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Skorohod's Spaces

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This paper presents a general concept of Skorohod's spaces. The space of all functions defined on a nonempty set is considered with Skorohod distance. Completeness of such a space is shown. The compacts are described for bounded functions which is closely connected with the construction due by Straf (1970).

Estimation theory as well as data processing and other branches of mathematical statistics naturally require a deeper knowledge on discontinuous functions. Namely, convenient topological spaces of real-valued functions are investigated. A pioneer work was done by Skorohod (1956), who considers the càdlàg space of functions on the interval $[0, 1]$. His idea was generalized by Neuhaus (1971) to the functions on the rectangle $[0, 1]^k$. Straf (1970) developed a general construction for functions on a nonempty set. His overlooking description is followed in our paper.

We will consider a non-empty set T and R^T will be the set of all functions defined on T. The space R^T is naturally equipped with the supremal norm $\|.\|$, i.e. $\|x\| = \sup_{t\in T}|x(t)|$. Let us denote the set of all finite partitions of T by the symbol $\Gamma(T)$ and the set of all bounded functions on T by $B(T)$.

We assume a given set Λ of automorphisms of T such that

$$(\Lambda, \circ, |||.|||) \text{ is a complete normed group, i.e. group with properties:} \qquad (1)$$

- $0 \leq |||\lambda^{-1}||| = |||\lambda||| < +\infty$
- $|||\lambda||| = 0 \Leftrightarrow \lambda = Id_T$
- $|||\lambda \circ \phi||| \leq |||\lambda||| + |||\phi|||$
- If λ_n is a Cauchy sequence in Λ, i.e. $\lim_{n\to+\infty} \sup_{k,l\geq n} |||\lambda_k \circ \lambda_l^{-1}||| = 0$, then there exists an automorphism $\lambda \in \Lambda$ such that $\lim_{n\to+\infty} |||\lambda_k \circ \lambda^{-1}||| = 0$.

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We establish Skorohod's distance on R^T.

Definition 1. *Let x, y be a couple of functions defined on T. The Skorohod's distance of these functions is*

$$d_\Lambda(x,y) = \inf\{\|x - y \circ \lambda\| \vee |||\lambda||| : \lambda \in \Lambda\}. \tag{2}$$

Skorohod's distance induces Skorohod's topology on R^T.

The Skorohod's distance is almost a pseudometric, but reaches infinity. Actually, it is a pseudometric on the space of all bounded functions $B(T)$.

Proposition 1. *Skorohod's distance fulfills*

1. $d_\Lambda(x,y) = d_\Lambda(y,x) \geq 0$,
2. $d_\Lambda(x,x) = 0$,
3. $d_\Lambda(x,y) \leq d_\Lambda(x,z) + d_\Lambda(z,y)$,
4. $|\,\|x\| - \|y\|\,| \leq d_\Lambda(x,y) \leq \|x - y\|$,
5. $d_\Lambda(x,0) = \|x\|$.

Unfortunately, if $d_\Lambda(x,y) = 0$ we cannot conclude $x = y$. The following example presents such a case.

Example 1. *Let $T = \mathscr{R}$ and Λ be the set of all shifts; i.e. $\lambda = Id_R + r$ for some real r. Define the norm $|||.|||$ on Λ by $|||\lambda||| = |\lambda(0)|$.*

Consider the following collection of functions

$$x_r(t) = \begin{cases} 1 & \text{if } t - r \text{ is rational} \\ 0 & \text{if } t - r \text{ is irrational} \end{cases}$$

indexed by reals. One can see that $d_\Lambda(x_r, x_s) = 0$ for any couple of reals r, s.

Fortunately, the space (R^T, d_Λ) is complete and its quotient space $R^T|_{d_\Lambda}$ is a Hausdorff space. Hence, the space $B(T)$ is a complete pseudometric space.

Theorem 1. *The space (R^T, d_Λ) is complete.*

Proof. Let x_n be a Cauchy sequence; i.e. $\lim_{n\to+\infty} \sup_{k,l\geq n} d_\Lambda(x_k, x_l) = 0$.
Without any loss of generality, we may assume $d_\Lambda(x_n, x_{n+1}) < 2^{-n}$.
Accordingly to the definition of Skorohod's distance, we have $\lambda_n \in \Lambda$ such that $|||\lambda_n||| < 2^{-n}$ and $\|x_n \circ \lambda_n - x_{n+1}\| < 2^{-n}$.
Setting $\phi_{n,k} = \lambda_k^{-1} \circ \lambda_{k-1}^{-1} \circ \ldots \lambda_n^{-1}$, we receive a Cauchy sequence of automorphisms, since for $k < l$ we have $|||\phi_{n,k} \circ \phi_{n,l}^{-1}||| = |||\lambda_{k+1} \circ \lambda_{k+2} \circ \ldots \lambda_l||| < \sum_{i=k+1}^{+\infty} 2^{-1} = 2^{-k}$.
But Λ is a complete normed group, therefore there is $\phi_n \in \Lambda$ such that $|||\phi_{n,k} \circ \phi_n^{-1}|||$ is vanishing if k tends to infinity.
Hence, $|||\phi_n||| \leq |||\phi_{n,k}||| + |||\phi_{n,k} \circ \phi_n^{-1}|||$ for any $k > n$.
Consequently, $|||\phi_n||| \leq \lim_{k\to+\infty} |||\phi_{n,k}||| \leq \sum_{i=n}^{+\infty} 2^{-i} = 2^{1-n}$.
Moreover, $\phi_n^{-1} = \lambda_n \circ \phi_{n+1}^{-1}$ and thus $\|x_n \circ \phi_n^{-1} - x_{n+1} \circ \phi_{n+1}^{-1}\| < 2^{-n}$.

Hence, we have a limit $x \in R^T$ such that $x_n \circ \phi_n^{-1}(t) \to x(t)$ uniformly on T. Moreover, $\|x_n \circ \phi_n^{-1} - x\| \leq \sum_{i=n}^{+\infty} \|x_n \circ \phi_n^{-1} - x_{n+1} \circ \phi_{n+1}^{-1}\| \leq 2^{1-n}$. Thus, we have proven $d_\Lambda(x_n, x) \to 0$.

Q.E.D.

We would like to describe compact and Radon probabilities. Define for that reason the modulus $w(x, \Delta, \Lambda) = \inf_{\lambda \in \Lambda} \max_{D \in \Delta} \sup_{t,s \in D}\{|x(\lambda t) - x(s)| \vee |||\lambda|||\}$ for any finite partition $\Delta \in \Gamma(T)$ and $x \in R^T$. But unfortunately, we have solved this question just partially. Our description is valid in the space $(B(T), d_\Lambda)$, only.

Proposition 2. *Let $A \subset B(T)$. Then the following is equivalent:*

1. *The set A is relatively compact in $(B(T), d_\Lambda)$.*
2. *There exists a sequence of finite partitions $\Delta_n \in \Gamma(T)$, Δ_{n+1} is a refinement of Δ_n, such that*

$$\lim_{n \to +\infty} \sup_{x \in A} w(x, \Delta_n, \Lambda) = 0 \text{ and } \sup_{x \in A} \|x\| < +\infty .$$

3. *There exist a totally bounded pseudometric ϱ on T and finite sets M_n of ϱ-equicontinuous functions such*

$$\lim_{n \to +\infty} \sup_{x \in A} \inf_{y \in M_n} d_\Lambda(x, y) = 0 .$$

Proof. 1. Let A be a relatively compact set.
There is a sequence $x_n \in A$ such that $\|x_n\| \to \sup_{x \in A} \|x\|$. Since $\bar{A}$ is a compact set, there exists a convergent subnet $x_{n_\alpha} \to \hat{x} \in \bar{A} \subset B(T)$. Therefore, $|\,\|x_{n_\alpha}\| - \|\hat{x}\|\,| \leq d_\Lambda(x_{n_\alpha}, \hat{x}) \to 0$ and hence, $\|\hat{x}\| = \sup_{x \in A} \|x\| < +\infty$.
For any $\Delta \in \Gamma(T)$, one can select $x_\Delta \in A$ such that

$$2w(x_\Delta, \Delta, \Lambda) \geq \sup_{x \in A} w(x, \Delta, \Lambda) \wedge 1 .$$

Since $\bar{A}$ is compact, there is a convergent subnet x_{Δ_α}, $x_{\Delta_\alpha} \to y \in \bar{A} \subset B(T)$.
Fix $\varepsilon > 0$.
There are λ_α, $\phi_\alpha \in \Lambda$

$$|||\lambda_\alpha||| < d_\Lambda(x_{\Delta_\alpha}, y) + \varepsilon, \; \|x_{\Delta_\alpha} \circ \lambda_\alpha - y\| < d_\Lambda(x_{\Delta_\alpha}, y) + \varepsilon ,$$

$$|||\phi_\alpha||| < w(y, \Delta_\alpha, \Lambda) + \varepsilon \text{ and } \max_{D \in \Delta_\alpha} \sup_{t,s \in D} |y \circ \phi_\alpha(t) - y \circ \phi_\alpha(s)| < w(y, \Delta_\alpha, \Lambda) + \varepsilon .$$

Hence,

$$\sup_{x \in A} w(x, \Delta_\alpha, \Lambda) \wedge 1 \leq 2w(x_{\Delta_\alpha}, \Delta_\alpha, \Lambda) \leq$$

$$\leq 2\Big(|||\lambda_\alpha \circ \phi_\alpha||| \vee \max_{D \in \Delta_\alpha} \sup_{t,s \in D} |x_{\Delta_\alpha} \circ \lambda_\alpha \circ \phi_\alpha(t) - x_{\Delta_\alpha} \circ \lambda_\alpha \circ \lambda_\alpha \circ \varphi_\alpha(s)|\Big) \leq$$

$$\leq 2\Big(\big[d_\Lambda(x_{\Delta_\alpha}, y) + w(y, \Delta_\alpha, \Lambda) + 2\varepsilon\big] \vee$$

$$\vee \Big[\max_{D\in\Delta_\alpha} \sup_{t,s\in D} |y \circ \phi_\alpha(t) - y \circ \phi_\alpha(s)| + 2\|x_{\Delta_\alpha} \circ \lambda_\alpha - y\|\Big]\Big) <$$

$$< 2w(y, \Delta_\alpha, \Lambda) + 4d_\Lambda(x_{\Delta_\alpha}, y) + 6\varepsilon\,.$$

Since y is bounded, we have $\lim_\alpha w(y, \Delta_\alpha, \Lambda) = 0$ and therefore, we may conclude $\lim_{\Delta\in\Gamma(T)} \sup_{x\in A} w(x, \Delta, \Lambda) = 0$.
Hence, one can select $\Delta_n \in \Gamma(T)$ such that $\sup_{x\in A} w(x, \Delta_n, \Lambda) < 2^{-n}$ and Δ_{n+1} is a refinement of Δ_n.

2. Let $\Delta_n \in \Gamma(T)$, Δ_{n+1} be a refinement of Δ_n, $\sup_{x\in A} w(x, \Delta_n, \Lambda) < 2^{-n}$ and $\sup_{x\in A} \|x\| < +\infty$.
We define pseudometric $\varrho(s,t) = \max\{2^{-n} : t = s \bmod \Delta_n\}$. Evidently, ϱ is totally bounded.
Putting $= \sup_{x\in A} \|x\|$, the set $M_n = \{\Sigma_{A\in\Delta_\alpha} \frac{k_A}{n} QI_A : k_A = -n, 1-n, \ldots, n-1, n\}$ fulfills the required condition.

3. Let ϱ be totally bounded pseudometric on T, M_n be finite sets of ϱ-equicontinuous functions with property $\lim_{n\to+\infty} \sup_{x\in A} \inf_{y\in M_n} d_\Lambda(x,y) = 0$.
Hence, A is totally bounded. But the space $(B(T), d_\Lambda)$ is complete pseudometric, therefore A is relatively compact.

Q.E.D.

Theorem 2. *Let μ be a probability on $(B(T), d_\Lambda)$. Then μ is Radon probability if and only if there exists a totally bounded pseudometric ϱ on T such that $\mu(\overline{C(T,\varrho)}) = 1$, where $C(T,\varrho)$ denotes the set of all ϱ-equicontinuous functions.*

Proof. Let μ be a Radon probability.
Hence, $\mu(\bigcup_n K_n) = 1$ for some compacts K_n. Each compact is characterized by a totally bounded pseudometric ϱ_n, $0 \leq \varrho_n \leq 1$.
Putting $\varrho = \Sigma_{n=1}^{+\infty} 2^{-n}\varrho_n$, we receive $\mu(\overline{C(T,\varrho)}) = 1$.
The opposite assertion is straightforward, since $\overline{C(T,\varrho)}$ is a complete separable pseudometric space. Thus each probability on $\overline{C(T,\varrho)}$ is Radon.

Q.E.D.

In the end of the paper, let us explain the connection with Straf's construction. He assumes a nonempty set T and complete normed group of automorphism Λ, too. But, he moreover employs a given collection of finite partitions $\mathscr{D} \subset \Gamma(T)$ which is closed on actions of automorphism from Λ, i.e. $\lambda\Delta \in \mathscr{D}$ whenever $\lambda \in \Lambda$ and $\Delta \in \mathscr{D}$.

He defines the set of simple functions $J(\mathscr{D}) = \{\Sigma_{A\in\Delta} a_A I_A : a_A \text{ is real}, \Delta \in \mathscr{D}\}$. The closure of $J(\mathscr{D})$ in supremal topology is called $D(T, \mathscr{D}, \Lambda)$.

Straf (1970) proved that the space $(D(T, \mathscr{D}, \Lambda), d_\Lambda)$ is complete pseudometric space. This space is separable if and only if there is a sequence of finite partitions Δ_n such that $\lim_{n\to+\infty} w(x, \Delta_n, \Lambda) = 0$ for any $x \in D(T, \mathscr{D}, \Lambda)$. In other words, if and only if there is a totally bounded pseudometric ϱ on T such that $D(T, \mathscr{D}, \Lambda) \subset \overline{C(T,\varrho)}$.

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