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Dependent Random Variables with a Given Marginal Distribution

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Some models for generating dependent random variables with a given covariance function and with a given marginal distribution are presented in the paper. The classical autoregressive process of the first order and general autoregressive processes with random coefficients are used for this purpose.

V práci jsou uvedeny modely, pomocí nichž lze generovat závislé náhodné veličiny s danou kovarianční funkcí a s daným marginálním rozdělením. Užívá se k tomu klasického autoregresního procesu prvního řádu a obecných autoregresních procesů s náhodnými koeficienty.

В статье рассматриваются модели для конструкции зависимых случайных величин с заданной ковариационной функцией и с заданным частным распределением. Используются классический процесс авторегрессии первого порядка и общие процессы авторегрессии со случайными параметрами.

1. Introduction

Pseudorandom variables with a given distribution are used in Monte Carlo methods. One of the important features of such variables is their statistical independence. However, for modelling real situations dependent pseudorandom variables with a given covariance function would be often more desirable. For example, the lengths of time intervals between the arrivals of customers can be exponentially distributed, but dependent. In a few last years the methods for simulating dependent random variables with the given distribution have been developed. Usually, the paper by Bernier [3] is quoted as one of the first works where the problem of non-normal dependent variables was considered.

One of the simplest models for creating dependent variables is the autoregressive model of the first order – AR(1). We shall assume that $\{Y_s\}$ are independent identically distributed (i.i.d.) random variables such that $EY_s^2 < \infty$ and $Var\ Y_s = \sigma^2 > 0$. Let $\varrho \in (-1, 1)$ be a given number. Then the process $\{X_s\}$ satisfying the relation

$$X_s = \varrho X_{s-1} + Y_s \qquad (1.1)$$

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having the form

$$(1.2) \qquad X_s = \sum_{k=0}^{\infty} c_k Y_{s-k}, \quad \sum_{k=0}^{\infty} c_k^2 < \infty,$$

is called an AR(1) process. From (1.1) and (1.2) we have

$$X_s = \sum_{k=0}^{\infty} \varrho^k Y_{s-k}.$$

It is well known that the process $\{X_s\}$ has the variance $Var\, X_s = \sigma^2/(1 - \varrho^2)$ and the correlation function $\varrho_s = \varrho^{|s|}$. The problem can be then formulated as follows. Let a distribution of the variables X_s in (1.1) be given. It is necessary to find the corresponding distribution of Y_s. Generally, it can happen, that the formulated problem has no solution.

Especially, exponential and gamma distributions of X_s are very important in point processes, particularly in queuing theory. Some papers were devoted to this problem – see Gaver and Lewis [5] and references given there.

Denote

$$\psi(t) = E e^{itY_s} \quad \text{and} \quad \omega(t) = E e^{itX_s}$$

the characteristic functions of Y_s and X_s, respectively. From (1.1) we have the fundamental relation

$$(1.3) \qquad \omega(t) = \omega(\varrho t)\, \psi(t).$$

If X is a random variable with characteristic function $\omega(t)$ such that for every $\varrho \in (0, 1)$ there exists a characteristic function $\psi(t)$ satisfying (1.3), we say that X belongs to the class L, or that X is a self-decomposable random variable – see Feller [4], p. 588. Some results concerning the self-decomposable random variables are given in Shanbhag, Pestana and Sreehari [10], Shanbhag and Sreehari [11], Thorin [12] and Thorin [13].

2. Classical AR(1) model

The basic tool for investigating AR(1) models is the relation (1.3), which immediately gives

$$(2.1) \qquad \psi(t) = \omega(t)/\omega(\varrho t).$$

This allows to calculate the characteristic function $\psi(t)$ in cases when the solution exists. However, the ratio $\omega(t)/\omega(\varrho t)$ may not be a characteristic funtion. Now, we shall briefly apply (2.1) to some frequently occurring distributions.

a. Normal distribution

If $X_s \sim N(0, v^2)$, then $\omega(t) = \exp\{-v^2 t^2/2\}$ and (2.1) gives $\psi(t) =$

$\exp . \{-v^2t^2/2\}/\exp\{-v^2\varrho^2t^2/2\} = \exp\{-v^2(1-\varrho^2)\,t^2/2\}$. Thus we have the known result $Y_s \sim N[0, (1-\varrho^2)\,v^2]$.

b. Exponential distribution (Gaver and Lewis [5])

Let X_s have an exponential distribution with parameter a, i.e. the density

$$f(x) = a^{-1}\exp\{-x/a\} \quad \text{for} \quad x > 0$$

and $f(x) = 0$ for $x \leqq 0$. This distribution will be denoted $Ex(a)$. Then $\omega(t) = (1 - iat)^{-1}$ and from (2.1)

$$\psi(t) = \varrho + (1-\varrho)(1-iat)^{-1}\,.$$

For $0 \leqq \varrho < 1$ the function $\psi(t)$ corresponds to a random variable which is zero with probability (w.p.) ϱ and equals to a random variable having $Ex(a)$ with probability $1-\varrho$. If E_s is a sequence of i.i.d. variables with $Ex(a)$, then the model (1.1) takes on the form

$$X_s = \begin{cases} \varrho X_{s-1} & \text{w.p.} \quad \varrho\,, \\ \varrho X_{s-1} + E_s & \text{w.p.} \quad 1-\varrho\,. \end{cases} \tag{2.2}$$

c. Gamma distribution (Gaver and Lewis [5])

Let X_s have $\Gamma(a, p)$ distribution with the density

$$f(x) = \frac{1}{a^p\,\Gamma(p)}\,e^{-x/a}x^{p-1} \quad \text{for} \quad x > 0$$

and $f(x) = 0$ for $x \leqq 0$, where $a > 0$ and $p > 0$ are given parameters. The characteristic function of $\Gamma(a, p)$ is $\omega(t) = (1 - iat)^{-p}$. Then

$$\psi(t) = [\varrho + (1-\varrho)(1-iat)^{-1}]^p\,.$$

Consider the case $0 \leqq \varrho < 1$. The result for $p = 1$ is given above in (2.2), since $\Gamma(a, 1) = Ex(a)$. If p is an integer, then we can obtain explicit results. For $p = 2$ we get

$$\psi(t) = \varrho^2 + 2\varrho(1-\varrho)(1-iat)^{-1} + (1-\varrho)^2(1-iat)^{-2}\,,$$

in the case $p = 3$ we have

$$\psi(t) = \varrho^3 + 3\varrho^2(1-\varrho)(1-iat)^{-1} + 3\varrho(1-\varrho)^2(1-iat)^{-2} + (1-\varrho)^3(1-iat)^{-3}\,.$$

For $p = 2$, the distribution corresponding to $\psi(t)$ is a mixture of zero, $\Gamma(a, 1)$ and $\Gamma(a, 2)$ with weights ϱ^2, $2\varrho(1-\varrho)$ and $(1-\varrho)^2$, respectively. Similar result follows for $p = 3$ etc.

d. Laplace (double exponential) distribution (Anděl [2])

Let X_s have Laplace distribution $L(b)$ with the density

$$f(x) = (2b)^{-1} \exp\{-|x|/b\}, \quad -\infty < x < \infty,$$

where $b > 0$ is a parameter. This distribution has the characteristic function

$$\omega(t) = (1 + b^2t^2)^{-1}.$$

Therefore, (2.1) gives

$$\psi(t) = \varrho^2 + (1 - \varrho^2)(1 + b^2t^2)^{-1}.$$

This corresponds to a mixture of zero and $L(b)$ with the weights ϱ^2 and $1 - \varrho^2$, respectively. Thus

$$X_s = \begin{cases} \varrho X_{s-1} & \text{w.p. } \varrho^2, \\ \varrho X_{s-1} + L_s & \text{w.p. } 1 - \varrho^2, \end{cases} \tag{2.3}$$

where $\{L_s\}$ are i.i.d. random variables with $L(b)$.

e. Continuous rectangular distribution (Anděl [2])

Let X_s have a continuous rectangular distribution on $[-a, a]$, where $a > 0$. The corresponding characteristic function is

$$\omega(t) = (at)^{-1} \sin at.$$

We shall consider only the non-trivial case $\varrho \neq 0$. From (2.1)

$$\psi(t) = \varrho \frac{\sin at}{\sin \varrho at}. \tag{2.4}$$

The function $\psi(t)$ does not depend on the sign of ϱ and thus we shall investigate only $\varrho > 0$. There are three possible situations.

(i) Let $\varrho = 1/(2n)$, $n = 1, 2, \ldots$
Then

$$\psi(t) = \frac{1}{2n} \frac{\sin at}{\sin(at/2n)} = \frac{1}{2n} \sum_{k=1}^{n} \left[\exp\left\{\frac{i(2k-1)at}{2n}\right\} + \exp\left\{-\frac{i(2k-1)at}{2n}\right\}\right]$$

is the characteristic function of the discrete rectangular distribution concentrated at the points

$$-\frac{2n-1}{2n}a, \; -\frac{2n-3}{2n}a, \; \ldots, \; -\frac{3}{2n}a, \; -\frac{1}{2n}a, \; \frac{1}{2n}a, \; \frac{3}{2n}a, \; \ldots,$$

$$\ldots \frac{2n-3}{2n}a, \; \frac{2n-1}{2n}a.$$

Each point has probability $1/(2n)$.

(ii) Let $\varrho = 1/(2n + 1)$, $n = 1, 2, \ldots$
In this case

$$\psi(t) = \frac{1}{2n+1} \sum_{k=-n}^{n} \exp\left\{i \frac{2kat}{2n+1}\right\},$$

which is the characteristic function of the discrete rectangular distribution concentrated at the points

$$-\frac{2n}{2n+1}a,\ -\frac{2n-2}{2n+1}a, \ldots,\ -\frac{2}{2n+1}a,\ 0,\ \frac{2}{2n+1}a,\ \ldots,\ \frac{2n-2}{2n+1}a,$$

$$\frac{2n+1}{2n}a\,.$$

The probability of each point is $1/(2n+1)$.

(iii) Let $\varrho \neq 1/n$, $n = 1, 2, \ldots$

If $t \to \pi/\varrho a$, then $\sin \varrho at \to 0$ and $\sin at \to \sin \pi/\varrho \neq 0$. From (2.4) we can see that $|\psi(t)| \to \infty$. Obviousiy, $\psi(t)$ cannot be a characteristic function (the absolute value of any characteristic function cannot exceed 1). Therefore, for $\varrho \neq 1/n$ there exists no distribution of Y_s which would lead to continuous rectangular distribution of X_s in model (1.1).

f. Mixed exponential distribution

Let X_s have the density

$$f(x) = p_1 a_1^{-1} e^{-x/a_1} + p_2 a_2^{-1} e^{-x/a_2} \quad \text{for} \quad x > 0$$

and $f(x) = 0$ for $x \leqq 0$, where $p_1 = 1 - p_2$ and $a_1 > a_2 > 0$ are some parameters. The results are derived in Gaver and Lewis [5] and in Lawrance [6].

g. Cauchy distribution

Let X_s have a Cauchy distribution $C(a, b)$ with the density

$$f(x) = \frac{1}{\pi} \frac{b}{b^2 + (x-a)^2}, \quad -\infty < x < \infty\,,$$

where a is a real and b a positive number. Since

$$\omega(t) = \exp\{iat/b - b|t|\}\,,$$

we obtain

$$\psi(t) = \exp\{iat(1-\varrho)/b - b(1-|\varrho|)\,|t|\}\,.$$

Thus Y_s has

$$C[a(1-\varrho)(1-|\varrho|),\ b(1-|\varrho|)]\,.$$

It is a little surprising that the results of the type (2.2) and (2.3) are not satisfactory. We shall consider (2.2) in detail. First of all, it is easy to see that

$$E(X_{s+1} \mid X_s = x) = \varrho x + (1-\varrho)\,a\,.$$

If $x > a$, then $\varrho x + (1 - \varrho) a < x$ and similarly for $x < a$. Generally, a typical realization of (2.2) consists of decreasing variables with the coefficient ϱ and only from time to time a shock E_s causes a jump to higher values. If we denote by R the number of runs down of the type $X_{s-1} = \varrho X_s$, then

$$ER = \varrho/(1 - \varrho), \quad Var\, R = \varrho/(1 - \varrho)^2 .$$

A key to understanding the bad behaviour of the model is given in the following two assertions.

Lemma 2.1. Let X and Y be independent $Ex(a)$ variables. If $c > 0$, then $P(X < cY) = c/(1 + c)$.

Proof is obvious.

Theorem 2.2. Let X_s be defined by (2.2). Then

$$P(X_s < X_{s-1}) = 1/(2 - \varrho) > 0.5 .$$

Proof. Using Lemma 2.1 we get

$$P(X_s < X_{s-1}) = \varrho + (1 - \varrho)\, P(\varrho X_{s-1} + E_s < X_{s-1}) =$$

$$= \varrho + (1 - \varrho)\, P[E_s < (1 - \varrho) X_{s-1}] = \varrho + (1 - \varrho) \frac{1 - \varrho}{2 - \varrho} = \frac{1}{2 - \varrho} .$$

The problem is to construct such models in which $P(X_s < X_{s-1}) = 0.5$. New models were proposed by Lawrance and Lewis [8].

3. Modified AR(1) models

Consider the following three models, where E_s are independent $Ex(a)$ variables.

Model I: $$X_s = \varrho X_{s-1} + \begin{cases} 0 & \text{w.p. } \varrho , \\ E_s & \text{w.p. } 1 - \varrho , \end{cases} \quad 0 \leqq \varrho < 1 .$$

Model II: $$X_s = (1 - \alpha) E_s + \begin{cases} X_{s-1} & \text{w.p. } \alpha , \\ 0 & \text{w.p. } 1 - \alpha , \end{cases} \quad 0 < \alpha < 1 .$$

Model III: $$X_s = \varepsilon_s + \begin{cases} \beta X_{s-1} & \text{w.p. } \alpha , \\ 0 & \text{w.p. } 1 - \alpha , \end{cases} \quad 0 \leqq \alpha \leqq 1 ,$$

where $0 \leqq \beta \leqq 1$, $\alpha\beta \neq 1$ and ε_s are i.i.d. random variables with a distribution which will be derived later.

Model I is known from Section 2. It was derived that X_s has $Ex(a)$ distribution. Since model I is the classical AR(1), its correlation function is $\varrho_s = \varrho^{|s|}$.

Theorem 3.1. Consider Model II. Then X_s has $Ex(a)$ distribution and $\{X_s\}$ is a stationary process with the correlation function $\varrho_s = \alpha^{|s|}$.

Proof. In the first part it is sufficient to show that if $X_{s-1} \sim Ex(a)$, then $X_s \sim \sim Ex(a)$. Introduce a random variable ξ by

$$\xi = \begin{cases} X_{s-1} & \text{w.p. } \alpha\,, \\ 0 & \text{w.p. } 1-\alpha\,. \end{cases}$$

Then ξ has the characteristic function

$$1 - \alpha + \frac{\alpha}{1 - iat}\,,$$

whereas $(1-\alpha)\,E_s$ has the characteristic function $[1 - ia(1-\alpha)\,t]^{-1}$. Then $(1-\alpha)\,E_s + \xi$ has the characteristic function

$$[1 - ia(1-\alpha)\,t]^{-1}\left[1 - \alpha + \frac{\alpha}{1 - iat}\right] = (1 - iat)^{-1}\,,$$

which corresponds to $Ex(a)$.

It is important to notice that model II is the special case of an autoregressive model with random parameters (ARRP). The assertion about the correlation function follows from general theory of ARRP – see Anděl [1] and Nicholls and Quinn [9].

Theorem 3.2. Consider model III. Then $X_s \sim Ex(a)$ if and only if

$$\varepsilon_s = \begin{cases} E_s & \text{w.p. } (1-\beta)\,[1 - (1-\alpha)\,\beta]^{-1}\,, \\ (1-\alpha)\,\beta E_s & \text{w.p. } \alpha\beta[1 - (1-\alpha)\,\beta]^{-1}\,. \end{cases} \tag{3.1}$$

Proof. Denote $\psi(t)$ the characteristic function of ε_s. Then X_{s-1} and X_s are $Ex(a)$ variables if and only if the relation

$$(1 - iat)^{-1} = \psi(t)\left[1 - \alpha + \frac{\alpha}{1 - ia\beta t}\right]$$

holds. From here we obtain

$$\psi(t) = \frac{1 - ia\beta t}{(1 - iat)\,[1 - ia(1-\alpha)\,\beta t]} =$$

$$= \frac{1-\beta}{1-(1-\alpha)\beta}\,\frac{1}{1 - iat} + \frac{\alpha\beta}{1 - (1-\alpha)\,\beta}\,\frac{1}{1 - ia(1-\alpha)\,\beta t}\,,$$

which is the characteristic function corresponding to (3.1).

Let us remark that model I is the special case of model III when $\alpha = 1$. Then we have $\varrho = \beta$. Also model II is the special case of model III, when $\beta = 1$. Model III also belongs to ARRP and the correlation function of $\{X_s\}$ is $\varrho_s = (\alpha\beta)^{|s|}$.

Theorem 3.3. Consider model III. Let $\alpha \neq 1$, $\beta \neq 1$. Then

$$P(X_s < X_{s-1}) = \frac{(1-\alpha)(1+\beta)}{2[1+(1-\alpha)\beta]} + \frac{\alpha(1-\beta)}{(2-\beta)(1-\alpha\beta)}.$$

Proof. We have

$$P(X_s < X_{s-1}) = (1-\alpha)\, P(X_{s-1} > \varepsilon_s) + \alpha P(X_{s-1} > \varepsilon_s + \beta X_{s-1}).$$

It can be calculated that

$$P(X_{s-1} > \varepsilon_s) = \frac{1}{1-(1-\alpha)\beta}\left[\frac{1-\beta}{2} + \frac{\alpha\beta}{1+(1-\alpha)\beta}\right],$$

$$P[X_{s-1} > (1-\beta)^{-1}\varepsilon_s] = \frac{1}{1-(1-\alpha)\beta}\left[\frac{1}{1+\dfrac{1}{1-\beta}} + \frac{\alpha\beta}{1+\dfrac{(1-\alpha)\beta}{1-\beta}}\right].$$

From here we get the result.

A similar calculation or the limit procedure gives

$$P(X_s < X_{s-1}) = (2-\beta)^{-1} > 0.5 \qquad \text{for} \quad \alpha = 1,$$

$$P(X_s < X_{s-1}) = (1-\alpha)/(2-\alpha) < 0.5 \quad \text{for} \quad \beta = 1.$$

Therefore, $P(X_s < X_{s-1}) = 0.5$ in the following cases:
(i) $\alpha = 0$; (ii) $\beta = 0$; (iii) $\beta = 1/(2-\alpha)$.

4. Models of higher order

The attempts to generalize model (1.1) to AR(n) with $n \geqq 2$ for obtaining dependent random variables were not succesful. Lawrance and Lewis [7] proposed the model

(4.1)
$$X_s = \begin{Bmatrix} \alpha_1 X_{s-1} & \text{w.p.} & 1-\alpha_2 \\ \alpha_2 X_{s-2} & \text{w.p.} & \alpha_2 \end{Bmatrix} + \varepsilon_s.$$

For $\alpha_2 = 0$ we get a model of type (2.2). Let us look for such a distribution of ε_s that $X_s \sim Ex(a)$. If $\psi(t)$ is the characteristic function of ε_s, then (4.1) leads to the condition

$$(1 - iat)^{-1} = \left(\frac{1-\alpha_2}{1-ia\alpha_1 t} + \frac{\alpha_2}{1-ia\alpha_2 t}\right)\psi(t).$$

From here

$$\psi(t) = A + B(1-iat)^{-1} + C[1 - ia\alpha_2(1+\alpha_1-\alpha_2)\, t]^{-1},$$

where

$$A = \alpha_1/(1 + \alpha_1 - \alpha_2)\,, \quad B = (1 - \alpha_1)(1 - \alpha_2)/[1 - \alpha_2(1 + \alpha_1 - \alpha_2)]\,,$$
$$C = (1 - \alpha_2)(\alpha_1 - \alpha_2)^2/\{(1 + \alpha_1 - \alpha_2)[1 - \alpha_2(1 - \alpha_1 - \alpha_2)]\}\,.$$

Therefore, X_s will have $Ex(a)$ if and only if

$$\varepsilon_s = \begin{cases} 0 & \text{w.p. } A\,, \\ Ex(a) & \text{w.p. } B\,, \\ \alpha_2(1 + \alpha_1 - \alpha_2)\,Ex(a) & \text{w.p. } C\,. \end{cases}$$

Again, (4.1) is ARRP model. Its correlation function is given by the relation

$$\varrho_r = \alpha_1(1 - \alpha_2)\,\varrho_{r-1} + \alpha_2^2\varrho_{r-2}\,, \quad r \geqq 2\,.$$

It is possible to generalize model (4.1) to a higher order.

5. Other models

Lawrance and Lewis [7] proposed also MA and ARMA models with random coefficients for calculating dependent random variables with $Ex(a)$ distribution. The simplest model is

$$X_s = \begin{cases} \beta E_s & \text{w.p. } \beta\,, \\ \beta E_s + E_{s-1} & \text{w.p. } 1 - \beta\,, \end{cases}$$

where $E_s \sim Ex(a)$ are i.i.d. variables.

However, let us remark that is possible to use also classical MA models. For example, put

$$X_s = Y_s + Y_{s-1}\,.$$

If Y_s are i.i.d. variables, $Y_s \sim \Gamma(a, 0.5)$, then $X_s \sim Ex(a)$.

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