SIMULTANEOUS STRATEGIES AND BOOLEAN GAMES OF UNCOUNTABLE LENGTH

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The paper is devoted to the study of the existence of a λ-closed dense subsets of a Boolean algebra under certain game-theoretical properties. T.Jech in [3] introduced the following game. Let B be a complete Boolean algebra and α an ordinal number. The transfinite game $\mathcal{G}(B,\alpha)$ is played between two players White and Black. Let White and Black define a decreasing sequence

$$w_0 \geq b_1 \geq w_2 \geq b_3 \geq \ldots \geq w_{\beta+2n} \geq b_{\beta+2n+1} \geq \ldots \qquad (1)$$

of nonzero elements of B of the length $\leq \alpha$ by taking turns defining its entries. (White chooses $w_{\beta+2n}$, for β limit $< \alpha$, $n \in \omega$; Black chooses $b_{\beta+2n+1}$) The play is won by Black if the sequence (1) has nonzero intersection and length α, and by White if the intersection is 0. A winning strategy for Black in the game $\mathcal{G}(B,\alpha)$ is a function $\sigma : \bigcup\{{}^{\beta}B : \beta < \alpha\} \longrightarrow B$ with the property that Black wins every play (1) in which he follows σ.

Theorem 1 (T.Jech [3]). Assume B is a complete Boolean algebra and λ an uncountable cardinal number. Then (a) $\rightarrow$ (c) $\rightarrow$ (d), where

(a) Algebra B has a λ-closed dense subset.

(c) Black has a winning strategy in the game $\mathcal{G}(B,\tau)$ for each $\tau < \lambda$.

(d) Algebra B is $(\tau,\infty,2)$ distributive for each $\tau < \lambda$.

Basically, our research was motivated by the question whether (c) $\rightarrow$ (a), i.e. whether the existence of winning strategy for Black implies that B has a λ-closed dense subset. The following definition concerns the structure of the set of all strategies and is powerfull also for limit cardinal numbers.

Definition 2. We say that Black has a simultaneous winning strategy of the length λ in the algebra B if there is one strategy $\sigma : \bigcup\{{}^{\alpha}B : \alpha < \lambda\} \longrightarrow B$ such that σ is winning for Black in each game $\mathcal{G}(B,\alpha)$ for $\alpha < \lambda$.

Lemma 3. (a) $\rightarrow$ (b) $\rightarrow$ (c), where

(a) and (c) are as in the Theorem 1, and

(b) Black has simultaneous strategy for B of the length λ.
PROOF. Is obvious.

In what follows we describe a type of algebras for which implication (b)$\longrightarrow$(a) holds, namely the ones which has a tree-base. We give some conditions under which a boolean algebra has a tree-base. As a consequence we get an characterization of the algebra Col (λ,κ). At the end of this paper we give some historical comments.

§1. Notations, definitions, constructions. Let B be a complete Boolean algebra, $\leq$ is the canonical ordering of the algebra B, λ,κ,τ are cardinal and α,β,γ ordinal numbers, Lim denotes the class of limit ordinal numbers. $B^+ = B - \{0\}$, $B\restriction u$ is a partial algebra, hsat(B) denotes the hereditarily saturatedness. By P and Q we denote a maximal partition of B, system $\mathcal{H} = \{P_\alpha : \alpha<\lambda\}$ is called a matrix, P_α's are columns of $\mathcal{H}$, $x \in P_\alpha$ is an element of the matrix $\mathcal{H}$, $P \ll Q$ denotes that P refines Q, $P \ll \mathcal{H}$ if P refines each P_α. $\mathcal{H}$ is said to be monotone provided $\alpha<\beta$ implies $P_\beta \ll P_\alpha$. Remark, that if $\mathcal{H}$ is monotone, then $(\bigcup\mathcal{H}, \leq)$ forms a tree and P_α is the α-th level of this tree. Let $\Omega = \{Q_\alpha : \alpha<\lambda\}$. Then Ω refines $\mathcal{H}$ if each Q_α refines P_α. For $x\in B^+$, $x\wedge\wedge P = \{y\wedge x : y\in P_\alpha\}\cap B^+$. The algebra B is said to be (λ,∞,κ)-distributive provided for any matrix $\mathcal{H} = \{P_\alpha : \alpha<\lambda\}$ there is a maximal partition P of B such that $(\forall x\in P)(\forall\alpha<\lambda)(|x\wedge\wedge P_\alpha| < \lambda)$. Algebra B is called (λ,∞,κ)-nowhere distributive if for each $x\in B^+$ the algebra $B\restriction x$ is not (λ,∞,κ)-distributive. Recall that B is (λ,∞,κ)-nowhere distributive iff there is a matrix $\mathcal{H} = \{P_\alpha : \alpha<\lambda\}$ such that for each $x\in B^+$ there is some $\alpha<\lambda$ with $|x\wedge\wedge P_\alpha| \geq \kappa$. In this case we say that $\mathcal{H}$ is a matrix witnessing to (λ,∞,κ)-nowhere distributivity and if B is $(\tau,\infty,2)$-distributive for all $\tau<\lambda$ (e.g. if B has simultaneous winning strategy for Black of length λ) then $\mathcal{H}$ will be assumed to be monotone. We say that $D\subseteq B^+$ is a λ-closed dense subset of algebra B (we say sometimes base instead of dense subset) if $(\forall x\in B^+)(\exists y\in D)(y\leq x)$ and for every decreasing sequence $\{a_\alpha : \alpha<\tau\}\subseteq D$ of the length $\tau<\lambda$ there is an $y\in D$ such that $y\leq a_\alpha$ for each $\alpha<\tau$. d(B) denotes the density of B. A matrix $\mathcal{H} = \{P_\alpha : \alpha<\lambda\}$ is said to be a tree-base of the algebra B of the length λ if $\bigcup\mathcal{H}$ is a base (i.e. elements of $\mathcal{H}$ form a base) and $\mathcal{H}$ is monotone (i.e. $(\bigcup\mathcal{H},\leq)$ is a tree). For unexplained notation we refer to [2].

§2. Tree-base, game-tree and simultaneous strategy. The idea to construct a λ-closed dense subset of a Boolean algebra from a tree appeared independently in [1] and [4]. We develope the technics of [4] to obtain more general results.

Theorem 4. Assume B is a complete Boolean algebra which has a tree-base of the length λ and B has simultaneous winning strategy for Black of length λ. Then B has a λ-closed dense subset.

PROOF. The idea of the proof is following. We introduce a natural notion of a game tree according to strategy σ for Black in which every branch is a play of game $\mathcal{G}$ in which Black follows σ and the elements of the tree split on even levels (turns of the White) and odd levels are determined by even ones and σ. Then by transfinite induction we construct an game-tree according to σ which refines the base $\mathcal{H}$ and this is the desired λ-closed base.

Let $\mathcal{H} = \{P_\alpha : \alpha < \lambda\}$ be a tree-base of B and σ is a simultaneous winning strategy. A tree $(T,\leq)$ is called a game-tree of length λ according to the strategy σ if

(i) $T \subseteq B^+$, $\leq$ is the canonical ordering of B, T has length λ

(ii) $(\forall \alpha \in \lambda \cap \mathrm{Lim})(\forall n \in \omega) T_{\alpha+2n+1}$ is a maximal partition of B (where T_β denotes the β-th level of the tree T)

(iii) $(\forall x \in T_{\alpha+2n+1})(x = \sigma(\mathrm{pr}(x)))$ (where pr(x) is the sequence of predecessors of x in the tree T).

Note that any branch of T is a play of game $\mathcal{G}$ in which Black follows σ, so it is of the length λ. So if T is a base of algebra B it is a λ-closed base. To this end we have to construct a game-tree T which refines the tree base $\mathcal{H}$ i.e.

(iv) $(\forall \alpha \in \lambda \cap \mathrm{Lim})(\forall n \in \omega)(T_{\alpha+2n+1} \ll P_{\alpha+n})$

Assume that $T_{<\alpha+2n}$ is constructed already. For $T_{\alpha+2n}$ take the maximal system such that

(v) $T_{\alpha+2n}$ refines $T_{<\alpha+2n}$

(vi) $\{\sigma(\mathrm{pr}(x),x) : x \in T_{\alpha+2n}\}$ is a maximal partition of B which refines $P_{\alpha+n}$.

This is possible because $T_{<\alpha+2n}$ is a game tree of the length $\alpha+2n$ according to the simultaneous strategy σ of the length $\lambda > \alpha+2n$, algebra B is complete and $(\alpha,\infty,2)$ distributive.

q.e.d.

Now we consider the question, when does there exists a tree-base for a Boolean algebra. We need the following technical lemma.

Lemma 5. Assume λ is an uncountable cardinal number, B is a complete Boolean algebra which has a simultaneous winning strategy for Black of length λ and $\mathcal{H} = \{P_\alpha : \alpha < \lambda\}$ is a monotone matrix wittnessing to (λ,∞,κ)-nowhere distributivity of B. Then for each $x \in B^+$ the following holds

(i) $|\{(y,\alpha) : y \in P_\alpha \& x \wedge y \neq \mathbb{O}\}| \geq \kappa^\lambda$ (i.e. x intersects many elements of the matrix $\mathcal{H}$)
and (ii) $\mathrm{hsat}(B) > \kappa^\lambda$

PROOF. The main idea again is to construct something like a game tree of type $({}^\lambda\kappa, \subseteq)$, but its relation to the matrix $\mathcal{H}$ is a little bit more complicated as in the Theorem 4.

Assume w.l.o.g. that $x = 1$. In fact for (i) we have to prove that $|\bigcup \mathcal{H}| \geq \kappa^\lambda$. The existence of simultaneous strategy of length λ implies $(\tau, \infty, 2)$-distributivity for $\tau < \lambda$ and B is $(\lambda, \infty, \kappa)$-nowhere distributive - we have that λ is regular. By transfinite induction we construct F and T such that

(i) $T \subseteq \bigcup \mathcal{H}$ is a Boolean tree (i.e. (i) of Theorem 4 holds)

(ii) $F : \bigcup\{{}^\alpha\kappa : \alpha < \lambda\} \to T$ is an tree isomorphism

(iii) for each $f \in {}^\lambda\kappa$ the sequence $F(f \restriction 0) = w_0$, $\sigma(w_0) = b_1$, $w_2 = b_1 \wedge F(f \restriction 1)$, $b_3 = \sigma(w_2), \ldots, w_{\alpha+2n} = \bigwedge\{b_\beta : \beta < \alpha + 2n\} \wedge F(f \restriction \alpha+2n)$, $b_{\alpha+2n+1} = \sigma(w_{\alpha+2n}), \ldots$ is a play in which Black follows σ - the simultaneous strategy of the length λ .

As $T \subseteq \bigcup \mathcal{H}$ and $\mathcal{H}$ is monotone and λ is regular we have immediately (i). For $\alpha < \lambda$ T_α is a disjoint system, so $\mathrm{hsat}(B) \geq \kappa^\lambda$. If $\kappa^\lambda > \lambda$ then κ^λ is a singular cardinal number and so $\mathrm{hsat}(B) > \kappa^\lambda$. If $\kappa^\lambda = \lambda$ then using σ we can construct a strictly decreasing tower $\{t_\alpha : \alpha < \lambda\}$ and then $\{t_\alpha - t_{\alpha+1} : \alpha < \lambda\}$ is the desired disjoint system of power $\kappa^\lambda = \lambda$.

q.e.d.

<u>Theorem 6</u>. Let B be a $(\lambda, \infty, \kappa)$-nowhere distributive complete Boolean algebra with density $d(B) = \kappa^\lambda$ for which the Black has simultaneous winning strategy of the length λ . Then B has a tree base of length λ .

PROOF. Let $D \subseteq B^+$ be a base of size κ^λ. As each element of D intersects κ^λ elements of $\Omega = \{Q_\alpha : \alpha < \lambda\}$ - a monotone matrix witnessing to $(\lambda, \infty, \kappa)$-nowhere distributivity - there is a one to one mapping $f : D \to \bigcup\Omega$ such that $x \wedge f(x) \neq \mathbb{O}$. Split each $y \in \mathrm{rng}(f)$ into two elements $y \wedge f^{-1}(y)$ and $y - f^{-1}(y)$. We obtain a matrix Ω' which refines D , i.e. $\bigcup\Omega'$ is a base. Using $(\tau, \infty, 2)$ distributivity for each $\tau < \lambda$ we obtain a monotone matrix $\mathcal{H}$ - the desired tree-base.

q.e.d.

<u>Corollary 7</u>. Let B be a $(\lambda, \infty, \kappa)$-nowhere distributive complete Boolean algebra with density $d(B) = \kappa^\lambda$ which has a simultaneous winning strategy for Black of length λ . Then

$$B \cong \mathrm{Col}(\lambda, \kappa^\lambda) = \mathrm{Col}(\lambda, \kappa) .$$

PROOF. From Lemma 5 we have $hsat(B) = (\kappa^{\vartheta})^+$, so in the construction of the Theorem 4 we can add the condition

$$(\forall x \in T_{\alpha+2n+1})(|\{y \in T_{\alpha+2n+2} : y \leq x\}| \geq \kappa^{\vartheta}) .$$

The base we obtain is isomorphic to the base of the algebra $Col(\lambda, \kappa^{\vartheta})$.

q.e.d.

We have seen that the existence of a tree base is important. Observe that if B has a tree base of the length λ and $\kappa <$ hsat(B) then B is $(\lambda, \infty, \kappa)$-nowhere distributive. About inverse implication we know only what Theorem 6 says.

The case $\lambda = \omega_1$ was studied in [4]. This paper is a generalization of the results of [4] for $\lambda = \omega_1$. For nonlimit cardinal numbers M.Foreman in [1] proved that if $\kappa = \lambda^+$, $\lambda^{\gamma} = \lambda$, B is $(\lambda, \infty, 2)$-distributive, $d(B) = \lambda^+$ and Black wins $\mathcal{G}(B, \gamma)$ then B has a γ^+-closed dense subset. Our technics works without cardinal assumptions and also for limit cardinals, but needs stronger strategical assumption.

The notion of the simultaneous strategy gives a view into the structure of all strategies. The problem, whether (c) $\rightarrow$ (a), ((a), (b), (c) are as in Theorem 1 and Lemma 2) can be splited into two: whether (c) $\rightarrow$ (b) and (b) $\rightarrow$ (a).

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