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The Set of Hypergroups With Operators Which Are Constructed From a Set With Two Elements*)

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Introduction

It is known [2 p. 167], that an hypergroup $\langle H, \perp \rangle$ is a nonempty set H in which

1^{st} an hyperoperation is defined

$\perp : H \times H \to \mathscr{P}(H) : (a, b) \mapsto a \perp b, \quad a \perp b \neq \emptyset$

if $A, B \subset H$ then $A \perp B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a \perp b$

(we write $a \perp B$ for $A = \{a\}$), $\mathscr{P}(H)$ is the power set of H,

2^{nd} the hyperoperation is associative

$$a \perp (b \perp c) = (a \perp b) \perp c, \quad \forall\, a, b, c \in H \tag{1}$$

this means that the sets on the two sides coincide,

$$3^{rd} \quad a \perp H = H \perp a = H, \quad \forall a \in H \tag{2}$$

Also, [3] an hypergroup with operators, or a W-hypergroup, is a set H equipped with an hyperoperation $\perp$ such that $\langle H, \perp \rangle$ is an hypergroup and there exists an external hyperoperation

$$* : W \times H \to \mathscr{P}(H) : (w, a) \mapsto w * a$$

which is distributive i.e.

$$w * (a \perp b) \subseteq (w * a) \perp (w * b) \tag{3}$$

or

$$\bigcup_{p \in a \perp b} w * p \subseteq \bigcup_{\substack{u \in w * a \\ v \in w * b}} u \perp v, \quad \forall w \in W, \; a, b \in H$$

If in (3) the equality is valid we call the distributive law strong otherwise we call it weak.

*) The results of the paper were presented during a lecture at the Charles University, Prague in January 1980.

The aim of this paper is to find out the set of hypergroups with operators that we can construct using a set $H = \{a, b\}$ with only two elements. In the first section of this paper we find the set of all hypergroups that we can construct in H and, in the second section, the set of hypergroups with operators in H.

The study of sets with more than two elements is exceptionally laborious, as we can see in the first section of this paper.

1. The set of hypergroups in $H = \{a, b\}$

Let H be a set with n elements. To define an hyperoperation in this set we have to answer n^2 questions. The number of subsets that we can give as a result in an hyperoperation is equal to $2^n - 1$, so we have as many hyperoperations as the number of all arrangements of $2^n - 1$ things taken n^2 with replacement i.e. $(2^n - 1)^{n^2}$.

So for $n = 2$ we get 81 hyperoperations. For $n = 3$ we get 40,353,607 and so on.

We shall study now the set of hypergroups in $H = \{a, b\}$.

Here the set of results for hyperoperations is $\mathscr{P}(H) - \{\emptyset\}$, but interchanging the roles of elements a and b we always get the same cases except the case in which the answer in every couple is the set $\{a, b\}$. So we study $81 - 40 = 41$ different cases of hyperoperations $\perp$. After that it is easy to check that only in 18 cases condition (2) is valid. Finally after some calculations, for associative law (1), we obtain the following:

Proposition 1. For every set $H = \{a, b\}$ consisting of two elements we can construct only the following eight hypergroups, where the hyperoperations $\perp$ are defined from every column of the table

	i	ii	iii	iv	v	vi	vii	viii
$a \perp a =$	$\{a\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
$a \perp b =$	$\{b\}$	$\{a\}$	$\{a, b\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
$b \perp a =$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
$b \dotplus b =$	$\{a\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$

(4)

Remarks 1. The hypergroups (iii) and (iv) are the only noncommutative in H.

2. Only the first hypergroup (i) can be called a group because it has, as a result: sets with one element each.

3. We did not check the associative law for the subsets of H because it is sufficient to check only for single elements, i.e. $\forall A, B, C \subset H$ we get

$$A \perp (B \perp C) = \bigcup_{a \in A} [a \perp (\bigcup_{\substack{b \in B \\ c \in C}} b \perp c)] = \bigcup_{\substack{a \in A \\ b \in B \\ c \in C}} [a \perp (b \perp c)] \,.$$

2. The set of hypergroups with operators in $H = \{a, b\}$

An operator acts in an hypergroup $\langle H, \perp \rangle$ with operators like a mapping from H into the power set $\mathscr{P}(H)$ [1 p. 223]. So every operator $w \in W$ defines a mapping

$$f : H \to \mathscr{P}(H) : x \mapsto f(x) = w * x \tag{5}$$

where

$$f(x \perp y) \subseteq f(x) \perp f(y), \quad \forall x, y \in H \tag{6}$$

Therefore, it is sufficient to find all those mappings f which satisfy (6) and we shall call them distributively weak, or the equation

$$f(x \perp y) = f(x) \perp f(y), \quad \forall x, y \in H \tag{7}$$

holds, in which case we shall call them distributively strong mappings.

We observe that (6), (7) are valid even if, instead of $x, y \in H$, we set subsets X and Y of H, because

$$f(X \perp Y) = \bigcup_{\substack{x \in X \\ y \in Y}} f(x \perp y), \quad \forall X, Y \subset H .$$

In our case we can get only the following mappings from the columns of the table

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
$f(a)=$	$\{a\}$	$\{a\}$	$\{a\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
$f(b)=$	$\{a\}$	$\{b\}$	$\{a, b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$

For all hypergroups from (4) and for the above mappings, we check the distributive law from (6) and (7) and we get the table

(8)

	i	ii	iii	iv	v	vi	vii	viii
f_1	S	W	W	W	S	W	W	W
f_2	S	S	S	S	S	S	S	S
f_3	W	W	W	W	S	W	S	S
f_4	—	—	—	—	S	—	—	S
f_5	—	S	S	S	S	—	S	W
f_6	—	—	—	—	S	—	—	S
f_7	—	W	W	W	S	S	W	S
f_8	—	S	S	S	S	—	S	S
f_9	S	S	S	S	S	S	S	S

We used – when f_λ, $\lambda = 1, \ldots, 9$ is not distributive to the corresponding hyperoperation, S when f_λ is distributively strong and, W when f_λ is distributively weak.

So we have the following:

Proposition 2. The set of hypergroups with operators, which are constructed from the set $H = \{a, b\}$ is equal to the family of sets which are mapped in the unique sets of mappings:

$F_\varkappa = \{f_{\varkappa\lambda} / f_{\varkappa\lambda} = f_\lambda$, $\lambda = 1, \ldots, 9$ and f_λ have the letter S or W in column $\varkappa$ on table (8)$\}$

for $\varkappa = \text{i, ii}, \ldots, \text{viii}$.

If $F_\varkappa = \{f_{\varkappa\lambda} / f_{\varkappa\lambda} = f_\lambda$, $\lambda = 1, \ldots, 9$ and f_λ have the letter S in column $\varkappa$ on table (8)$\}$ we get the set of distributively strong operators.

Remarks 1. In $H = \{a, b\}$ only the (v) and (viii) have the distributive property $\forall f : H \to \mathscr{P}(H)$ (and only (v) strongly), so every set that is mapped on the mappings f is a set of operators.

2. The identity mapping f_2 and the mapping f_9 are the only ones strongly distributive to every hypergroup in H.

References

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