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Exchangeable Partial Groupoids I

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In the paper, the problem of distances between finite quasigroups and groups is studied.

V článku se studuje problém vzdáleností konečných kvazigrup od grup.

В статье изучается проблемма расстояния между конечными квазигруппами и группами.

1. Introduction

By a partial groupoid we mean a non-empty set together with a partial binary operation. Let K be a partial groupoid. For $a \in K$, we put $M(K, a) = \{(b, c); a = bc\}$, so that the operation of K is defined just for ordered pairs from the set $M = M(K) = \bigcup M(K, a)$. Further, let $T(K) = \{(b, c, a);\ bc = a\}$, $B(K) = \{b; (b, c) \in M\}$, $C(K) = \{c; (b, c) \in M\}$, $A(K) = B(K) \cup C(K)$, $D(K) = \{bc; (b, c) \in M\}$, $m = m(K) = \operatorname{card}(M)$, $p = p(K) = \operatorname{card}(B)$, $q = q(K) = \operatorname{card}(C)$, $o = o(K) = \operatorname{card}(D)$. The number m will be called the rank of K. For all $b, c, d \in K$, let $p(b) = p(K, b) = \operatorname{card}(\{a; (b, a) \in M\})$, $q(c) = q(K, c) = \operatorname{card}(\{a; (b, a) \in M\})$, $o(d) = o(K, d) = \operatorname{card}(M(K, d))$. Finally, if K is finite, then let $\delta = \delta(K) = 3m - 2(p + q + o)$, $\delta(p) = \delta(K, p) = m - 2p$, $\delta(q) = \delta(K, q) = m - 2q$, $\delta(o) = \delta(K, o) = m - 2o$.

1.1. Lemma. Let K be a finite partial groupoid. Then:

(i) $m = \sum p(a) = \sum q(a) = \sum o(a)$, $a \in K$.

(ii) $m \leqq pq$, $\max(p(b)) \leqq q$, $\max(q(c)) \leqq p$.

(iii) $\delta = \delta(p) + \delta(q) + \delta(o)$.

(iv) $\delta(p) = \sum_{b \in B(K)} (p(b) - 2)$, $\delta(q) = \sum_{c \in C(K)} (p(c) - 2)$, $\delta(o) = \sum_{d \in D(K)} (o(d) - 2)$.

(v) The numbers $m, \delta, \delta(p), \delta(q), \delta(o)$ have the same parity.

Proof. Obvious.

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A partial groupoid K is said to be

- balanced if the sets B, C and D are pair-wise disjoint;
- reduced if $K = A \cup D$;
- cancellative if $(a, b), (a, c) \in M$ (resp. $(b, a), (c, a) \in M$) and $ab = ac$ (resp. $ba = ca$) implies $b = c$.

Let K and L be partial groupoids. A mapping f of K into L is said to be a homomorphism if $(f(a), f(b)) \in M(L)$ and $f(ab) = f(a) f(b)$ for all $(a, b) \in M(K)$. Furthermore, we shall say that f is strong if for all $(x, y) \in M(L)$, $x, y \in f(K)$, there exists a pair $(a, b) \in M(K)$ with $f(a) = x$ and $f(b) = y$. A homomorphism f of K into L is said to be a pseudoimmersion (resp. an immersion) if the restrictions $f \mid B(K)$, $f \mid C(K)$ (and $f \mid D(K)$) are injective mappings.

Let K and L be partial groupoids. We shall say that L is a (strong) partial subgroupoid of K if $L \subseteq K$ and this inclusion is a (strong) homomorphism.

Let K be a partial groupoid and N a non-empty subset of $M(K)$. Define a partial groupoid $L = S[K, N]$ by $L = \{a, b, c; (a, b) \in N, ab = c\}$ and $M(L) = N$. Clearly, L is reduced and L is a partial subgroupoid of K.

1.2. Lemma. Let K be a cancellative partial groupoid. Then $\max(p(K, b), o(K, d)) \leqq q(K)$, $\max(q(K, c), o(K, d)) \leqq p(K)$ and $\max(p(K, b), q(K, c)) \leqq o(K)$.

Proof. Obvious.

2. Parastrophes of cancellative partial groupoids

Let K be a partial groupoid. Define a partial groupoid $\bar{K} = K(\circ)$ (the opposite of K) by $(a, b) \in M(\bar{K})$ iff $(b, a) \in M(K)$ and then $a \circ b = ba$.

2.1. Lemma. Let K be a partial groupoid. Then:

(i) $A(\bar{K}) = A(K)$, $B(\bar{K}) = C(K)$, $C(\bar{K}) = B(K)$, $D(\bar{K}) = D(K)$, $m(\bar{K}) = m(K)$, $p(\bar{K}) = q(K)$, $q(\bar{K}) = p(K)$, $o(\bar{K}) = o(K)$, $p(\bar{K}, b) = q(K, b)$, $q(\bar{K}, c) = p(K, c)$ and $o(\bar{K}, d) = o(K, d)$.

(ii) $\delta(\bar{K}) = \delta(K)$, $\delta(\bar{K}, p) = \delta(K, q)$, $\delta(\bar{K}, q) = \delta(K, p)$ and $\delta(\bar{K}, o) = \delta(K, o)$, provided K is finite.

(iii) $\bar{\bar{K}} = K$ and $\bar{K}$ is balanced (resp. cancellative, reduced) iff K is so.

(iv) A mapping f of K into L is a (strong) homomorphism iff it is a (strong) homomorphism of $\bar{K}$ into $\bar{L}$.

(v) A homomorphism f of K into L is a (pseudo)immersion iff it is a (pseudo)immersion of $\bar{K}$ into $\bar{L}$.

Proof. Obvious.

Let K be a cancellative partial groupoid. Define two partial groupoids $K^{-1} = K(+)$ (the right inverse of K) nad $^{-1}K = K(-)$ (the left inverse of K) as follows:

$(a, b) \in M(K^{-1})$ and $a + b = c$ iff $(a, c) \in M(K)$ and $ac = b$; $(a, b) \in M(K^{-1})$ and $a - b = c$ iff $(c, b) \in M(K)$ and $cb = a$.

2.2. Lemma. Let K be a cancellative partial groupoid. Then:

(i) $A(K^{-1}) = B(K) \cup D(K)$, $B(K^{-1}) = B(K)$, $C(K^{-1}) = D(K)$, $D(K^{-1}) = C(K)$, $m(K^{-1}) = m(K)$, $p(K^{-1}) = p(K)$, $q(K^{-1}) = o(K)$, $o(K^{-1}) = q(K)$, $p(K^{-1}, b) = p(K, b)$, $q(K^{-1}, c) = o(K, c)$ and $o(K^{-1}, d) = q(K, d)$.

(ii) $\delta(K^{-1}) = \delta(K)$, $\delta(K^{-1}, p) = \delta(K, p)$, $\delta(K^{-1}, q) = \delta(K, o)$ and $\delta(K^{-1}, o) = \delta(K, q)$, provided K is finite.

(iii) $(K^{-1})^{-1} = K$, K^{-1} is cancellative and K^{-1} is balanced (resp. reduced) iff K is so.

(iv) A mapping f of K into L is a (strong) homomorphism iff it is a (strong) homomorphism of K^{-1} into L^{-1}.

(v) A homomorphism f of K into L is an immersion iff it is an immersion of K^{-1} into L^{-1}.

Proof. Obvious.

2.3. Lemma. Let K be a cancellative partial groupoid. Then:

(i) $A(^{-1}K) = C(K) \cup D(K)$, $B(^{-1}K) = D(K)$, $C(^{-1}K) = C(K)$, $D(^{-1}K) = B(K)$, $m(^{-1}K) = m(K)$, $p(^{-1}K) = o(K)$, $q(^{-1}K) = q(K)$, $o(^{-1}K) = p(K)$, $p(^{-1}K, b) = o(K, b)$, $q(^{-1}K, c) = q(K, c)$ and $o(^{-1}K, d) = p(K, d)$.

(ii) $\delta(^{-1}K) = \delta(K)$, $\delta(^{-1}K, p) = \delta(K, o)$, $\delta(^{-1}K, q) = \delta(K, q)$ and $\delta(^{-1}K, o) = \delta(K, p)$, provided K is finite.

(iii) $^{-1}(^{-1}K) = K$, ^{-1}K is cancellative and ^{-1}K is balanced (resp. reduced) iff K is so.

(iv) A mapping f of K into L is a (strong) homomorphism iff it is a homorphism of ^{-1}K into ^{-1}L.

(v) A homomorphism f of K into L is an immersion iff it is an immersion of ^{-1}K into ^{-1}L.

Proof. Obvious.

2.4. Lemma. Let K be a cancellative partial groupoid. Then $(\bar{K})^{-1} = (\overline{^{-1}K}) = {}^{-1}(K^{-1})$ and $^{-1}(\bar{K}) = (\overline{K^{-1}}) = (^{-1}K)^{-1}$.

Proof. Obvious.

A partial groupoid L is said to be a parastrophe of a cancellative partial groupoid K if L can be obtained from K by a (possibly repeated) application of the operators $P \to \bar{P}$, $P \to P^{-1}$, $P \to {}^{-1}P$.

2.5. Lemma. Let L be a parastrophe of a cancellative partial groupoid K. Then:

(i) L is cancellative, K is a parastrophe of L and L is equal to at least one of the partial groupoids $K, K, K^{-1}, {}^{-1}K, {}^{-1}(K^{-1}), (^{-1}K)^{-1}$.

(ii) L is balanced (resp. reduced) iff K is so.
(iii) $m(L) = m(K)$.
(iv) $\delta(L) = \delta(K)$, provided K is finite.

Proof. Use 2.1, 2.2, 2.3 and 2.4.

3. Isotopes of reduced partial groupoids

A reduced partial groupoid L is said to be an isotope of a reduced partial groupoid K if there exist bijections f, g and h of $B(K)$ onto $B(L)$, $C(K)$ onto $C(L)$ and $D(K)$ onto $D(L)$, resp., such that $M(L) = \{(f(a), g(b);\ (a, b) \in M(K)\}$ and $h(ab) = f(a)\, g(b)$ for all $(a, b) \in M(K)$.

3.1. Lemma. Let a reduced partial groupoid L be an isotope of a reduced partial groupoid K. Then:
(i) L is cancellative iff K is so.
(ii) $m(L) = m(K)$, $p(L) = p(K)$, $q(L) = q(K)$ and $o(L) = o(K)$.
(iii) $\delta(L) = \delta(K)$, $\delta(L, p) = \delta(K, p)$, $\delta(L, q) = \delta(K, q)$ and $\delta(L, o) = \delta(K, o)$, provided K is finite.
(iv) K is an isotope of L and $\bar{L}$ is an isotope of $\bar{K}$.
(v) L^{-1} (resp. ${}^{-1}L$) is an isotope of K^{-1} (resp. ${}^{-1}K$), provided K is cancellative.
(vi) The partial groupoids K and L are isomorphic, provided both K and L are balanced.

Proof. Easy.

4. Group modifications of partial groupoids

Let $K = K(\circ)$ be a reduced partial groupoid and $F = F(K)$ the free group of words over the set K. Denote by $N = N(K)$ the normal subgroup of F generated by all the words bca^{-1}, $(b, c, a) \in T(K)$, by $f = f_K$ the natural homormophism of F onto $G = G(K) = F/N$ and by $g = g_K$ the restriction of f to K. Then g is a homomorphism of K into G and it is the modification of K into the category of groups. A normal subgroup R of G is said to be (pseudo)regular if the homomorphism kg of K into G/R, k being the natural homomorphism of G onto G/R, is a (pseudo)immersion. The partial groupoid K is said to be (pseudo)regular if the unit subgroup of G is so.

4.1. Lemma. Let $K(\circ)$ be a reduced partial groupoid and $K(*) = \overline{K(\circ)}$. Then there exists an isomorphism i of $G(K(\circ))$ onto $G(K(*))$ such that a normal subgroup R

of $G(K(\circ))$ is (pseudo) regular iff the same is true for the normal subgroup $i(R)$ of $G(K(*))$.

Proof. There is an automorphism j of $F(K(\circ)) = F(K(*))$ such that $j(a) = a^{-1}$ for every $a \in K$. Obviously, $j(N(K(\circ))) = N(K(*))$, and hence j induces the isomorphism i.

4.2. Lemma. Let $K(\circ)$ be a cancellative reduced partial groupoid with $B(K(\circ)) \cap \cap (C(K(\circ)) \cup D(K(\circ))) = \emptyset$. Put $K(*) = K(\circ)^{-1}$. Then there exists an isomorphism i of $G(K(\circ))$ onto $G(K(*))$ such that a normal subgroup R of $G(K(\circ))$ is regular iff the normal subgroup $i(R)$ of $G(K(*))$ is so.

Proof. There is an automorphism j of $F(K(\circ)) = F(K(*))$ such that $j(a) = a$ and $j(b) = b^{-1}$ for all $a \in C(K(\circ)) \cup D(K(\circ))$ and $b \in B(K(\circ))$. Obviously $j(N(K(\circ)) = = N(K(*))$.

4.3. Lemma. Let $K(\circ)$ be a balanced cancellative reduced partial groupoid and $K(*)$ a parastrophe of $K(\circ)$. Then there exists an isomorphism i of $G(K(\circ))$ onto $G(K(*))$ such that a normal subgroup R of $G(K(\circ))$ is regular iff so is the normal subgroup $i(R)$ of $G(K(*))$.

Proof. Apply 2.4, 2.5(i), 4.1 and 4.2.

4.4. Corollary. Let K be a regular balanced cancellative reduced partial groupoid. Then every parastrophe of K is regular.

Let $K = K(\circ)$ be a reduced partial groupoid, $g = g_K$ and $x = (a, b) \in M(K)$. Denote by $H(K, x)$ the subgroup of $G = G(K)$ generated by all $g(a)^{-1}\, g(c)$ and $g(d)\, g(b)^{-1}$ where $(c, d) \in M(K)$ and let $P(K, x)$ be the normal subgroup of G generated by $g(a)$ and $g(b)$. Denote by p_x the natural projection of G onto $G/P(K, x)$. By [1, Lemma 3.3], $H(K, x) = H(K, y) = H(K)$ for all $x, y \in M(K)$ and $p_x(H(K)) = = G/P(K, x)$. Moreover, if K is balanced then there exists an isomorphism f_x of $G/P(K, x)$ onto $H(K)$ with $t_x \mid H(K) = id$, $t_x = f_x p_x$ ([1, Lemma 4.3]). A normal subgroup S of $H(K)$ is said to be (pseudo)regular if so is the normal subgroup $t_x^{-1}(S)$ of G ([1, Lemma 4.5]).

4.5. Lemma. Let $K(\circ)$ be a reduced partial groupoid, $K(*) = \overline{K(\circ)}$ and let i be the isomorphism of $G(K(\circ))$ onto $G(K(*))$ by 4.1. Then $i(H(K(\circ)) = H(K(*))$ and a normal subgroup S of $H(K(\circ))$ is (pseudo)regular iff so is the normal subgroup $i(S)$ of $H(K(*))$.

Proof. Easy.

4.6. Lemma. Let $K(\circ)$ be a balanced cancellative reduced partial groupoid, $K(*) = = K(\circ)^{-1}$ and let i be the isomorphism of $G(K(\circ))$ onto $G(K(*))$ by 4.2. Further, let $x = (a, b) \in M(K(\circ))$ and $a \circ b = c$. Then:

(i) $i(H(K(\circ))) = g_{K(*)}(a)\, H(K(*))\, g_{K(*)}(a)^{-1}$.

(ii) $i(P(K(\circ), x)) = P(K(*), y)$ where $y = (a, c) \in M(K(*))$.

(iii) A normal subgroup S of $H(K(\circ))$ is regular iff so is the normal subgroup $g_{K(*)}(a)^{-1}\, i(S)\, g_{K(*)}$ of $H(K(*))$.

Proof. Easy.

4.7. Lemma. Let $K(\circ)$ be a balanced cancellative reduced partial groupoid and $K(*)$ a parastrophe of $K(\circ)$. Then there exists an isomorphism k of $H(K(\circ))$ onto $H(K(*))$ such that a normal subgroup S of $H(K(\circ))$ is regular iff so is the normal subgroup $k(S)$ of $H(K(*))$.

Proof. Apply 2.4, 2.5 (i), 4.5 and 4.6.

5. Couples of companions

Let $K(\circ)$ and $K(*)$ be partial groupoids with the same underlying set. We shall say that the ordered pair $(K(\circ), K(*))$ is a couple of companions if both the partial groupoids are reduced and cancellative, $M(K(\circ)) = M(K(*)) = M$ and for all $(a, b) \in M$ there exist $(a, c), (d, b), (a, e), (f, b) \in M$ such that $c \neq b \neq e$, $d \neq a \neq f$ and $a \circ b = a * c = d * b$, $a * b = a \circ e = f \circ b$.

A partial groupoid $K(\circ)$ is called (strictly) exchangeable if there exists at least one (just one) partial groupoid $K(*)$ such that $(K(\circ), K(*))$ is a couple of companions.

5.1. Lemma. Let $I = (K(\circ), K(*))$ be a couple of companions. Then:

(i) $M(I) = M(K(\circ)) = M(K(*))$, $A(I) = A(K(\circ)) = A(K(*))$, $B(I) = B(K(\circ)) = B(K(*))$, $C(I) = C(K(\circ)) = C(K(*))$ and $D(I) = D(K(\circ)) = D(K(*))$.

(ii) $m(I) = m(K(\circ)) = m(K(*))$, $p(I) = p(K(\circ)) = p(K(*))$, $q(I) = q(K(\circ)) = q(K(*))$ and $o(I) = o(K(\circ)) = o(K(*))$.

(iii) $p(I, b) = p(K(\circ), b) = p(K(*), b)$, $q(I, c) = q(K(\circ), c) = q(K(*), c)$ and $o(I, d) = o(K(\circ), d) = o(K(*), d)$.

(iv) $\delta(I) = \delta(K(\circ)) = \delta(K(*))$, $\delta(I, p) = \delta(K(\circ), p) = \delta(K(*), p)$, $\delta(I, q) = \delta(K(\circ), q) = \delta(K(*), q)$ and $\delta(I, o) = \delta(K(\circ), o) = \delta(K(*), o)$, provided K is finite.

(v) The ordered pair $\tilde{I} = (K(*), K(\circ))$ is a couple of companions (the converse couple).

(vi) Both $K(\circ)$ and $K(*)$ are exchangeable.

(vii) $K(\circ)$ is balanced iff $K(*)$ is so.

(viii) $a \circ b \neq a * b$ for all $(a, b) \in M(I)$.

(ix) $a \circ K = a * K$ and $K \circ a = K * a$ for every $a \in K$.

Proof. Obvious.

Let $I = (K(\circ), K(*))$ and $J = (L(\circ), L(*))$ be couples of companions. A mapping f of K into L is said to be a (strong) homomorphism of I into J if f is simultaneously a (strong) homomorphism of $K(\circ)$ into $L(\circ)$ and of $K(*)$ into $L(*)$.

5.2. Lemma. Let $I = (K(\circ), K(*))$ be a couple of companions. Then:

(i) The ordered pairs $\bar{I} = (\overline{K(\circ)}, \overline{K(*)})$, $I^{-1} = (K(\circ)^{-1}, K(*)^{-1})$, ${}^{-1}I = ({}^{-1}K(\circ), {}^{-1}K(*))$, etc., are couples of companions.

(ii) All the parastrophes of $K(\circ)$ are eaxchangeable.

Proof. Easy.

Let $I = (K(\circ), K(*))$ and $J = (L(\circ), L(*))$ be couples of companions. We shall say that J is an isotope of I if there exist bijections f, g and h of $B(I)$ onto $B(J)$, $C(I)$ onto $C(J)$ and $D(I)$ onto $D(J)$, resp., such that $M(J) = \{(f(a), g(b));\ (a, b) \in M(I)\}$ and $h(a \circ b) = f(a) \circ g(b)$, $h(a * b) = f(a) * g(b)$ for all $(a, b) \in M(I)$.

5.3. Lemma. Let I and J be couples of balanced companions such that J is an isotope of I. Then these couples are isomorphic.

Proof. Easy.

Let $I = (K(\circ), K(*))$ be a couple of companions. A non-empty subset N of $M(I)$ is said to be admissible if the ordered pair $(S[K(\circ), N], S[K(*), N])$ is again a couple of companions. The couple I is called simple if $M(I)$ contains no proper admissible subset.

5.4. Lemma. Let I be a simple couple of companions. Then $\bar{I}$ and all the parastrophes of I are simple.

Proof. Easy.

A partial groupoid K is said to be (strongly) primary if it is exchangeable and every pseudoimmersion of an exchangeable partial groupoid into K is a surjective strong pseudoimmersion (immersion).

5.5. Lemma. An exchangeable partial groupoid K is primary iff every immersion of an exchangeable partial groupoid into K is surjective and strong.

Proof. The direct implication is clear. As for the converse one, let f be a pseudoimmersion of an exchangeable partial groupoid L into K. Put $P = S[K, f(M(L))]$. By [1, Lemma 6.2], P is an exchangeable partial groupoid and the identity mapping of P into K is an immersion. Consequently, $P = K$ and f is surjective and strong.

5.6. Corollary. Every parastrophe of a primary exchangeable partial groupoid is primary and exchangeable.

5.7. Lemma. Let K be an exchangeable partial groupoid. Then K is primary iff for every non-empty and proper subset N of $M(K)$ the partial groupoid $S[K, N]$ is not exchangeable.

5.8. Corollary. Let $I = (K(\circ), K(*))$ be a couple of companions such that at least one of the partial groupoids $K(\circ)$ and $K(*)$ is primary. Then I is simple.

A couple $I = (K(\circ), K(*))$ of companions is said to be left (resp. right) (strongly) primary if $K(\circ)$ (resp. $K(*)$) is (strongly) primary. It is said to be (strongly) primary if it is both left and right (strongly) primary. Further, I is said to be left (resp. right) (pseudo)regular if $K(\circ)$ (resp. $K(*)$) is (pseudo)regular and I is said to be (pseudo)regular if it is both left and right (pseudo)regular.

6. Several inequalities

6.1. Lemma. Let K be a finite exchangeable partial groupoid, $\delta = \delta(K)$, $m = m(K)$, $p = p(K)$, $q = q(K)$ and $o = o(K)$. Then:

(i) $2 \leqq \min(p, q, o) \leqq \max(p, q, o) \leqq m/2$.

(ii) $2 \leqq o(d) \leqq \min(p, q)$, $2 \leqq p(b) \leqq \min(o, q)$ and $2 \leqq q(c) \leqq \min(o, p)$ for all $b \in B(K)$, $c \in C(K)$ and $d \in D(K)$.

(iii) $4 \leqq m \leqq \min(pq, po, qo)$.

(iv) $\delta \leqq 3(m - 2\sqrt{m})$.

(v) $\delta/3 + 2 + 2\sqrt{[(\delta + 3)/3]} \leqq m$.

(vi) $p + q + o = (3m - \delta)/2$ and $3\sqrt{m} \leqq p + q + o$.

Proof. The assertions (i), (ii), (iii) are easy. (iv) By (iii), $9m \leqq 3(pq + qo + po) \leqq$ $\leqq 2(pq + qo + po) + p^2 + q^2 + o^2 = (p + q + o)^2$.
(v) and (vi). These are an easy consequence of (iv).

6.2. Lemma. Let K be a finite exchangeable partial groupoid. If $\delta(K) = 0$ (resp. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) then $m(K) \geqq 4$ (resp. 5, 6, 7, 8, 9, 10, 9, 10, 11, 12, 11).

Proof. Apply 1.1 (v) and 6.1 (v).

6.3. Lemma. Let K be a finite exchangeable partial groupoid, $\delta = \delta(K)$, $m = m(K)$, $p = p(K)$, $q = q(K)$ and $o = o(K)$.

(i) If $\delta = 0$ then $m \geqq 4$ is even and $p = q = o = m/2$.

(ii) $\delta \neq 1$.

(iii) If $\delta = 2$ then $m \geqq 6$ is even and $\{p, q, o\} = \{m/2, m/2, (m/2) - 1\}$.

(iv) If $\delta = 3$ then $m \geqq 7$ is odd and $p = q = o = (m - 1)/2$.

(v) If $\delta = 4$ then $m \geqq 8$ is even and if $m = 8$ then either $\{p, q, o\} = \{4, 2, 2\}$ or $\{p, q, o\} = \{4, 3, 3\}$.

(vi) If $\delta = 5$ then $m \geqq 9$ is odd and $\{p, q, o\} = \{(m - 1)/2, (m - 1)/2, (m - 3)/2\}$.

(vii) If $\delta = 6$ them $m \geqq 8$ is even and if $m = 8$ then $p = q = o = 3$.

(viii) If $\delta = 7$ then $m \geqq 9$ is odd and if $m = 9$ then $\{p, q, o\} = \{4, 3, 3\}$.

(ix) If $\delta = 8$ then $m \geqq 10$ is even and if $m = 10$ then $\{p, q, o\} = \{4, 4, 3\}$.

(x) If $\delta = 9$ then $m \geqq 9$ is odd and if $m = 9$ then $p = q = o = 3$.

(xi) If $\delta = 10$ then $m \geqq 10$ is even and if $m = 10$ then $\{p, q, o\} = \{4, 3, 3\}$.

(xii) If $\delta = 11$ then $m \geqq 11$ is odd and if $m = 11$ then $\{p, q, o\} = \{4, 4, 3\}$.

(xiii) If $\delta = 12$ then $m \geqq 12$ is even and if $m = 12$ then either $\{p, q, o\} = \{5, 4, 3\}$ or $p = q = o = 4$.

(xiv) If $\delta = 13$ them $m \geqq 13$ is odd and if $m = 13$ then either $\{p, q, o\} = \{5, 5, 3\}$ or $\{p, q, o\} = \{5, 4, 4\}$.

Proof. Apply 1.1 (v), 6.1 and 6.2.

6.4. Lemma. Let K be a finite exchangeable partial groupoid, $m = m(K)$ and $\delta = \delta(K)$.

(i) If $m = 4$ then $\delta = 0$.

(ii) $m \neq 5$.

(iii) If $m = 6$ then $\delta = 0, 2$.

(iv) If $m = 7$ then $\delta = 3$ and $p = q = o = 3$.

(v) If $m = 8$ then $\delta = 0, 2, 4, 6$.

(vi) If $m = 9$ then $\delta = 3, 5, 7, 9$.

(vii) If $m = 10$ then $\delta = 0, 2, 4, 6, 8, 10$.

(viii) If $m = 11$ then $\delta = 3, 5, 7, 9, 11$.

Proof. Apply 1.1 (v), 6.1 and 6.3.

7. An example

Let $k \geqq 2$, $B = \{a, b\}$, $C = \{c_1, \ldots, c_k\}$ and $K = \{1, 2, \ldots, k\}$. Define a balanced partial groupoid $Z = Z(\circ) = Z(k, \circ)$ as follows: $Z = B \cup C \cup K$; $a \circ c_i = i$ for every $i \in K$; $b \circ c_j = j + 1$ for every $j \in K$, $j \neq k$; $b \circ c_k = 1$. Put $m = m(Z)$, etc.

7.1. Proposition.

(i) Z is a balanced cancellative reduced partial groupoid.

(ii) card $(Z) = 2k + 2$, $m = 2k$, $p = q(c_i) = 2$, $q = o = o(i) = k$, $\delta = \delta(p) = 2k - 4$, $\delta(q) = \delta(o) = 0$.

(iii) Z is regular, $H(Z)$ is a cyclic group of order k and no non-trivial subgroup of $H(Z)$ is pseudoregular.

(iv) The partial groupoids Z and Z^{-1} are isomorphic and the partial groupoids Z, $\bar{Z}$ and ${}^{-1}Z$ are pair-wise non-isomorphic, provided $k \geqq 3$.

(v) If $k = 2$ then the partial groupoids $Z, \bar{Z}, Z^{-1}, {}^{-1}Z$ are isomorphic.

Proof. Easy; for (iii), use eventually some results from [1, §5].

Now define a partial operation $*$ on Z as follows: $b * c_i = i$ for every $i \in K$; $a * c_j = j + 1$ for every $j \in K$, $j \neq k$; $a * c_k = 1$. Put $I = I(k) = (Z(\circ), Z(*))$.

7.2. Proposition.

(i) I is a strongly primary regular couple of balanced companions.

(ii) The couples I and $\tilde{I}$ are isomorphic and the partial groupoids $Z(\circ)$ and $Z(*)$ are isomorphic.

(iii) The partial groupoid $Z(\circ)$ is strictly exchangeable and strongly primary.

(iv) The couples I and I^{-1} are isomorphic and the couples $I, \bar{I}$ and ${}^{-1}I$ are pair-wise non-isomorphic, provided $k \geqq 3$.

(v) If $k = 2$ then the couples $I, \bar{I}, I^{-1}$ and ${}^{-1}I$ are isomorphic.

Proof. Easy.

8. Auxiliary results

8.1. Lemma. Let $(K(\circ), K(*))$ be a couple of companions, $a, b, c, d, e \in K$, $a \neq b$, $a \circ c = e = b \circ d$ and $o(e) = 2$. Then $a * d = e = b * c$.

Proof. Obvious.

Let K be a partial groupoid. For $c \in C(K)$, put $B(c) = B(K, c) = \{b;\ (b, c) \in M(K)\}$, $E(c) = E(K, c) = \{b \in B(c);\ o(bc) = 2\}$ and define a mapping $s_c = s_{K,c}$ of $E(c)$ into $B(K)$ by $s_c(b) = a$ where $(a, d) \in M(K)$, $bc = ad$ and $(b, c) \neq (a, d)$.

8.2. Lemma. Let K be an exchangeable partial groupoid and $c \in C(K)$. Then:

(i) s_c is an injective mapping of $E(c)$ into $B(c)$ and $s_c(b) \neq b$ for every $b \in E(c)$.

(ii) s_c is a permutation of $B(c)$, provided $E(c) = B(c)$.

Proof. Use 8.1.

8.3. Proposition. Let $K(\circ)$ be a reduced cancellative partial groupoid such that $o(d) = 2$ for every $d \in D(K(\circ))$. Then $K(\circ)$ is exchangeable iff the mapping s_c is a permutation of $B(c)$ for every $c \in C(K(\circ))$. In this case, $K(\circ)$ is strictly exchangeable.

Proof. The direct implication follows from 8.2. Now, assume that $o(d) = 2$ for every $d \in D(K(\circ))$ and put $b * c = s_c^{-1}(b) \circ c$ for every $(b, c) \in M(K(\circ))$. Clearly, $K(*)$

is a reduced partial groupoid and $M(K(*)) = M(K(\circ))$. Let us show that $K(*)$ is left cancellative. Let $b * c = b * d = f$. Then $f = s_c^{-1}(b) \circ c = s_d^{-1}(b) \circ d$ and and $s_c^{-1}(b) \circ c = b \circ e$. Since $o(f) = 2$, either $c = d$ or $d = e$, $s_d^{-1}(b) = b$, a contradiction. The rest is similar.

8.4. Lemma. Let $I = (K(\circ), K(*))$ be a simple couple of finite companions and let $L = L(\circ)$ be a partial subgroupoid of $K(\circ)$ such that $p(L, b) = p(I, b)$ and $q(L, c) = q(I, c)$ for all $b \in B(L)$ and $c \in C(L)$. Then $L(\circ) = K(\circ)$.

Proof. Let $(b, c) \in M(L)$ and $b * c = d$. There are $e, f \in K$ with $d = b \circ e = f \circ c$, $e \neq c$, $f \neq b$. Now, with respect to the hypothesis, $(b, e), (f, c) \in M(L)$ and it is easy to see that $(S[K(\circ), M(L)], S[K(*), M(L)]$ is a couple of companions.

8.5. Corollary. Let $I = (K(\circ), K(*))$ be a simple couple of finite companions and let $L = L(\circ)$ be a partial subgroupoid of $K(\circ)$ such that $\delta(L, p) = \delta(I, p)$ and $\delta(L, q) = \delta(I, q)$. Then $L(\circ) = K(\circ)$.

8.6. Lemma. Let K be a primary exchangeable partial groupoid such that K is not strongly primary. Then $o(K, d) \geqq 4$ for at least one $d \in D(K)$ and hence $p(K), q(K) \geqq 4$.

Proof. Easy.

Let $I = (K(\circ), K(*))$ be a couple of companions and suppose that $(a, c), (b, c) \in M(I)$, $a \neq b$, $a \circ c = d$ and $o(d) = q(c) = 2$. A finite sequence $x = (c_1, \ldots, c_r)$ is called admissible if $r \geqq 2$, $c_1, \ldots, c_r$ are pair-wise different elements of $C(I)$, $c_1 = c$, $a \circ c_i = b \circ c_{i+1}$ and $q(c_i) = o(a \circ c_i) = 2$ for every $1 \leqq i < r$. Further, we shall say that x is maximal if it has no admissible prolongation.

8.7. Lemma. Let $(c_1, \ldots, c_r)$ be an admissible sequence. Then $b * c_i = a * c_{i+1} = a \circ c_i$ for every $1 \leqq i < r$.

Proof. Use 8.1.

8.8. Lemma. Let $(c_1, \ldots, c_r)$ be a maximal admissible sequence. Then at least one of the following three conditions is satisfied:

(1) $q(c_r) = 2$, $b \circ c_1 = a \circ c_r = a * c_1 = b * c_r$.
(2) $q(c_r) \geqq 3$ and $p(I) \geqq 3$.
(3) $o(a \circ c_r) \geqq 3$ and $p(I) \geqq 3$.

Proof. Suppose that none of these conditions is satisfied. By 8.7, $(a, c_r) \in M(I)$ and $a * c_r = b * c_{r-1}$. By 8.1, $a \circ c_r = b \circ c_{r+1}$ for some $c_{r+1} \in K$. Since (1) is not saisfied, $c_{r+1} \neq c_1$ and the sequence $(c_1, \ldots, c_{r+1})$ is admissible, a contradiction.

8.9. Lemma. Suppose that I is simple and $(c_1, \ldots, c_r)$ is a maximal admissible sequence satisfying (1). Then $M(I) = \{a, b\} \times \{c_1, \ldots, c_r\}$.

Proof. Use 8.7.

8.10. Lemma. There exists $e \in C(I)$ such that the sequence (c, e) is admissible.

Proof. Use 8.1.

8.11. Lemma. Suppose that K is finite. Then there exists at least one maximal admissible sequence.

Proof. Use 8.10.

9. Couples of companions with $p(I) = 2$

9.1. Proposition. Let $I = (K(\circ), K(*))$ be a simple couple of finite balanced companions and $m = m(I)$. The following conditions are equivalent:

(i) m is even and I is isomorphic to one of the couples $I(m/2)$, $\overline{I(m/2)}$, ${}^{-1}I(m/2)$.

(ii) At least two of the numbers $\delta(I, p)$, $\delta(I, q)$, $\delta(I, o)$ are equal to 0.

(iii) m is even and at least two of the numbers $p(I)$, $q(I)$, $o(I)$ are equal to $m/2$.

(iv) At least one of the numbers $p(I)$, $q(I)$, $o(I)$ is equal to 2.

Proof. The implications (i) implies (ii) and (ii) implies (iii) are clear.

(iii) implies (iv). This implication is easy (use 8.1).

(iv) implies (i). Without loss of generality, we can assume that $p(I) = 2$. Then $q(c) = o(d) = 2$ for all $c \in C(I)$ and $d \in D(I)$. The result follows now easily from 8.7, 8.8(1), 8.9 and 8.11.

9.2. Proposition. Let $I = (K(\circ), K(*))$ be a simple couple of finite balanced companions. Suppose that there are $a, b \in B(I)$ such that $a \neq b$ and $p(I, a) \geqq p(I, b) \geqq$ $\geqq q(I) \geqq o(I)$. Then $m(I)$ is even and I is isomorphic to $I(m(I)/2)$.

Proof. Clearly, $p(a) = p(b) = q = o$ and there are two bijections f and g of $C(I)$ such that $a \circ c = f(c)$ and $b \circ C = g(c)$ for every $c \in C(I)$. Now, the subset $\{a, b\} \times C(I)$ of $M(I)$ is admissible, $p(I) = 2$ and the result follows from 9.1.

9.3. Proposition. Let I be a simple couple of finite balanced companions, $m = m(I)$ and $\delta = \delta(I)$.

(i) If $\delta = 0$ them $m = 4$ and I is isomorphic to the couple $I(2)$.

(ii) If $\delta = 2$ then $m = 6$ and I is isomorphic to one of the couples $I(3)$, $I(3)$, ${}^{-1}I(3)$.

(iii) If $m = 4$ then $\delta = 0$ and if $m = 6$ then $\delta = 2$.

(iv) If $m = 7$ then $\delta = 3$ and $p = q = o = 3$.

(v) If $m = 8$ then $\delta = 4$ and either $\{p, q, o\} = \{4, 3, 3\}$ or I is isomorphic to one of the couples $I(4)$, $I(4)$, ${}^{-1}I(4)$.

(vi) If $m = 9$ then $\delta = 5$ and $\{p, q, o\} = \{4, 4, 3\}$.

Proof. For (i), ..., (v), use 6.3, 6.4 and 9.2. (vi) First, let $p = q = 3$. Then, by 1.2 and 6.1, $B(I) = \{a, b, c\}$, $C(I) = \{e, f, g\}$ and $p(a) = p(b) = p(c) = q(e) = q(f) = q(g) = 3$. It is easy to see that then $o = 3$, a contradiction with 9.2. Now, according to 6.3 and 6.4, $\delta = 5$ and $\{p, q, o\} = \{4, 4, 3\}$. For the rest, use the parastrophy.

10. Examples

Consider the following two balanced partial groupoids $K_1(\circ)$ and $K_1(*) : K_1 = \{a, b, c, e, f, g, 1, 2, 3\}$, $a \circ e = b \circ f = 1$, $a \circ f = b \circ e = c \circ g = 2$, $a \circ g = c \circ e = 3$, $a * f = b * e = 1$, $a * g = b * f = c * e = 2$, $a * e = c * g = 3$. Put $I_1 = (K_1(\circ), K_1(*))$.

10.1. Proposition.

(i) $K_1 = K_1(\circ)$ is a balanced reduced cancellative partial groupoid, $K_1(\circ)$ is regular, $H(K_1)$ is a cyclic group of order 4 and no non-trivial subgroup of $H(K_1)$ is regular.

(ii) $\operatorname{card}(K_1) = 9$, $m(K_1) = 7$, $p(K_1) = q(K_1) = o(K_1) = 3$, $p(a) = q(e) = o(2) = 3$, $p(b) = p(c) = q(f) = q(g) = o(1) = o(3) = 2$, $\delta(K_1) = 3$, $\delta(p) = \delta(q) = \delta(o) = 1$.

(iii) The partial groupoids $K_1(\circ)$ and $K_1(*)$ are isomorphic.

(iv) Every parastrophe of $K_1(\circ)$ is isomorphic to $K_1(\circ)$.

Proof. Easy.

10.2. Proposition.

(i) I_1 is a simple regular couple of balanced companions and I_1 is not primary.

(ii) All parastrophes of I_1 and $\tilde{I}_1$ are isomorphic to I_1.

(iii) The partial groupoid $K_1(\circ)$ is exchangeable, but not primary.

Proof. Easy.

Consider the following two balanced partial groupoids $K_2(\circ)$ and $K_2(*) : K_2 = \{a, b, c, e, f, g, 1, 2, 3, 4\}$, $a \circ e = b \circ g = 1$, $a \circ f = c \circ e = 2$, $a \circ g = b \circ f = 3$, $b \circ e = c \circ f = 4$, $a * g = b * e = 1$, $a * e = c * f = 2$, $a * f = b * g = 3$, $b * f = c * e = 4$. Put $I_2 = (K_2(\circ), K_2(*))$.

10.3. Proposition.

(i) $K_2 = K_2(\circ)$ is a balanced reduced cancellative partial groupoid, $K_2(\circ)$ is regular, $H(K_2)$ is a cyclic group of order 5.

(ii) $\operatorname{card}(K_2) = 10$, $m(K_2) = 8$, $p(K_2) = q(K_2) = 3$, $o(K_2) = 4$, $p(a) = p(b) =$

$= q(e) = q(f) = 3$, $p(c) = q(g) = o(1) = o(2) = o(3) = o(4) = 2$, $\delta(K_2) = 4$, $\delta(p) = \delta(q) = 2$, $\delta(o) = 0$.

(iii) The partial groupoids $K_2(\circ), \overline{K_2(\circ)}$ and $K_2(*)$ are isomorphic. The partial groupoids $K_2(\circ), K_2(\circ)^{-1}$ and ${}^{-1}K_2(\circ)$ are pair-wise non-isomorphic.

Proof. Easy.

10.4. Proposition.

(i) I_2 is a strongly primary regular couple of balanced companions.

(ii) The couples $I_2, \bar{I}_2$ and $\tilde{I}_2$ are isomorphic.

(iii) The couples I_2, I_2^{-1} and ${}^{-1}I_2$ are pair-wise non-isomorphic.

(iv) The partial groupoid $K_2(\circ)$ is strictly exchangeable and strongly primary.

Proof. Easy.

11. Primary groupoids with $m(K) \leq 8$

11.1. Proposition.

(i) Let I be a couple of companions of rank at most 8. Then $m(I) = 4, 6, 7, 8$.

(ii) $I(2)$ is a simple couple of balanced companions of rank 4 and every couple of balanced companions of rank 4 is isomorphic to $I(2)$.

(iii) $I(3), \overline{I(3)}$ and ${}^{-1}I(3)$ are pair-wise non-isomorphic simple couples of balanced companions of rank 6 and every couple of balanced companions of rank 6 is isomorphic to one of them.

(iv) I_1 is a couple of balanced companions of rank 7 and every couple of balanced companions of rank 7 is isomorphic to I_1.

(v) $I(4), \overline{I(4)}, {}^{-1}I(4), I_2, I_2^{-1}$ and ${}^{-1}I_2$ are simple couples of balanced companions of rank 8 and every simple couple of balanced companions is isomorphic to one of them.

Proof. (i), (ii) and (iii). See 6.4(ii) and 9.3.

(iv) Let $I = (K(\circ), K(*))$ be a couple of balanced companions with $m(I) = 7$ by 9.3(iv), $p(I) = q(I) = o(I) = 3$ and we can assume that $B(I) = \{a, b, c\}$, $C(I) = \{e, f, g\}$ and $D(I) = \{1, 2, 3\}$. Further, it is clear that $\{p(a), p(b), p(c)\} = \{q(e), q(f), q(g)\} = \{3, 2, 2\}$ and we can assume $p(a) = q(e) = 3$, $a \circ e = 1$, $a \circ f = b \circ e = 2$, $a \circ g = c \circ e = 3$ (see 5.3). If $(b, g) \in M(I)$ then $b \circ g = 1$, since $K(\circ)$ is cancellative. But $p(b) = 2$ and, by 5.1(viii), (ix), we must have $b * e = 1$, $b * g = 2$. Similarly, $c * e = 2$, $a * e = 3$, $a * f = 1$, $a * g = 2 = b * g$, a contradiction. We have proved that $(b, f) \in M(I)$, and so $(c, g) \in M(I)$. If $b \circ f = 1$ then it is easy to see that $I = I_1$. If $b \circ f = 3$ then I is an isotope of I_1.

(v) Let $I = (K(\circ), K(*))$ be a simple couple of balanced companions with $m(I) = 8$.

With respect to 9.3(iv), we can assume that $p(I) = q(I) = 3$, $o(I) = 4$, $B(I) = \{a, b, c\}$, $C(I) = \{e, f, g\}$ and $D(I) = \{1, 2, 3, 4\}$. Clearly, $\{p(a), p(b), p(c)\} = \{q(e), q(f), q(g)\} = \{3, 3, 2\}$ and we can assume that $p(a) = p(b) = q(e) = q(f) = 3$, $a \circ e = 1$, $a \circ f = 2$, $a \circ g = 3$. Since $4 \in D(I)$, $4 \in b \circ K$. If $b \circ g = 4$ then $4 \in a * K = a \circ K$, a contradiction. Now, we can assume that $b \circ e = 4$. Clearly, $b * g = 3$ and $c * e = 4$, and so $b * e = 1$, $c \circ f = 4$, $b * f = 4$, $b \circ f = 3$, $b \circ g = 1$, $a * f = 3$, $c * f = 2$, $a * e = 2$, $a * g = 1$, $c \circ e = 2$ and $I = I_2$.

11.2. Corollary.

(i) $Z(2, \circ)$ is up to isomorphism the only primary balanced partial groupoid of rank 4.

(ii) $Z(3, \circ)$, $\overline{Z(3, \circ)}$ and ${}^{-1}Z(3, \circ)$ are up to isomorphism the only primary balanced partial groupoids of rank 6.

(iii) There are no primary partial groupoids of ranks 5 and 7.

(iv) $Z(4, \circ)$, $\overline{Z(4, \circ)}$, ${}^{-1}Z(4, \circ)$, $K_2(\circ)$, $K_2(\circ)^{-1}$ and ${}^{-1}K_2(\circ)$ are up to isomorphism the only primary balanced partial groupoids of rank 8.

All these partial groupoids are strongly primary regular.

12. Examples

Consider the following four balanced partial groupoids $K_3(\circ)$, $K_3(*)$, $K_4(\circ)$ and $K_4(*) : K_3 = \{a, b, c, d, e, f, g, h, 1, 2, 3, 4\}$, $a \circ e = b \circ f = a * f = b * e = 1$, $a \circ f = b \circ h = d \circ g = a * g = b * f = d * h = 2$, $a \circ g = b \circ e = c \circ h = a * e = b * h = c * g = 3$, $c \circ g = d \circ h = c * h = d * g = 4$; $K_4 = \{a, b, c, d, e, f, g, h, 1, 2, 3\}$, $a \circ e = b \circ f = c \circ g = d \circ h = a * f = b * e = c * h = d * g = 1$, $a \circ f = b \circ h = d \circ g = a * g = b * f = d * h = 2$, $a \circ g = b \circ e = c \circ h = a * e = b * h = c * g = 3$. Put $I_3 = (K_3(\circ), K_3(*))$ and $I_4 = (K_4(\circ), K_4(*))$.

12.1. Proposition.

(i) I_3 is a strongly primary regular couple of balanced companions.

(ii) Both $K_3(\circ)$ and $K_3(*)$ are strongly primary and regular and the groups $H(K_3(\circ))$ and $H(K_3(*))$ are cyclic groups of order 8.

(iii) There exists surjective strong pseudoimmersion t of I_3 onto I_4 such that t is not an immersion.

(iv) I_4 is a primary regular couple of balanced companions and I_4 is not strongly primary.

(v) Both $K_4(\circ)$ and $K_4(*)$ are primary and regular, they are not strongly primary and the groups $H(K_4(\circ))$ and $H(K_4(*))$ are cyclic groups of order 4.

Proof. Easy.

Consider the following two balanced partial groupoids $K_5(\circ)$ and $K_5(*)$: $K_5 = \{a, b, c, e, f, g, h, 1, 2, 3, 4, 5\}$, $a \circ e = b \circ h = c \circ g = a * h = b * g = c * e = 1$, $a \circ f = b \circ e = a * e = b * f = 2$, $a \circ g = c \circ f = a * f = c * g = 3$, $a \circ h = b \circ g = a * g = b * h = 4$, $b \circ f = c \circ e = b * e = c * f = 5$. Put $I_5 = (K_5(\circ), K_5(*))$.

12.2. Proposition.

(i) I_5 is a left strongly primary left regular couple of balanced companions and I_5 is neither right primary nor right pseudoregular.

(ii) $K_5(\circ)$ is strongly primary, regular and $H(K_5(\circ))$ is a cyclic group of order 5.

(iii) $K_5(*)$ is neither primary nor pseudoregular and $H(K_5(*))$ is a cyclic group or order 2.

Proof. Easy.

Reference

[1] DRÁPAL, A. and KEPKA, T.: Group modifications of some partial groupoids (to appear).