson* [1986], *Spiegelman and McKenzie* [1987], *Choblet and Parmentier* [2001], *Šrámek et al.* [2007]

representing a partially molten region below the upper surface. The equations (25–28) can be reduced into the following form [4]

$$\frac{\partial\phi(z,t)}{\partial t} + \frac{\partial}{\partial z}\Big[F_2\phi^3(z,t) - D_1\phi^2(z,t) - F_2\phi^2(0,t)\phi(z,t) - \\ -F_1\phi(z,t)\int_0^z r_f(z',t)dz'\Big] = A_1 r_f(z,t)\ , \tag{34}$$

$$v_m(z,t) = F_2[\phi^2(z,t) - \phi^2(0,t)] - F_1\int_0^z r_f(z',t)dz'\ , \tag{35}$$

$$v_{dif}(z,t) = -\phi(z,t)\ , \tag{36}$$

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[4] $F_1 = \frac{\boldsymbol{\Delta\varrho}}{\boldsymbol{\varrho}}$, $F_2 = \widetilde{\mathscr{R}}_a\boldsymbol{\chi}\boldsymbol{\Delta\varrho}\mathscr{M}^{-1}\boldsymbol{\phi}_0^2$, $c(\phi(z,t)) = \frac{1-\boldsymbol{\phi}_0\phi}{\boldsymbol{\phi}_0\phi} + \frac{4}{3}(1+\boldsymbol{\gamma}\boldsymbol{\phi}_0\phi)(1-\boldsymbol{\phi}_0\phi)$

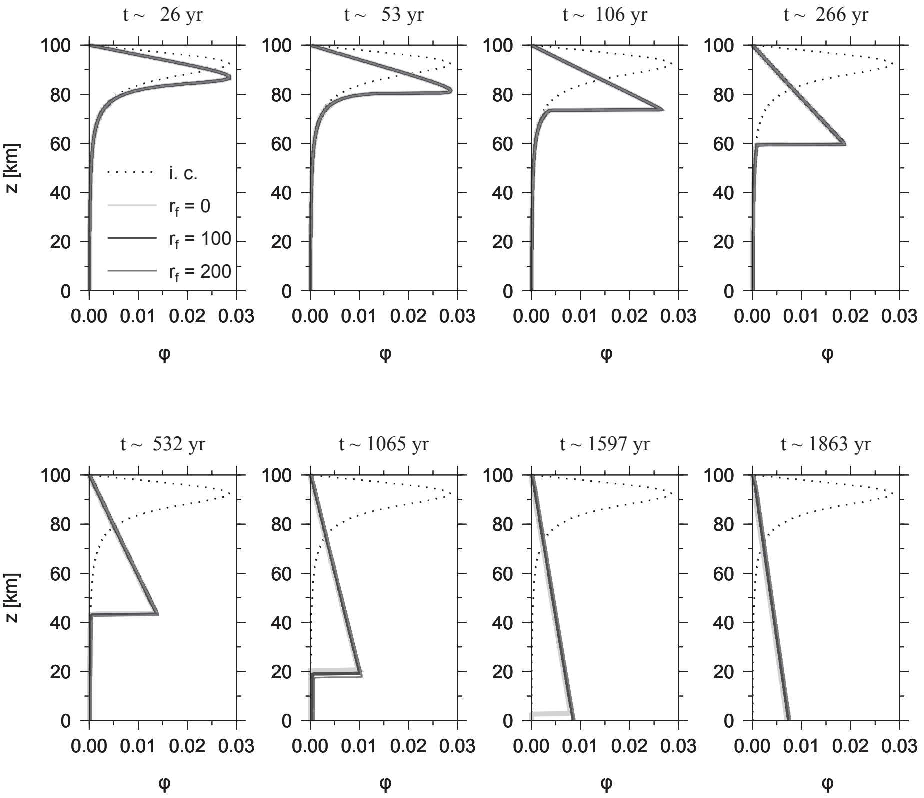

FIGURE 1. Evolution of porosity in 1D Cartesian case. The initial condition is plotted as a black dotted line. Three cases are considered: no melting ($r_f = 0$ green) and constant melting with $r_f = 100$ (blue) and $r_f = 200$ (red).

$$0 = -D_1 \frac{\partial \Pi(z,t)}{\partial z} - C_1 \phi(z,t) + \frac{\partial}{\partial z}\left[c(\phi(z,t)) \frac{\partial v_m(z,t)}{\partial z}\right]. \tag{37}$$

Note that eq. (34) is decoupled from the rest. Solving first for the porosity ϕ, the remaining unknowns can be easily obtained.

3. Results

The nonlinear PDE for porosity advection (34) is solved using the so-called ENO (Essentially Non-Oscillatory) schemes for hyperbolic conservation equations [*Shu*, 1998] in the spatial domain (in the Finite-Volume formalism) and using the explicit Euler integration in the time domain.

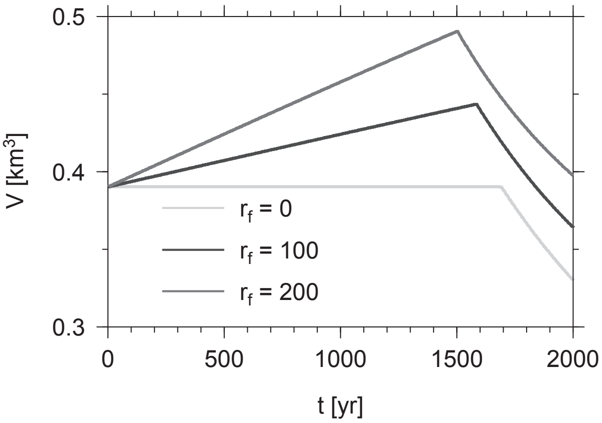

FIGURE 2. Evolution of water volume. The colors are the same as in Fig. 1.

Figure 1 shows the evolution of porosity in time for three different values of melting rate, namely $r_f(z,t) = 0$ (no melting), $r_f(z,t) = 100$, and $r_f(z,t) = 200$. The individual panels in Fig. 1 illustrate the gradual formation of a porosity shock wave [see also *Spiegelman*, 1993, for discussion of the conditions under which shock waves in 1D can be developed] and its subsequent decay until the moment when the evolving wavefront arrives at the bottom boundary. At that moment the outflow of water from the domain begins. The scenarios of wave evolution are very similar in all computed cases and differ mostly in the velocity of wavefront propagation: in the case of constant melting the wave propagates quicker than in the case without melting. Figure 2 shows the volume of fluid evolving in time. As expected, the volume of fluid is conserved (in the case of no melting) or rises (in the case of constant non-zero melting rate) until the arrival of the wavefront at the bottom boundary and the beginning of the outflow. Starting from this moment, the upper boundary descends which would require to include an evolving geometry of the domain. This case is not discussed here and it will be a subject of further study.

4. Conclusions

Using the approach of *Souček* [2005], we have derived the equations governing a two phase flow in water-ice mixture and we compared the resultant equations with the traditional formulation of the problem [e.g., *McKenzie*, 1984, *Šrámek et al.*, 2007]. Assuming that the system is close to a thermodynamic equilibrium and linearizing the equations in terms of water content, we demonstrated that the equations derived here can be related to those of *McKenzie* [1984] and *Šrámek et al.* [2007]. Using the dimensionless analysis we introduced 18 dimensionless numbers fully characterizing the two-phase system and we determined their values for parameters typical of ice satellites. This allows us to further simplify the governing equations and to omit the terms of small importance. Subsequently, we have solved a special case in the approximation of a 1D Cartesian geometry with constant material densities, with shear viscosity of ice dependent only on porosity and without the energy balance.

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