Acta Universitatis Carolinae. Mathematica et Physica

O. V. Kucher; A. M. Plichko

The Wiener transformation on the limits of symmetric spaces

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 36 (1995), No. 2, 39--52

Persistent URL: http://dml.cz/dmlcz/702024

The Wiener Transformation on the Limits of Symmetric Spaces

O. V. KUCHER and A. M. PLICHKO

Lvov*)

Received 15. March 1995

By analogy with the known constructions of the spaces M^p, $\mathcal{M}^p$ and V^p, $\mathcal{V}^p$ which are generated by the space L^p, for symmetric function spaces E on a segment and F on the real line we construct the corresponding "limit" spaces M_E, $\mathcal{M}_E$ and spaces V_F, $\mathcal{V}_F$ of bounded F-variation. We prove that V_F, $\mathcal{V}_F$ are complete and investigate the action of the Wiener transformation between the spaces M_E and V_F. In particular, we give conditions under which this operator is bounded, injective and non-strictly singular.

For a complex valued Borel measurable function $x(t)$ on $\mathbb{R}$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt$$

exists, N. Wiener [19] defined the integrated Fourier transformation $y = Wx$ of x as

$$y(s) = \lim_{T\to\infty}\frac{1}{2\pi}\left(\int_{-T}^{-1} + \int_{1}^{T}\right)x(t)\frac{e^{-ist}}{-it}\,dt + \frac{1}{2\pi}\int_{-1}^{1}x(t)\frac{e^{-ist}-1}{-it}\,dt \tag{1}$$

We call W the *Wiener transformation.* N. Wiener has proved that the mean square modulus of the above function $x(t)$ equals quadratic variation of its transformation $y(s)$, i.e.

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{\varepsilon\to+0}\frac{2}{2\varepsilon}\int_{-\infty}^{\infty}|y(s+\varepsilon)-y(s-\varepsilon)|^2\,ds\,. \tag{2}$$

But the sets of functions for which the limits in (2) exist do not form linear spaces. Therefore the following linear spaces have been introduced

$$\mathcal{M}^p = \left\{x : \|x\|_{\mathcal{M}^p} = \overline{\lim_{T\to\infty}}\left(\frac{1}{2T}\int_{-T}^{T}|x(t)|^p\,dt\right)^{1/p} < \infty\right\},$$

$$M^p = \left\{x : \|x\|_{M^p} = \sup_{1\le T<\infty}\left(\frac{1}{2T}\int_{-T}^{T}|x(t)|^p\,dt\right)^{1/p} < \infty\right\},$$

*) Institute of Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, Naukova 3B, 290601 L'viv, Ukraine

$$\mathscr{V}^p = \left\{ y : \|y\|_{\mathscr{V}^p} = \overline{\lim_{\varepsilon \to +0}} \left(\frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |y(t+\varepsilon) - y(t-\varepsilon)|^p \, dt \right)^{1/p} < \infty \right\},$$

$$V^p = \left\{ y : \|y\|_{V^p} = \sup_{0<\varepsilon<1} \left(\frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |y(t+\varepsilon) - y(t-\varepsilon)|^p \, dt \right)^{1/p} < \infty \right\},$$

where $x(t)$, $y(t)$ are measurable functions, $1 < p < \infty$; and the Wiener transformation acts between these spaces naturally. Marcinkiewicz [13] and independently Bohr and Følner [1] showed that the space $\mathscr{M}^p$ is complete. Banach properties of the spaces $\mathscr{M}^p$ and M^p have been studied in detail (see, [4], [9], [10]). The structure of the spaces $\mathscr{V}^p$ and V^p has turned out to be more complicated and now we do not know much about it except for the case $p = 2$. Completeness of the space $\mathscr{V}^p$ has been proved with the help of the theory of helixes in [10]. We do not know whether the proof of completeness of the space V^p, $p \neq 2$ was published anywhere, but as it will be seen below its idea is like the proof for the space $\mathscr{V}^p$. In [10, 3] it has been shown that the Wiener transformation is an isomorphism between $\mathscr{M}^2$ and $\mathscr{V}^2$ and between M^2 and V^2 and also is a bounded operator from $\mathscr{M}^p$ into $\mathscr{V}^q$, $1 < p < 2$, $1/p + 1/p = 1$. The predual space to V^2 is described in [3]. Injectivity of the Wiener transformation from M^p into V^q follows from results of the papers [10, 2], but its injectivity from $\mathscr{M}^p$ into $\mathscr{V}^q$ is unknown [11].

By analogy with the known construction of the spaces M^p and $\mathscr{M}^p$, which are generated by the space $L^p[-1, 1]$, in the paper [6] for every symmetric function space E on a segment, we construct the corresponding "limit" spaces M_E and $\mathscr{M}_E$ on the real line and investigate some of their Banach properties. These investigations has been continued in [8]. We recall some definitions from [6].

Let (Ω, Σ, μ) be a measure space with a positive measure μ. A Banach space E of (classes of) measurable functions on Ω will be called *symmetric* if:

1. $y \in E$ and $|x(\omega)| \leq |y(\omega)|$ for almost all $\omega \in \Omega$ imply $x \in E$ and $\|x\| \leq \|y\|$;
2. $y \in E$ and $d_{|x|}(t) = d_{|y|}(t)$ for all $t > 0$ imply $x \in E$ and $\|x\| = \|y\|$, where $d_{|x|}(t) = \mu\{\omega : |x(\omega)| > t\}$ is the distribution function of $|x(\omega)|$.

The norm $\|\cdot\|$ of a symmetric space E is said to be *absolutely continuous* if for every function $x \in E$ and every decreasing sequence of measurable sets Ω_n with empty intersection $\|x\chi_{\Omega_n}\| \to 0$ as $n \to \infty$, where χ_{Ω_n} is the characteristic function of a subset $\Omega_n \subset \Omega$. Note that a symmetric space with an absolutely continuous norm is rearrangement invariant in the sense of [12]. For a number $T > 0$ denote by ψ_T the linear map of the segment $[-T, T]$ onto $[-1, 1]$ and $\psi_T(-T) = -1$, $\psi_T(T) = 1$. Let E be a symmetric space on $[-1, 1]$ with the normalized Lebesgue measure $\lambda : \lambda([-1, 1]) = 1$. Then all functions $x(\psi_T(t))$, where x runs through E, form a symmetric space E_T on $[-T, T]$ with the norm $\|x(\psi_T(t))\|_T := \|x\|_E$. Every function on the segment $[-T, T]$ we identify with a function on the real line, defining it outside of $[-T, T]$ by zero. Denote by M_E the set of (also classes of) complex measurable functions $x(t)$ on the real line for which $\|x\|_{M_E} =$

$\sup_{T\geq 1} \|x\|_T < \infty$, and by $\mathscr{M}_E$ the set of (classes of also) elements of M_E such that $\|x\|_{\mathscr{M}_E} = \overline{\lim}_{T\to\infty} \|x\|_T < \infty$. In the same way, for a symmetric space F, we introduce the spaces V_F and $\mathscr{V}_F$ of bounded F-variation and investigate the action of the Wiener transformation between the spaces M_E and V_F. In particular, we establish conditions under which this operator is bounded, injective and non-strictly singular.

§1. The space V_F and its completeness

Let F be a complex symmetric space on the real line with absolutely continuous norm $\|\cdot\|$, and $\varphi(\varepsilon) = \|\chi_{[0,\varepsilon]}\|$ be its fundamental function, we may take $\varphi(1) = 1$. Let $\tau_\varepsilon(y) = y(t+\varepsilon)$, $\varepsilon \in \mathbb{R}$ be a translation operator and $\bar{\tau}_\varepsilon(y) := \tau_\varepsilon y - y$. Denote by V_F the space of (classes of) measurable functions $y(t)$ such that $\|y\|_{V_F} = \sup_{0<\varepsilon<1} \varphi^{-1}(\varepsilon)\, \|\bar{\tau}_\varepsilon y\| < \infty$. Obviously, it is a normed space. It is easy to see that

(3) $$\|y\|_{V_F} \leq \sup_{0<\varepsilon<1} \varphi^{-1}(2\varepsilon)\, \|(\tau_\varepsilon - \tau_{-\varepsilon})\, y\| \leq 2\|y\|_{V_F}$$

for every $y \in V_F$. Thus for $F = L^p(\mathbb{R})$ the space V_F is the same as V^p up to an equivalent norm. Our proof of completeness of V_F is similar to the Nelson's proof for the space of functions of finite upper p-variation [16] and to the proof for $\mathscr{V}^p$ in [10] and is based on the theory of helixes [14, 15].

Definition 1. A continuous function $f_{(\cdot)}$ on $\mathbb{R}$ to a Banach space X is a helix if there exists a strongly continuous group of isometries $(U_s : s \in \mathbb{R})$ on the closed linear span $H_f = [f_b - f_a : a, b \in \mathbb{R}] \subset X$ onto itself such that $U_s(f_b - f_a) = f_{b+s} - f_{a+s}$ for any s, a, b. The set $(U_s : s \in \mathbb{R})$ is called the shift group of the helix $f_{(\cdot)}$.

The following theorem is basic for us.

Theorem (Masani [14]). *Let $f_{(\cdot)}$ be a helix in X with shift group (U_s). Then $a_f = \int_0^\infty e^{-s}(f_0 - f_s)\, ds$ (Bochner integral) exists and is in H_f. Moreover, for any a and b*

(4) $$f_b - f_a = \left(U_b - U_a - \int_a^b U_s\, ds\right) a_f.$$

Lemma 1. *Let $y \in V_F$. Then the map $f_s^y = \bar{\tau}_s y$ is a helix in F with shift group $(\tau_s : s \in \mathbb{R})$.*

Proof. (see [10; Lemma 3.2]). Since $y \in V_F$, $f_s^y = \bar{\tau}_s y \in F$ for every fixed s and by the absolute continuity of the norm $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$, then $\|\bar{\tau}_\varepsilon y\| \to 0$ as $\varepsilon \to +0$. It follows that $\|f_{s+\varepsilon}^y - f_s^y\| = \|(\tau_{s+\varepsilon} - \tau_s)\, y\| = \|\bar{\tau}_\varepsilon y\| \to 0$ as $\varepsilon \to +0$ for any s. Hence the function $f_{(\cdot)}^y : \mathbb{R} \to F$ is continuous. By definition of f_s^y, we can show that $\tau_s(f_b^y - f_a^y) = f_{b+s}^y - f_{a+s}^y$ for any s, a, b. Then (τ_s) is a strongly continuous group of isometries of the helix f_s^y.

Corollary 1. *The averaging operator* $Ay = \int_0^\infty e^{-s}\bar{\tau}_s y \, ds$ *acts from the space* V_F *into F, moreover* $Ay \in [(\tau_a - \tau_b)\, y : a, b \in \mathbb{R}] \subset F$.

Lemma 2. $\|Ay\| \le \alpha \|y\|_{V_F}$ *for any element* $y \in V_F$, *where* $\alpha = e(e-1)^{-1} < 2$.

Proof. (see [16; Lemma 4.5 (a)].

$$\|Ay\| = \left\| \int_0^\infty e^{-s}\bar{\tau}_s y \, ds \right\| \le \text{(by [5, p. 65])}$$

$$\le \int_0^\infty e^{-s}\|\bar{\tau}_s y\| \, ds = \sum_{n=0}^\infty \int_n^{n+1} e^{-s}\|\bar{\tau}_s y\| \, ds \le$$

(using that for $s \in [n, n+1]$ we have $\|\bar{\tau}_s y\| = \|((\tau_s - \tau_n) + (\tau_n - \tau_{n-1}) + \ldots + (\tau_1 - 1))\, y\| \le (n+1) \sup\limits_{0<\varepsilon<1} \|\bar{\tau}_\varepsilon y\|$)

$$\le \left(\sum_{n=0}^\infty \int_n^{n+1} e^{-s}(n+1)\, ds \right) \|y\|_{V_F} \le \alpha \|y\|_{V_F}.$$

Lemma 3. *For any* $y \in V_F$ *the element* $Ay \in V_F$ *and* $\|Ay - y\|_{V_F} \le \|Ay\|$.

Proof. (see [16; Lemma 4.5 (b)]). Putting $x = Ay$ by (4) we have $\bar{\tau}_\varepsilon y = \bar{\tau}_\varepsilon x - \int_0^\varepsilon U_s(x)\, ds$, that is $\bar{\tau}_\varepsilon(y - x) = \int_0^\varepsilon x(t+s)\, ds$. Since the function $\varphi(\varepsilon)$ is quasiconvex [5, p. 70], for any $\varepsilon \in (0, 1)$ $\varphi^{-1}(\varepsilon)\, \|\bar{\tau}_\varepsilon(y - x)\| = \varphi^{-1}(\varepsilon)\, \|\int_0^\varepsilon x(t+s)\, ds\| \le \varphi^{-1}(\varepsilon)\, \varepsilon \|x\| \le \|x\|$. It remains to take supremum over $\varepsilon \in (0, 1)$.

Combining Lemmas 2 and 3, we have

Corollary 2. *For any* $y \in V_F$

$$\|Ay\|_{V_F} \le \|Ay\| + \|y\|_{V_F} \le 3\|y\|_{V_F} \quad \text{and} \quad \|y\|_{V_F} \le \|Ay\| + \|Ay\|_{V_F}.$$

Lemma 4. *(a particular case of Theorem 3.4 in [16]). Let* $y(t)$ *be a complex valued measurable function such that for each* $\varepsilon \in (0, 1)$ $y(t + \varepsilon) = y(t)$, $t \in \mathbb{R} \setminus N_\varepsilon$, *where* N_ε *is a Lebesgue-negligible set. Then* $y(t) \equiv c$ *a.e. for some constant c.*

Lemma 5. *(see [16; Lemma 3.5]). Let* $\hat{y}$ *be the equivalence class of functions y in* V_F. *Then* $\hat{y} = \{z : \exists c \in \mathbb{C};\, z(t) = y(t) + c \text{ a.e.}\}$.

Proof. It is sufficient to show that $\hat{0} = \{z : \exists c : z(t) \equiv c \text{ a.e.}\}$. Obviously if $z(t) \equiv c$ a.e., then $z \in \hat{0}$. Let $z \in \hat{0}$. Hence $\|z\|_{V_F} = \sup\limits_{0<\varepsilon<1} \varphi^{-1}(\varepsilon)\, \|\bar{\tau}_\varepsilon z\| = 0$. Thus $\|\bar{\tau}_\varepsilon z\| = 0$ for each $\varepsilon \in (0, 1)$ and $z(t + \varepsilon) = z(t)$ a.e. It follows from Lemma 4 that there exists number c that $z(t) \equiv c$ a.e.

The following theorem is crucial for the proof of completeness of the space V_F.

Theorem 1. *a) The averaging operator A is linear, continuous and injective from* V_F *into itself and from* V_F *into F*;
b) $AV_F = V_F \cap F$.

Proof. It follows from Corollary 1 and Lemma 3 that

$$AV_F \subseteq V_F \cap F\,. \tag{5}$$

By Lemma 5, $Az = 0$ for every $z \in \hat{0}$. Hence A is a one-to-one operator. By Corollary 2 it is bounded from V_F into V_F, and by Lemma 2 from V_F into F. Let us show its injectivity. Let $Az = \hat{0}$. Since $Az \in F$, using Lemma 5 we have $(Az)(t) = 0$ a.e. Thus $\|Az\| = 0$. By Lemma 3 $\|z\|_{V_F} = \|z - Az\|_{V_F} \le \|Az\| = 0$ and a) is proved.

b) Let $x \in V_F \cap F$. In view of (5) we have only to show that there exists an element $y \in V_F$ such that $Ay = x$. Since $x \in F$, x is a locally Lebesgue integrable function [12, p. 118]. Let $\bar{x}(u) = \int_0^u x(t)\,\mathrm{d}t$, $u \in \mathbb{R}$. Then for any u and $\varepsilon > 0$

$$\bar{x}(u + \varepsilon) - \bar{x}(u) = \int_u^{u+\varepsilon} x(t)\,\mathrm{d}t = \int_0^{\varepsilon} x(u + t)\,\mathrm{d}t\,. \tag{6}$$

By [5, p. 65] $\|\int_0^{\varepsilon} x(u + t)\,\mathrm{d}t\| \le \int_0^{\varepsilon} \|x(u + t)\|\,\mathrm{d}t \le \varepsilon\|x\|$. Hence

$$\|\bar{x}\|_{V_l} = \sup_{0<\varepsilon<1} \varphi^{-1}(\varepsilon)\,\|\bar{\tau}_{\varepsilon}\bar{x}\| \le \sup_{0<\varepsilon<1} \varepsilon\varphi^{-1}(\varepsilon)\,\|x\| \le (\text{by [5, p. 70]}) \le \|x\|\,.$$

Thus $\bar{x} \in V_F$ and $y = x - \bar{x} \in V_F$. Now we will show that $Ay = x$. Observe that Lemma 4.4 from [16] is true for x, i.e.

$$(Ax)(u) = x(u) - \int_0^{\infty} \mathrm{e}^{-s} x(u + s)\,\mathrm{d}s \ \text{ a.e.} \tag{7}$$

Then by (6) and Dirichlet's formula

$$(A\bar{x})(u) = -\int_0^{\infty} \mathrm{e}^{-s} \left\{\int_0^{\infty} x(u + t)\,\mathrm{d}t\right\} \mathrm{d}s = \tag{8}$$

$$= -\int_0^{\infty} \left\{\int_t^{\infty} \mathrm{e}^{-s}\,\mathrm{d}s\right\} x(u + t)\,\mathrm{d}t = -\int_0^{\infty} \mathrm{e}^{-t} x(u + t)\,\mathrm{d}t \ \text{ a.e.}$$

Therefore, by (7) and (8), $Ay = Ax - A\bar{x} = x$.

Theorem 2. *The space V_F is complete.*

Proof. Let $(y_n)_1^{\infty}$ be a Cauchy sequence in V_F and $x_n = Ay_n$. By Lemma 2, $\|x_n - x_m\| = \|A(y_n - y_m)\| \le 2\|y_n - y_m\|_{V_F}$ for any n and m, so that (x_n) is a Cauchy sequence in the space F. Since F is complete, (x_n) converges in the norm $\|\cdot\|$ to some element $x \in F$.

We will show that $\|x_n - x\|_{V_F} \to 0$ as $n \to \infty$, that is, for each $\delta > 0$ there exists a number N such that $\|x_n - x\|_{V_F} \le \delta$ as $n > N$. Take the number N such that $\|y_n - y_m\|_{V_F} < \delta/3$ for $n, m > N$. Then, by Corollary 2, $\|x_n - x_m\|_{V_F} < \delta$, hence $\forall \varepsilon \in (0, 1)$ we have $\varphi^{-1}(\varepsilon)\,\|\bar{\tau}_{\varepsilon}(x_n - x_m)\| < \delta$. Fixing n and passing to the limit as $m \to \infty$, we obtain $\varphi^{-1}(\varepsilon)\,\|\bar{\tau}_{\varepsilon}(x_n - x)\| \le \delta$ for every $\varepsilon \in (0, 1)$ i.e.

$\|x_n - x\|_{V_F} \le \delta$. In particular, we have shown that $x_n - x$ and hence x belong to the space V_F.

Then, by Theorem 1 b), $x = Ay$ for some $y \in V_F$. Finally, $\|y_n - y\|_{V_F} \le$ (by Corollary 2) $\le \|A(y_n - y)\| + \|A(y_n - y)\|_{V_F} \le \|x_n - x\| + 3\|x_n - x\|_{V_F} \to 0$ as $n \to \infty$. The theorem is proved.

Denote by $\mathscr{V}_F$ the space of (classes of) measurable functions $y(t)$ on $\mathbb{R}$ for which

$$\|y\|_{\mathscr{V}_F} = \varlimsup_{\varepsilon \to +0} \varphi^{-1}(\varepsilon) \|\bar{\tau}_\varepsilon y\| < \infty .$$

Put $V_F^0 = \{y \in V_F : \varphi^{-1}(\varepsilon) \|\bar{\tau}_\varepsilon y\| \to 0 \text{ as } \varepsilon \to +0\}$.

Proposition 1. *The set V_F^0 is a closed linear subspace of V_F. Hence $\mathscr{V}_F = V_F / V_F^0$.*

Proof. The linearity of the set V_F^0 is obvious. Let us verify that it is closed. Let a sequence $y_n \in V_F^0$ converge to an element $y \in V_F$ as $n \to \infty$. Then

$$\varlimsup_{\varepsilon \to +0} \varphi^{-1}(\varepsilon) \|\bar{\tau}_\varepsilon y\| \le \varlimsup_{\varepsilon \to +0} \varphi^{-1}(\varepsilon) \|\bar{\tau}_\varepsilon (y_n - y)\| + \varlimsup_{\varepsilon \to +0} \varphi^{-1}(\varepsilon) \|\bar{\tau}_\varepsilon y_n\| \le$$

$$\le \sup_{0<\varepsilon<1} \varphi^{-1}(\varepsilon) \|\bar{\tau}_\varepsilon (y_n - y)\| \to 0 \quad \text{as} \quad n \to \infty .$$

Then $y \in V_F^0$.

Using Theorem 2 and Proposition 1 we have

Corollary 3. *The space $\mathscr{V}_F$ is complete.*

Proposition 2. *The space V_F is not separable.*

Proof. Let us consider a continuum power set of characteristic functions of half-intervals $\{\chi_{(a, \infty)} : a \in \mathbb{R}\}$. We will show that for every two real numbers a, b, $\|\chi_{(b, \infty)} - \chi_{(a, \infty)}\|_{V_F} \ge 1$. It suffices to consider the case $a < b$. Then

$$\|\chi_{(b, \infty)} - \chi_{(a, \infty)}\|_{V_F} = \sup_{0<\varepsilon<1} \varphi^{-1}(\varepsilon) \|\chi_{(a, b)}(t + \varepsilon) - \chi_{(a, b)}(t)\|_F \ge$$

$$\ge \sup_{\substack{\varepsilon < b-a \\ 0<\varepsilon<1}} \varphi^{-1}(\varepsilon) \|\chi_{(a-\varepsilon, a)}(t) - \chi_{(b-\varepsilon, b)}(t)\|_F =$$

since the norm of function is equal to the norm of its rearrangement and functions with equal modulus have equal norms, hence

$$= \sup_{\substack{\varepsilon < b-a \\ 0<\varepsilon<1}} \varphi^{-1}(\varepsilon) \|\chi_{(0, 2\varepsilon)}\|_F = \sup_{\substack{\varepsilon < b-a \\ 0<\varepsilon<1}} \varphi^{-1}(\varepsilon) \varphi(2\varepsilon) \ge 1$$

and the result follows.

Let F be a symmetric space on the real line with absolutely continuous norm. Since A is a linear continuous and injective operator from V_F into the separable space F (by Theorem 1), we have the following corollary.

Corollary 4. *The dual V_F^* is weakly* separable. Hence, V_F does not contain non-separable reflexive (and even non-separable weakly compactly generated) subspaces.*

Denote by F_{loc} the class of measurable functions $y(t)$ on $\mathbb{R}$ such that for each compact subset K of $\mathbb{R}$, $y\chi_K \in F$.

Corollary 5. $V_F \subset F_{loc} \subset L^1_{loc}(\mathbb{R})$.

Proof. Let $y \in V_F$, $x = Ay$ and $\bar{x}(u) = \int_0^u x(t)\,dt$. By arguments of part b) of the proof of Theorem 1, $x - \bar{x} \in V_F$ and $A(x - \bar{x}) = x$. Then by Theorem 1 a) and Lemma 5 we have $y(t) = x(t) - \bar{x}(t) + c$ a.e. for some number c. By Theorem 1 b) $x \in F \subseteq F_{loc}$. Since the function $\bar{x}(t)$ is continuous, $\bar{x}(t) \in F_{loc}$. Hence $y(t) = x(t) - \bar{x}(t) + c \in F_{loc}$. The second inclusion is well known [12, p. 118].

§2. Boundedness of the Wiener transformation

First let us recall some definitions and facts of the interpolation theory of linear operators in symmetric spaces [5]. Let $E(\mathbb{R})$ be a symmetric function space on the real line with the norm $\|\cdot\|_1$. The dilation operator $D_T x(t) = x(t/T)$, $T > 0$, acts in this space and for $T \geq 1$ its norm is at most T [5, p. 131]. The lower and upper Boyd indices of the space $E(\mathbb{R})$ are defined by

$$p = \lim_{T \to \infty} (\log T)/\log \|D_T\|_1, \qquad q = \lim_{T \to +0} (\log T)/\log \|D_T\|_1, \text{ respectively.}$$

For a complex measurable function $y(s)$ we denote by $y^*(s)$ its non-increasing rearrangement: $y^*(s) = \inf\{t > 0 : d_{|y|}(t) < s\}, 0 \leq s < \infty$ [5, p. 83] and $y^{**}(s) = s^{-1}\int_0^s y^*(u)\,du$, $0 < s < \infty$ [5, p. 169]. Let $E(0, \infty)$ be a subspace of $E(\mathbb{R})$ which consists of functions supported on $(0, \infty)$. Let ν, μ be real numbers. By $E_{\nu,\mu}(\mathbb{R})$ denote the space of all functions $y(s) \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ such that $\|y\|_{E_{\nu,\mu}} := \|s^\nu y^{**}(s^\mu)\|_1 < \infty$.

Let us consider the ordinary Fourier transform

$$(\mathcal{F}x)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-ist}\, dt$$

and its interpolation in symmetric spaces. Since it is bounded as an operator from $L^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ [18, 7.5, 7.9], for the symmetric space $E(\mathbb{R})$ with the Boyd indices $1 < p \leq q \leq 2$ we may apply the known Krein and Semyonov generalization on the Marcinkiewicz interpolation theorem [5, Theorem 6.12, p. 196]. By formula (6.44) of the book [5, p. 195] (more precisely, by its equivalent formula (6.5) [5, p. 174]) we may find the symmetric space $E_{\nu,\mu}(\mathbb{R})$ such that the Fourier transform is a bounded mapping from $E(\mathbb{R})$ into this space. Namely, putting $p_0 = q_0 = 2$, $p_1 = 1$, $q_1 = \infty$ we found $\mu = (1/p_1 - 1/p_0)/(1/q_1 - 1/q_0) = -1$; $\nu = (1/p_1 q_0 - 1/p_0 q_1)/(1/q_1 - 1/q_0) = -1$ (see [5, p. 195]).

Let us illustrate this in the case $E(\mathbb{R}) = L^p(\mathbb{R})$, $1 < p < 2$. Then the space $E_{\nu,\mu}(\mathbb{R})$ is denoted by $L^{q,p}(\mathbb{R})$ and its norm is $\|x\|_{L^{q,p}(R)} = (q^{-2}(q-1)\int_0^\infty [x^{**}(s)]^p s^{p\,q-1}ds)^{1\,p}$, where $1/p + 1/q = 1$ [5, p. 197]. The Fourier transform being bounded on $L^p(\mathbb{R})$ into $L^{q,p}(\mathbb{R})$. Remark, that the space $L^{q,p}(\mathbb{R})$ is included into $L^{q,q}(\mathbb{R}) = L^q(\mathbb{R})$ [5, p. 197]. Thus it is making the Hausdorff-Young classical theorem more precise.

The following two lemmas will be needed for our next theorem.

Lemma 6. *The fundamental function $\varphi(s)$ of the space $F = E_{-1,-1}(\mathbb{R})$ satisfies the following condition $\varphi(s) \geq s\varphi_{E(R)}(s^{-1})$.*

Proof. Indeed,

$$\varphi(s) = \|\chi_{[0,s]}\|_F = \|t^{-1}\chi^{**}_{[0,s]}(t^{-1})\|_1 = \|t^{-1}\chi_{[1/s,\infty)}(t) + s\chi_{[0,1/s]}(t)\|_1 \geq s\varphi_{E(R)}(s^{-1}).$$

Lemma 7. *Let the space $(E(\mathbb{R}), \|\cdot\|_1)$ has the lower Boyd index $p > 1$ and let E be its subspace consisting of functions supported on the segment $[-1,1]$. Let $0 < \varepsilon < 1$ and $h(t)$ be an arbitrary measurable function such that $|h(t)| \leq \min(\varepsilon,|t|^{-1})$ for any t. Then $h(t)x(t) \in E(\mathbb{R})$ for any function $x \in M_E$ and $\|hx\|_1 \leq K_1\,\varepsilon\|D_{1/\varepsilon}\|_1\|x\|_{M_E}$, where the constant K_1 depends on p only.*

Proof. By the definition of the dilation operator

(9)

$$\|x\chi_{[-T,T]}\|_1 = \|x(Tt/T)\chi_{[-T,T]}(Tt/t)\|_1 \leq \|D_T\|_1\|x(Tt)\chi_{[-1,1]}(t)\|_1 = \|D_T\|_1\|x\|_T.$$

Let χ_0 be the characteristic function of the segment $[-1/\varepsilon, 1/\varepsilon]$ and n_1 be the first integer for which $2^{n_1} > 1/\varepsilon$, let χ_1 be the characteristic function of the set $\{t: 1/\varepsilon \leq |t| < 2^{n_1}\}$, and let χ_n, $n < n_1$ be the characteristic function of $\{t: 2^{n-1} \leq |t| < 2^n\}$. From the definition of the Boyd index it follows that for every $1 < p' < p$ there exists a constant K such that $\|D_T\|_1 \leq KT^{1/p'}$ for every $T \geq 1$ (see [12, p. 133]). Then

$$\|hx\|_1 \leq \|hx\chi_0\|_1 + \sum_{n\geq n_1}\|hx\chi_n\|_1 \leq \varepsilon\|x\chi_0\|_1 + \sum_{n\geq n_1}2^{-n}\|x\chi_n\|_1 \leq$$

$$\leq \varepsilon\|x\chi_0\|_1 + \sum_{n\geq n_1}2^{-n}\|x\chi_{[2^{-n},2^n]}\|_1 \leq \text{(by (9))}$$

$$\leq \varepsilon\|D_{1/\varepsilon}\|_1\|x\|_{1/\varepsilon} + \sum_{n\geq n_1}2^{-n}\|D_{2^n}\|_1\,\|x\|_{2^n} \leq$$

$$\leq \left(\varepsilon\|D_{1/\varepsilon}\|_1 + \sum_{n\geq n_1}2^{-n}\|D_{2^n\varepsilon/\varepsilon}\|_1\right)\|x\|_{M_E} \leq \text{([5, p.132])}$$

$$\leq \left(\varepsilon\|D_{1/\varepsilon}\|_1 + \sum_{n\geq n_1}2^{-n}\|D_{2^n\varepsilon}\|_1\,\|D_{1/\varepsilon}\|_1\right)\|x\|_{M_E} \leq$$

$$\leq \left(\varepsilon + \sum_{n\geq n_1}2^{-n}K(2^n\varepsilon)^{1/p'}\right)\|D_{1/\varepsilon}\|_1\|x\|_{M_E} \leq$$

$$\leq \left(\varepsilon + \frac{K\varepsilon^{1/p'}}{2^{n_1(1-1/p')}} \sum_{k=0}^{\infty} 2^{-k(1-1/p')}\right) \|D_{1/\varepsilon}\|_1 \|x\|_{M_E} \leq$$

$$\leq \left(\varepsilon + K\varepsilon \frac{2^{1-1/p'}}{2^{(1-1/p')}-1}\right) \|D_{1/\varepsilon}\|_1 \|x\|_{M_E} \leq K_1\, \varepsilon \|D_{1/\varepsilon}\|_1 \|x\|_{M_E},$$

where $K_1 = 1 + K\, 2^{1-1/p'}(2^{(1-1/p')} - 1)^{-1}$. The lemma is proved.

Now we will consider the Wiener transformation.

Theorem 3. *Let the space $(E(\mathbb{R}), \|\cdot\|_1)$ have the Boyd indices $1 < p \leq q \leq 2$, and let $E \subset E(\mathbb{R})$ be its subspace consisting of functions supported on the segment $[-1, 1]$, and $F = E_{-1,-1}(\mathbb{R})$. If there exists a constant $b < \infty$ such that for every $T > 1$*

$$\varphi^{-1}_{\bar{E}(\mathbb{R})}(T)\, \|D_T\|_1 < b\,, \tag{19}$$

then the Wiener transformation W defined by (1) is a bounded linear operator from M_E into V_F.

Proof. From Lemma 7 it follows at once that for $x \in M_E$ the function $t^{-1}(t)\chi_{\{s:|s|>1\}}(t) \in E(\mathbb{R})$, hence by the Krein and Semyonov generalization of the Marcinkiewicz interpolation theorem [5, Theorem 6.12, p. 196] its Fourier transformation and therefore the first integral in (1) belongs to F. Next, the function $\frac{e^{-ist}-1}{-it}$ of the variable t is bounded on $[-1, 1]$ for every fixed s, and the restriction of $x \in M_E$ to $[-1, 1]$ belongs to E, hence it belongs to $L^1[-1,1]$, too [12, p.118]. Therefore the second integral in (1) has the ordinary Lebesgue sense.

Note that for any $\varepsilon > 0$ and $y = Wx$

$$y(s+\varepsilon) - y(s-\varepsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\frac{e^{i\varepsilon t} - s^{-i\varepsilon t}}{it} e^{-ist}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\frac{2\sin(\varepsilon t)}{t} e^{-ist}dt.$$

Thus $y(s+\varepsilon) - y(s-\varepsilon) = \mathscr{F}(x(t)h_\varepsilon(t))$, where $h_\varepsilon(t) = \sqrt{\frac{\pi}{2}}\sin(\varepsilon t)/t$. By Lemma 7, $\|xh_\varepsilon\|_1 \leq K_1\sqrt{\frac{\pi}{2}}\varepsilon\|D_{1/\varepsilon}\|_1\|x\|_{M_E}$, and by the Krein-Semyonov interpolation theorem [5, p.196] we have $\|\mathscr{F}(xh_\varepsilon\|_F \leq C\|xh_\varepsilon\|_E \leq K_1\, C\sqrt{\frac{\pi}{2}}\varepsilon\|D_{1/\varepsilon}\|_1\, \|x\|_{M_E}$, where the constant C depends only on the space $E(\mathbb{R})$. Putting $C' = K_1\, C\sqrt{\frac{\pi}{2}}$ we have

$$\varphi^{-1}(2\varepsilon)\|y(s+\varepsilon) - y(s-\varepsilon)\|_F \leq C'\varphi^{-1}(\varepsilon)\|D_{1/\varepsilon}\|_1\|x\|_{M_E} \leq \text{(by Lemma 6)}$$

$$\leq C'\varphi^{-1}_{\bar{E}(\mathbb{R})}(\varepsilon^{-1})\|D_{1/\varepsilon}\|_1\|x\|_{M_E}.$$

Taking the supremum over $0 < \varepsilon < 1$, the result follows.

Remark. Equalities (4.20) and (4.21) [5, p.134] imply that for example a Lorentz L_ϕ and Marcinkiewicz M_ϕ spaces with a semimultiplicative fundamental function ϕ [5, p.74] and obviously $L^p(\mathbb{R})$ satisfy condition (10) of Theorem 3. Note that in this case for the space $E = L^p$, L_ϕ or M_ϕ the space F is continuously embedded in the space E^* and the Wiener transformation is a bounded linear operator from M_E into V_{E^*}.

Denote by I_E the (closed [6]) subspace of functions $x \in M_E$ for which $\lim_{T\to\infty} \|x\|_T = 0$.

Lemma 8. *Under the conditions of Theorem 3, the Wiener transformation maps the subspace I_E into the subspace V_F^0.*

Proof. It is necessary to show that if $x \in I_E$ then $Wx \in V_F^0$. Suppose that the function $x(t)$ has a bounded support on $[-T, T]$. Then $x \in E(\mathbb{R})$ and for a sufficiently small ε on $[-T, T]$ we have $|x(t)\frac{\sin(\varepsilon t)}{t}| \le \varepsilon |x(t)|$. Let $h_\varepsilon(t)$ be the function from Theorem 3. Then for a sufficiently small ε, $\|xh_\varepsilon\|_1 \le \varepsilon\|x\|_1 \le$(by (9)) $\le \varepsilon\|D_T\|_1\|x\|_T \le \varepsilon\,\|x\|_{M_E}$. Hence, there exists a constant $a = Tb$ independent on x and ε such that for $y = Wx$ we have

$$\varphi^{-1}(2\varepsilon)\,\|y(s+\varepsilon) - y(s-\varepsilon)\|_F \le T\varphi^{-1}(\varepsilon)\,\varepsilon\|x\|_{M_E} \le \text{(by Lemma 6)} \le$$

$\le T\varphi_{E(R)}^{-1}(\varepsilon^{-1})\|x\|_{M_E} \le$ (by (10)) $\le Tb(\|D_{1/\varepsilon}\|_1)^{-1}\|x\|_{M_E} \le$ (by the definition of the Boyd index) $\le a\varepsilon^{1/p}\|x\|_{M_E} \to 0$ as $\varepsilon \to 0$, because $p < \infty$.

Let now $x \in I_E$ be an arbitrary function. Then for any $\delta > 0$ there exists a number $T > 1$ such that $\|x - x\chi_{[-T,T]}\| < \delta$. Put $y = Wx$ and $y_T = W(x\chi_{[-T,T]})$. Hence we have,

$$\overline{\lim_{\varepsilon\to+0}}\,\varphi^{-1}(2\,\varepsilon)\|(\tau_\varepsilon - \tau_{-\varepsilon})y\|_F \le \sup_{0<\varepsilon<1}\varphi^{-1}(2\,\varepsilon)\|(\tau_\varepsilon - \tau_{-\varepsilon})(y - y_T)\|_F +$$

$$+ \lim_{\varepsilon\to+0}\varphi^{-1}(2\,\varepsilon)\|(\tau_\varepsilon - \tau_{-\varepsilon})\,y_T\|_F \le$$

$$\le 2\|W(x - x\chi_{[-T,T]})\|_{V_F} < 2\|W\|\delta.$$

Since δ is arbitrary, $y \in V_F^0$. The lemma is proved.

Theorem 4. *Under the conditions of Theorem 3, the Wiener transformation W is a bounded linear operator from $\mathcal{M}_E$ into $\mathcal{V}_F$.*

In the case of the spaces $\mathcal{M}^p$ and $\mathcal{V}^q$ Theorem 4 has been proved in [10] and that proof uses the Tauberian type theorem which is certain version of the equality (2). Our proof will be based on Theorem 3 that states some general results of the interpolation theory of linear operators in symmetric spaces.

Proof of Theorem 4. The correctness of the mapping W follows from Lemma 8. Let $\hat{x} \in \mathcal{M}_E$ and $x \in \hat{x}$, $x \in M_E$, $a > 1$ and $\|x\|_{M_E} \le a\|\hat{x}\|_{\mathcal{M}_E}$. As it has been proved in Theorem 3, there exists the independent on ε constant C_1 such that for $y = Wx$ we have $\varphi^{-1}(2\varepsilon)\|y(s+\varepsilon) - y(s-\varepsilon)\|_F \le C_1\,\|x\|_{M_E} \le aC_1\|\hat{x}\|$. Since $a > 1$ is arbitrary, passing to the limit as $\varepsilon \to 0$, we obtain the required assertion.

As has been stated above, if $E(\mathbb{R}) = L^p(\mathbb{R})$, then $F = L^{q,p}(\mathbb{R})$, where $1/p + 1/q = 1$. The following corollary makes the known results about boundedness of the Wiener transformation from the spaces $\mathscr{M}^p$, M^p into the spaces $\mathscr{V}^q$, V^q respectively [10, 3] more precise.

Corollary 6. *For $1 < p \leq 2$ the Wiener transformation is a bounded linear operator from M^p into $V_{L^{q,p}(R)}$ and from $\mathscr{M}^p$ into $\mathscr{V}_{L^{q,p}(R)}$.*

Corollary 7. *For $1 < p \leq 2$ the Wiener transformation is a bounded linear operator from M^p into V^q and from $\mathscr{M}^p$ into $\mathscr{V}^q$, $1/p + 1/q = 1$.*

§ 3. Injectivity, non-isomorphism and non-strict singularity of the Wiener transformation

Theorem 5. *Under the conditions of Theorem 3, the Wiener transformation is an injective operator from M_E into V_F, where M_E and V_F are the spaces as in the previous paragraph.*

Proof. Let us consider the space S_∞ of infinitely differentiable on the real line functions which are decreasing at infinity together with all its derivatives more rapidly than an arbitrary power of $1/|t|$. Let S'_∞ be its dual space. As it is known [18, 7.15] the Fourier transform maps S'_∞ onto S'_∞ one-to-one and continuously. Since the space $E(\mathbb{R})$ has the Boyd indices $1 < p, q \leq 2$, for every $1 < r < p$ any every $T \leq 1$, E_T is continuously and injectively imbedded into $L^r[-T, T]$ and the imbedding constants are uniformly bounded. Hence $M_E \subset M^r$. It is known that M^r is continuously and injectively imbedded into $L^r(1/(1 + t^2))$ [10] and $L^r(1/(1 + t^2)) \subset S'_\infty$ [18, p.7.12]. Therefore functions from the space M_E may be considered as distributions, i.e. elements of S'_∞. Since $Wx \in S'_\infty$ and $(Wx)'$ is the Fourier transform of x [3], the identity $\|Wx\|_{V_F} = 0$ (i.e. $Wx \equiv c$ a.e. (by Lemma 5)) implies $(Wx)' = \mathscr{F}x = 0$ and by injectivity of $\mathscr{F}$ we have $x = 0$. The theorem is proved.

Theorem 6. *Let the space $E(\mathbb{R})$ satisfy the assumptions of Theorem 3 and let the Wiener transformation continuously map M_E into V_{E^*}. If the upper Boyd index q of the space E is less than 2, then the Wiener transformation $W\colon I_E \to V^0_{E^*}$ is not an isomorphism.*

Proof. At first show that the Fourier transform is not an isomorphism from E into E^*. The assumption on the Boyd index implies that the space E has the lower r-estimate for some $r < 2$ [12, p.132]. By Proposition 2.b.2. of [12], $2 < p_{E^*} \leq q_{E^*} < \infty$ and hence the space E^* has the upper s-estimate for some $s > 2$ and the lower p'-estimate for some $p' < \infty$ [12, p.132]. Next, combining the Theorem 1.f.7 and Proposition 1.f.3 of [12], we get that the space E^* is of type 2. It remains to apply Corollary 6 from [7]. Thus the Fourier transform is a strictly singular operator from E into E^*. Hence there exists a sequence of numbers $\delta_n \to 0$

and a sequence $x_n \in E_{\delta_n}$, $\|x_n\| = 1$, supp $x_n \subset (-\delta_n, \delta_n)$ such that $\|\mathcal{F}(x_n)\|_{E^*} \to 0$ as $n \to \infty$. Since $(\sin \varepsilon t)/(\varepsilon t)$ tends to 1 as $t \to 0$ uniformly for $\varepsilon \in (0, 1)$, $\|x_n - x_n(\sin \varepsilon t)/(\varepsilon t)\|_E$ tends to zero as $n \to \infty$ uniformly for ε. Therefore, uniformly by ε $\|\mathcal{F}(x_n(\sin \varepsilon t)/(\varepsilon t)\|_{E^*} \to 0$ as $n \to \infty$. Thus,

$$\varphi^{-1}(2\varepsilon)\|W(x_n(s+\varepsilon)) - W(x_n(s-\varepsilon))\|_{E^*} \le \sqrt{\frac{\pi}{2}}\,\|\mathcal{F}\left(x_n\frac{\sin(\varepsilon t)}{\varepsilon t}\right)\|_{E^*} \to 0$$

as $n \to \infty$. Hence $\|Wx_n\|_{V_{E^*}} \to 0$ as $n \to \infty$.

Corollary 8. *Let $1 < p < 2, 1/p + 1/q = 1$. Then the Wiener transformation $W: M^p \to V^q$ is not an isomorphism.*

We recall that a bounded linear operator U acting from a Banach space X into a Banach space Y is called strictly singular if the restriction $U|E$ of U to every infinite dimensional subspace E of X is not an isomorphism. As in § 2 by I_E we denote the subspace of functions $x \in M_E$ for which $\lim\limits_{T\to\infty} \|x\|_T = 0$. Let the space $E(\mathbb{R})$ have the Boyd indices $1 < p \le q \le 2$, E, F are the spaces constructed by $E(\mathbb{R})$ in § 2.

Theorem 7. *Under the conditions of Theorem 3, the Wiener transformation $W: I_E \to V_F^0$ is not strictly singular (moreover, non-compact).*

Proof. By Lemma 8, the Wiener transformation maps the space I_E into the space V_F^0. Let x_n be the characteristic function of the interval $(2^{n-1}, 2^n)$. In [6, the proof of Corollary 7] it has been shown that the subspaces $E^n = \{x\chi_{\{t: 2^{n-1} \le |t| < 2^n\}} : x \in I_E\} \subset I_E$ form the c_0-decomposition provided $q < \infty$. Then to show equivalence of the sequence x_n to the standard basis of c_0, it is sufficient to show that (x_n) is bounded and separate from zero. By Lemma following Proposition 7 of [6] and by Lemma 1 following Corollary 7 from [6] we have

$$\forall S, T \quad S < T \quad (S/T)\|y\|_S \le \|y\|_T \le (S/T)^{1/q}\ \|y\|_S$$

for $y \in E_S$. Then $2^{-1} \le \|x_n\|_{M_E} \le (1/2)^{1/q}$.

According [17], if an operator U is defined on c_0 with the standard basis (x_n), then either there exists an infinite subset $N \subset \mathbb{N}$ such that $U|c_0(N)$ is an isomorphism or $Ux_n \to 0$ as $n \to \infty$. Therefore, to prove the theorem, it suffices to show that $Wx_n \not\to 0$.

As it is well known [5, p.126], $\varphi^{-1}(2\varepsilon)\|y\|_F \ge (2\varepsilon)^{-1}\int_{-\varepsilon}^{\varepsilon} y(s)ds$ for any $0 < \varepsilon < 1$ and every $y \in F$. Then for each n,

$$\varphi^{-1}(2\varepsilon)\left\|\mathcal{F}\left(x_n(t)\frac{\sin(\varepsilon t)}{t}\right)\right\|_F \ge \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x_n(t)\frac{\sin(\varepsilon t)}{t}e^{-ist}dt\right)ds =$$

(both the integrals have finite limits and we can change the order of integration)

$$= \frac{1}{2\varepsilon\sqrt{2\pi}} \left(\int_{2^{n-1}}^{2^n} \frac{\sin(\varepsilon t)}{t} \int_{-\varepsilon}^{\varepsilon} e^{-ist} ds \right) dt =$$

(the second integral is equal to $\int_{-\varepsilon}^{\varepsilon} \cos(st) ds = \frac{\sin(st)}{t} |_{s=-\varepsilon}^{s=\varepsilon} = 2 \sin(\varepsilon t)/t$)

$$= \frac{1}{\varepsilon\sqrt{2\pi}} \int_{2^{n-1}}^{2^n} \frac{\sin^2(\varepsilon t)}{t^2} dt =$$

(putting $u = 2^{-n}t$ and $\varepsilon = 2^{-n}$)

$$= \frac{2^n}{\sqrt{2\pi}} \int_{1/2}^{1} \frac{\sin^2 u}{(2^n u)^2} 2^n du = a,$$

where $a > 0$ is independent of n. Hence, recalling the proof of Theorem 3 and (3), we have

$$\|Wx_n\|_{V_F} \geq \sup_{0<\varepsilon<1} \varphi^{-1}(2\varepsilon) \sqrt{\frac{\pi}{2}} \left\| \mathcal{F}\left(x_n(t) \frac{\sin(\varepsilon t)}{t} \right) \right\|_F \geq a \sqrt{\frac{\pi}{2}}.$$

Corollary 9. *The Wiener transformation is not strictly singular from I_E into V_F^0, from M_E into V_F, and from I^p into V^q, $1/p + 1/q = 1$, $1 < p \leq 2$.*

Corollary 10. *The space V_F^0 contains a (complemented) subspace isomorphic to c_0.*

Unfortnately, we do not know whether the Wiener transformation is injective from $\mathcal{M}_E$ into $\mathcal{V}_F$ and even from $\mathcal{M}^p$ into $\mathcal{V}^q$. The following proposition shows that its injectivity would imply non-strict singularity.

Proposition 3. *Let E be a symmetric reflexive space on a segment, let Y be a Banach space, and let $U : \mathcal{M}_E \to Y$ be a linear continuous injective operator. Then U is an isomorphism on continuum weight subspaces.*

Proof. By [6], $\mathcal{M}_E$ contains a subspace isomorphic tto l_∞/c_0. It is well known (see for example [17]) that l_∞/c_0 contains a subspace isomorphic to $c_0(\Gamma)$, card $\Gamma = c$. Then, by Remark 1 following Theorem 3.4 [17], there exists a subset $\Gamma' \subset \Gamma$ such that card $\Gamma =$ card Γ' and $U|\, c_0(\Gamma')$ is an isomorphism.

References

[1] Bohr, H., Følner, E., *On some typs of function spaces,* Acta Math. **76** (1945), 31–155.

[2] Chen, Y. and Lau, K., *Harmonic analysis on functions with bounded means,* Contemporary Math. **91** (1989), 165–175.

[3] ———, *Wiener transformation on function with bounded averages,* Proc. Amer. Math. Soc. **108** (1990), no. 2, 411–421.

[4] Feichtinger, H. G., *An elementary approach to Wiener's third Tauberian theorem on Euclidean n-space,* Symposia Math. **29** (1987), 267–301.

[5] KREIN, S. G., PETUNIN, YU, I., SEMYONOV, E. M., *Interpolation of linear operators,* Moscow; „Nauka", 1978. (Russian)

[6] KUCHER, O. V., PLICHKO, A. M., *Limits on the real line of symmetric spaces on segments,* Ukr. Math. Jour. **47** (1995), no. 1, 46–55.

[7] ———, *On strict singularity of locally integral operators in ideal Banach spaces,* Questiones Math. (to appear).

[8] KUCHER, O. V., *On Grothendieck property in the space $l_\infty(E)$ and the p-Banach-Saks property in* c_0, Dopovidi AN Ukrainy (to appear). (Ukrainian)

[9] LAU, K., *On the Banach spaces of functions with bounded upper means*, Pacif. J. Math. **91** (1980), 153–172.

[10] LAU, K. and LEE, J., *On generalized harmonic analysis,* Trans. Amer. Math. Soc. **259** (1980), 75–97.

[11] LAU, K., *Extensions of Wiener's Tauberian identity and multiplies on the Marcinkiewicz space,* Trans. Amer. Math. Soc. **277** (1983), 489–506.

[12] LINDENSTRAUSS, J., TZAFRIRI, L., *Classical Banach spaces. II,* Springer, Berlin e.a., 1979.

[13] MARCINKIEWICZ, J., *Une remarque sur les espaces de M. Besicowitch,* C. R. Acad. Sci. Paris **208** (1939), 157–159.

[14] MASANI, K., *On helixes in Banach spaces,* Sänkya Ser. **A38** (1976), 1–27.

[15] ———, *On helixes in Hilbert space II,* Theor. Probability Appl. USSR **17** (1972), 3–20.

[16] NELSON, D. R., *The spaces of functions of finite upper p-variation,* Trans. Amer. Math. Soc. **253** (1979), 171–190.

[17] ROSENTHAL, H. P., *On relatively disjoint families of measures, with some applications to Banach space theory,* Studia Math. **37** (1970), 13–36.

[18] RUDIN, W., *Functional analysis,* Mc Graw-Hill Book Company, New York e.a., 1973.

[19] WIENER, N., *Generalized harmonic Analysis,* Acta Math. **55** (1930), 117–258.