Czechoslovak Mathematical Journal

Chengfu Qin; Xiaofeng Guo; Weihua Yang
The contractible subgraph of 5-connected graphs

Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 3, 671–677

Persistent URL: http://dml.cz/dmlcz/143483

Czechoslovak Mathematical Journal, 63 (138) (2013), 671–677

THE CONTRACTIBLE SUBGRAPH OF 5-CONNECTED GRAPHS

Chengfu Qin, Nanning, Xiaofeng Guo, Xiamen,
Weihua Yang, Taiyuan

(Received April 13, 2012)

Abstract. An edge e of a k-connected graph G is said to be k-removable if $G-e$ is still k-connected. A subgraph H of a k-connected graph is said to be k-contractible if its contraction results still in a k-connected graph. A k-connected graph with neither removable edge nor contractible subgraph is said to be minor minimally k-connected. In this paper, we show that there is a contractible subgraph in a 5-connected graph which contains a vertex who is not contained in any triangles. Hence, every vertex of minor minimally 5-connected graph is contained in some triangle.

Keywords: 5-connected graph; contractible subgraph; minor minimally k-connected

MSC 2010: 05C40, 05C83

1. Introduction

An edge of a k-connected graph G is said to be *k-removable* if $G-e$ is still k-connected. A subgraph H of a k-connected graph is said to be *k-contractible* if its contraction, that is, identification of every component of H to a single vertex, results still in a k-connected graph. Further, H is called *contractible edges* if $H \cong K_2$. The existence of k-removable edge or k-contractible subgraph can give an inductive proof of some topics related to the connectivity of graph. Tutte's ([8]) famous wheel theorem implies that every 3-connected graph on more than four vertices contains an edge whose contraction yields a new 3-connected graph. One can give an inductive proof of Kuratowski's theorem by the wheel theorem. So the existence and the distribution of k-removable edges or k-contractible subgraphs is an attractive research area within graph connectivity theory.

This work was supported by NSFC (No. 11126321, No. 11171279), Natural Sciences Foundation of Guangxi Province (No. 2012GXNSFBA053005), the Scientific Research Foundation of Guangxi Education Committee (No. 200103YB069)

For k-connected graphs with $k \geqslant 4$, it is difficult to perform an induction proof by using single edge contraction as there are infinitely many nonisomorphic k-connected graphs which do not contain any k-contractible edge. These graphs are called *contraction critically k-connected*.

However, every 4-connected graph on at least seven vertices can be reduced to a smaller 4-connected graph by contracting one or two edges subsequently. So, naturally, for $k \geqslant 1$, one can expect that there are b and h such that every k-connected graph on more than b vertices can be reduced to a more smaller k-connected graph by contracting less than h edges ([5]). This is true for $k = 1, 2, 3, 4$. But for $k \geqslant 6$, such a statement fails since toroidal triangulations of large face width are a counterexample ([5]).

The question is still open for $k = 5$.

Conjecture 1 ([5])**.** *There exist b, h such that every 5-connected graph G with at least b vertices can be contracted to a 5-connected graph H such that $0 < |V(G)| - |V(H)| < h$.*

The icosahedron shows that $b \geqslant 13$. A k-connected graph which can not be reduced to a smaller k-connected graph by contracting or deleting any number of edges is said to be a *minor minimally k-connected graph*. So, in order to deal with Conjecture 1, we must find all the minor minimally 5-connected graphs. From the graph minor theorem, it follows that there are only finitely many minor minimally 5-connected graphs. Determining the minor minimally 5-connected graphs should be a hard task, G. Fijavž posted the following conjecture in [3].

Conjecture 2 ([3])**.** *Every 5-connected graph contains a minor which is isomorphic to one of the graphs K_6, $K_{2,2,2,1}$, $C_5 * \bar{K}_3$, I, $\bar{I}$ or G_0.*

Here K_6 is the complete graph on six vertices, the Turan graph $K_{2,2,2,1}$ is obtained from the complete graph on seven vertices by deleting three independent edges, $C_5 * \bar{K}_3$ is obtained from the cycle C_5 by adding three new vertices and making them adjacent to all vertices of C_5. Denote the icosahedron by I and $\bar{I}$ is the graph obtained from I by replacing the edges of a cycle $abcdea$ induced by the neighborhood of some vertex with the edges of the cycle $abceda$. G_0 is the graph obtained from the icosahedron by deleting a vertex w, replacing the edge ab of the cycle $abcdea$ induced by the neighborhood of w with the two edges ac and ad, and, finally, identifying b and e.

The statement is true when restricted to minor minimally 5-connected projective graphs. It is true for all graphs on at most 10 vertices and all 5-regular graphs on at most 12 vertices (see [3]).

Let G be a graph with $\kappa(G)=k$. A separating set of G with cardinality k is called a smallest separator. Let G be a graph with $\kappa(G)=5$, T be a smallest separator of G. We say that T is quasi-trivial if $T=N(x)$ for some $x\in V(G)$. We call a graph with $\kappa(G)=5$ a super 5-connected graph if every smallest separator set is quasi-trivial. Further, a graph G with $\kappa(G)=5$ is called essentially 6-connected if for every smallest separator T, $G-T$ has exactly two components and one of them is an isolated vertex. In [5], M. Kriesell characterized a special kind of minor minimally 5-connected graphs as follows.

Theorem A ([5]). *Let G be a minor minimally 5-connected graph. If G is essentially 6-connected, then G has at most 12 vertices.*

In this paper, we show the following two theorems.

Theorem 1. *Let G be a 5-connected graph which contains a vertex that is not contained in any triangles, then G has a contractible subgraph.*

Theorem 2. *Let G be a minor minimally 5-connected graph, then every vertex of G is contained in some triangle.*

Obviously, Theorem 2 is just a corollary of Theorem 1.

2. Terminology

All graphs considered here are supposed to be finite, simple and undirected.

For terms not defined here we refer the reader to [2]. Let $G=(V(G),E(G))$ be a graph, where $V(G)$ denote the vertex set and $E(G)$ the edge set. Let $e(G)=|E(G)|$ and $\kappa(G)$ denotes the vertex connectivity of G. An edge joining the vertices x and y will be written as xy. For $x\in F\subseteq V(G)$, we define $N_G(x)=\{y\colon xy\in E(G)\}$, $N_G(F)=\bigcup_{y\in F}N_G(y)-F$. By $d_G(x)=|N_G(x)|$ we denote the degree of x and $V_k(G)$ stands for the set of vertices with degree k. For $A\subseteq V(G)$, $G[A]$ denotes the subgraph induced by A and $G-A$ denotes the graph obtained from G by deleting the vertices of A together with the edges incident with them. A set $T\subseteq V(G)$ is called a separating set of a connected graph G, if $G-T$ has at least two connected components. A separating set with $\kappa(G)$ vertices is called a k-separator. Let G be a k-connected non-complete graph, T be a k-separator. The union of at least one but not of all the components of $G-T$ is called a T-fragment. Let F be a T-fragment. Then, $\bar{F}=V(G)-(F\cup T)\neq\emptyset$, and $\bar{F}$ is also a T-fragment and $N_G(F)=T=N_G(\bar{F})$. The set of all k-separators of G will be denoted by $\mathcal{T}_G$. For $N_G(x)$, $d_G(x)$, $N_G(F)$ and $\mathcal{T}_G$, we often omit the index G if it is clear from the context.

Moreover, for contraction critical k-connected graph, we need the following notations. For a graph G, let $\mathcal{H}$ be a non-empty set of subsets of $E(G)$. An *$\mathcal{H}$-fragment* of G is a T-fragment of G for any $T \in \mathcal{T}_G$ such that there is an $H \in \mathcal{H}$ with $H \subseteq T$. An inclusion-minimal $\mathcal{H}$-fragment of G is called an *$\mathcal{H}$-end* and one of the least vertex numbers is an *$\mathcal{H}$-atom*. The following properties of fragments are well known (for the proof see [6]), we will use them without any further reference.

Let $T, T' \in \mathcal{T}_G$, and F, F' be the T, T'-fragment of G, respectively. If $F \cap F' \neq \emptyset$, then

$$(1) \qquad |F \cap T'| \geqslant |\overline{F'} \cap T|, \quad |F' \cap T| \geqslant |\bar{F} \cap T'|.$$

If $F \cap F' \neq \emptyset \neq \bar{F} \cap \overline{F'}$, then both $F \cap F'$ and $\bar{F} \cap \overline{F'}$ are fragments of G, and $N(F \cap F') = (F' \cap T) \cup (T' \cap T) \cup (F \cap T')$. If $F \cap F' \neq \emptyset$ and $F \cap F'$ is not a fragment of G, then $\bar{F} \cap \overline{F'} = \emptyset$ and

$$(2) \qquad |F \cap T'| > |\overline{F'} \cap T|, \quad |F' \cap T| > |\bar{F} \cap T'|.$$

Also, by definition, the two endvertices of an edge which is not k-contractible are contained in some k-separator. For an edge e of G, a fragment A of G is said to be a *fragment with respect to* e if $V(e) \subseteq N(A)$.

3. Some lemmas

Lemma 1 ([4]). *Let A be a fragment of cardinality 2 in a contraction critically 5-connected graph, and let $t_1 \neq t_2$ in $N(A)$ be such that $|N(t_1) \cap A| = |N(t_2) \cap A| = 1$. Then, one of t_1, t_2 has a neighbor of degree 5, say t_3, in $N(A) - \{t_1, t_2\}$ and $A \subseteq N(t_3)$.*

Lemma 2 ([1]). *Let G be a contraction critically 5-connected graph. Let A be a fragment with $x \in N(A)$ such that $|A| \geqslant 3$ and $|\bar{A}| \geqslant 2$. If $|N(x) \cap A| = 1$, then there exists a vertex $y \in N(x) \cap N(A) \cap V_5(G)$ such that $N(x) \cap A \subseteq N(y) \cap A$ and $|N(y) \cap A| \geqslant 2$.*

Here we call y is *an admissible vertex* of (x, A).

Lemma 3. *Let G be a contraction critically 5-connected graph, and $A = \{x, y\}$ a fragment of G. Let B be a fragment with respect to xz, where $z \in N(A)$. If $N(A) - \{z\} \subseteq N(y)$, then $A \subseteq N(B)$.*

P r o o f. Assume that $A \nsubseteq N(B)$, we may let $y \in A \cap B$. Then we can see that $\bar{B} \cap N(A) = \emptyset$, since $N(A) - \{z\} \subseteq N(y)$. Further, it can be seen that $\bar{B} \cap A = \emptyset$ since $|A| = 2$. On the other hand, the facts that $A \cap N(B) \neq \emptyset$ and $\bar{B} \cap N(A) = \emptyset$ show that $\bar{B} \cap \bar{A} = \emptyset$. It follows that $\bar{B} = \emptyset$, a contradiction. $\square$

Lemma 4 ([7]). *Let G be a contraction critically 5-connected graph and x is a vertex of G which does not contained in any triangles. Let $\mathcal{H} = \{xy\colon\ y \in N(x)\}$, then every $\mathcal{H}$-end has cardinality* 2.

4. Proof of Theorem 1

If G has some contractible edges, then we are done. So we may assume that G is contraction critically 5-connected. Suppose $x \in V(G)$ and x is not contained in any triangle. Let $\mathcal{H} = \{xy\colon\ y \in N(x)\}$ and A be an $\mathcal{H}$-atom. By Lemma 4, we have $|A| = 2$. Let $A = \{a, b\}$, $N(A) = \{x, y, w_1, w_2, w_3\}$ and $xy \in E(G)$. So we may assume that $ax \in E(G)$, then, as x is not contained in any triangle, we have $N(x) \cap A = \{a\}$, $N(y) \cap A = \{b\}$ and $N(A) - \{x, y\} \subseteq N(a) \cap N(b)$. By Lemma 1 and the fact that x is not contained in any triangle, we may assume $d(w_1) = 5$ and $w_1y \in E(G)$. Let C be a fragment with respect to xa, then by Lemma 3, we have $A \subseteq N(C)$. Further, as x is not contained in any triangle, we have $|C \cap N(A)| = |\bar{C} \cap N(A)| = 2$. We may assume that $C \cap N(A) = \{y, w_1\}$, $\bar{C} \cap N(A) = \{w_2, w_3\}$. Thus there is no edge connecting the vertex set $\{w_2, w_3\}$ to the set $\{x, y, w_1\}$.

Let $G_1 = G/\{aw_2, bw_3\}$ and w_2', w_3' be the new vertices got by contracting aw_2, bw_3, respectively. Next we will show that G_1 is 5-connected.

Claim 1. $\delta(G_1) \geqslant 5$.

Proof. By the fact that there is no edge connecting the vertex set $\{w_2, w_3\}$ to the set $\{x, y, w_1\}$, we can see that for any $t \in V(G_1) - \{w_2', w_3'\}$, $d_{G_1}(t) = d_G(t) \geqslant 5$. Further, for w_2' and w_3', we find that $\{x, w_1, w_3'\} \subseteq N(w_2')$ and $\{y, w_1, w_2'\} \subseteq N(w_3')$. On the other hand, we see that, in G, both w_2 and w_3 have at least two neighbors in $\bar{A}$. It follows that $d_{G_1}(w_2') \geqslant 5$ and $d_{G_1}(w_3') \geqslant 5$. Hence Claim 1 holds. □

Claim 2. $\kappa(G_1) \geqslant 4$.

Proof. Assume that $\kappa(G_1) \leqslant 3$ and let T' be a separator of cardinality 3. Then, obviously, by the fact $\kappa(G) = 5$, we have $\{w_2', w_3'\} \subseteq T'$. Let B' be a T'-fragment in G_1. Then, as $\delta(G_1) \geqslant 5$, we have $|B'| \geqslant 3$ and $|\overline{B'}| \geqslant 3$. Let T be the corresponding original state of T' in G and B be the corresponding original state of B' in G. Clearly, $|T| = 5$, $\{a, w_2, b, w_3\} \subseteq T$ and, further, we can see that $B = B'$ and $\bar{B} = \overline{B'}$. It follows that $|N(b) \cap T| \geqslant 3$. This implies that $|N(b) \cap B| = |N(b) \cap \bar{B}| = 1$. We may assume that $N(b) \cap B = \{w_1\}$, then $N(b) \cap \bar{B} = \{y\}$, which is a contradiction, as $w_1y \in E(G)$. Thus we have $\kappa(G_1) \geqslant 4$. □

Claim 3. If $\kappa(G_1) = 4$, then $\{w_2', w_3'\}$ is contained in every smallest separating set.

P r o o f. Suppose $\kappa(G_1) = 4$ and let T' be a separator of cardinality 4, and B' be a T'-fragment in G_1. Then, as $\delta(G_1) \geqslant 5$, we have $|B'| \geqslant 2$, $|\overline{B'}| \geqslant 2$. Let T be the corresponding original state of T' in G and B be the corresponding original state of B' in G. Hence B is a fragment of T. Further, as $\kappa(G) = 5$, $\{w_2', w_3'\} \cap T' \neq \emptyset$. If $\{w_2', w_3'\} \nsubseteq T'$, then we distinguish two cases according to the position of w_2'.

Subcase 3.1. $w_2' \in T'$ and $w_3' \in B'$.

Clearly, $|T| = 5$, $\{a, w_2\} \subseteq T$, $\{b, w_3\} \subseteq B$ and $|B| \geqslant 3$. Further, B and $\bar{B} = \overline{B'}$ are also the fragments of T. As $b \in B$ and, for $i = 1, 2, 3$, $w_i \in N(b)$, we can see that $N(a) \cap \bar{B} = \{x\}$ and $y \in T$. If $|\bar{B}| = 2$, then, obviously, x is contained in some triangle, a contradiction.

So we may assume that $|\bar{B}| \geqslant 3$, thus, again by Lemma 2, there is an admissible vertex t of $(a, \bar{B})$ and x is contained in some triangle, a contradiction.

Subcase 3.2. $w_3' \in T'$ and $w_2' \in B'$.

Similar to Subcase 3.1, we can see that $|T| = 5$, $\{b, w_3\} \subseteq T$, $\{a, w_2\} \subseteq B$ and $|B| \geqslant 3$. Further, we have $\bar{B} = \overline{B'}$. As $a \in B$ and, for $i = 1, 2, 3$, $w_i \in N(a)$, we can see that $N(b) \cap \bar{B} = \{y\}$ and $x \in T$. Thus $\{x, w_1\} \subseteq T$. If $|\bar{B}| = 2$, then let $\bar{B} = \{y, t\}$. As x is not contained in any triangles, $xt \notin E(G)$. It follows that $d(t) \leqslant 4$, a contradiction.

So we may assume that $|\bar{B}| \geqslant 3$. Now focusing on A and B, we find that $a \in A \cap B$, $b \in A \cap T$, $w_2 \in N(A) \cap B$, $\{x, w_1, w_3\} \subseteq T \cap N(A)$ and $y \in N(A) \cap \bar{B}$.

Clearly, $\bar{B} \cap A = \emptyset$ since $|A| = 2$. Now as $N(A) \cap \bar{B} = \{y\}$ and $|\bar{B}| \geqslant 3$, we can see that $\bar{B} \cap \bar{A} \neq \emptyset$ and $|\bar{B} \cap \bar{A}| \geqslant 2$. Next, by Lemma 2, there is an admissible vertex of $(b, \bar{B})$. It must be w_1, as $w_3 y \notin E(G)$ and x is not contained in any triangle. So $|\bar{B} \cap N(w_1)| \geqslant 2$. Similarly, as $A \cap B = \{a\}$, $N(A) \cap B = \{w_2\}$ and $|B| \geqslant 3$, we have $B \cap \bar{A} \neq \emptyset$. So $B \cap \bar{A}$ is a fragment and $|N(w_1) \cap (B \cap \bar{A})| = 1$ and $N(w_1) \cap N(B \cap \bar{A}) = \emptyset$. Thus, by Lemma 2, we have $|B \cap \bar{A}| \leqslant 2$.

If $|B \cap \bar{A}| = 1$, let $B \cap \bar{A} = \{t\}$, then we have $|B| = 3$ and $\{w_1, w_2, w_3, x\} \subseteq N(t)$. Now focusing on B and C, we find that $a \in B \cap N(C)$, $\{b, x\} \subseteq N(B) \cap N(C)$, $w_1 \in C \cap N(B)$, $y \in C \cap \bar{B}$, $w_3 \in \bar{C} \cap N(B)$ and $w_2 \in \bar{C} \cap B$. Now we can see that $|B \cap \bar{C}| \geqslant 2$, since $xw_2 \notin E(G)$. It follows that $B \cap C = \emptyset$, as $|B| = 3$. It follows that $B \cap \bar{C} = \{w_2, t\}$. This is a contradiction, since $w_1 \in N(t)$.

So we may assume that $|B \cap \bar{A}| = 2$. Now, as $|N(w_1) \cap (B \cap \bar{A})| = 1$, we can see that $B \cap \bar{A}$ is connected. Hence, by the fact that x is not contained in any triangles, we can see that $|N(x) \cap (B \cap \bar{A})| = 1$ and, clearly, $N(x) \cap (B \cap \bar{A}) \neq N(w_1) \cap (B \cap \bar{A})$. Now, by Lemma 1, there is a vertex of degree 5 in $N(B \cap \bar{A}) - \{x, w_1\}$ which adjacent to one of $\{x, w_1\}$. On the other hand, $N(w_1) \cap N(B \cap \bar{A}) = \emptyset$. It follows that there is a vertex of degree 5 in $N(B \cap \bar{A}) - \{x, w_1\}$ which is adjacent to x, thus x is contained in some triangle, a contradiction. Thus Claim 3 holds. $\square$

Now we are ready to show that G_1 is 5-connected. For otherwise, let T' be a separator of cardinality 4, B' be a T'-fragment in G_1. Then, as $\delta(G_1) \geqslant 5$, we have $|B'| \geqslant 2$, $|\overline{B'}| \geqslant 2$. Let T be the corresponding original state of T' in G and B be the corresponding original state of B' in G. By Claim 3, we have $\{w_2', w_3'\} \subseteq T'$ and thus $|T| = 6$ and $\{a, b, w_2, w_3\} \subseteq T$. We have $B = B'$ and $\bar{B} = \overline{B'}$.

We first show that $N(b) \cap B \neq \emptyset$, $N(b) \cap \bar{B} \neq \emptyset$. Assume $N(b) \cap B = \emptyset$. Let $T_0 = T - \{b\}$, then T_0 is a smallest separator set of G. Let $B_0 = B$, clearly, B_0 is a fragment of T_0. Further, $\bar{B}_0 = \bar{B} \cup \{b\}$ (obviously, $|\bar{B}_0| \geqslant 3$). Now we have $N(a) \cap B_0 = \{x\}$, $|B_0| = 2$, then, obviously, x is contained in some triangles, a contradiction. So $|B_0| \geqslant 3$; then, by Lemma 2, x is contained in some triangles, a contradiction.

So $N(b) \cap B \neq \emptyset$ and, similarly, $N(b) \cap \bar{B} \neq \emptyset$. Without loss of generality, let $N(b) \cap B = \{w_1\}$ and $N(b) \cap \bar{B} = \{y\}$. This is a contradiction, as $w_1 y \in E(G)$. $\square$

Acknowledgement. The authors would like to thank the referee for his/her valuable suggestions.

References

[1] *K. Ando, C. Qin*: Some structural properties of minimally contraction-critically 5-connected graphs. Discrete Math. *311* (2011), 1084–1097.

[2] *J. A. Bondy, U. S. R. Murty*: Graph Theory with Applications. American Elsevier Publishing, New York, 1976.

[3] *G. Fijavž*: Graph Minors and Connectivity. Ph.D. Thesis. University of Ljubljana, 2001.

[4] *M. Kriesell*: Triangle density and contractibility. Comb. Probab. Comput. *14* (2005), 133–146.

[5] *M. Kriesell*: How to contract an essentially 6-connected graph to a 5-connected graph. Discrete Math. *307* (2007), 494–510.

[6] *W. Mader*: Generalizations of critical connectivity of graphs. Discrete Mathematics *72*, (1988), 267–283, Proceedings of the first Japan conference on graph theory and applications. Hakone, Japan, June 1–5, 1986. (J. Akiyama, Y. Egawa, H. Enomoto, eds.). North-Holland, Amsterdam.

[7] *C. Qin, X. Yuan, J. Su*: Triangles in contraction critical 5-connected graphs. Australas. J. Comb. *33* (2005), 139–146.

[8] *W. T. Tutte*: A theory of 3-connected graphs. Nederl. Akad. Wet., Proc., Ser. A *64* (1961), 441–455.

Authors' addresses: Chengfu Qin, School of Mathematics Science, Guangxi Teachers Education University, 530001, Nanning, Guangxi, P. R. China, e-mail: qtclf@163.com; Xiaofeng Guo, School of Mathematics Science, Xiamen University, 310065, Xiamen, Fujian, P. R. China, e-mail: xfg@xmu.edu.cn; Weihua Yang, Department of Mathematics, Taiyuan University of Technology, 030024, Taiyuan, Shanxi, P. R. China, e-mail: ywh222@163.com.