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Upper and Lower Solutions Method for Darboux Problem for Fractional Order Implicit Impulsive Partial Hyperbolic Differential Equations

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Abstract

In this paper we investigate the existence of solutions for the initial value problems (IVP for short), for a class of implicit impulsive hyperbolic differential equations by using the lower and upper solutions method combined with Schauder's fixed point theorem.

Key words: partial hyperbolic differential equation, fractional order, left-sided mixed, Riemann–Liouville integral, mixed regularized derivative, impulse, upper solution, lower solution, fixed point

2000 Mathematics Subject Classification: 26A33

1 Introduction

The subject of fractional calculus is as old as the differential calculus since, starting from some speculations of G. W. Leibniz (1697) and L. Euler (1730), it has been developed up to nowadays. We can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [17, 19, 24, 25, 27]. There has been a significant development in ordinary and

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partial fractional differential equations in recent years; see the monographs of Abbas et al. [7], Kilbas et al. [20], Miller and Ross [26], Samko et al. [30], the papers of Abbas et al. [1, 2, 3, 4, 6, 8, 9], Agarwal et al. [10], Belarbi et al. [11], Benchohra et al. [12, 13, 15], Diethelm [16], Kilbas and Marzan [21], Mainardi [24], [29], Podlubny et al. [29], Staněk [31], Vityuk [32], Vityuk and Golushkov [33], Vityuk and Mykhailenko [34, 35], and the references therein.

There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Benchohra et al. [14], and Lakshmikantham et al. [22]. Recently some results on the Darboux problem for fractional order impulsive hyperbolic differential equations and inclusions have been obtained by Abbas et al. [1, 4, 5, 6].

The method of upper and lower solutions plays an important role in the investigation of solutions for differential and partial differential equations and inclusions. We refer to the monographs by Benchohra et al. [14], and the papers of Lakshmikantham and Pandit [23], Pandit [28] and the references cited therein. In [3, 4, 5] the authors applied the method of upper and lower solutions for some classes of Darboux problem for hyperbolic fractional order differential equations and inclusions.

In the present article we are concerning by the existence of solutions to fractional order IVP for the system

$$\overline{D}^r_{\theta_k} u(x,y) = f(x,y,u(x,y),\overline{D}^r_{\theta_k} u(x,y)); \quad \text{if } (x,y)\in J_k,\ k=0,\dots,m, \tag{1}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)); \quad \text{if } y\in[0,b],\ k=1,\dots,m, \tag{2}$$

$$\begin{cases} u(x,0)=\varphi(x);\ x\in[0,a],\\ u(0,y)=\psi(y);\ y\in[0,b],\\ \varphi(0)=\psi(0), \end{cases} \tag{3}$$

where $J_0 = [0,x_1]\times[0,b]$, $J_k := (x_k,x_{k+1}]\times[0,b]$; $k=1,\dots,m$, $a,b>0$, $\theta_k=(x_k,0)$; $k=0,\dots,m$, $\overline{D}^r_{\theta_k}$ is the mixed regularized derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $0=x_0<x_1<\cdots<x_m<x_{m+1}=a$, $f\colon J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $J=[0,a]\times[0,b]$, $I_k\colon\mathbb{R}^n\to\mathbb{R}^n$, $k=1,\dots,m$ are given functions, $\varphi\colon[0,a]\to\mathbb{R}^n$ and $\psi\colon[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions. Here $u(x_k^+,y)$ and $u(x_k^-,y)$ denote the right and left limits of $u(x,y)$ at $x=x_k$, respectively.

In this paper we initiate the application of the method of upper and lower solutions for impulsive hyperbolic implicit differential equations. These results are based on Schauder's fixed point theorem [18].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$. As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebegue-integrable functions $w\colon J \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x,y)\|\, dydx.$$

Definition 2.1 [20, 30] Let $\alpha \in (0,\infty)$ and $u \in L^1(J)$. The partial Riemann–Liouville integral of order α of $u(x,y)$ with respect to x is defined by the expression

$$I^\alpha_{0,x} u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y)\, ds,$$
$$\text{for almost all } x \in [0,a] \text{ and almost all } y \in [0,b],$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t}\, dt$; $\varsigma > 0$.

Analogously, we define the integral

$$I^\alpha_{0,y} u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-s)^{\alpha-1} u(x,s)\, ds,$$
$$\text{for almost all } x \in [0,a] \text{ and almost all } y \in [0,b].$$

Definition 2.2 [20, 30] Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Riemann–Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$$(D^\alpha_{0,x} u)(x,y) = \frac{\partial}{\partial x} I^{1-\alpha}_{0,x} u(x,y), \text{ for almost all } x \in [0,a] \text{ and almost all } y \in [0,b].$$

Analogously, we define the derivative

$$(D^\alpha_{0,y} u)(x,y) = \frac{\partial}{\partial y} I^{1-\alpha}_{0,y} u(x,y), \text{ for almost all } x \in [0,a] \text{ and almost all } y \in [0,b].$$

Definition 2.3 [20, 30] Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Caputo fractional derivative of order α of $u(x,y)$ with respect to x is defined by the expression

$${}^cD^\alpha_{0,x} u(x,y) = I^{1-\alpha}_{0,x} \frac{\partial}{\partial x} u(x,y), \text{ for almost all } x \in [0,a] \text{ and almost all } y \in [0,b].$$

Analogously, we define the derivative

$${}^cD^\alpha_{0,y} u(x,y) = I^{1-\alpha}_{0,y} \frac{\partial}{\partial y} u(x,y), \text{ for almost all } x \in [0,a] \text{ and almost all } y \in [0,b].$$

Definition 2.4 [33] Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1(J)$. The left-sided mixed Riemann–Liouville integral of order r of u is defined by

$$(I_\theta^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} u(s, t)\, dt ds.$$

In particular,

$$(I_\theta^\theta u)(x, y) = u(x, y), \qquad (I_\theta^\sigma u)(x, y) = \int_0^x \int_0^y u(s, t)\, dt ds;$$
$$\text{for almost all } (x, y) \in J,$$

where $\sigma = (1, 1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(x, 0) = (I_\theta^r u)(0, y) = 0; \quad x \in [0, a],\ y \in [0, b].$$

Example 2.5 Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1 + \lambda)\Gamma(1 + \omega)}{\Gamma(1 + \lambda + r_1)\Gamma(1 + \omega + r_2)} x^{\lambda + r_1} y^{\omega + r_2}, \quad \text{for almost all } (x, y) \in J.$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.6 [33] Let $r \in (0, 1] \times (0, 1]$ and $u \in L^1(J)$. The mixed fractional Riemann–Liouville derivative of order r of u is defined by the expression $D_\theta^r u(x, y) = (D_{xy}^2 I_\theta^{1-r} u)(x, y)$ and the Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(x, y) = (I_\theta^{1-r} D_{xy}^2 u)(x, y)$.

The case $\sigma = (1, 1)$ is included and we have

$$(D_\theta^\sigma u)(x, y) = ({}^cD_\theta^\sigma u)(x, y) = (D_{xy}^2 u)(x, y), \quad \text{for almost all } (x, y) \in J.$$

Example 2.7 Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1 + \lambda)\Gamma(1 + \omega)}{\Gamma(1 + \lambda - r_1)\Gamma(1 + \omega - r_2)} x^{\lambda - r_1} y^{\omega - r_2}, \quad \text{for almost all } (x, y) \in J.$$

Definition 2.8 [35] For a function $u\colon J \to \mathbb{R}^n$, we set

$$q(x, y) = u(x, y) - u(x, 0) - u(0, y) + u(0, 0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$ of a function $u(x, y)$, we name the function

$$\overline{D}_\theta^r u(x, y) = D_\theta^r q(x, y).$$

The function

$$\overline{D}_{0,x}^{r_1}u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

is called the partial r_1-order regularized derivative of the function $u(x,y)\colon J \to \mathbb{R}^n$ with respect to the variable x. Analogously, we define the derivative

$$\overline{D}_{0,y}^{r_2}u(x,y) = D_{0,y}^{r_2}[u(x,y) - u(x,0)].$$

Let $a_1 \in [0,a]$, $z^+ = (a_1,0) \in J$, $J_z = (a_1,a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I_{z^+}^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)\,dtds,$$

is called the left-sided mixed Riemann–Liouville integral of order r of u.

Definition 2.9 [33]. For $u \in L^1(J_z, \mathbb{R}^n)$ where $D_{xy}^2 u$ is Lebesque integrable on $[x_k, x_{k+1}] \times [0,b]$, $k = 0,\dots,m$, the Caputo fractional-order derivative of order r of u is defined by the expression $({}^cD_{z^+}^r f)(x,y) = (I_{z^+}^{1-r} D_{xy}^2 f)(x,y)$. The Riemann–Liouville fractional-order derivative of order r of u is defined by $(D_{z^+}^r f)(x,y) = (D_{xy}^2 I_{z^+}^{1-r} f)(x,y)$.

Analogously, we define the derivatives

$$\overline{D}_{z^+}^{r}u(x,y) = D_{z^+}^{r}q(x,y),$$

$$\overline{D}_{a_1,x}^{r_1}u(x,y) = D_{a_1,x}^{r_1}[u(x,y) - u(0,y)],$$

and

$$\overline{D}_{a_1,y}^{r_2}u(x,y) = D_{a_1,y}^{r_2}[u(x,y) - u(x,0)].$$

3 Existence of solutions

To define the solutions of problems (1)–(3), we shall consider the space

$$\begin{aligned} PC(J) = \big\{ u\colon J \to \mathbb{R}^n : {} & u \in C(J_k);\ k = 0,1,\dots,m, \text{ and} \\ & \text{there exist } u(x_k^-, y) \text{ and } u(x_k^+, y);\ k = 1,\dots,m, \\ & \text{with } u(x_k^-, y) = u(x_k, y) \text{ for each } y \in [0,b] \big\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x,y)\|.$$

Definition 3.1 A function $u \in PC(J) \bigcap \bigcup_{k=0}^m C^1((x_k, x_{k+1}) \times [0,b])$ such that u, $\overline{D}_{x_k,x}^{r_1}u$, $\overline{D}_{x_k,y}^{r_2}u$, $\overline{D}_{\theta_k}^{r}u$; $k = 0,\dots,m$, are continuous on J_k and $I_{\theta_k}^{1-r}u \in AC(J_k)$ is said to be a solution of (1)–(3) if u satisfies equation (1) on J_k, and conditions (2), (3) are satisfied.

Let $z,\ \bar{z} \in C(J)$ be such that

$$z(x,y) = (z_1(x,y), z_2(x,y), \dots, z_n(x,y)); \quad (x,y) \in J,$$

and

$$\bar{z}(x,y) = (\bar{z}_1(x,y), \bar{z}_2(x,y), \dots, \bar{z}_n(x,y)); \quad (x,y) \in J.$$

The notation $z \leq \bar{z}$ means that

$$z_i(x,y) \leq \bar{z}_i(x,y); \quad i = 1, \dots, n,$$

and

$$\max_{(x,y)\in J} z(x,y) := \left(\max_{(x,y)\in J} z_1(x,y), \max_{(x,y)\in J} z_2(x,y), \dots, \max_{(x,y)\in J} z_n(x,y) \right).$$

Definition 3.2 A function $z \in PC(J) \bigcap \bigcup_{k=0}^{m} C^1((x_k, x_{k+1}) \times [0,b])$ is said to be a lower solution of (1)–(3) if z satisfies

$$\begin{gathered}
\overline{D}_{\theta_k}^{r} z(x,y) \leq f(x,y,z(x,y), \overline{D}_{\theta_k}^{r} z(x,y)), \\
z(x,0) \leq \varphi(x),\ z(0,y) \leq \psi(y) \text{ on } J_k, \\
z(x_k^+, y) \leq z(x_k^-, y) + I_k(z(x_k^-, y)), \ \text{ if } y \in [0,b];\ k = 1, \dots, m, \\
z(x,0) \leq \varphi(x),\ z(0,y) \leq \psi(y) \text{ on } J, \text{ and } z(0,0) \leq \varphi(0).
\end{gathered}$$

The function z is said to be an upper solution of (1)–(3) if the reversed inequalities hold.

For the existence of solutions for the problem (1)–(3) we need the following lemmas

Lemma 3.3 [35] *Let a function $f\colon J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem*

$$\overline{D}_{\theta_0}^{r} u(x,y) = f(x,y,u(x,y), \overline{D}_{\theta_0}^{r} u(x,y)); \quad \text{if } (x,y) \in J := [0,a] \times [0,b], \tag{4}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{5}$$

is equivalent to the equation

$$g(x,y) = f(x,y,\mu(x,y) + I_{\theta_0}^{r} g(x,y), g(x,y)), \tag{6}$$

and if $g \in C(J)$ is the solution of (6)*, then $u(x,y) = \mu(x,y) + I_{\theta_0}^{r} g(x,y)$, where*

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Lemma 3.4 [6] *Let $0 < r_1, r_2 \leq 1$ and let $h\colon J \to \mathbb{R}^n$ be continuous. A function $u \in PC(J)$ is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds; \\ \text{if } (x,y) \in [0,x_1] \times [0,b], \\ \mu(x,y) + \displaystyle\sum_{i=1}^{k} (I_i(u(x_i^-,y)) - I_i(u(x_i^-,0))) \\ + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds \\ + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)\,dtds; \\ \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1, \dots, m, \end{cases} \tag{7}$$

if and only if u is a solution of the fractional IVP

$$^cD_{\theta_k}^r u(x,y) = h(x,y); \quad (x,y) \in J_k,\ k = 0, \dots, m, \tag{8}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)); \quad y \in [0,b],\ k = 1, \dots, m. \tag{9}$$

By Lemmas 3.3 and 3.4, we have

Lemma 3.5 *Let a function $f\colon J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem* (1)–(3) *is equivalent to the problem of the solution of the equation*

$$g(x,y) = f(x,y,\xi(x,y),g(x,y)), \tag{10}$$

where

$$\xi(x,y) = \begin{cases} \mu(x,y) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,dtds; \\ \text{if } (x,y) \in [0,x_1] \times [0,b], \\ \mu(x,y) + \displaystyle\sum_{i=1}^{k} (I_i(u(x_i^-,y)) - I_i(u(x_i^-,0))) \\ + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,dtds \\ + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,dtds; \\ \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1, \dots, m, \end{cases}$$

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

And if $g \in C(J)$ is the solution of (10)*, then $u(x,y) = \xi(x,y)$.*

Further, we present conditions for the existence of solutions of problem (1)–(3).

Theorem 3.6 *Assume*

(H_1) *The function* $f\colon J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *is continuous,*

(H_2) *There exist* v *and* $w\in PC\bigcap C^1((x_k,x_{k+1})\times[0,b])$, $k=0,\dots,m$ *lower and upper solutions for the problem* (1)–(3) *such that* $v\le w$,

(H_3) *The functions* $I_k\colon\mathbb{R}^n\to\mathbb{R}^n$; $k=1,\dots,m$ *are continuous and for each* $y\in[0,b]$, *we have*

$$v(x_k^+,y)\le\min_{u\in[v(x_k^-,y),w(x_k^-,y)]}I_k(u)\le\max_{u\in[v(x_k^-,y),w(x_k^-,y)]}I_k(u)\le w(x_k^+,y),$$
$$k=1,\dots,m.$$

Then the problem (1)–(3) *has at least one solution* u *such that*

$$v(x,y)\le u(x,y)\le w(x,y);\quad \textit{for all } (x,y)\in J.$$

Proof Transform the problem (1)–(3) into a fixed point problem. Consider the following modified problem,

$$\overline{D}^r_{\theta_k}u(x,y)=g(x,y);\quad \text{if } (x,y)\in J_k;\ k=0,\dots,m, \tag{11}$$

$$u(x_k^+,y)=u(x_k^-,y)+I_k(h(x_k^-,y,u(x_k^-,y)));\quad \text{if } y\in[0,b];\ k=1,\dots,m, \tag{12}$$

$$u(x,0)=\varphi(x),\ u(0,y)=\psi(y);\ x\in[0,a],\ y\in[0,b], \tag{13}$$

where $g\in C(J)$ such that for each $(x,y)\in J$

$$g(x,y)=f(x,y,h(x,y,u(x,y)),\overline{D}^r_{\theta_k}h(x,y,u(x,y)));\quad k=0,\dots,m,$$

and

$$h(x,y,u(x,y))=\max\{v(x,y),\min\{u(x,y),w(x,y)\}\}.$$

A solution to (11)–(13) is a fixed point of the operator $N\colon PC(J)\to PC(J)$ defined by,

$$\begin{aligned}N(u)(x,y)&=\mu(x,y)+\sum_{0<x_k<x}(I_k(h(x_k^-,y,u(x_k^-,y)))-I_k(h(x_k^-,0,u(x_k^-,0))))\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dtds\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dtds.\end{aligned}$$

Notice that g is a continuous function, and from (H_2) there exists $M>0$ such that

$$\|g(x,y)\|\le M,\quad \text{for each } (x,y)\in J. \tag{14}$$

Also, by the definition of h and from (H_3) we have

$$v(x_k^+, y) \le I_k(h(x_k, y, u(x_k, y))) \le w(x_k^+, y); \quad y \in [0, b];\ k = 1, \dots, m, \tag{15}$$

Set

$$\eta = \|\mu\|_\infty + 2\sum_{k=1}^{m} \max_{y\in[0,b]} (\|v(x_k^+, y)\|, \|w(x_k^+, y)\|) + \frac{2Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)},$$

and

$$D = \{u \in PC(J) \colon \|u\|_{PC} \le \eta\}.$$

Clearly D is a closed convex subset of $PC(J)$ and that N maps D into D. We shall show that N satisfies the assumptions of Schauder's fixed point theorem [18]. The proof will be given in several steps.

Step 1: *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in D. Then, for each $(x, y) \in J$, we have

$$\begin{aligned}
&\|N(u_n)(x, y) - N(u)(x, y)\| \\
&\le \sum_{k=1}^{m} \Big(\|I_k(h(x_k^-, y, u_n(x_k^-, y)))) - I_k(h(x_k^-, y, u(x_k^-, y)))\| \\
&+ \|I_k(h(x_k^-, 0, u_n(x_k^-, 0))) - I_k(h(x_k^-, 0, u(x_k^-, 0)))\| \Big) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y - t)^{r_2-1} \|g_n(s, t) - g(s, t)\|\, dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1} \|g_n(s, t) - g(s, t)\,\| dtds,
\end{aligned} \tag{16}$$

where $g_n, g \in C(J)$ such that

$$g_n(x, y) = f(x, y, h(x, y, u_n(x, y)), \overline{D}_{\theta_k}^{r} h(x, y, u_n(x, y))); \quad k = 0, \dots, m,$$

and

$$g(x, y) = f(x, y, h(x, y, u(x, y)), \overline{D}_{\theta_k}^{r} h(x, y, u(x, y))); \quad k = 0, \dots, m.$$

Since $u_n \to u$ as $n \to \infty$ and f, h are continuous functions, we get

$$g_n(x, y) \to g(x, y) \text{ as } n \to \infty, \text{ for each } (x, y) \in J.$$

Also $I_k;\ k = 1, \dots, m$ are continuous functions. Hence, (16) gives

$$\|N(u_n) - N(u)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Step 2: *$N(D)$ is bounded.*
This is clear since $N(D) \subset D$ and D is bounded.

Step 3: *$N(D)$ is equicontinuous.*
Let $(\tau_1,y_1),(\tau_2,y_2)\in J$, $\tau_1<\tau_2$ and $y_1<y_2$ and let $u\in D$. Then

$$\begin{aligned}
&\|N(u)(\tau_2,y_2)-N(u)(\tau_1,y_1)\|\leq\|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|\\
&+\sum_{k=1}^{m}\left\|I_k(h(x_k^-,y_1,u(x_k^-,y_1)))-I_k(h(x_k^-,y_2,u(x_k^-,y_2)))\right\|\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\|g(s,t)\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\\
&\times\|g(s,t)\|\,dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dtds,
\end{aligned}$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,h(x,y,u(x,y)),\overline{D}_{\theta_k}^{r}h(x,y,u(x,y)));\quad k=0,\dots,m.$$

Then,

$$\begin{aligned}
&\|N(u)(\tau_2,y_2)-N(u)(\tau_1,y_1)\|\leq\|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|\\
&+\sum_{k=1}^{m}\left\|I_k(h(x_k^-,y_1,u(x_k^-,y_1)))-I_k(h(x_k^-,y_2,u(x_k^-,y_2)))\right\|\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\,dtds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}dtds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]dtds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dtds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|N(u)(\tau_2,y_2)-N(u)(\tau_1,y_1)\| \le \|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\| \\
&+\sum_{k=1}^{m}\left\|I_k(h(x_k^-,y_1,u(x_k^-,y_1)))-I_k(h(x_k^-,y_2,u(x_k^-,y_2)))\right\| \\
&+\frac{2M}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big|2y_2^{r_2}(\tau_2-\tau_1)^{r_1}+2\tau_2^{r_1}(y_2-y_1)^{r_2} \\
&+\tau_1^{r_1}y_1^{r_2}-\tau_2^{r_1}y_2^{r_2}-2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}\Big|.
\end{aligned}$$

Since the functions μ, h and I_k; $k=1,\dots,m$ are continuous, then; for any $\epsilon>0$, there exists $\delta>0$, such that $\max\{\tau_2-\tau_1,y_2-y_1\}<\delta$ implies

$$\|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|<\frac{\epsilon}{3},$$

$$\sum_{k=1}^{m}\left\|I_k(h(x_k^-,y_1,u(x_k^-,y_1)))-I_k(h(x_k^-,y_2,u(x_k^-,y_2)))\right\|<\frac{\epsilon}{3},$$

and

$$\begin{aligned}
&\left|2y_2^{r_2}(\tau_2-\tau_1)^{r_1}+2\tau_2^{r_1}(y_2-y_1)^{r_2}+\tau_1^{r_1}y_1^{r_2}-\tau_2^{r_1}y_2^{r_2}-2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}\right| \\
&\qquad <\frac{\epsilon\Gamma(1+r_1)\Gamma(1+r_2)}{6M}.
\end{aligned}$$

Hence, for any $\epsilon>0$, there exists $\delta>0$, such that $\max\{\tau_2-\tau_1,y_2-y_1\}<\delta$ implies

$$\|N(u)(\tau_2,y_2)-N(u)(\tau_1,y_1)\|<\epsilon.$$

As a consequence of Steps 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $N\colon D\to D$ is continuous and compact. From an application of Schauder's theorem [18], we deduce that N has a fixed point u which is a solution of the problem (11)–(13).

Step 4: *The solution u of* (11)–(13) *satisfies*

$$v(x,y)\le u(x,y)\le w(x,y);\quad \text{for all } (x,y)\in J.$$

Let u be the above solution to (11)–(13). We prove that

$$u(x,y)\le w(x,y)\quad \text{for all } (x,y)\in J.$$

Assume that $u-w$ attains a positive maximum on $[x_k,x_{k+1}]\times[0,b]$ at $(\overline{x}_k,\overline{y})\in[x_k,x_{k+1}]\times[0,b]$ for some $k=0,\dots,m$, that is,

$$(u-w)(\overline{x}_k,\overline{y})=\max\{u(x,y)-w(x,y):(x,y)\in[x_k,x_{k+1}]\times[0,b]\}>0,$$
$$\text{for some } k=0,\dots,m.$$

We distinguish the following cases.

Case 1. If $(\overline{x}_k, \overline{y}) \in (x_k, x_{k+1}) \times [0, b]$ there exists $(x_k^*, y^*) \in (x_k, x_{k+1}) \times [0, b]$ such that

$$[u(x, y^*) - w(x, y^*)] + [u(x_k^*, y) - w(x_k^*, y)] - [u(x_k^*, y^*) - w(x_k^*, y^*)] \leq 0;$$
$$\text{for all } (x, y) \in ([x_k^*, \overline{x}_k] \times \{y^*\}) \cup (\{x_k^*\} \times [y^*, b]), \tag{17}$$

and

$$u(x, y) - w(x, y) > 0; \quad \text{for all } (x, y) \in (x_k^*, \overline{x}_k] \times (y^*, b]. \tag{18}$$

By the definition of h one has

$$\overline{D}^r_{\theta_k^*} u(x, y) = g(x, y); \quad \text{for all } (x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b], \tag{19}$$

where $\theta_k^* = (x_k^*, 0)$ and

$$g(x, y) = f(x, y, w(x, y), \overline{D}^r_{\theta_k^*} w(x, y)); \quad \text{for all } (x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b].$$

An integration of (19) on $[x_k^*, x] \times [y^*, y]$ for each $(x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b]$ yields

$$u(x, y) + u(x_k^*, y^*) - u(x, y^*) - u(x_k^*, y)$$
$$= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k^*}^{x} \int_{y^*}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} g(s, t)\, dt ds. \tag{20}$$

From (20) and using the fact that w is an upper solution to (1)–(3) we get

$$u(x, y) + u(x_k^*, y^*) - u(x, y^*) - u(x_k^*, y) \leq w(x, y) + w(x_k^*, y^*) - w(x, y^*) - w(x_k^*, y),$$

which gives,

$$[u(x, y) - w(x, y)]$$
$$\leq [u(x, y^*) - w(x, y^*)] + [u(x_k^*, y) - w(x_k^*, y)] - [u(x_k^*, y^*) - w(x_k^*, y^*)]. \tag{21}$$

Thus from (17), (18) and (21) we obtain the contradiction

$$0 < [u(x, y) - w(x, y)]$$
$$\leq [u(x, y^*) - w(x, y^*)] + [u(x_k^*, y) - w(x_k^*, y)] - [u(x_k^*, y^*) - w(x_k^*, y^*)] \leq 0;$$
$$\text{for all } (x, y) \in [x_k^*, \overline{x}_k] \times [y^*, b].$$

Case 2. If $\overline{x}_k = x_k,\ k = 1, \ldots, m$, then

$$w(x_k^+, \overline{y}) < I_k(h(x_k^-, u(x_k^-, \overline{y}))) \leq w(x_k^+, \overline{y}),$$

which is a contradiction. Thus

$$u(x, y) \leq w(x, y), \quad \text{for all } (x, y) \in J.$$

Analogously, we can prove that

$$u(x, y) \geq v(x, y), \quad \text{for all } (x, y) \in J.$$

This shows that the problem (11)–(13) has a solution u satisfying $v \leq u \leq w$ which is solution of (1)–(3). □

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