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THE FIRST DIRICHLET EIGENVALUE OF BICYCLIC GRAPHS

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Abstract. In this paper, we have investigated some properties of the first Dirichlet eigenvalue of a bicyclic graph with boundary condition. These results can be used to characterize the extremal bicyclic graphs with the smallest first Dirichlet eigenvalue among all the bicyclic graphs with a given graphic bicyclic degree sequence with minor conditions. Moreover, the extremal bicyclic graphs with the smallest first Dirichlet eigenvalue among all the bicycle graphs with fixed k interior vertices of degree at least 3 are obtained.

Keywords: first Dirichlet eigenvalue, bicyclic graph, degree sequence

MSC 2010: 05C50, 05C35

1. Introduction

In this paper, we only consider connected, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(v)$ denote the degree of a vertex v in G and $L(G) = D(G) - A(G)$ be the Laplacian matrix of G, where $D(G)$ is the diagonal matrix whose diagonal entries are vertex degrees and $A(G)$ is the adjacency matrix of G. A sequence of non-negative integers $(d_0, d_1, \ldots, d_{n-1})$ is called *graphic degree sequence* if there exists a simple connected graph having π as its vertex degree sequence. Zhang [6] determined the extremal graphs which have largest Laplacian eigenvalues among all trees with a given graphic tree degree. Recently, the Dirichlet eigenvalues of graphs with boundary have received much attention (see [1], [3]–[5]), since the graph Laplacian can be regarded the discrete analog of the continuous Laplace operator on manifolds. Bıyıkoğlu and Leydold in [1] characterized the graphs with the smallest first Dirichlet eigenvalue among

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the trees with a given degree sequence with the spectral geometry methods. Further, the authors of this paper in [5] determined the graphs with the smallest first Dirichlet eigenvalue among the unicyclic graphs with a give n-degree sequence with minor conditions. These results may be called discrete Faber-Krahn type theorems, since they can be regarded the discrete analog of the celebrated Faber-Krahn Theorem concerning Dirichlet eigenvalues among all bounded domains of the same volume in $\mathbb{R}^s$.

A connected graph is said to be *bicyclic* if its number of edges is equal to its number of vertices plus one. Motivated by the above results, we will investigate the first Dirichlet eigenvalue of bicyclic graphs with a given degree sequence with minor conditions. Let π be a graphic bicyclic degree sequence. In this paper we always assume that the frequency of 2 in π is 0. Let $\mathcal{B}_\pi$ be the set of bicyclic graphs with given degree sequence π. In this paper, we are interested in the first Dirichlet eigenvalue of bicyclic graphs. The rest of this paper is organized as follows: In Section 2, we recall some properties of the first Dirichlet eigenvalue of a graph with boundary. In Section 3, we investigate the structure of bicyclic graphs with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$. In Section 4, we characterize the extremal graphs with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and investigate the relationship between the first Dirichlet eigenvalues of bicyclic graphs having the smallest Dirichlet eigenvalue with different graphic bicyclic degree sequences.

2. Preliminaries

In this section, we introduce some notations and lemmas. The definition of a graph with boundary differs from that given in [2] where the set of vertices is just partitioned. In a further step the set of boundary vertices is then the set of pendant vertices. A graph with boundary is a graph $G = (V_0 \cup \partial V,\, E_0 \cup \partial E)$ such that degree of any vertex in V_0 (interior vertex) is at least 2, a vertex in ∂V (boundary vertex) is a *pendant vertex*, the edge in E_0 (interior edge) connects two interior vertices, and the edge in ∂E (boundary edge) connects an interior vertex and a boundary vertex. Let S be the set of all real-valued functions on $V(G)$ with $f(u) = 0$ for $u \in \partial V$. The *first Dirichlet eigenvalue*, denoted by $\lambda(G)$, is the smallest real number λ such that there exists a function $f \in S$ such that $L(G)f(u) = \lambda f(u)$ for $u \in V(G)$. Let $R_G(f)$ denote the Rayleigh quotient of $L(G)$ on real-valued function f on $V(G)$. Then

$$\lambda(G) = \min_{f \in S} R_G(f) > 0,$$

and there exists a unique eigenfunction f of $\lambda(G)$ with $\|f\| = 1$ and $f(v) > 0$ for $v \in V_0$ (see [2]). Such an eigenfunction f is called *the first Dirichlet eigenfunction* of G.

Let $G - uv$ denote the graph obtained from G by deleting an edge uv in G and $G + uv$ denote the graph obtained from G by adding an edge uv. The following lemma is well-known.

Lemma 2.1 ([1])**.** *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected graph with boundary. Suppose that u, v, $x \in V_0$ and $y \in V(G)$ satisfy uv, $xy \in E(G)$ and $ux, vy \notin E(G)$. Let f be the first Dirichlet eigenfunction of G and $G' = G - uv - xy + ux + vy$. If $f(u) \geqslant f(y)$ and $f(x) \geqslant f(v)$, then*

$$\lambda(G') \leqslant \lambda(G).$$

Moreover, the inequality is strict if one of the two inequalities is strict.

By Lemma 2.1, we can get the following corollary:

Corollary 2.2. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$. Suppose that u, v, $x \in V_0$ and $y \in V(G)$ satisfy uv, $xy \in E(G)$ and $ux, vy \notin E(G)$. Let f be the first eigenfunction of G and $G' = G - uv - xy + ux + vy$. If $G' \in \mathcal{B}_\pi$, then the following holds:*
(1) *if $f(u) = f(y)$, then $f(v) = f(x)$;*
(2) *if $f(u) > f(y)$, then $f(v) > f(x)$.*

Let $uv \in E(G)$ be a cut edge of a graph G. If a connected component of $G - uv$ is a tree T with $v \in V(T)$, then the induced subgraph of G induced by $V(T) \cup \{u\}$ is called a *branch tree* at the vertex u of G.

Lemma 2.3. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and f be the first Dirichlet eigenfunction of G. If there is a branch tree T at a vertex u, then $f(x_2) > f(u)$, for any edge x_1x_2 contained in any cycle C in G with $u \neq x_1$, $u \neq x_2$, and $ux_1 \notin E(G)$.*

Proof. Since there is a branch tree T at u, there exists a path $P = uv_1 \dots v_s$ with $s \geqslant 1$, $v_k \in V(T)$ for $1 \leqslant k \leqslant s$ and $v_s \in \partial V$. Clearly, $x_1v_k, x_2v_k \notin E(G)$ for $1 \leqslant k \leqslant s$. Suppose that $f(x_2) \leqslant f(u)$. Let $G_1 = G - x_1x_2 - uv_1 + x_1u + x_2v_1$. Then $G_1 \in \mathcal{B}_\pi$ and $f(x_1) \leqslant f(v_1)$ by Corollary 2.2. Further, let $G_2 = G - x_1x_2 - v_1v_2 + x_1v_2 + x_2v_1$. Then $G_2 \in \mathcal{B}_\pi$ and $f(x_2) \leqslant f(v_2)$ by Corollary 2.2. Hence by repeating above methods, we get $f(v_s) \geqslant \min\{f(x_1), f(x_2)\}$. But $f(v_s) = 0$, since $v_s \in \partial V$. Therefore $f(x_1) = 0$ or $f(x_2) = 0$. This is a contradiction with $x_1, x_2 \in V_0$. So the assertion holds. $\square$

Corollary 2.4. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and f be the first Dirichlet eigenfunction of G. If a vertex v is not contained in any cycle of G, then there exist at least three vertices v_0, v_1, v_2 in some cycle such that $f(v_i) > f(v)$ for $i = 0, 1, 2$.*

Proof. If v is a boundary vertex, the assertion obviously holds. Assume that v is an interior vertex not contained in any cycle. Then there is a branch tree T at the vertex v. Further there exist at least three edges v_0u_0, v_1u_1, v_2u_2 such that $v_0 \neq v_1 \neq v_2$ and $vu_i \notin E(G)$ for $i = 0, 1, 2$. Hence by Lemma 2.3, $f(v_i) > f(v)$ for $i = 0, 1, 2$. $\square$

Lemma 2.5 (see also [5])**.** *Let $G = (V_0 \cup \partial V,\ E_0 \cup \partial E) \in \mathcal{B}_\pi$ and $v_1, v_2 \in V_0$. Suppose that $u_tv_1 \in E(G)$, $u_tv_2 \notin E(G)$ and u_t is not on any path from v_1 to v_2 for $t = 1, 2, \ldots, k$, where $k \leqslant d_G(v_1) - 2$. Let $f \in S$ with $\|f\| = 1$ and $G_1 = G - u_1v_1 - \ldots - u_kv_1 + u_1v_2 + \ldots + u_kv_2$. Then G_1 and G have the same boundary, and $R_{G_1}(f) \leqslant R_G(f)$ if $f(v_1) \geqslant f(v_2) \geqslant f(u_t)$ for $t = 1, 2, \ldots, k$.*

Proof. Clearly, G_1 and G have the same boundary. By

$$\begin{aligned}
R_{G_1}(f) - R_G(f) &= \sum_{t=1}^{k}(f(v_2) - f(u_t))^2 - \sum_{t=1}^{k}(f(v_1) - f(u_t))^2 \\
&= \sum_{t=1}^{k}(f(v_2) - f(v_1))(f(v_1) + f(v_2) - 2f(u_t)) \\
&\leqslant 0,
\end{aligned}$$

the assertion holds. $\square$

3. The structure of graphs with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$

In this section we will investigate the structure of the graphs with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$. Let $B(P_k, P_l, P_m)$ be the bicyclic graph obtained from three pairwise internal disjoint paths P_k, P_l, and P_m from a vertex x to a vertex y with length k, l and m, respectively, where at most one of k, l, m is equal to 1. It is well known that there exist two edge-disjoint cycles or some induced subgraph $B(P_k, P_l, P_m)$ in any bicycle graph. Then we have the following:

Lemma 3.1. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph of order $n \geqslant 4$ with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and f be the first Dirichlet eigenfunction of G. Then there exist three vertices v_0, v_1 and v_2 contained in some cycles of G such that $f(v_0) \geqslant f(v_1) \geqslant f(v_2) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2\}$.*

Proof. Let v_0, v_1 and v_2 be three vertices such that $f(v_0) \geqslant f(v_1) \geqslant f(v_2) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2\}$. By Corollary 2.4, it is easy to see that v_0, v_1, v_2 are contained in some cycles of G. $\square$

Lemma 3.2. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$. Then G contains some induced subgraph $B(P_k, P_l, P_m)$.*

Proof. Suppose that G does not contain any induced subgraph $B(P_k, P_l, P_m)$. Then there exist two edge-disjoint cycles C_1 and C_2. Let f be the first Dirichlet eigenfunction of G. By Lemma 3.1, there exists three vertices v_0, v_1 and v_2 contained in cycles of G such that $f(v_0) \geqslant f(v_1) \geqslant f(v_2) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2\}$. We consider the following two cases:

Case 1: C_1 and C_2 have no common vertices. Then there exists at least one vertex in $\{v_0, v_1, v_2\}$ such that there is a branch tree at some vertex, say $v_2 \in V(C_1)$, since $d(v_2) \geqslant 3$ and v_2 is an interior vertex. Let $x_1x_2 \in E(C_2)$ be such that $x_1v_2 \notin E(G)$ and $x_2 \neq v_0, v_1$. Then $f(x_2) > f(v_2)$ by Lemma 2.3. This is a contradiction.

Case 2: C_1 and C_2 have a common vertex. If $v_0 \in V(C_1) \cap V(C_2)$, then there is a branch tree at v_1 since $d(v_1) \geqslant 3$. Without loss of generality, assume $v_1 \in V(C_1)$ and let $y_1y_2 \in E(C_2)$ be such that $y_1v_1 \notin E(G)$ and $y_2 \neq v_0$. Then $f(y_2) > f(v_1)$ by Lemma 2.3, which is a contradiction. If $v_0 \notin V(C_1) \cap V(C_2)$, then there is a branch tree at v_0, since $d(v_0) \geqslant 3$. By Lemma 2.3, we can also get a contradiction. So the assertion holds. $\square$

Lemma 3.3. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a bicyclic graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and f be the first Dirichlet eigenfunction of G. Then there are two cycles $v_0v_1v_2$ and $v_0v_1v_3$ in G such that $f(v_0) \geqslant f(v_1) \geqslant f(v_2) \geqslant f(v_3) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2, v_3\}$.*

Proof. By Lemma 3.2, G contains an induced subgraph $B(P_k, P_l, P_m)$, denoted by H. By Lemma 3.1, there exist three vertices v_0, v_1 and v_2 such that $v_0, v_1, v_2 \in V(H)$ and $f(v_0) \geqslant f(v_1) \geqslant f(v_2) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2\}$.

Claim 1: $d_H(v_0) = d_H(v_1) = 3$. If $d_H(v_0) = 2$, then there is a branch tree at v_0 since $d_G(v_0) \geqslant 3$. Let C_1 and C_2 be two cycles in G such that $v_0 \in V(C_1)$ and $v_0 \notin V(C_2)$. Clearly, there exists an edge $x_1x_2 \in E(C_2)$ such that $x_1v_0 \notin E(G)$. Then we have $f(x_2) > f(v_0)$ by Lemma 2.3. This is impossible. So $d_H(v_0) = 3$. Similarly, we can show that $d_H(v_1) = 3$.

Claim 2: $v_1v_2, v_0v_2 \in E(G)$. Suppose that $v_1v_2 \notin E(G)$. By Claim 1, we have $d_H(v_2) = 2$. Further there is a branch tree at v_2 since $d_G(v_2) \geqslant 3$. On the other hand, there exists an edge $v_1y \in E(H)$ such that $y \neq v_0$. Then $f(y) > f(v_2)$ by Lemma 2.3. This is impossible. So $v_1v_2 \in E(G)$. By the same proof, we have $v_0v_2 \in E(G)$.

Let $v_3 \in V(G)$ be such that $f(v_3) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2, v_3\}$. Then $v_3 \in V(H)$. Otherwise, let C be a cycle of the graph $G - v_0v_2$. Then there exists an edge $z_1z_2 \in E(C)$ such that $z_1v_3 \notin E(G)$ and $z_2 \neq v_0, v_1$. Note that there is a branch tree at v_3 and $z_2 \neq v_2$. Then $f(v_3) < f(z_2)$ by Lemma 2.3, a contradiction.

Claim 3: $v_1v_3, v_0v_3 \in E(G)$. Suppose that $v_1v_3 \notin E(G)$. Note that $d_H(v_3) = 2$ and there is a branch tree at v_3. Moreover, clearly there exists an edge $v_1w \in E(C)$ such that $w \neq v_0$ and $w \neq v_2$. Then $f(w) > f(v_3)$ by Lemma 2.3. This is impossible. So $v_1v_3 \in E(G)$. Further we have $v_0v_3 \in E(G)$ by the same proof.

At last, we prove $v_0v_1 \in E(G)$. If $v_0v_1 \notin E(G)$, then there exists a vertex $u \in V(G)$ such that $v_0u \in E(H)$ and $u \neq v_2, v_3$. Clearly, $uv_2 \notin E(G)$. Since $d_H(u) = 2$, there is a branch tree at u. By Lemma 2.3, $f(v_1) > f(u)$. Let $G_1 = G - v_0u - v_1v_2 + v_0v_1 + uv_2$. Then G_1 is connected. Further $G_1 \in \mathcal{B}_\pi$ and $f(v_2) > f(v_0)$ by Corollary 2.2. This is a contradiction. So $v_0v_1 \in E(G)$. The proof is completed. $\square$

Theorem 3.4. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and f be the first Dirichlet eigenfunction of G. Then there exists a numeration of the vertices of G with root v_0 such that the following hold:*

(1) $f(v_0) \geqslant f(v_1) \geqslant \ldots \geqslant f(v_{n-1})$;

(2) $h(v_0) \leqslant h(v_1) \leqslant \ldots \leqslant h(v_{n-1})$, *where $h(v)$ is the distance between a vertex v and the root v_0;*

(3) *if v_iv_j, $v_sv_t \in E(G)$ with $i < s$ and $h(v_j) = h(v_t) = h(v_i) + 1 = h(v_s) + 1$, then $j < t$.*

Proof. By Lemma 3.3, there are two cycles $v_0v_1v_2$ and $v_0v_1v_3$ in G such that $f(v_0) \geqslant f(v_1) \geqslant f(v_2) \geqslant f(v_3) \geqslant f(x)$ for $x \in V(G) \setminus \{v_0, v_1, v_2, v_3\}$. Let v_0 be the root of G and $h(v)$ be the distance between a vertex v and the root v_0. Suppose $h(G) = \max\limits_{v \in V(G)} h(v)$. Let $V_i = \{v \in V(G) | h(v) = i\}$ and $D_i = | V_i |$ for $0 \leqslant i \leqslant h(G)$. In order to prove that the assertion holds, we relabel the vertices of G. Let $V_0 = \{v_{0,1}\}$, where $v_{0,1} = v_0$. Clearly, $D_1 = d_G(v_0)$. The vertices in V_1 are relabeled $v_{1,1}, v_{1,2}, \ldots, v_{1,D_1}$ such that $f(v_{1,1}) \geqslant f(v_{1,2}) \geqslant \ldots \geqslant f(v_{1,D_1})$ and $v_{1,1}v_{1,2}$, $v_{1,1}v_{1,3} \in E(G)$. Assume that the vertices in V_t have already been relabeled $v_{t,1}, v_{t,2}, \ldots, v_{t,D_t}$. The vertices in V_{t+1} can be relabeled $v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,D_{t+1}}$ such that they satisfy the following conditions: If $v_{t,k}v_{t+1,i}, v_{t,k}v_{t+1,j} \in E(G)$ and $i < j$, then $f(v_{t+1,i}) \geqslant f(v_{t+1,j})$; if $v_{t,k}v_{t+1,i}, v_{t,l}v_{t+1,j} \in E(G)$ and $k < l$, then $i < j$.

Claim: $f(v_{t,1}) \geqslant f(v_{t,2}) \geqslant \ldots \geqslant f(v_{t,D_t}) \geqslant f(v_{t+1,1})$ for $0 \leqslant t \leqslant h(G)$.

We will prove that the Claim holds by the induction on t. Clearly, the Claim holds for $t=0$ and $f(v_{1,1}) \geqslant f(v_{1,2}) \geqslant \ldots \geqslant f(v_{1,D_1})$. If $d_G(v_{0,1}) = 3$, then $D_1 = 3$ and $v_{1,1} = v_1, v_{1,2} = v_2, v_{1,3} = v_3$. It is easy to see that $f(v_{1,3}) \geqslant f(v_{2,1})$. Assume now $d_G(v_{0,1}) > 3$. Clearly, $v_{1,D_1}v_{1,1}, v_{0,1}v_{2,1} \notin E(G)$. Since $d_G(v_{1,2}) \geqslant 3$, we have $v_{1,1}v_{2,1} \in E(G)$ or $v_{1,2}v_{2,1} \in E(G)$. Without loss of generality, assume $v_{1,1}v_{2,1} \in E(G)$. Let $G_1 = G - v_{0,1}v_{1,D_1} - v_{1,1}v_{2,1} + v_{0,1}v_{2,1} + v_{1,1}v_{1,D_1}$. Clearly, $G_1 \in \mathcal{B}_\pi$. Since $f(v_{0,1}) \geqslant f(v_{1,1})$, we get $f(v_{1,D_1}) \geqslant f(v_{2,1})$ by Corollary 2.2. So the Claim also holds for $t=1$. Now assume that the Claim holds for $t = s-1$. In the following we prove that the Claim holds for $t=s$. If there are two vertices $v_{s,i}, v_{s,j} \in V_s$ with $i<j$ and $f(v_{s,i}) < f(v_{s,j})$, then there exist two vertices $v_{s-1,k}, v_{s-1,l} \in V_{s-1}$ with $k<l$ such that $v_{s-1,k}v_{s,i}, v_{s-1,l}v_{s,j} \in E(G)$. By the induction hypothesis, $f(v_{s-1,k}) \geqslant f(v_{s-1,l})$. Let $G_2 = G - v_{s-1,k}v_{s,i} - v_{s-1,l}v_{s,j} + v_{s-1,k}v_{s,j} + v_{s-1,l}v_{s,i}$. Clearly, $G_2 \in \mathcal{B}_\pi$. By Lemma 2.1, $\lambda(G_1) < \lambda(G)$. This is impossible. So $f(v_{s,i}) \geqslant f(v_{s,j})$. By a similar proof we have $f(v_{s,D_s}) \geqslant f(v_{s+1,1})$. So the Claim holds. Therefore, we finish our proof. $\square$

4. The extremal graphs with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$

In this section, we will characterize the extremal graphs with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and give the majorization theorem for bicyclic graphs with different graphic bicyclic degree sequences. The following lemma is a well known result.

Lemma 4.1. *Let $G = (V_0 \cup \partial V, E_0 \cup \partial E)$ be a graph with boundary and f be the first Dirichlet eigenfunction of G. Then for any vertex $v \in V_0$, there exists a vertex $u \in V(G)$ such that $uv \in E(G)$ and $f(u) < f(v)$.*

Let $\pi = (d_0, d_1, \ldots, d_{k-1}, d_k, \ldots, d_{n-1})$ be a graphic bicyclic degree sequence with $3 \leqslant d_0 \leqslant d_1 \leqslant \ldots \leqslant d_{k-1}$ and $d_k = \ldots = d_{n-1} = 1$ for $k \geqslant 4$. In order to present our main result, we need to construct a special bicyclic graph B_π^* with degree sequence π. Select a vertex v_{01} a root and begin with v_{01} of the zero-th layer. Let $s_1 = d_0$ and select s_1 vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,s_1}$ of the first layer such that they are adjacent to $v_{0,1}$ and $v_{1,1}$ is adjacent to $v_{1,2}$ and $v_{1,3}$. Next we construct the second layer as follows. Let $s_2 = \sum_{i=1}^{s_1} d_i - s_1 - 4$ and select s_2 vertices $v_{2,1}, v_{2,2}, \ldots, v_{2,s_2}$ such that $v_{1,1}$ is adjacent to $v_{2,1}, \ldots, v_{2,d_1-3}$, $v_{1,2}$ is adjacent to $v_{2,d_1-2}, \ldots, v_{2,d_1+d_2-5}$, $v_{1,3}$ is adjacent to $v_{2,d_1+d_2-4}, \ldots, v_{2,d_1+d_2+d_3-7}, \ldots,$

$v_{1,j}$ is adjacent to $v_{2,d_1+\ldots+d_{j-1}-j-2}, \ldots, v_{2,d_1+\ldots+d_j-j-4}, \ldots,$ v_{1,s_1} is adjacent to $v_{2,d_1+\ldots+d_{s_1-1}-s_1-2}, \ldots, v_{2,d_1+\ldots+d_{s_1}-s_1-4} = v_{2,s_2}$. In general, assume that all vertices of the t-th layer have been constructed and are denoted by $v_{t,1}, v_{t,2}, \ldots, v_{t,s_t}$. We construct all the vertices of the $(t+1)$-st layer by the induction hypothesis. Let $s_{t+1} = d_{s_1+\ldots+s_{t-1}+1} + \ldots + d_{s_1+\ldots+s_t} - s_t$ and select s_{t+1} vertices $v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,s_{t+1}}$ of the $(t+1)$-st layer such that $v_{t,1}$ is adjacent to $v_{t+1,1}, \ldots, v_{t+1,d_{s_1+\ldots+s_{t-1}+1}-1}, \ldots,$ v_{t,s_t} is adjacent to $v_{t+1,s_{t+1}-d_{s_1+\ldots+s_t}+2}, \ldots, v_{t+1,s_{t+1}}$. In this way, we obtain only one bicyclic graph B_π^* with degree sequence π (see Figure 1 for an example).

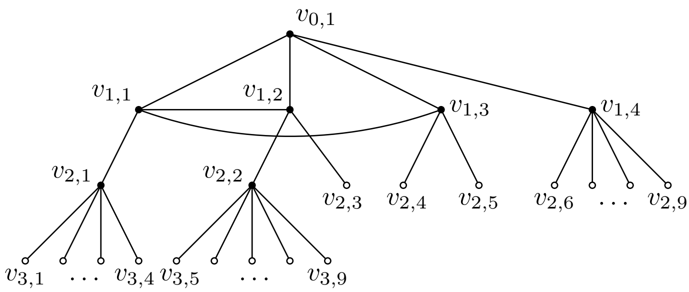

Figure 1. B_π^* with $\pi = (4,4,4,4,5,5,6,1,\ldots,1)$ ($\circ \ldots$ boundary vertex)

Theorem 4.2. *Let $\pi = (d_0, d_1, \ldots, d_{k-1}, 1, 1, \ldots, 1)$ be a graphic bicyclic degree sequence with $3 \leqslant d_0 \leqslant d_1 \leqslant \ldots \leqslant d_{k-1}$ for $k \geqslant 4$. Then every extremal graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ is isomorphic to B_π^*.*

Proof. Let G be a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$ and f be the first Dirichlet eigenfunction of G. Without loss of generality, assume $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that the three assertions in Theorem 3.4 hold and G has two cycles $v_0v_1v_2$ and $v_0v_1v_3$. In order to prove that the assertion holds, we only need to prove $d_G(v_t) = d_t$ for $0 \leqslant t \leqslant k-1$.

Since $d_{i+1} \geqslant d_i$ for $0 \leqslant i \leqslant k-2$, we have $d_G(v_0) \geqslant d_0$. If $d_G(v_0) = d_0$, there is nothing to do and let $G_1 = G$. Otherwise, let $G_1 = G - v_0v_{d_0+1} - \ldots - v_0v_{d_G(v_0)} + v_1v_{d_0+1} + \ldots + v_1v_{d_G(v_0)}$. Since $f(v_0) \geqslant f(v_1) \geqslant f(v_{d_0+1}) \geqslant \ldots \geqslant f(v_{d_G(v_0)})$, we have $R_{G_1}(f) \leqslant R_G(f)$ by Lemma 2.5. Clearly, $d_{G_1}(v_0) = d_0$ and $d_{G_1}(v_1) \geqslant d_1$. If $d_{G_1}(v_1) = d_1$, there is nothing to do and let $G_2 = G_1$. Otherwise, let $G_2 = G_1 - v_1v_{d_0+d_1-2} - \ldots - v_1v_{d_G(v_0)+d_G(v_1)-3} + v_2v_{d_0+d_1-2} + \ldots + v_2v_{d_G(v_0)+d_G(v_1)-3}$. Then $d_{G_2}(v_1) = d_1$ and $R_{G_2}(f) \leqslant R_{G_1}(f)$ by Lemma 2.5, since $f(v_1) \geqslant f(v_2) \geqslant f(v_{d_0+d_1-2}) \geqslant \ldots \geqslant f(v_{d_G(v_0)+d_G(v_1)-3})$. Obviously, $d_{G_2}(v_2) \geqslant d_2$ and $v_2v_s \in E(G_2)$ for $d_0 + d_1 - 2 \leqslant s \leqslant d_G(v_0) + d_G(v_1) + d_G(v_2) - 5$. If $d_{G_2}(v_2) = d_2$, let $G_3 = G_2$. Otherwise, let $G_3 = G_2 - v_2v_{d_0+d_1+d_2-4} - \ldots - v_2v_{d_G(v_0)+d_G(v_1)+d_G(v_2)-5} + v_3v_{d_0+d_1+d_2-4} + \ldots + v_3v_{d_G(v_0)+d_G(v_1)+d_G(v_2)-5}$. Note that $f(v_1) \geqslant f(v_2) \geqslant f(v_{d_0+d_1+d_2-4}) \geqslant \ldots \geqslant f(v_{d_G(v_0)+d_G(v_1)+d_G(v_2)-5})$. We have

$d_{G_3}(v_2) = d_2$ and $R_{G_3}(f) \leqslant R_{G_2}(f)$ by Lemma 2.5. Similarly, if $d_{G_3}(v_3) = d_3$, let $G_4 = G_3$. Otherwise, we can also get graph G_4 by exchanging edges such that $d_{G_4}(v_3) = d_3$ and $R_{G_4}(f) \leqslant R_{G_3}(f)$. We proceed in the same way with $v_4, v_5, \dots, v_{i-1}$ and get a sequence of bicyclic graphs $G_1, G_2, \dots, G_i$ with the same set of boundary vertices such that $d_{G_i}(v_t) = d_t$ for $t = 0, 1, \dots, i-1$, and

$$\lambda(G) = R_G(f) \geqslant R_{G_1}(f) \geqslant R_{G_2}(f) \geqslant \ldots \geqslant R_{G_i}(f),$$

where $i \geqslant 4$. Since

$$\sum_{t \leqslant i} d_{G_i}(v_t) = \sum_{t \leqslant i} d_G(v_t) \geqslant \sum_{t \leqslant i} d_t$$

and

$$\sum_{t<i} d_{G_i}(v_t) = \sum_{t<i} d_t,$$

we have

$$d_{G_i}(v_i) = \sum_{t \leqslant i} d_{G_i}(v_t) - \sum_{t<i} d_{G_i}(v_t) \geqslant \sum_{t \leqslant i} d_t - \sum_{t<i} d_t = d_i.$$

If $d_{G_i}(v_i) = d_i$, there is nothing to do and let $G_{i+1} = G_i$. Otherwise, let x_1, x_2, $\dots$, $x_{d_{G_i}(v_i)-1}$ be all vertices in G_i which are adjacent to v_i with $f(x_s) \geqslant f(x_{s+1})$ for $1 \leqslant s \leqslant d_{G_i}(v_i) - 2$ and $h(v_i) < h(x_s)$ for $1 \leqslant s \leqslant d_{G_i}(v_i) - 1$. Let $G_{i+1} = G_i - v_i x_{d_i} - \ldots - v_i x_{d_{G_i}(v_i)-1} + v_{i+1} x_{d_i} + \ldots + v_{i+1} x_{d_{G_i}(v_i)-1}$. Then $d_{G_{i+1}}(v_i) = d_i$ and $R_{G_i}(f) \geqslant R_{G_{i+1}}(f)$ by Lemma 2.5, since $f(v_i) \geqslant f(v_{i+1}) \geqslant f(x_s)$ for $1 \leqslant s \leqslant d_{G_i}(v_i) - 1$. We continue in this way and get a sequence of bicyclic graphs

$$G \to G_1 \to G_2 \to \ldots \to G_k$$

such that $d_{G_s}(v_{s-1}) = d_{s-1}$ for $1 \leqslant s \leqslant k$ and

$$\lambda(G) = R_G(f) \geqslant R_{G_1}(f) \geqslant R_{G_2}(f) \geqslant \ldots \geqslant R_{G_k}(f) \geqslant \lambda(G_k).$$

Clearly, $G_k = B^*_\pi$. Since G is a graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_\pi$, we have $\lambda(G) = \lambda(B^*_\pi)$ and f is also the first Dirichlet eigenfunction of B^*_π. If $G \neq B^*_\pi$, then there exists a vertex v_p such that $d_G(v_s) = d_s$ for $0 \leqslant s \leqslant p-1$ and $d_G(v_p) > d_p$. Let $y_1, y_2, \dots, y_m$ be all vertices which are adjacent to v_p in G and are not adjacent to v_p in B^*_π such that $f(y_m) \leqslant f(y_{m-1}) \leqslant \ldots \leqslant f(y_1) \leqslant f(v_p)$ and $h(y_s) > h(v_p)$ for $1 \leqslant s \leqslant m$. Then by the construction of G_{p+1}, we have $f(y_m) \leqslant f(u) \leqslant f(v_p)$ for $u \in V(G)$ such that $uv_p \in E(G)$ and $h(u) > h(v_p)$. By

$$\begin{aligned} \lambda(B^*_\pi) f(v_p) &= \sum_{xv_p \in E(B^*_\pi)} (f(v_p) - f(x)) = \lambda(G) f(v_p) \\ &= \sum_{xv_p \in E(B^*_\pi)} (f(v_p) - f(x)) + \sum_{s=1}^{m} (f(v_p) - f(y_s)), \end{aligned}$$

we have $f(v_p) = f(y_s)$ for $1 \leqslant s \leqslant m$. Then $f(w) \geqslant f(v_p)$ for any vertex w adjacent to v_p. This is a contradiction with Lemma 4.1. So $G = B^*_\pi$. The proof is completed. $\square$

In order to present other results, we need a notation and a lemma. Let $\sigma = (a_0, a_1, \ldots, a_{n-1})$ and $\tau = (b_0, b_1, \ldots, b_{n-1})$ be two positive sequences. If the sum of the largest t terms in $\{b_0, \ldots, b_{n-1}\}$ is no less than the sum of the largest t terms in $\{a_0, \ldots, a_{n-1}\}$ for $t = 0, 1, \ldots, n-2$ and $\sum\limits_{i=0}^{n-1} a_i = \sum\limits_{i=0}^{n-1} b_i$, then τ is said to *majorize* σ, denoted by $\sigma \trianglelefteq \tau$. The following result is well known.

Lemma 4.3 (for example, see [7]). *Let $\pi = (d_0, \ldots, d_{n-1})$ and $\pi' = (d'_0, \ldots, d'_{n-1})$ be two graphic degree sequences. If $\pi \trianglelefteq \pi'$, then there exists a series of graphic degree sequences $\pi_1, \ldots, \pi_k$ such that $\pi \trianglelefteq \pi_1 \trianglelefteq \ldots \trianglelefteq \pi_k \trianglelefteq \pi'$, and π_i and π_{i+1} differ in exactly two components and these differ by* 1.

Theorem 4.4. *Let $\pi_1 = (d_0, d_1, \ldots, d_{k-1}, 1, \ldots, 1)$ and $\pi_2 = (d'_0, d'_1, \ldots, d'_{k-1}, 1, \ldots, 1)$ be two graphic bicyclic degree sequences with $3 \leqslant d_0 \leqslant d_1 \leqslant \ldots \leqslant d_{k-1}$ and $3 \leqslant d'_0 \leqslant d'_1 \leqslant \ldots \leqslant d'_{k-1}$. If $\pi_2 \trianglelefteq \pi_1$, then $\lambda(B^*_{\pi_1}) \leqslant \lambda(B^*_{\pi_2})$ with equality if and only if $\pi_1 = \pi_2$.*

Proof. By Lemma 4.3, without loss of generality, we may assume that $\pi_1 = (d_0, \ldots, d_p - 1, \ldots, d_q + 1, \ldots, d_{k-1}, 1, \ldots, 1)$ and $\pi_2 = (d_0, \ldots, d_{k-1}, 1, \ldots, 1)$ for $0 \leqslant p < q \leqslant k-1$. Then $d_p \geqslant 4$. In fact, $B^*_{\pi_1}$ can be obtained from $B^*_{\pi_2}$ by the same method of constructing B^*_π in Theorem 4.2 such that $\lambda(B^*_{\pi_1}) \leqslant \lambda(B^*_{\pi_2})$. If $\lambda(B^*_{\pi_1}) = \lambda(B^*_{\pi_2})$, it is easy to see that $\pi_1 = \pi_2$ by last part of the proof of Theorem 4.2. Therefore the assertion holds. $\square$

Let $\mathcal{B}_{n,k}$ be the set of all bicyclic graphs with order n and k interior vertices whose degree is at least 3. Then we have the following

Corollary 4.5. *Let $\pi_1 = (3, 3, \ldots, 3, n-2k+5, 1, \ldots, 1)$ with the frequencies of 3 and 1 being $k-1$ and $n-k$, respectively, where $n \geqslant 2k-2$. Then $B^*_{\pi_1}$ is the unique graph with the smallest first Dirichlet eigenvalue in $\mathcal{B}_{n,k}$.*

Proof. Clearly, $B^*_{\pi_1} \in \mathcal{B}_{n,k}$. Let G be any bicyclic graph in $\mathcal{B}_{n,k}$ with degree sequence $\pi = (d_0, d_1, \ldots, d_{k-1}, 1, \ldots, 1)$. It is easy to see that $\pi \trianglelefteq \pi_1$. By Theorems 4.2 and 4.4, we have $\lambda(B^*_{\pi_1}) \leqslant \lambda(B^*_\pi) \leqslant \lambda(G)$ with equality if and only if G is $B^*_{\pi_1}$. The proof is completed. $\square$

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