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In abstract analysis we encounter some types of convergences that cannot be topologized, e.g., convergence almost everywhere with respect to a measure or order convergence in a Boolean algebra. There have been several attempts to define completeness-type conditions for nontopological convergences. We deal here with the following condition of that kind introduced by the team of Prof. J. Mikusiński (Katowice). We say that an Abelian convergence group X has property (K) if every sequence (x_n) in X with $x_n \to 0$ contains a subsequence (x_{n_k}) such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is convergent (see [3]). We present here some results, taken from [1] and [4], concerning property (K) in the context of topological convergences.

Suppose that the convergence in X is induced by a metrizable complete group topology. Then, clearly, (K) holds. The converse fails as shown by Kliś ([3], Theorem 3) who constructed, under the continuum hypothesis, a noncomplete inner product space with property (K). We shall give more general results to this effect not relying on the continuum hypothesis (Theorems 1 and 2).

Given a subseries convergent series $\sum_{n=1}^{\infty} x_n$ in X, we denote by $r((x_n))$ the set

$$\{ \sum_{k=1}^{\infty} x_{n_k} : \ n_1 < n_2 < \dots \}.$$

PROPOSITION 1 ([4], Proposition 2 and Corollary 1). If (x_n) is a linearly independent sequence in a topological linear space such that $\sum_{n=1}^{\infty} x_n$ is subseries convergent, then there exists a subsequence (x_{n_k}) for which $\sum_{k=1}^{\infty} \lambda_k x_{n_k} = 0$ implies $(\lambda_k) = 0$ whenever (λ_k) is a bounded sequence of scalars. In particular, $\dim \operatorname{lin} r((x_n)) = 2^{\aleph_0}$.

THEOREM 1 (cf. [4], Theorem 2). Let X be a metrizable linear space with $\aleph_0 \leq \dim X \leq 2^{\aleph_0}$. Then X contains dense subspaces X_1 and X_2 with the following properties:

(i) $X_1 \cap X_2 = \{0\}$.

(ii) $r((x_n)) \cap X_i \neq \emptyset$ for every linearly independent sequence (x_n) in X such that $\sum_{n=1}^{\infty} x_n$ is subseries convergent and $i = 1, 2$.

In particular, X_i has property (K) provided X is complete.

SKETCH OF PROOF. Let $\mathcal{U}$ be a base of the topology of X with card $\mathcal{U} \leq \dim X$. (This is the only place where the metrizability assumption is used.) Denote by $\mathcal{V}$ the family of all sets $r((x_n))$, where (x_n) satisfies the conditions of (ii). Clearly, $\mathcal{V}$ is either empty or card $\mathcal{V} = 2^{\aleph_0}$. It follows that card $(\mathcal{U} \cup \mathcal{V}) \leq \dim X$. Arrange $\mathcal{U} \cup \mathcal{V}$ into a transfinite sequence $(S_\alpha)_{\alpha<\varphi}$, where φ is the least ordinal with card $\varphi =$ card $(\mathcal{U} \cup \mathcal{V})$. As $\dim \operatorname{lin} S_\alpha = \dim X$ for every $\alpha < \varphi$ (Proposition 1), it is easy to construct a linearly independent set $\{x_\alpha^i : \ i = 1, 2 \text{ and } \alpha < \varphi\} \subset X$ such that

$$x_\alpha^i \in S_\alpha \quad \text{for } i = 1, 2 \text{ and } \alpha < \varphi .$$

Now, it is enough to put $X_i = \operatorname{lin} \{x_\alpha^i : \alpha < \varphi\}$ for $i = 1, 2$.

The last assertion is clear. Indeed, if (x_n) is a sequence in X with $x_n \to 0$, then $\sum_{k=1}^{\infty} |x_{n_k}| < \infty$ for some $n_1 < n_2 < \dots$, where $|\cdot|$ is an F-norm in X. In case (x_{n_k}) contains no linearly independent subsequence, $\sum_{k=1}^{\infty} x_{n_k} \in X_i$. In the other case, we use (ii).

PROPOSITION 2 ([1], Proposition 1). If (x_n) is a sequence of nonzero elements in a Hausdorff Abelian group such that $\sum_{n=1}^{\infty} x_n$ is subseries convergent, then there exists a subsequence (x_{n_k}) for which $\sum_{k=1}^{\infty} \delta_k x_{n_k} = 0$ implies $(\delta_k) = 0$ whenever $\delta_k \in \{-1, 0, 1\}$. In particular, card $r((x_n)) = 2^{\aleph_0}$.

Applying Proposition 2 instead of Proposition 1, one can prove

THEOREM 2 (cf. [1], Theorem 1). Let X be a nondiscrete metrizable complete Abelian group with card $X = 2^{\aleph_0}$ such that the equation $nx = z$ has (at most) countably many solutions given $n \in N$ and $z \in X$ with $z \neq 0$. Then X contains dense subgroups X_1 and X_2 with the following properties:

(i) $X_1 \cap X_2 = \{0\}$.

(ii) $r((x_n)) \cap X_i \neq \emptyset$ for every sequence (x_n) in X of nonzero elements such that $\sum_{n=1}^{\infty} x_n$ is subseries convergent and $i = 1, 2$.

In particular, X_i has property (K).

Even though property (K) is weaker than completeness, it turns out to be strong enough to imply the Baire category theorem.

THEOREM 3 ([1], Theorem 2). Every metrizable Abelian group with property (K) is a Baire space.

On the other hand, there exist metrizable Baire linear spaces without property (K) ([1], Theorem 3).

Theorem 3 and a result of Christensen ([2], Theorem 5.4) yield the nontrivial part of the following

COROLLARY ([1], Corollary). Let X be an analytic metrizable Abelian group. Then X has property (K) if and only if X is complete.

This corollary seems to suggest that an example of a noncomplete metrizable Abelian group (or a linear space) with property (K) cannot be constructed effectively.

For an application of Theorems 1 and 3 see L. Drewnowski, Solution to a problem of De Wilde – Tsirulnikov, this volume.

References

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