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Zdeněk Frolík and the Descriptive Theory of Sets and Spaces

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The aim of these notes is to pay a tribute to Zdeněk Frolík, the founder of the Winter Schools on Abstract Analysis, with the emphasis on his favourite field – the descriptive theory of topological spaces. He contributed to this area with many basic notions and ideas, and his papers have become classic.

We are not only going to present his main results but also to single out several periods of Frolík's interest in this field and to clarify their mutual relationship. Notions of various analytic spaces have recently found an application to non-separable Banach spaces equipped with the weak topology. The goal of this survey is to draw the attention, explicitely, but also implicitely, to some open problems. We hope it would encourage further research in this area which intersects border lines between set-theory, topology, and analysis.

I met Z. Frolík first in 1972 when he returned from his stay in the United States. I attended his seminar on the measure theory at Charles University in Prague. Those of us who had known him before only due to his scientific reputation were surprised by his sportlike and youthful appearance and the corresponding enthusiasm and attitude to younger participants.

This seminar helped to the scientific growth of a great number of mathematicians, by no means only students. More than one generation of mathematicians in Prague and elsewhere was influenced by "Winter schools on abstract analysis" organized by Z. Frolík. They began as schools on vector measures for the participants of that seminar and took place in Štefanová in Slovakia. They have been transformed into a conference, in fact, with many foreign participants. New results of measure theory, functional analysis, and, in the separate part, of topology, graph theory, and combinatorics were presented. Z. Frolík paid great attention to maintain the informal and working atmosphere of these schools all the time. The problem sessions held during first days of these events. The solutions were being searched

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by many participants, and the successful ones got the opportunity to present their solutions still at the same school.

The Winter schools passed of course through many changes. The nineteenth one and the second after Frolík's death is just over. Especially the former schools were undoubteldy useful for a number of young Czechoslovak (but also Polish, German, ...) mathematicians who found there often the only opportunity for the personal contact with foreign scientists from other countries.

We have got to know Z. Frolík not only as an excellent mathematician and organizer during those schools, we found that he was a very sociable man. He did not miss the only opportunity for skiing, he initiated football matches between Bohemia (and later, after the number of foreign participants had increased, Czechoslovakia) and the "rest of the world". I always admired his ability to switch his full concentration from the sport to mathematics again.

Z. Frolík was a well known mathematician when I met him first. Among others he was one of the creators of the modern descriptive theory of sets and spaces. After solving a problem of the descriptive theory posed by Z. Frolík I got the opportunity to cooperate with him in this area of mathematics for several years. Our contacts did not confine to the problem solving, we would get many ideas on tennis courts, we were meeting a couple of years at the Sunday volleyball matches. In last years our meetings were unfortunately less frequent. Z. Frolík paid a lot of his attention to Fremlin's notion of Čech analytic space. He formulated the problem of preserving of the propserty to be Čech analytic by perfect mappings but found the affirmative answer for maps with metrizable range only. Due to this topic we started to meet often again. Our discussion on this theme was unfortunately the last one. However, it confirms the fact that Z. Frolík achieved not only an impressive number of results but he formulated problems of basic importance. The fact that he left some of them unsolved could perhaps be a challenge for possible successors.

In the following we try to touch those parts of the descriptive theory which Z. Folík dealt mostly with. We stress the notion of analytic spaces. To study descriptive theory systematically, we should not omit the results of mathematicians who were influenced by or who influenced Frolík's work, or who were developing other parts of that theory. Anyway we introduce several historical remarks for better understanding.

1. On the development of descriptive theory

The rise of descriptive theory of sets is connected with H. Lebesgue, M. Suslin, and N. N. Luzin. H. Lebesgue claimed in [HL], pp. 191–192, that the projection of a Borel subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is Borel for $m < n$. M. Suslin showed that this assertion is false, and the solution of the question under which additional conditions are the images of Borel sets of reals Borel or at least Lebesgue measurable gave rise to the "classical descriptive theory". M. Suslin introduced in [S] the notion of an analytic set of reals which coincides with the continuous image of a Borel subset of the real line. Each analytic set of reals is Lebesgue measurable but not necessarily Borel. Exactly those analytic sets are Borel complement of which is analytic. This assertion, together with more detailed investigation of analytic sets, enables for example to prove that the one-to-one continuous images of Borel sets are Borel. Some essential results on analytic sets and measurable functions were published by N. N. Luzin [L]. Further deep assertions with proofs of the former results may be found in [L – S]. All this "classical" theory works without essential changes in separable metric spaces and was formed mainly by Polish and Russian mathematicians. We may recommend the corresponding chapters of [K] for some details. A rather different language is used in a more recent monograph [M].

In [$\check{S}_{1,2}$] V. E. Šnejder extended the descriptive theory to compact spaces. G. Choquet (see [Ch_{1-3}]) introduced and used the concept of the K-analytic topological space. Another characterization was given by M. Sion in [MS]. But only the Frolík's definition of the analytic space (e.g. [F_1]) enabled to get results comparable with the classical theory. The role of Borel sets from the classical theory is played mostly by Baire sets which coincide with Borel ones in metric spaces. It turns out that the Baire sets of a compact space are exactly those analytical subspaces which have analytical complements. The analytical spaces from [F_1] are Lindelöf. In spite of the fact that Lindelöf spaces need not be separable, it is used to call the corresponding theory "separable". The survey of separable theory can be found in [F_2], the monograph [AS] is well suited for the detailed study. The notion of Lindelöf analytic spaces was succesfully used not only for the study of measurability of sets but also for the investigation of topological properties of some spaces or mappings. Separable descriptive theory is applied e.g. to derive a selection theorem in [N], to the theory of Hausdorff measures in [D], to the theory of capacities in [Ch_3], to the theory of locally compact groups in [GWM], to improve the theorem on the closed graph in [LS] and [F_3], to show that the weakly compactly generated Banach spaces are Lindelöf with their weak topology [T], etc.

The Borel sets in nonseparable complete metric spaces were investigated first by D. Montgomery ([DM]), A. H. Stone ([AHS]), and by R. W. Hansell ([H_{1-3}]).

The theories of Lindelöf and metric analytic spaces do not contain each other. Both are contained in the theory developed in [PH] and [$F - H_{1-3}$]. Moreover it includes e.g. Baire subsets of the product of a compact and a complete metric space. The analytic spaces defined in this extended sense are paracompact. It is possible to study similarly more general spaces, e.g. a definition, mentioned already in [PH], was studied in [$H - J - R_{1,2}$]. The analytic spaces there are subparacompact. To study the weak topology of some Banach spaces [H_4], another generalization of [F_4] was used.

All the extensions of the separable theory we just mentioned do not enable the direct study of Borel or Baire measurability. They give the possibility to study some other topological properties of spaces and the analogy to the classical results on Borel measurability have some counterparts for suitably extended classes of measurable sets.

The notion of analytic sets is replaced in any of mentioned extensions by a new concept of "analytic" spaces. At the end of seventies, D. Fremlin [DF] introduced Čech analytic spaces which are still more general, have also some good topological properties, but it is not clear if it is possible to make a theory analogical to the older ones for it.

2. Lindelöf analytic spaces and separable descriptive theory

For the sake of simplicity, we have in mind only the Hausorff completely regular topological spaces. The notations $\mathscr{G}(X)$, $\mathscr{F}(X)$, $\mathscr{K}(X)$, or mostly just $\mathscr{G}$, $\mathscr{F}$, $\mathscr{K}$ stand for the respective families of all open, closed, or compact subsets of the topological space X, the symbol $\mathscr{Z}(X) = \mathscr{Z}$ is used for all zero sets of continuous real functions and $\mathscr{C}(X) = \mathscr{C}$ for their complements.

Definition 1. For any collection $\mathscr{M}$ of subsets of X, we use $\mathbf{B}(\mathscr{M})$ to denote the smallest family of subsets of X which contains $\mathscr{M}$ and is closed to countable intersections and unions. The symbols $\mathscr{M}_\sigma$ and $\mathscr{M}_\delta$ stand for families of all countable unions, or intersections of the sets from $\mathscr{M}$, respectively.

Basic facts about B

(a) Whenever $\mathscr{M}$ contains the complement of each element of $\mathscr{M}$, $\mathbf{B}(\mathscr{M})$ forms a σ-algebra; $\mathbf{B}(\mathscr{Z})$ is the σ-algebra of *Baire* sets and $\mathbf{B}(\mathscr{F} \cup \mathscr{G})$ the σ-algebra of *Borel* sets;

(b) $\mathbf{B}(\mathscr{Z}) \subset \mathbf{B}(\mathscr{F}) \subset \mathbf{B}(\mathscr{F} \cup \mathscr{G})$; of course, these three families coincide in metric spaces.

2.1. Suslin sets and analytic spaces

An important role in separable descriptive theory is played by the topological space of irrational numbers. It is well known that it is homeomorphic to $\mathscr{N} = \mathbb{N}^{\mathbf{N}}$, the space of all sequences of natural numbers endowed with the product topology. Obviously, $\mathscr{N}$ is homeomorphic to $\mathscr{N}^{\mathbf{N}}$. This fact can be used to show that the family of projections of closed subsets of $X \times \mathscr{N}$ to the topological space X is closed to countable intersections. Because it clearly contains all closed sets and countable unions of its elements, each set from $\mathbf{B}(\mathscr{F})$, and hence also from $\mathbf{B}(\mathscr{Z})$, is such a projection. This observation leads us to the following notion.

Definition 2. A set $S \subset X$ is called a *Suslin* subset of the topological space X if it is the projection of some closed $F \subset X \times \mathcal{N}$ to X.

We denote the family of all Suslin subsets of X by $S^X(\mathscr{F})$ or simply by $\mathbf{S}(\mathscr{F})$.

Basic facts about Suslin sets

(a) $\mathbf{B}(\mathscr{Z}) \subset \mathbf{B}(\mathscr{F}) \subset \mathbf{S}(\mathscr{F})$;

(b) $S \subset X$ is a Suslin set, $S \in \mathbf{S}(\mathscr{F})$, if and only if

$$S = \bigcup \{ \bigcap_{n \in \mathbb{N}} F_{\nu_1, \dots, \nu_n} \mid \nu \in \mathcal{N} \} \tag{1}$$

for some closed sets $F_{\nu_1,\dots,\nu_n}$; we use the standard notation $\nu|n$ for $(\nu_1, \dots, \nu_n)$; the map from the right-hand side of (1) which assigns to each indexed family of sets $(F_{\nu/n} | \nu \in \mathcal{N}, n \in \mathbb{N})$ the corresponding set S is called the *Suslin operation*;

(c) by $\mathbf{S}(\mathscr{M})$ we denote the collection of all sets which arise by the Suslin operation using sets $F_{\nu/n}$ from $\mathscr{M}$, and we have

$$\mathbf{B}(\mathbf{S}(\mathscr{F})) \subset \mathbf{S}(\mathbf{S}(\mathscr{F})) = \mathbf{S}(\mathscr{F})\,;$$

(d) if $\mathscr{M}_\sigma = \mathscr{M}$, $X \setminus M \in \mathscr{M}$ for $M \in \mathscr{M}$, and if for each $A \subset X$ there is a $B \in \mathscr{M}$ which contains A, and $D \in \mathscr{M}$ whenever $D \subset B \setminus C$ for some $C \in \mathscr{M}$ with $A \subset C$, then $\mathbf{S}(\mathscr{M}) = \mathscr{M}$ (see [K]);

this fact implies that Lebesque measurable subsets of $\mathbb{R}^n$ or the sets with the Baire property are closed to the Suslin operation;

(e) there is a Suslin subset of $\mathbb{R}$ which is not Borel.

One of the most important results of the classical descriptive theory is the characterization of Borel subsets of a separable complete metric space M as those subsets B of M for which both B and $M \setminus B$ are Suslin. To prove this, it is important to notice that Suslin sets in M have a special topological structure. Namely, let $\mathscr{U}_n$ be some countable cover of M by open balls with diameter $1/n$. Then

(2) each filter $\mathscr{P}$ in M for which $\mathscr{P} \cap \mathscr{U}_n \neq \emptyset$ for every $n \in \mathbb{N}$ has an accumulation point in M.

Even, $\mathscr{P}$ has a limit in our case. It is not difficult to realize that such a sequence of covers exists also in $M \times \mathcal{N}$, in $F \subset M \times \mathcal{N}$ for each closed F, and in the continuous image of F, especially in the projection S of F to X.

The following two definitions enable to express the above property of the Suslin subset S of M shortly by "S is analytic".

Definition 3 ([F_5]). We say that the sequence of covers $\mathscr{U}_n$ is a *complete sequence of covers* of the topological space M if (2) is valid in M.

Definiton 4 ([F_1]). A topological space X is called *analytic* if there is a complete sequence of *countable* covers in X.

Let $\mathscr{U}_n = \{U_{n,1}, U_{n,2}, \ldots\}$, $n = 1, \ldots$, form a complete sequence of countable covers of X. Let us define $\mathsf{F}(\nu) = \bigcap_k \overline{U_{1,\nu_1} \cap \ldots \cap U_{k,\nu_k}}$ for $\nu \in \mathscr{N}$. We get a characterization of analytic spaces.

Theorem 1 ([$\mathbf{F}_1$]). *The topological space X is analytic, if and only if there is a map* $\mathsf{F}\colon \mathscr{N} \to \mathscr{K}(X)$ *such that*

$$X = \bigcup \{\mathsf{F}(\nu) | \nu \in \mathscr{N}\} = \mathsf{F}(\mathscr{N}) \textit{ and}$$

$$\{\nu \in \mathscr{N} \mid \mathsf{F}(\nu) \subset G\} \textit{ is open in } \mathscr{N} \textit{ for } G \textit{ open in } X. \tag{3}$$

Definiton 5 ([$\mathbf{F}_1$]). The mapping F from a topological space X to subsets of a topological space Y is called *upper semi-continuous* if (3) holds. We write also that $\mathsf{F}\colon \mathscr{N} \to X$ is an upper semi-continuous *compact-valued* map from $\mathscr{N}$ to X for the upper semi-continuous map $\mathsf{F}\colon \mathscr{N} \to \mathscr{K}(X)$.

Hence, Theorem 1 says in other words that X is analytic, if and only if X is the image of $\mathscr{N}$ by some upper semi-continuous compact-valued map. Since the composition of two upper semi-continuous compact-valued maps is of the same kind, and since every separable complete metric space is analytic, X is analytic, if and only if X is the image of some separable complete metric space under an upper semi-continuous compact-valued map. Due to the above characterizations of analytic spaces, we get the following properties of them.

Basic facts about analytic spaces

(a) Every separable complete metric space is analytic;
(b) every compact space is analytic;
(c) the product of countably many analytic spaces is analytic;
(d) any continuous (even upper semi-continuous compact-valued) image of the analytic space is analytic;
(e) Suslin subsets of an analytic space are analytic.

Theorem 2 ([$\mathbf{F}_1$]). *Every analytic space is Lindelöf.*

Theorem 3 ([$\mathbf{F}_1$]). *Every analytic subspace A of the topological space X is a Suslin subset of X.*

Denoting $\mathscr{N}_{\mu/n} = \{\nu \in \mathscr{N} \mid \nu|n = \mu|\, n\}$ for $\mu \in \mathscr{N}$, $n \in \mathbb{N}$, and $\mathsf{F}\colon \mathscr{N} \to \mathscr{K}(A)$ being some upper semi-continuous map such that $\mathsf{F}(\mathscr{N}) = A$, we get obviously

$$A = \bigcup \{\bigcap_n \mathsf{F}(\mathscr{N}_{\nu/n}) \mid \nu \in \mathscr{N}\} = \bigcup \{\bigcap \overline{\mathsf{F}(\mathscr{N}_{\nu/n})} | \nu \in \mathscr{N}\} \tag{4}$$

This proves Theorem 3.

We may realize that, due to the preceding properties of analytic spaces, analytic spaces form the smallest class of topological spaces, which contains compact spaces, the countable discrete space $\mathbb{N}$, is closed to countable products and continuous images, and contains all closed subspaces of each its element. Namely, the analytic space X is the Suslin subset of some compactification, thus it is a projection of some closed subset of $K \times \mathscr{N}$ to K, and $\mathscr{N} = \mathbb{N}^{\mathbb{N}}$ is the countable product of countable discrete space $\mathbb{N}$.

Besides the definitions of analytic space by means of complete sequence of countable covers or upper semi-continuous compact-valued mappings, the original Choquet definition ([Ch$_1$]) or Sion's definition ([MS]), via continuous images of subsets from $\mathscr{K}_{\sigma\delta}$ in some topological space, exist. It can be shown that they coincide even within the framework of Hausdorff spaces ([J]). Recalling that a space is *Čech complete* if there is a complete sequence of *open* covers ([F$_5$]) we see that the Lindelöf Čech complete space is analytic, and we can summarize some characterizations of Lindelöf analytic spaces.

Theorem 4. *Each of the following properties of a topological space X is equivalent with the fact that X is analytic*:

(a) *X is the image of $\mathscr{N}$ under some upper semi-continuous compact-valeed map*;
(b) *X admits a complete sequence of countable covers*;
(c) *X is a Suslin subset of Y whenever it is a subspace of Y*;
(c′) *X is a Suslin subset of some Lindelöf Čech complete space*;
(d) *X is Lindelöf and it is a Suslin subset of some Čech complete space*;
(e) *X is the projection of some Lindelöf Čech complete subspace of $X \times \mathscr{N}$ onto X*;
(e′) *X is a continuous image of some Lindelöf Čech complete space.*

Since we are not sure that the characterizations (d), (e), (e′) are well known, we explain the reason why they hold.

Obviously, X is analytic implies (d); (e) and (e′) are sufficient for X to be analytic.

Let X be analytic. Then it is Suslin in some compactification K by (c). Thus there is a closed subset F of $K \times \mathscr{N}$ with X being its projection. Now F is Lindelöf and Čech complete. This proves (e). Obviously, (e) implies (e′).

It remains to show that (d) is sufficient. Let F be a closed subset of $Y \times \mathscr{N}$ with the projection X into Y and with Y being Čech complete. Now consider a complete sequence of open covers $\mathscr{U}_n$ in Y. The family $\{\mathscr{N}_{\sigma/n} \times U \mid U \in \mathscr{U}_n, \sigma|n \in \mathbb{N}^n\}$ forms an open cover of F for every n. Choose a countable cover from it and intersect each its element with F. It follows, from the fact that $(\mathscr{U}_n)$ is complete, that these new covers form a complete sequence of countable covers of F. Hence F is analytic and so is its projection X.

Warning. Lindelöf spaces which are continuous images of a Čech complete space need not be analytic. An example is mentioned after Theorem 19 below.

2.2. Connections to Baire and Borel sets

Although the study of analytic spaces and their properties is an interesting theory with a number of applications (we mentioned some of them above), we come back to the original motivation for the descriptive theory, namely to the study of images of Borel sets. The improvement of the classical theory is the characterization of Baire subsets of the analytic space X as such subsets B for which B and $X \setminus B$ are Suslin in X. It follows from the following theorem (a generalization of the "first Luzin separation principle"), which says moreover that every analytic set with Suslin complement is Baire in any (completely regular Hausdorff!) space.

Theorem 5 ([F_6]). *If A is an analytic and S a Suslin subset of the topological space X with $A \cap S = \emptyset$, then there is a Baire set $B \in \mathbf{B}(\mathscr{Z})$ such that*

$$A \subset B \subset X \setminus S .$$

Due to the definition of the analytic space which enables to partition it using countable covers which form a complete sequence, and due to the possibility to express S in the form (1) with $F_{\nu|(n+1)} \subset F_{\nu/n}$, we show that if A and S cannot be separated as in Theorem 5, then we cannot separate some compact subset of A and some closed subset of S. The validity of the above separation principle thus follows immediately from the complete regularity of the Hausdorff space X.

To formulate the next theorem one more notion is suitable for us.

Definition 6 (the definition of "*Borelian*" in [F_1]). The space X is called *Luzin* if there is an upper semi-continuous map $\mathsf{F}\colon \mathscr{N} \to \mathscr{K}(X)$ with $\mathsf{F}(\mathscr{N}) = X$ and $\mathsf{F}(\mu) \cap \mathsf{F}(\nu) = \emptyset$ for $\mu \neq \nu$ (i.e. F is *disjoint*).

Basic facts about Luzin spaces

(a) Every Luzin space is analytic;
(b) every separable complete metric space is Luzin;
(c) every compact space is Luzin;
(d) one-to-one continuous (or more generally disjoint upper semi-continuous compact-valued) image of a Luzin space is Luzin;
(e) each set from $\mathbf{B}_d(\mathscr{F} \cup \mathscr{C})$ in a Luzin space is a Luzin space ([F_1]) – here $\mathbf{B}_d(\mathscr{M})$ denotes the smallest class of sets containing $\mathscr{M}$ and closed to countable intersections and countable disjoint unions, and $\mathbf{B}(\mathscr{Z}) \subset \mathbf{B}_d(\mathscr{C})$ holds;
(f) every Lindelöf Čech complete X is the countable intersection of some open subset of a compactification K, $K \setminus X$ is therefore Suslin in K; Theorem 4(e′) says that X is analytic and Theorem 5 implies that X is a Baire subset of K and, by (e), it is Luzin;
(g) the countable union of Luzin subspaces need not be Luzin ([F_1]), but a non-trivial fact holds:

Theorem 6 ([F_7]). *The union of two Luzin subspace is a Luzin subspace.*

Because the one-to-one image of a Luzin space is Luzin, and Borel subsets of separable complete metric spaces are Luzin (see (e)), the following theorem includes the classical result mentioned in the historical introduction, namely, that one-to-one continuous image of any Borel subset of separable complete metric space is Borel.

Theorem 7 ([F_1]). *Luzin subspace L of the topological space X is in $\mathbf{B}_d(\mathscr{F} \cup \mathscr{C})$ in X. Hence L is Borel in X.*

The space L can be described by the formula (4). Due to the separation principle we find $B_{\nu/n} \in \mathrm{B}(\mathscr{Z})$ which separate $\mathrm{F}(\mathscr{N}_{\nu/n})$. Then

$$L = \bigcup \{\bigcap_n B_{\nu/n} \cap \overline{\mathrm{F}(\mathscr{N}_{\nu/n})}\} = \bigcap_n \bigcup_{\nu|n} (B_{\nu/n} \cap \overline{\mathrm{F}(\mathscr{N}_{\nu/n})}) \in \mathrm{B}_d(\mathscr{F} \cup \mathscr{C})\,.$$

Theorem 7 and the property (e) above give

Theorem 8 ([F_1]). *Luzin spaces are just those spaces X which are in $\mathbf{B}_d(\mathscr{F} \cup \mathscr{C})$ in some compactification of X.*

The next theorem gives the analogy of Luzin theorem on the Borel measurability of the inverse mapping of one-to-one Borel measurable map. Let us notice that it says also that the disjoint collection of subsets of a topological space is countable if the union of each its subcollection is analytic.

Theorem 9 ([F_8]). *Let A be analytic, M metric, and $f\colon A \to M$ Baire measurable (or Suslin measurable only, i.e. the preimages of open sets are Suslin). Then the graph of f and $f(A)$ are analytic. The set $B \subset M$ is Baire in $f(A)$ if $f^{-1}(B)$ is Baire in A.*

The last assertion of Theorem 9 follows from the former one because Baire sets are analytic in A, thus B with its complement in $f(A)$ are analytic, and hence B is Baire due to the separation principle.

The first assertion can be first shown for separable M. One may prove that the graph of f is Suslin in $A \times M$, where M stands for the completion of M. Therefore the graph of f is analytic and so is its projection $f(A)$ in M.

This assertion can be used to show that $f(A)$ cannot be nonseparable.

3. Nonseparable descriptive theory

To study the Baire or Suslin subsets of complete metric spaces, or of products of a compact and a complete metric space, or of some Banach spaces with their weak topology (if it is not Lindelöf), or to study measurable maps between such spaces, the notion of analytic spaces as defined above is not sufficient. In spite of that we may await that some suitably chosen concept might help us to get similar results as for Lindelöf spaces. Let us examine the characterizations in Theorem 4. By excluding, in (c′), (d), (e), or (e′), the assumption that X is Lindelöf, we get notions which will be mentioned later. Since most of the separable theory is based on the definitions (a) and (b) from Theorem 4, we first explain how to extend those definitions some non-Lindelöf spaces.

It seems natural to replace the space $\mathscr{N} = \mathbb{N}^{\mathrm{N}}$ in (a) by the space $\mathbb{D}^{\mathrm{N}}$ with $\mathbb{D}$ being some discrete topological space. Since every topological space is a one-to-one continuous image of a discrete space, we do not get a meaningful definition. Similarly, leaving out the assumption on countability of the covers in (b), we get all spaces (the covers by singletons form a complete sequence).

We find a motivation for the possible reasonable definition of "nonseparable analytic" spaces in the following theorem which was originally proved for complete metric spaces and disjoint families of sets by R. W. Hansell in [H_1]. Realize first that if a one-to-one continuous map $f: M \to P$ between metric spaces has a Suslin measurable inversion, then for every discrete system $\{D_a | a \in A\}$ of subsets of M, the set $\bigcup \{f(\overline{D_a}) | a \in B\}$ is Suslin for every $B \subset A$.

Definition 7. We say that a family $\{E_a | a \in A\}$ is *completely $\mathscr{M}$-additive* if

$$\bigcup \{E_a | a \in B\} \in \mathscr{M} \tag{5}$$

for every $B \subset A$.

Theorem 10 ([F—H_1]). *If $F : M \to \mathscr{K}(X)$ is upper semi-ontinuous, if F maps the complete metric space M into the topological space X, and if $\{S_a | a \in A\}$ is completely $\mathbf{S}(\mathscr{F})$-additive family of subsets of X such that $\bigcap \{S_a | a \in B\} = \emptyset$ for every infinite $B \subset A$, then the family*

$$\{f^{-1}(S_a) = \{m \in M | f(m) \cap S_a \neq \emptyset\} | a \in A\}$$

has the following properties:

(6) *there are $F_{an} \subset M$ such that $f^{-1}(S_a) = \bigcup \{F_{an} | n \in \mathbb{N}\}$ and $\{F_{an} | a \in A\}$ is discrete in M.*

Definition 8. If $\mathscr{D}$ is some set of collections of subsets of X, we say that the collection $\mathscr{M}$ of subsets of X is *countably-$\mathscr{D}$* if there are $\mathscr{M}_n \subset \mathscr{M}$ such that $\mathscr{M}_n \in \mathscr{D}$ and $\mathscr{M} = \bigcup \{\mathscr{M}_n | n \in \mathbb{N}\}$. A family $\mathscr{M} = \{M_a | a \in A\}$ of subsets of X is called (countably) *$\mathscr{D}$-decomposable* if there are sets $M_{an} \subset X$ such that

$$M_a = \bigcup \{M_{an} | n \in \mathbb{N}\} \text{ for each } a \in A \text{ and}$$
$$\{M_{an} | a \in A\} \in \mathscr{D} .$$

If the space X is metric and $\mathscr{M}$ fulfils the conditions analogous to (6), we say that $\mathscr{M}$ is discretely decomposable or $\mathscr{D}_\varrho$-decomposable, where $\mathscr{D}_\varrho$ stands for the set of all discrete collections of subsets of the metric space X.

We state now a rather general definition of an "analytic" space motivated by Theorem 10.

Definition 9. The topological space X is called *$\mathscr{D}$-analytic*, if there is a complete metric space M and an upper semi-continuous map $\mathsf{F} : M \to \mathscr{K}(X)$ with

$$X = \mathsf{F}(M) \, (= \bigcup \{\mathsf{F}(m) | m \in M\}) \text{ and} \tag{7}$$

$$\{\mathsf{F}(D_a) | a \in A\} \text{ is } \mathscr{D}\text{-decomposable whenever } \{D_a | a \in A\} \text{ is discrete in } M.$$

In case that $\mathscr{D}$ is the set $\mathscr{D}_\omega$ of all countable collections of subsets of X, the space X is $\mathscr{D}_\omega$-analytic obviously iff it is analytic according to Definition 4.

Theorem 11. *Let $\mathscr{D}$ be a set of collections of subsets of the topological space X fulfilling*

($\mathscr{D}_1$) *$\{E_a | a \in A\}$ is $\mathscr{D}$-decomposable if $E_a \subset D_a$ for each $a \in A$ and for $\{D_a | a \in A\}$ in $\mathscr{D}$.*

Let X be $\mathscr{D}$-analytic. Then every open cover X has a countably-$\mathscr{D}$ refinement.

For $\mathscr{D} = \mathscr{D}_\omega$ Theorem 11 says that analytic spaces are Lindelöf. This fact we know from Theorem 2 already.

If $\mathscr{D} = \mathscr{D}_\varrho$ is the set of all collections of subsets of X which are discrete with respect to some continuous pseudometric, then Theorem 11 gives that X is paracompact whenever it is $\mathscr{D}_\varrho$-analytic.

Let $\mathscr{D} = \mathscr{D}_\tau$ be the set of all collections $\mathscr{M}$ of subsets of X which are discrete with respect to the topology of X (i.e. for each $x \in X$ there is an open neighbourhood of x which meets at most one set from $\mathscr{M}$). The property from Theorem 11 for $\mathscr{D}_\tau$-analytic spaces is usually called subparacompactness by topologists.

If $\mathscr{D} = \mathscr{D}_\iota$ is the set of all isolated collections of subsets of the topological space X (i.e. collections $\mathscr{M}$ which are topologically discrete in $\bigcup \mathscr{M}$), then the property of Theorem 11 for $\mathscr{D}_\iota$-analytic space is the property studied already in the topology under the name weak θ-refinability.

It is obvious that analytic spaces are $\mathscr{D}_\varrho$-analytic, $\mathscr{D}_\varrho$-analytic spaces are $\mathscr{D}_\tau$-analytic and $\mathscr{D}_\tau$-analytic spaces are $\mathscr{D}_\iota$-analytic. In the class of metric spaces all these notions coincide. Moreover, Suslin subsets of products $K \times M$ of a compact space and a complete metric space M are $\mathscr{D}_\varrho$-analytic, paracompact Čech complete spaces are $\mathscr{D}_\varrho$-analytic, subparacompact Čech complete spaces are $\mathscr{D}_\tau$-analytic. We do not introduce further properties and examples of $\mathscr{D}$-analytic spaces now. Many properties and examples follow immediately from the characterizations included in the paragraph 3.2. From Theorem 10 we get

Theorem 12. *In any $\mathscr{D}$-analytic space, every completely $\mathbf{S}(\mathscr{F})$-additive family $\{S_a | a \in A\}$ with $\bigcap \{S_a | a \in B\} = \emptyset$ for any infinite subset $B \subset A$ is $\mathscr{D}$-decomposable.*

The characterizations (a) and (b) from Theorem 4 are close to the following assertions.

Theorem 13. *The space X is $\mathscr{D}$-analytic, if and only if there is a discrete topological space $\mathbb{D}$ and an upper semi-continuous map $\mathbf{F}$ of the space $\mathbb{D}^{\mathbf{N}}$ with compact values in X such that $\mathbf{F}(\mathbb{D}^{\mathbf{N}}) = X$ and $\{\mathbf{F}(D_a) | a \in A\}$ is $\mathscr{D}$-decomposable whenever. $\{D_a | a \in A\}$ is discrete in $\mathbb{D}^{\mathbf{N}}$.*

Theorem 14. *If $\mathscr{D}$ fulfils ($\mathscr{D}_1$) and moreover*

($\mathscr{D}_a$) *$\{\bar{S}_a | a \in A\}$ is $\mathscr{D}$-decomposable, whenever $\{S_a | a \in A\}$ is in $\mathscr{D}$,*

then X is $\mathscr{D}$-analytic, if and only if there is a complete sequence of $\mathscr{D}$-decomposable covers of X. $\mathscr{D}$-analytic space X is Suslin in any topological space.

Remark. Notice that $(\mathscr{D}_1)$ and $(\mathscr{D}_2)$ are fulfilled for $\mathscr{D}_\omega$, $\mathscr{D}_\varrho$, $\mathscr{D}_\tau$ or e.g. for the set of all locally countable collections $\mathscr{D}_{e\omega}$, etc. $(\mathscr{D}_2)$ is not fulfilled for $\mathscr{D}_\iota$!

3.1. Separation principle

We are not able to verify whether a set is Baire or Borel using the analyticity in non-Lindelöf spaces in contrast with the separable theory. Those classes of sets which are closed also to discrete unions are more convenient for such a description. Such generalizations are not too strange because the unions of topologically discrete families of Suslin sets are Suslin. Topologically discrete union of Suslin subsets of a topological space are in $\mathbf{S}(\mathscr{F} \cup \mathscr{G})$ but they need not be Suslin. This indicates that $\mathscr{D}_\iota$-analytic spaces differ essentially from $\mathscr{D}_\varrho$-analytic and $\mathscr{D}_\tau$-analytic spaces. This is confirmed e.g. by Theorem 19.

Problem. For which (one-to-one continuous) maps (between complete metric spaces) the image of Borel sets are Borel? A partial answer can be found in [J – R].

We will now introduce the extended classes of measurable sets as mentioned above.

Definition 10. Let $\mathscr{M}$ be some collection of subsets of X and $\mathscr{D}$ be some set of collections of subsets of X. Then $\mathrm{B}_{\mathscr{D}}(\mathscr{M})$ stands for the smallest class of subsets of X which contains $\mathscr{M}$ and is closed to countable intersections, countable unions, and unions of collections from $\mathscr{D}$. We say that B is *$\mathscr{D}$-Baire* or *$\mathscr{D}$-Borel* if it is in $\mathbf{B}_{\mathscr{D}}(\mathscr{Z})$ or in $\mathbf{B}_{\mathscr{D}}(\mathscr{F} \cup \mathscr{G})$.

In the following example of a separation theorem, we suppose $(\mathscr{D}_1)$, that $\mathscr{B}$ stands for a class of subsets of X with $\mathbf{B}_{\mathscr{D}}(\mathscr{B}) = \mathscr{B}$, and that each family $\{X_a\}$ from $\mathscr{D}$ can be enlarged to a family $\{Y_a\}$ from $\mathscr{D}$ with $X_a \subset Y_a$, $Y_a \in \mathscr{B}$.

Theorem 15. *If A is a $\mathscr{D}$-analytic subspace of the topological space X and $S \subset X$ is Suslin in X with $A \cap S = \emptyset$, then there is a set $B \in \mathscr{B}$ such that*

$$A \subset B \subset X \setminus S .$$

This theorem has a great number of interesting corollaries similarly as in the separable theory. We introduce here one, the separable analogy of which was not mentioned. The separable version follows from it immediately and characterizes all "Borel bimeasurable" maps. In the nonseparable case only a special class of maps is studied here.

Theorem 16 ([F—H$_2$]). *Let f be a $\mathscr{D}_\varrho$-Borel measurable map from a complete metric space M to a complete metric space P, and let f fulfils the following condition*

(8) $\{f(D_a) | a \in A\}$ *is $\mathscr{D}_\varrho$-decomposable in P if $\{D_a | a \in A\}$ is discrete in M.*

Then the images of $\mathscr{D}_\varrho$-Borel subsets of M by f are $\mathscr{D}_\varrho$-Borel, if and only if

(9) *the set of those $p \in P$ for which $f^{-1}(p)$ is uncountable is σ-discrete.*

Problem. Which mappings between complete metric spaces have the property that they are $\mathscr{D}_\varrho$-Borel measurable and images of $\mathscr{D}_\varrho$-Borel sets are $\mathscr{D}_\varrho$-Borel? Are the conditions (8) and (9) necessary?

3.2. Characterizations of nonseparable analytic spaces

We introduce now some characterizations of $\mathscr{D}$-analytic spaces in the case of $\mathscr{D}$ equal to $\mathscr{D}_\varrho$, $\mathscr{D}_\tau$, and $\mathscr{D}_\iota$. We indicate the connection to various characterizations from Theorem 4 by our notation clearly. The excluded analogies does not hold, or they may yield interesting problems.

Theorem 17 ([F—H$_3$]). *The space X is $\mathscr{D}_\varrho$-analytic, if and only if one of the following conditions holds.*

(a_ϱ) *There is a discrete topological space $\mathbb{D}$ and an upper semi-continuous compact-valued map $\mathsf{F}:\mathbb{D}^{\mathbf{N}} \to X$ such that $\mathsf{F}(\mathbb{D}^{\mathbf{N}}) = X$ and $\{\mathsf{F}(D_a)|a \in A\}$ is $\mathscr{D}_\varrho$-decomposable whenever $\{D_a|a \in A\}$ is discrete in $\mathbb{D}^{\mathbf{N}}$.*

(b_ϱ) *There is a complete sequence of countably-$\mathscr{D}_\varrho$ covers in X.*

(c'_ϱ) *X is a Suslin subset of some paracompact Čech complete space.*

(d_ϱ) *X is paracompact, and it is a Suslin subset of some Čech complete space.*

The equivalences of (a_ϱ) and (b_ϱ) with the definition follow directly from Theorems 13 and 14. To prove the other assertions one may use Theorem 11, the characterization of Čech complete spaces with the help of complete sequences of open covers from [F$_5$] and the assertion that for a paracompact subspace P of a Čech complete subspace C there is a paracompact Čech complete subspace $Q \subset C$ which contains P ([F$_{11}$]). One may show that any $\mathscr{D}_\varrho$-analytic space X is a projection of some paracompact Čech complete subspace of $X \times \mathscr{N}$.

Theorem 18 ([H—J—R$_1$]). *The following conditions are equivalent for a space X to be $\mathscr{D}_\tau$-analytic.*

(a_τ) *There is a discrete topological space $\mathbb{D}$ and an upper semi-continuous compact-valued map $\mathsf{F}:\mathbb{D}^{\mathbf{N}} \to X$ where $\mathsf{F}(\mathbb{D}^{\mathbf{N}}) = X$ and $\{\mathsf{F}(D_a)|a \in A\}$ is $\mathscr{D}_\tau$-decomposable whenever $\{D_a|a \in A\}$ is discrete in $\mathbb{D}^{\mathbf{N}}$.*

(b_τ) *There is a complete sequence of countably-$\mathscr{D}_\tau$ covers in X.*

(d_τ) *X is subparacompact and it is a Suslin subset of some Čech complete space.*

The assertion of the preceding theorem follow from Theorems 11, 13, and 14, similarly as above. One may again show that the $\mathscr{D}_\tau$-analytic space X can be presented as a projection of some paracompact Čech subspace of $X \times \mathscr{N}$ so as the $\mathscr{D}_\varrho$-analytic spaces.

However, one may show that any such projection is $\mathscr{D}_\iota$-analytic. It does not seem that the corresponding characterizations for $\mathscr{D}_\iota$-analytic spaces can be verified similarly as in the above cases. E.g. H. Junilla and J. Pelant showed in [J–P] that the existence of complete sequence of countably-$\mathscr{D}_\iota$ families is not sufficient for a space to be $\mathscr{D}_\iota$-analytic!

From what we have stated until now we conclude that the following connections hold true.

Theorem 19. *A $\mathscr{D}_\varrho$-analytic space is analytic, if and only if it is Lindelof. A $\mathscr{D}_\tau$-analytic space is $\mathscr{D}_\varrho$-analytic, if and only if it is paracompact. However, there is a Lindelof $\mathscr{D}_\iota$-analytic space that is not $\mathscr{D}_\tau$-analytic.*

We define a Lindelöf space which serves as the example from Theorem 19. Let us consider any uncountable set X. Denote $Y = X \cup \{y\}$ for some $y \notin X$. The open sets of Y are all subsets of X and all complements of countable subsets of X in Y.

In [F_9] Z. Frolík defined "*absolutely Suslin*" space as Suslin subsets of some Čech complete space. (Notice the connnection to the characterizations (c′) or (d) of Theorem 4.) He used the notion to characterize all metric spaces that are Suslin in any metric space in which they are imbedded. These are exactly all metric spaces which are absolutely Suslin. Every $\mathscr{D}_\tau$-analytic space is absolutely Suslin.

The example after Theorem 19 above and the characterization (d_τ) from Theorem 18 show that $\mathscr{D}_\iota$-analytic spaces need not be absolutely Suslin.

In 1980 the notion of Čech analytic space was introduced by D. Fremlin in [DF]. The following definition is close to the characterization (e) from Theorem 4.

Definition 11. The space is *Čech analytic* if it is the projection of some Čech complete subspace of $X \times \mathscr{N}$.

Because a set S is in $\mathbf{S}(\mathscr{F} \cup \mathscr{G})$ iff it is the projection of a set $F \cap G \subset X \times \mathscr{N}$ into X where F is closed and G is the intersection of countably many open subsets of $X \times \mathscr{N}$, the following characterization of Čech analytic spaces holds.

Theorem 20. *The space is Čech analytic, if and only if it is in $\mathbf{S}(\mathscr{F} \cup \mathscr{G})$ in some compact space (equivalently, in some Čech complete space).*

Hence we know that absolutely Suslin or $\mathscr{D}_\tau$-analytic spaces are Čech analytic. A problem of characterization of Čech analytic spaces which are (K-)analytic was posed by D. Fremlin. It is probably not easier to characterize $\mathscr{D}_\tau$-analytic spaces among $\mathscr{D}_\iota$-analytic spaces.

Every $\mathscr{D}_\iota$-analytic space is Čech analytic ([J_4]). Some examples of Banach spaces that are Čech analytic in their weak topology can be found in the same paper. Interesting topological properties of some Čech analytic Banach spaces may be also found in [J–N–R]. The Čech analytic space containing a nonempty perfect set is described in [GK]. This property of Čech analytic spaces is used to show that the spaces $\mathbb{R}^n$ with the density topology for $n \geqq 2$, studied in the real analysis, and all spaces $\mathbb{R}^n$ with the fine topology, studied in the potential theory, are not Čech analytic ([F–N]).

Z. Frolík posed the problem whether the perfect images of Čech analytic spaces are Čech analytic. We recall that a map is perfect if it is continuous, closed, and preimages of points are compact. He found a partial nontrivial solution to this problem.

Theorem 21 ([F_{10}]). *The perfect image of any Čech analytic space to a metric space is Čech analytic.*

I would like to express my hope that this paper not only recalls some Frolík's results but may also help to draw the attention of further researchers to this topic. I would like to express my gratitude to all who helped me with the preparation of these notes, namely to B. Balcar, D. Preiss, P. Simon, L. Zajíček, and especially to J. Pelant.

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