WSAA 13

Dalibor Volný

Determinism and martingale decompositions of strictly stationary random processes

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Analysis. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 10. pp. [185]--192.

Persistent URL: http://dml.cz/dmlcz/701873

Terms of use:

© Circolo Matematico di Palermo, 1985

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

DETERMINISM AND MARTINGALE DECOMPOSITIONS OF STRICTLY STATIONARY RANDOM PROCESSES

Dalibor Volný

1. Introduction. Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system where $(\Omega, \mathcal{A}, \mu)$ is a probability space and T is a 1 - 1 bimeasurable and measure preserving transformation of Ω onto itself. If for each $A \in \mathcal{A}$ such that $TA=A$ it is $\mu A=0$ or $\mu A=1$, we shall say that μ is ergodic. If $\mathcal{M} \subset \mathcal{A}$ is a σ-algebra and $\mathcal{M} \subset T^{-1}\mathcal{M}$, we shall say that $\mathcal{M}$ is invariant. $L^2(\mathcal{A})$ assigns the Hilbert space of square integrable functions on Ω. If $\mathcal{E} \subset \mathcal{A}$ is a σ-algebra, $L^2(\mathcal{E})$ is the Hilbert space of functions $f \in L^2(\mathcal{A})$ for which $E(f|\mathcal{E}) = f$ modμ. For each invariant σ-algebra $\mathcal{M}$ and $i \in \mathbb{Z}$, $L^2(T^{-i}\mathcal{M})$ is a subspace of $L^2(T^{-i-1}\mathcal{M})$ and the projection operator onto $L^2(T^{-i-1}\mathcal{M}) \ominus L^2(T^{-i}\mathcal{M})$ will be called a difference projection operator and denoted by P_i. The unitary operator sending $f \in L^2(\mathcal{A})$ to $f \circ T$ is denoted by U. For each σ-algebra $\mathcal{E} \subset \mathcal{A}$ and $f \in L^2(\mathcal{A})$, it is $U\,E(f|\mathcal{E}) = E(Uf|T^{-1}\mathcal{E})$ modμ. From this we obtain that $UP_if = U(E(f|T^{-i-1}\mathcal{M})-E(f|T^{-i}\mathcal{M})) = P_{i+1}Uf$ modμ. If $f = P_kf$ modμ for some $k \in \mathbb{Z}$, then $(f\circ T^i; i \in \mathbb{Z})$ is a martingale difference sequence (i.e. for each $i \in \mathbb{Z}$, the sums $\sum_{j=0}^{n} f\circ T^{i+j}$, $n=0,1,\ldots$ form a martingale). Putting $\mathcal{M} = \sigma\{f\circ T^i : i < 0\}$ we can express each martingale difference sequence in this form.

For each $\mathcal{A}$-measurable function f on Ω, the sequence $(f\circ T^i; i \in \mathbb{Z})$ is strictly stationary. Moreover, for each strictly stationary sequence of random variables $(X_i; i \in \mathbb{Z})$, a dynamical system $(\Omega, \mathcal{A}, T, \mu)$ and a function f can be found in such a manner that (X_i) and $(f\circ T^i)$ have the same distribution. If the measure μ is ergodic, we shall say that the process (X_i) is ergodic. By the central limit problem we shall mean the problem of weak convergence of measures $\mu s_n^{-1}(f)$ where $s_n(f) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f\circ T^j$, $n=1,2,\ldots$. In 1961 P.Billingsley and in 1963 I.A.Ibragimov proved the central limit theorem for ergodic martingale difference sequences (see [1], [4]). In 1969 M.I.Gordin

([2]) published a generalization of this theorem based on properties of difference projection operators. Gordin's result was followed by contributions of other authors (some of them are collected in the monography [3]). This development of the central limit problem evokes a question of possibility of decomposition of a function $f \in L^2(\mathcal{A})$ by difference projection operators (i.e. a question of decomposition of $(f \circ T^i; i \in \mathbb{Z})$ into martingale difference sequences). In the following two parts of this article, two ways of decomposition of $(f \circ T^i)$ into a sum of martingale difference sequences and of a sequence which is in some sense deterministic will be shown.

2. Invariant σ-algebras generated by the process. Investigating the possibility of decomposition of $f \in L^2(\mathcal{A})$ by difference projection operators, we have to choose an appropriate invariant σ-algebra. In many cases it is convenient to use the fact that the σ-algebra $\mathcal{E}(f) = \sigma\{f \circ T^i : i < 0\}$ is invariant. If we have $\mathcal{E}(f) = T^{-1}\mathcal{E}(f) \bmod \mu$, we shall say that the process $(f \circ T^i)$ is deterministic (in the theory of stochastic processes, the deterministic process is defined in somehow stronger sense as instead of $L^2(\mathcal{E}(f))$ the closed linear envelop of $\{f \circ T^i : i < 0\}$ is used there). Let us put $\mathcal{E}_{-\infty}(f) = \bigcap_{i \in \mathbb{Z}} T^i \mathcal{E}(f)$, $f_1 = E(f|\mathcal{E}_{-\infty}(f))$, $\mathcal{E}^1(f) = \sigma\{f_1 \circ T^i : i < 0\}$; in the same way we can obtain $\mathcal{E}^1_{-\infty}(f)$, f_2, $\mathcal{E}^2(f)$ etc. The main aim of the following example is to show that the process $(f_1 \circ T^i)$ need not be deterministic.

Example. Let $(\overline{X}, \overline{\mathcal{S}}, \overline{\lambda})$ be a probability space where $\overline{X} = \{-1, 1\}$, $\overline{\mathcal{S}} = \exp \overline{X}$ and $\overline{\lambda}(-1) = \frac{1}{2} = \overline{\lambda}(1)$; we put $X = \overline{X}^{\mathbb{Z}}$, $\mathcal{S} = \overline{\mathcal{S}}^{\mathbb{Z}}$, $\lambda = \overline{\lambda}^{\mathbb{Z}}$ and for $\omega \in X$ we define $(S\omega)_i = \omega_{i+1}$. Let us define $\Omega = X \times X$, $\mathcal{A} = \mathcal{S} \times \mathcal{S}$, $\mu = \lambda \times \lambda$ and for $(\omega', \omega'') \in \Omega$ we put $T(\omega', \omega'') = (S\omega', S\omega'')$. Thus, $(X, \mathcal{S}, S, \lambda)$ and $(\Omega, \mathcal{A}, T, \mu)$ are dynamical systems. For $i \in \mathbb{Z}$ and $\omega \in X$ we define $q_i(\omega) = \omega_i$, for $(\omega', \omega'') \in \Omega$ we define $p'(\omega', \omega'') = \omega'$, $p''(\omega', \omega'') = \omega''$. By $g = q_0 + \sum_{n=1}^{\infty} \left(\frac{1}{3^{2n-1}} q_n + \frac{1}{3^{2n}} q_{-n}\right)$ we define a bounded $\mathcal{S}$-measurable function on X. Let $q = q_0$ and $t = 2 + g$ be functions on X, $r = (q \circ p') \cdot (t \circ p'')$, $s = 8 q \circ p''$ and $f = r + s$ be functions on Ω. We shall show that for $f_1 = E(f|\mathcal{E}_{-\infty}(f)) \bmod \mu$ it is $f_1 = s \bmod \mu$. It follows that $(f_1 \circ T^i)$ is a sequence of independent random variables. Thus, $f_2 = E(f_1|\mathcal{E}^1_{-\infty}(f)) = Ef_1 = 0 \bmod \mu$.

First of all, we shall show that $\sigma\{g\} = \mathcal{S}$. We have $\left|\sum_{n=1}^{\infty} \left(\frac{1}{3^{2n-1}} q_n + \frac{1}{3^{2n}} q_{-n}\right)\right| \leq \frac{1}{2}$, so q_0 is $\sigma\{g\}$-measurable.

If $k \geq 0$ and the functions $q_{-k},\dots,q_k$, resp. $q_{-k+1},\dots,q_k$ are $\sigma\{g\}$--measurable, then the function $\sum_{n=k+1}^{\infty}(\frac{1}{3^{2n-1}} q_n + \frac{1}{3^{2n}} q_{-n}) = Q_k$, resp. $\frac{1}{3^{2k}} q_{-k} + \sum_{n=k+1}^{\infty}(\frac{1}{3^{2n-1}} q_n + \frac{1}{3^{2n}} q_{-n}) = Q_k'$, is $\sigma\{g\}$-measurable, too.

From the fact that $|Q_k - \frac{1}{3^{2k+1}} q_{k+1}| \leq \frac{1}{2.3^{2k+1}}$, resp. $|Q_k' - \frac{1}{3^{2k}} q_{-k}| \leq \frac{1}{2.3^{2k}}$, we obtain $\sigma\{g\}$-measurability of q_{k+1}, resp. q_{-k}. Thus, the $\sigma\{g\}$-measurability of all q_i is proved and hence $\sigma\{g\} = \mathcal{S}$.

Let us put $\mathcal{M}' = \mathcal{E}(s)$, $\mathcal{M}'' = \mathcal{E}(\text{top}'')$ and $\mathcal{M}''' = \mathcal{E}(r)$. It can be easily seen that $\mathcal{M}' \subset \mathcal{M}'' \subset \mathcal{M}'''$, $\bigcap_{i\in\mathbb{Z}} T^i \mathcal{M}''' = \mathcal{M}'' \bmod \mu$ and $E(f|\mathcal{M}'') = s \bmod \mu$. From this and from the fact that $\mathcal{M}''' = \mathcal{E}(f)$ we obtain that $f_1 = s \bmod \mu$.

Theorem 1. Let $f \in L^2(\mathcal{A})$. There exists a countable ordinal number δ and a family of invariant σ-algebras $\mathcal{E}^\alpha(f)$ and functions f_α, $\alpha \leq \delta$ such that

1. $f_0 = f$,
2. for $\alpha \geq 1$, it is $f_\alpha = E(f|\bigcap_{\beta<\alpha} \mathcal{E}^\beta_{-\infty}(f)) \bmod \mu$,
3. $\mathcal{E}^\alpha(f) = \sigma\{f_\alpha \circ T^i : i < 0\}$ and
4. δ is the least ordinal number such that $T^{-1}\mathcal{E}^\alpha(f) = \mathcal{E}^\alpha(f) \bmod \mu$.

The family $(f_\alpha, \mathcal{E}^\alpha(f); \alpha \leq \delta)$ is determined by the function f uniquelly (with respect to equality $\bmod \mu$).

If $P_{\alpha,i}$, $\alpha \leq \delta$, $i \leq 0$ are difference projection operators generated by $\mathcal{E}^\alpha(f)$, then

$$f = \sum_{\alpha<\delta} \sum_{i=-\infty}^{0} P_{\alpha,i} f + E(f|\mathcal{E}^\delta(f)) \bmod \mu .$$

Let Λ be a linearly ordered set and let $\mathcal{M}_\alpha$, $\alpha\in\Lambda$ be invariant σ-algebras such that for $\alpha<\beta$ and $i,j \in \mathbb{Z}$ we have $T^i\mathcal{M}_\beta \subset T^j\mathcal{M}_\alpha$. We shall say then that $(\mathcal{M}_\alpha; \alpha\in\Lambda)$ is an <u>ordered family of σ-algebras</u>. The σ-algebras $\mathcal{E}^\alpha(f)$, $\alpha\leq\delta$ from Theorem 1 form a special case of such a family.

Let $P_{-\infty}$ be the projection operator onto $L^2(\bigcap_{\alpha\in\Lambda} \bigcap_{i\in\mathbb{Z}} T^i \mathcal{M}_\alpha)$ and $P_{\alpha,i}$ be the difference projection operator onto $L^2(T^{-i-1}\mathcal{M}_\alpha) \ominus L^2(T^{-i}\mathcal{M}_\alpha)$. We shall say then that $\Pi(\Lambda) = (P_{-\infty}, P_{\alpha,i}; \alpha\in\Lambda, i\in\mathbb{Z})$ is a <u>family of difference projections</u>. If $f = \sum_{\alpha\in\Lambda}\sum_{i\in\mathbb{Z}} P_{\alpha,i} f \bmod \mu$, we shall say that f is <u>difference decomposable with respect to $\Pi(\Lambda)$</u>. If $f \in L^2(\mathcal{A})$ is difference decomposable with respect to some family of difference projections, we shall say that f is <u>difference decomposable</u>. In [8], several propositions about difference decomposability of functions from $L^2(\mathcal{A})$ are proved.

If there exists a countable set $\mathcal{G} \subset \mathcal{A}$ such that $\sigma\mathcal{G} = \mathcal{A} \bmod \mu$, we shall say that $\mathcal{A}$ is countably generated. On the σ-algebra $\mathcal{A}$ we define a pseudometric d by $d(A,B) = \mu(A \triangle B)$ where $A \triangle B$ is the symmetric difference of A,B. If we identify sets $A=B \bmod \mu$, we obtain a metric space $(\mathcal{A}/\mu, d)$ If $\mathcal{A}$ is countably generated, the space $\mathcal{A}/\mu$ is separable.

<u>Lemma 1</u>. Let $\mathcal{C}^\alpha$ be sub-σ-algebras of $\mathcal{A}$, α being all countable ordinals, and for $\alpha < \beta$ let $\mathcal{C}^\beta \subset \mathcal{C}^\alpha$.
If $\mathcal{A}$ is countably generated, there exists α such that $\mathcal{C}^{\alpha+1} = \mathcal{C}^\alpha \bmod \mu$.

<u>Proof</u>. If $\mathcal{C}^\alpha \neq \mathcal{C}^{\alpha+1} \bmod \mu$, there exist $E(\alpha) \in \mathcal{C}^\alpha$ and $\varepsilon > 0$ such that $d(E(\alpha), C) > \varepsilon$ for all $C \in \mathcal{C}^{\alpha+1}$. Suppose that the Lemma does not hold. Then there exist an uncountable set Λ of (countable) ordinals and $\varepsilon > 0$ such that for each $\alpha \in \Lambda$, it is $d(E(\alpha), C) > \varepsilon$ for all $C \in \mathcal{C}^{\alpha+1}$. For every $\alpha \neq \beta$ from Λ we then have $d(E(\alpha), E(\beta)) \geq \varepsilon$ which contradicts the assumption that $\mathcal{A}/\mu$ is separable.

<u>Proof of Theorem 1</u>. For a countable ordinal number γ let $V(\gamma)$ denotes the statement that for each $\alpha \leq \gamma$ there exist a unique (mod μ) σ-algebra $\mathcal{C}^\alpha(f)$ and a function f_α such that the conditions 1,2,3 hold and that $f - f_\gamma = \sum_{\alpha<\gamma} \sum_{i=-\infty}^{0} P_{\alpha,i} f \bmod \mu$. We shall prove by transfinite induction that for each countable γ, $V(\gamma)$ holds.
It is evident that $V(0)$ holds.
Suppose that γ is a countable ordinal and that for each $\beta < \gamma$, $V(\beta)$ holds. The function f_γ and the σ-algebra $\mathcal{C}^\gamma(f)$ are then uniquelly determined by conditions 2,3.
If γ is not a limit ordinal, there exists $\bar{\gamma}$, $\gamma = \bar{\gamma} + 1$. From the fact that $f_{\bar{\gamma}}$ is $T^{-1}\mathcal{C}^{\bar{\gamma}}(f)$-measurable and that $T^{-1}\mathcal{C}^{\bar{\gamma}}(f) \subset \bigcap_{\alpha<\bar{\gamma}} \mathcal{C}^\alpha_{-\infty}(f)$ it follows that $f_{\bar{\gamma}} = E(f|T^{-1}\mathcal{C}^{\bar{\gamma}}(f)) \bmod \mu$. Thus, $f_{\bar{\gamma}} - f_\gamma = E(f|T^{-1}\mathcal{C}^{\bar{\gamma}}(f)) - E(f|\mathcal{C}^{\bar{\gamma}}_{-\infty}(f)) = \sum_{i=-\infty}^{0} P_{\bar{\gamma},i} f \bmod \mu$, hence $f - f_\gamma = \sum_{\alpha<\gamma} \sum_{i=-\infty}^{0} P_{\alpha,i} f \bmod \mu$.
If γ is a limit ordinal then there exists an increasing sequence $\alpha_1 < \alpha_2 < \ldots < \gamma$, $\alpha_k \uparrow \gamma$. It holds that $\mathcal{C}^{\alpha_k}_{-\infty}(f) \downarrow \bigcap_{\alpha<\gamma} \mathcal{C}^{\alpha}_{-\infty}(f)$.
From this and from the martingale limit theorem we obtain that

$f_{\alpha_k} \xrightarrow[k\to\infty]{} f_\gamma$ in $L^2(\mathcal{A})$. The last statement and $V(\alpha_k)$, $k=1,2,\dots$ imply that $f-f_\gamma = \sum_{\alpha<\gamma}\sum_{i=-\infty}^{0} P_{\alpha,i}f \mod \mu$.

The last thing left to do is to find δ. From Lemma 1 it follows that there exists an ordinal number γ such that $\mathcal{E}^\gamma(f) = T^{-1}\mathcal{E}^\gamma(f)$ mod μ. Choosing the minimal one from these ordinals for δ, the proof of the theorem is finished.

3. General invariant σ-algebras. In the sequel, $\mathcal{P}$ will denote the Pinsker σ-algebra (see [5] for the definition). In each dynamical system the Pinsker σ-algebra exists (see [5]) and in [8] it is proved that the Pinsker σ-algebra is fully characterized by the following three properties:

1. $\mathcal{P}$ is an invariant sub-σ-algebra of $\mathcal{A}$;
2. for each invariant σ-algebra $\mathcal{M}\subset\mathcal{P}$, it is $\mathcal{M} = T^{-1}\mathcal{M}$ mod μ;
3. $\mathcal{P}$ is the maximal σ-algebra satisfying conditions 1 and 2.

The function $f\in L^2(\mathcal{A})$ will be said to be absolutely undecomposable iff f is $\mathcal{P}$-measurable.

It follows directly from the above characterization of Pinsker σ-algebra that if f is absolutely undecomposable, then $(f\circ T^i; i\in\mathbb{Z})$ is deterministic. If f is finitely valued, the opposite implication holds, too. However, the opposite implication does not hold in general. As a counterexample can serve the sequence $(g\circ S^i; i\in\mathbb{Z})$ from the example from section 2. The dynamical system $(X,\mathcal{F},S,\lambda)$ is Bernoulli, hence its Pinsker σ-algebra is trivial (see [5]). Thus, g cannot be absolutely undecomposable.

Theorem 2. The difference decomposable functions from $L^2(\mathcal{A})$ form a Hilbert space $L^2(\mathcal{A}) \ominus L^2(\mathcal{P})$. Each function $f\in L^2(\mathcal{A})$ can be thus uniquelly expressed as a sum of a difference decomposable and an absolutely undecomposable function.
If $f\in L^2(\mathcal{A})$ is difference decomposable, it is decomposable with respect to some (single) invariant σ-algebra.

An analogous result can be obtained if T is surjective, but not 1-1 and bimeasurable. Such a transformation will be called an endomorphism; an endomorphism which is 1-1 and bimeasurable will be called an automorphism. Let us enlarge the definition of a dynamical system to the case when T is an endomorphism.

If T is an endomorphism, it is $\mathcal{A}\supset T^{-1}\mathcal{A}\supset\dots$; $\mathcal{R} = \bigcap_{n=0}^{\infty} T^{-n}\mathcal{A}$ is the Rohlin σ-algebra. In the definition of difference decomposable function we could use $T^{-1}\mathcal{M} \subset \mathcal{M}$ as the defining property of invariant σ-algebras (instead of the opposite inclusion) and we could define $P_i f = E(f|T^{-i}\mathcal{M}) - E(f|T^{-i-1}\mathcal{M}) \bmod \mu$. Each function $f \in L^2(\mathcal{A}) \ominus L^2(\mathcal{R})$ is thus (in this sense) difference decomposable. In the dynamical system $(\Omega, \mathcal{R}, T, \mu)$ the transformation T behaves as an automorphism (if we put points from Ω that are undistinguishable by $\mathcal{R}$, together, we really obtain an automorphism). According to Theorem 2 the following theorem thus holds.

Theorem 3. Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system where T is an endomorphism. Then each $f \in L^2(\mathcal{A})$ can be uniquelly (mod μ) expressed as a sum $f_1+f_2+f_3$ where f_1 is absolutely undecomposable, f_2 is difference decomposable with respect to a (single) invariant σ-algebra and $f_3 \in L^2(\mathcal{A}) \ominus L^2(\mathcal{R})$.

The Theorem 2 is an easy consequence of the following two propositions.

Proposition 1. (Rohlin-Sinai theorem). If $\mathcal{A}$ is countably generated and T is an automorphism then there exists an invariant σ-algebra $\mathcal{M}$ such that $\sigma(\bigcup_{i\in\mathbb{Z}} T^i\mathcal{M}) = \mathcal{A}$ and $\bigcap_{i\in\mathbb{Z}} T^i\mathcal{M} = \mathcal{P}$.

The proof can be found e.g. in [5].

Proposition 2. Let $(\mathcal{M}_\alpha;\ \alpha\in\Lambda)$ be an ordered system of σ-algebras. Let $P_{\alpha,i}$, $\alpha\in\Lambda$, $i\in\mathbb{Z}$ be the difference projection operators generated by $(\mathcal{M}_\alpha; \alpha\in\Lambda)$ and $P'_{\alpha,i}$, $\alpha\in\Lambda$, $i\in\mathbb{Z}$ be the difference projection operators generated by $(\mathcal{M}_\alpha \vee \mathcal{P}; \alpha\in\Lambda)$. If $f \in L^2(\mathcal{A})$ and $f = \sum_{\alpha\in\Lambda}\sum_{i\in\mathbb{Z}} P_{\alpha,i}f \bmod \mu$, then we have $P'_{\alpha,i}f = P_{\alpha,i}f \bmod \mu$ for all $\alpha\in\Lambda$ and $i\in\mathbb{Z}$.

This proposition is proved in [8]. In [8] it is also shown that without the assumption that $f = \sum_{\alpha\in\Lambda}\sum_{i\in\mathbb{Z}} P_{\alpha,i}f \bmod \mu$, the equalities need not hold.

As an immediate consequence of Proposition 2 we obtain the following lemma.

Lemma 2. If $f\in L^2(\mathcal{A})$ is a difference decomposable function then $E(f|\mathcal{P}) = 0 \bmod \mu$.

Proof of Theorem 2. Let $f \in L^2(\mathcal{A})$ and $\mathcal{E} = \sigma\{f\circ T^i;\ i\in\mathbb{Z}\}$. The σ-algebra $\mathcal{E}$ is countably generated, $(\Omega, \mathcal{E}, T, \mu/_{\mathcal{E}})$ is a dynamical system and by Proposition 1 there exists an invariant σ-algebra $\mathcal{M}$ such that $\sigma(\bigcup_{i\in\mathbb{Z}} T^i\mathcal{M}) = \mathcal{E}$, $\bigcap_{i\in\mathbb{Z}} T^i\mathcal{M} \subset \mathcal{P}$. From this we obtain that f is a sum of an absolutely undecomposable and a difference decomposable functions. From Proposition 2 it follows that this decomposition is unique (mod μ).

4. The central limit problem. At the end we shall introduce some information about the CLP for difference decomposable functions. In [8] it is proved that the Billingsley - Ibragimov theorem can be generalized for functions $f = \sum_{\alpha\in A} P_{\alpha,0} f$ mod μ and non-ergodic measure μ (the limit law is then a mixture of normals them), From this proposition, several sufficient conditions for the CLT for difference decomposable functions (analogous to those of M. I. Gordin and C. C. Heyde) can be obtained. Some information about the case of absolutely undecomposable functions can be found in [9] (however, it is known very little about the CLP for such functions). In [10] it is shown that in some cases from the central limit theorem for a difference decomposable function and a central limit theorem for an absolutely undecomposable function a CLT for their sum can be obtained. Our last theorem can speak for itself.

Theorem 4. In the space of difference decomposable functions $L^2(\mathcal{A}) \ominus L^2(\mathcal{P})$ there exists a dense set of functions f for which the measures $\mu s_n^{-1}(f)$, n=1,2,... weakly converge to some probability measure (i.e. the CLT holds) and there exists a dense subset of functions for which this sequence has at least two distinct limit points.

This theorem is proved in [8] (see also [7]).

REFERENCES

[1] BILLINGSLEY P. "The Lindeberg-Lévy theorem for martingales", Proc.Amer.Math.Soc., 2(1961), 788-792.
[2] GORDIN M.I. "The CLT for stationary processes", Soviet.Math. Doklady 10(1969), 1174-1176.
[3] HALL P., HEYDE C.C. "Martingale Limit Theory and Its Application", Academic Press, New York 1980.
[4] IBRAGIMOV I.A. "A CLT for a class of dependent random variables", Theory Probab.Appl., 8(1963), 83-89.

[5] PARRY W. "Topics in Ergodic Theory", Cambridge Univ.Press, Cambridge 1980.

[6] ROHLIN V.A. "Lectures on the entropy theory of transformations with invariant measure", Usp.Mat.Nauk, 22(1967), 3-56 (Russian) Russian Math.Surveys 22(1967), 1-52.

[7] VOLNÝ D. "A negative answer to the CLP for strictly stationary sequences", Proc.3rd.Prague Symp.Asymptot.Statistics, Elsevier Sci. Publishers, Amsterdam 1980, 433-41.

[8] VOLNÝ D. "The CLP for strictly stationary sequences", PhD. thesis, Charles University, Prague 1980.

[9] VOLNÝ D. "On the CLP for processes of zero entropy", Comment. Math.Univ.Carol., 25(1985), 253-258.

MATHEMATICAL INSTITUTE
CHARLES UNIVERSITY
SOKOLOVSKÁ 83
PRAHA 8
CZECHOSLOVAKIA