Thomas N. Vougiouklis
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Cyclicity in a Special Class of Hypergroups

T. VOUGIOUKLIS

Democritus University of Thrace, Mathematics Department, Xanthi (Greece)

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Let $\langle H, *\rangle$ be a multiplicative hypergroup as defined in [1], [2] i.e. the non-empty set H equipped with a non-degenerate hyperoperation

$$* : H \times H \to \mathscr{P}(H) : (x, y) \mapsto x * y \subset H\,, \quad x * y \neq \emptyset$$

(If $A, B \subset H$, we set $A * B = \bigcup_{\substack{a \in A \\ b \in B}} a * b$. If $A = \{a\}$, we write $A * B = a * B$.) which is associative: $x * (y * z) = (x * y) * z$, $\forall x, y, z \in H$, and the condition $a * H = H * a = H$, $\forall a \in H$, is valid.

For every integer $v > 0$, and $\forall s \in H$, we get the powers of $s : s^1 = \{s\}$, $s^{v+1} = s^v * s \subset H$.

Now, using the original definition of cyclic hypergroup as we can see in [3] as well, we give the following definitions.

Definitions. A hypergroup H is called cyclic, if

$$H = h^1 \cup h^2 \cup \ldots \cup h^n \cup \ldots, \quad \text{for some} \quad h \in H\,. \tag{1}$$

If there exists an integer $n > 0$, the minimum one with the following property

$$H = h^1 \cup h^2 \cup \ldots \cup h^n\,, \tag{2}$$

then we call H cyclic hypergroup with finite period and we call h generator of H with period n. If there is no number n for which (2) is valid, but (1) is valid, then we say that H has infinite period for h. If all generators of H have the same period, then we call H cyclic with period.

If there exists an integer $n > 0$, the minimum one with the following property

$$H = h^n\,, \tag{3}$$

then we call H single-power cyclic hypergroup and h generator of H with period n. If (1) is valid and also $\forall n \in \mathbb{N}_0$ and $n \geqq n_0$, for constant $n_0 \in \mathbb{N}_0$, the following condition is valid

$$h^1 \cup h^2 \cup \ldots \cup h^{n-1} \subsetneqq h^n\,, \tag{4}$$

then we call H single-power cyclic hypergroup with infinite period for h.

Obviously we can prove the following proposition.

Proposition 1. Let $(H, \cdot)$ be a commutative group and P a subset of H. Then $\left\langle H, \overset{P}{*}\right\rangle$ is a hypergroup, where the hyperoperation $\overset{P}{*}$ is defined by the relation

$$\overset{P}{*} : H \times H \to \mathscr{P}(H) : (x, y) \mapsto x \overset{P}{*} y = xy(\{e\} \cup P), \tag{5}$$

where e is the unit element of $(H, \cdot)$.

We shall call the above hypergroup P-hypergroup.

Proposition 2. Let $(H_n, \cdot)$ be a finite cyclic group $\# H_n = n$ and $P \subset H_n$. Then $\left\langle H_n, \overset{P}{*}\right\rangle$, where $\overset{P}{*}$ is defined by (5), is a cyclic hypergroup which we shall call P-cyclic hypergroup.

Proof. From now on we denote the powers of the elements of H_n for the hyperoperation in square brackets.

We can easily see that:

$$x^{[\nu]} = x^{[\nu]}(\{e\} \cup P \cup P^2 \cup \ldots \cup P^{\nu-1}), \quad \forall \nu \in \mathbb{N}_0 . \tag{6}$$

So if $a \in H_n$ is a generator of $(H_n, \cdot)$, all over in this paper, then

$$a^{[1]} \cup a^{[2]} \cup \ldots \cup a^{[n]} = H_n ,$$

so a is a generator of $\left\langle H_n, \overset{P}{*}\right\rangle$ with period at most n.

In the following, we shall prove some theorems which are valid in the special case of P-cyclic hypergroups, where $P = \{p\}$ is a set with only one element. We write it as $\left\langle H_n, \overset{P}{*}\right\rangle$.

Theorem 1. In the P-cyclic hypergroup $\left\langle H_n, \overset{a^\varkappa}{*}\right\rangle$, the element a^λ is a generator iff $(\lambda, \varkappa, n) = 1$, i.e. $\lambda, \varkappa, n$ are relatively prime.

Proof. The μ-th power of the element a^λ under the hyperoperation $\overset{a^\varkappa}{*}$, using the relation (6), is

$$a^{\lambda[\mu]} = \{a^{\lambda\mu}, a^{\lambda\mu+\varkappa}, \ldots, a^{\lambda\mu+(\mu-1)\varkappa}\} . \tag{7}$$

Therefore the elements of the powers of a^λ have the form

$$a^{\lambda s + t\varkappa}, \quad \text{where} \quad s \in \mathbb{N}_0 \quad \text{and} \quad t = 0, 1, \ldots, s-1 .$$

Also we have

$$\lambda s + t\varkappa \equiv 1 \bmod n \quad \text{iff} \quad \exists \varrho \in \mathbb{Z} : \lambda s + t\varkappa - \varrho^n = 1 \quad \text{iff} \quad (\lambda, \varkappa, n) = 1 .$$

So if we choose appropriate $s, t, \varrho \bmod n$, as we need above, the relation $a^{\lambda s + t\varkappa} = a^1 = a$ is valid iff $(\lambda, \varkappa, n) = 1$. Therefore the element $a \in H_n$ belongs to some power of a^λ iff $(\lambda, \varkappa, n) = 1$.

Now, if a belongs to some power of a^λ, then $\forall v \in \mathbb{N}_0$ the element $a^v \in H_n$ belongs to some power of a^λ, because

$$a^{\lambda(vs)+(vt)\varkappa} = a^v .$$

From the above, we obtain that the element a^λ is a generator of $\left\langle H_n, \overset{a^\varkappa}{*} \right\rangle$ iff $(\lambda, \varkappa, n) = 1$.

Theorem 2. In the P-cyclic hypergroup $\left\langle H_n, \overset{a^\varkappa}{*} \right\rangle$, $a^\varkappa \neq a^n = e$,

(i) the element $a^\varkappa$ is a generator with period $\mu = [n/2] + 1$ (where $[n/2] = z$, when $n = 2z$ or $n = 2z + 1$),

(ii) the element $a^{n-\varkappa}$ is a generator with period n iff $(n, \varkappa) = 1$.

Proof (i) From (7) $\forall \lambda \in \mathbb{N}_0$, we get

$$a^{\varkappa[\lambda]} = \{a^{\varkappa\lambda}, a^{\varkappa(\lambda+1)}, \ldots, a^{\varkappa(2\lambda-1)}\}$$

and

$$a^{\varkappa[\lambda+1]} = \{a^{\varkappa(\lambda+1)}, a^{\varkappa(\lambda+2)}, \ldots, a^{\varkappa(2\lambda-1)}, a^{\varkappa 2\lambda}, a^{\varkappa(2\lambda+1)}\} .$$

Therefore, increasing the power of $a^\varkappa$ from λ to $\lambda + 1$, there appear at most two new elements, i.e. $a^{\varkappa 2\lambda}$ and $a^{\varkappa(2\lambda+1)}$. Since $a^{\varkappa[1]} = \{a^\varkappa\}$ is a set with only one element, to cover H_n we need at least $[n/2]$ other successive powers of $a^\varkappa$. In either case, if n is odd or even, for $\mu = [n/2] + 1$ we get

$$a^{\varkappa[1]} \cup a^{\varkappa[2]} \cup \ldots \cup a^{\varkappa[\mu]} = \{a^\varkappa, a^{\varkappa 2}, \ldots, a^{\varkappa(n-1)}, e\} \tag{8}$$

and in every higher power of $a^\varkappa$ the same elements are appearing.

If $(n, \varkappa) = 1$, then the elements of the set (8) are different, so $a^\varkappa$ is a generator with period $[n/2] + 1$.

If $(n, \varkappa) \neq 1$, then $(\varkappa, \varkappa, n) \neq 1$; so from theorem 1 we get that $a^\varkappa$ is not a generator.

(ii) From (7), $\forall \lambda \in \mathbb{N}_0$ and $\lambda < n$, we get

$a^{(n-\varkappa)[\lambda]} = \{a^{(n-\varkappa)\lambda}, a^{(n-\varkappa)\lambda+\varkappa}, \ldots, a^{(n-\varkappa)\lambda+(\lambda-1)\varkappa}\}$ and

$a^{(n-\varkappa)[\lambda+1]} = \{a^{(n-\varkappa)(\lambda+1)}, a^{(n-\varkappa)(\lambda+1)+\varkappa}, \ldots, a^{(n-\varkappa)(\lambda+1)+\lambda\varkappa}\}$

from where we can see easily that

$$a^{(n-\varkappa)[\lambda+1]} = \{a^{(n-\varkappa)(\lambda+1)}\} \cup a^{(n-\varkappa)[\lambda]}, \quad \lambda < n .$$

Let $(n, \varkappa) = 1$, then

$$a^{(n-\varkappa)(\lambda+1)} \notin a^{(n-\varkappa)[\lambda]} ,$$

because, if there exists $t \in \{0, 1, \ldots, \lambda - 1\}$ such that $a^{(n-\varkappa)(\lambda+1)} \doteq a^{(n-\varkappa)\lambda+t\varkappa}$, then $\varkappa(t+1) \equiv 0 \bmod n$, which is a contradiction. Therefore the sequence of sets

$$a^{(n-\varkappa)[1]},\ a^{(n-\varkappa)[2]},\ \ldots,\ a^{(n-\varkappa)[n]}$$

is strictly increasing and also the set $a^{(n-\varkappa)[n]}$ has exactly n different elements of H_n, i.e. $a^{(n-\varkappa)[n]} = H_n$.

So the element $a^{n-\varkappa}$ is a generator with period n of $\left\langle H_n, \overset{a^{\varkappa}}{*} \right\rangle$, when $(n, \varkappa) = 1$.

Let now $(n, \varkappa) \neq 1$, then $(\varkappa, n - \varkappa, n) \neq 1$. Hence from theorem 1 we get that $a^{n-\varkappa}$ is not a generator. Q.E.D.

The above theorem states that from n P-cyclic hypergroups $\left\langle H_n, \overset{a^{\varkappa}}{*} \right\rangle$, $\varphi(n)$ elements $a^{\varkappa}$ and $\varphi(n)$ elements $a^{n-\varkappa}$ are generators, where $\varphi(n)$ is the Euler's phi-function.

Theorem 3. The P-cyclic hypergroup $\left\langle H_n, \overset{a^{\varkappa}}{*} \right\rangle$, $a^{\varkappa} \neq e$, is a single-power cyclic hypergroup iff $(\varkappa, n) = 1$ and in this case every element of H_n is a generator of $\left\langle H_n, \overset{a^{\varkappa}}{*} \right\rangle$ with period n.

Proof. In the relation (7) we have at most μ different elements, so in order $\left\langle H_n, \overset{a^{\varkappa}}{*} \right\rangle$ to be a P-cyclic hypergroup we must have $\mu \geqq n$.

For $\mu = n$, we have

$$a^{\lambda[n]} = \{a^{\lambda n}, a^{\lambda n + \varkappa}, \ldots, a^{\lambda n + (n-1)\varkappa}\} = \{e, a^{\varkappa}, \ldots, a^{(n-1)\varkappa}\},$$

while, for every $\sigma \in \mathbb{N}$, we get

$$a^{\lambda[n+\sigma]} = a^{\lambda\sigma} \, . \, a^{\lambda[n]} .$$

Therefore $\left\langle H_n, \overset{a^{\varkappa}}{*} \right\rangle$, $a^{\varkappa} \neq e$, is a single-power P-cyclic hypergroup with generator a^{λ} iff exactly the n-th power of a^{λ} is equal to H_n.

The n elements of $y^{\lambda[n]}$ are different iff $(\varkappa, n) = 1$, independently of λ, and the period of a^{λ} is n.

References

[1] Dubreil P.: Algèbre, Tome I. Gauthier-Villars, Paris 1963.
[2] Konguetsof L., Bull. Soc. Math. Belgique *25* 1973, 211.
[3] Wall H. S.: Amer. J. Math. *59* 1937, 77.