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WEIGHTED ENDPOINT ESTIMATES FOR COMMUTATORS OF MULTILINEAR FRACTIONAL INTEGRAL OPERATORS

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Abstract. Let m be a positive integer, $0 < \alpha < mn$, $\vec{b} = (b_1, \ldots, b_m) \in \mathrm{BMO}^m$. We give sufficient conditions on weights for the commutators of multilinear fractional integral operators $\mathcal{I}_\alpha^{\vec{b}}$ to satisfy a weighted endpoint inequality which extends the result in D. Cruz-Uribe, A. Fiorenza: Weighted endpoint estimates for commutators of fractional integrals, Czech. Math. J. 57 (2007), 153–160. We also give a weighted strong type inequality which improves the result in X. Chen, Q. Xue: Weighted estimates for a class of multilinear fractional type operators, J. Math. Anal. Appl., 362, (2010), 355–373.

Keywords: multilinear fractional integral operators, commutator, BMO, weight, maximal operators

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1. Introduction and main results

Let m be a positive integer, $0 < \alpha < mn$, and let $\vec{f} = (f_1, \ldots, f_m)$ be a collection of m locally integrable functions on $\mathbb{R}^n$. We define the multilinear fractional integral operator as

$$\mathcal{I}_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \ldots f_m(y_m)}{(|x-y_1| + \ldots + |x-y_m|)^{mn-\alpha}} \, \mathrm{d}y_1 \ldots \mathrm{d}y_m, \tag{1.1}$$

and the multilinear fractional maximal operator $\mathcal{M}_\alpha$ by

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| \, \mathrm{d}y_j,$$

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where the supremum is taken over all cubes Q containing x in $\mathbb{R}^n$ with the sides parallel to the axes.

Multilinear fractional integral operators were studied by Grafakos [7], Kenig and Stein [11], Grafakos and Kalton [8]. Recently, Chen and Xue [2], Moen [13] and Pradolini [14] studied the weighted norm inequalities for the multilinear fractional integral operators and the multilinear maximal operator.

Assuming that $\vec{b} = (b_1, \ldots, b_m)$ is a collection of locally integrable functions, multilinear commutators of $\mathcal{I}_\alpha$ with $\vec{b}$ are defined as

$$\mathcal{I}_\alpha^{\vec{b}}(\vec{f}\,) = \sum_{j=1}^{m} \mathcal{I}_\alpha^{\vec{b},j}(\vec{f}\,), \tag{1.2}$$

where $\mathcal{I}_\alpha^{\vec{b},j}(\vec{f}\,) = b_j \mathcal{I}_\alpha(f_1, \ldots, f_j, \ldots, f_m) - \mathcal{I}_\alpha(f_1, \ldots, b_j f_j, \ldots, f_m)$.

Chen and Xue [2] also studied the weighted norm inequalities for $\mathcal{I}_\alpha^{\vec{b}}$ with $\vec{b} \in \mathrm{BMO}^m$. For linear operators, Cruz-Uribe and Fiorenza [5] obtained weighted endpoint estimates for commutators of fractional integrals, Pérez and Pradolini [15] gave the sharp weighted endpoint estimates for commutators of singular integral operators. In this paper, we give weighted endpoint estimates and strong type weighted inequalities for the commutators $\mathcal{I}_\alpha^{\vec{b}}$.

Here we must point out that the multilinear operators have been studied by many authors, including Christ and Journé [3], Kenig and Stein [11], and Grafakos and Torres [9], [10]. In [12], Lerner, Ombrosi, Pérez and Torres Trujillo-González developed the multilinear weighted theory. For historical details and more information on multilinear theory see these papers and the related references.

To state our results, we need some notation. For $\vec{b} = (b_1, \ldots, b_m) \in \mathrm{BMO}^m$, denote $\|\vec{b}\|_{\mathrm{BMO}^m} = \sup_i \|b_i\|_{\mathrm{BMO}}$. For m exponents $1 \leqslant p_1, \ldots, p_m < \infty$, we will write p for the number given by $1/p = 1/p_1 + \ldots + 1/p_m$, and the vector $\vec{p} = (p_1, \ldots, p_m)$.

Let $\Psi(t)\colon\ [0,\infty) \to [0,\infty)$ be a Young function. That is a continuous, convex, increasing function with $\Psi(0) = 0$ and such that $\Psi(t) \to \infty$ as $t \to \infty$. The Luxemburg norm of a function f over a cube Q is defined by

$$\|f\|_{\Psi,Q} = \inf\Big\{\lambda > 0\colon\ \frac{1}{|Q|}\int_Q \Psi\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \leqslant 1\Big\}.$$

In particular, for the Young function $\Phi(t) = t(1 + \log^+ t)$, the Luxemburg norm will be denoted by $\|f\|_{L(\log L),Q}$.

For $i = 1, \ldots, m$, $0 < \alpha < mn$, multilinear fractional Orlicz maximal operators associated with Φ are defined as

$$\mathcal{M}^i_{\alpha,L(\log L)}(\vec{f}\,)(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \|f_i\|_{L(\log L),Q} \prod_{j \neq i} \frac{1}{|Q|}\int_Q |f_j(y_j)|\,\mathrm{d}y_j,$$

and

$$\mathcal{M}_{\alpha,L(\log L)}(\vec{f})(x) = \sup_{Q\ni x} |Q|^{\alpha/n} \prod_{j=1}^{m} \|f_j\|_{L(\log L),Q}.$$

Let $1 \leqslant p_1, \dots, p_m < \infty$ and q be such that $1/m \leqslant p \leqslant q < \infty$. Given $\vec{w} = (w_1, \dots, w_m)$, where w_j are nonnegative locally integrable functions on $\mathbb{R}^n$, $j = 1, \dots, m$, set $v_{\vec{w}} = \prod\limits_{j=1}^{m} w_j$. We say that $\vec{w}$ satisfies the $A_{\vec{p},q}$ condition, or that it is in the class $A_{\vec{p},q}$, if

$$[\vec{w}]_{A_{\vec{p},q}} = \sup_Q \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}^q\right)^{1/q} \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q \omega_j^{-p_j'}\,\mathrm{d}x\right)^{1/p_j'} < \infty.$$

If some $p_j = 1$, then $\left(\frac{1}{|Q|}\int_Q \omega_j^{-p_j'}\,\mathrm{d}x\right)^{1/p_j'}$ is understood to be $(\inf\limits_Q \omega_j)^{-1}$.

The following theorems are our main results.

Theorem 1.1. *Given* $0 < \alpha < mn$, $\vec{b} \in \mathrm{BMO}^m$, *let* $\Gamma(t) = [t(1+\log^+ t)]^{n/(mn-\alpha)}$, $\Theta(t) = t(1 + \log^+ t^{-1})$, $q = n/(mn - \alpha)$. *Then for each weight* $\vec{\omega} \in A_{(1,\dots,1),q}$ *there exists a constant* $C > 0$ *such that for any* $t > 0$,

$$\begin{aligned} &v_{\vec{\omega}}^q(\{x \in \mathbb{R}^n : |\mathcal{I}_\alpha^{\vec{b}}(\vec{f})(x)| > t^m\}) \\ &\qquad \leqslant C\Gamma(\Phi(\|\vec{b}\|_{\mathrm{BMO}}^{1/m})^m)\Gamma\left(\prod_{j=1}^{m}\int_{\mathbb{R}^n}\Phi\left(\frac{|f_j(y)|}{t}\right)\Theta(\omega_j(y))\,\mathrm{d}y\right). \end{aligned}$$

Theorem 1.2. *Let* $0 < \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $1/m < p < n/\alpha$, *and* $1/q = 1/p - \alpha/n$. *Then for each weight* $\vec{\omega} \in A_{\vec{p},q}$ *there exists a constant* $C > 0$ *such that*

$$\left(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha^{\vec{b}}(\vec{f})(x)|v_{\vec{w}}(x))^q\,\mathrm{d}x\right)^{1/q} \leqslant C\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}(|f_j(x)|\omega_j(x))^{p_j}\,\mathrm{d}x\right)^{1/p_j} \tag{1.3}$$

for all $\vec{f}$ *of bounded measurable functions with compact support.*

Remark. Chen and Xue [2] have obtained (1.3) under the following assumption on weights $\vec{\omega}$: there exists $r > 1$ such that $\vec{\omega}^r \in A_{\vec{p}/r,q/r}$. Hence Theorem 1.2 improves the result in [2].

2. Some lemmas

For $\delta > 0$, let M_δ be the maximal function and $M_\delta^\sharp$ the sharp maximal function

$$M_\delta f(x) = \sup_{x\in Q}\biggl(\frac{1}{|Q|}\int_Q |f(y)|^\delta \,\mathrm{d}y\biggr)^{1/\delta}$$
$$M_\delta^\sharp f(x) = \sup_{x\in Q}\inf_c\biggl(\frac{1}{|Q|}\int_Q \bigl||f(y)|^\delta - |c|^\delta\bigr| \,\mathrm{d}y\biggr)^{1/\delta}.$$

We will use the following form of the classical result of Fefferman and Stein [6]: Let $0 < p, \delta < \infty$ and ω be a weight in A_∞. If $\varphi\colon (0,\infty) \to (0,\infty)$ is doubling, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \omega(x)\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} (M_\delta^\sharp f(x))(x)^p \omega(x)\,\mathrm{d}x,$$

and

$$\sup_{\lambda>0}\varphi(\lambda)\omega(\{y\in\mathbb{R}^n\colon\ M_\delta f(y) > \lambda\}) \leqslant C\sup_{\lambda>0}\varphi(\lambda)\omega(\{y\in\mathbb{R}^n\colon\ M_\delta^\sharp f(y) > \lambda\}) \tag{2.1}$$

for every function f such that the left-hand side is finite.

We need the following lemmas in the proofs of our main results.

Lemma 2.1 ([11]). *Let $0 < \alpha < mn$, $1 \leqslant p_1,\dots,p_m < \infty$, $p < n/\alpha$, $1/q = 1/p - \alpha/n$.*

(a) *If $1 < p_j < \infty$, $j = 1,\dots,m$, then*

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q} \leqslant C\prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

(b) *If $1 \leqslant p_j < \infty$, $j = 1,\dots,m$, and at least one of the p_j equals 1, then*

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^{q,\infty}} \leqslant C\prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

Lemma 2.2 ([13]). *Let $0 < \alpha < mn$, $1 \leqslant p_1,\dots,p_m < \infty$ and $1/q = 1/p - \alpha/n$. Suppose $\vec{\omega} \in A_{\vec{p},q}$. Then*

$$v_{\vec{\omega}}^q \in A_{mq}, \quad \omega_i^{-p_j'} \in A_{mp_j'}.$$

When some p_j equals 1, $\omega_i^{-p_j'} \in A_{mp_j'}$ is understood as $\omega_j^{n/(nm-\alpha)} \in A_1$.

Lemma 2.3 ([2]). *Let $0 < \alpha < mn$ and $0 < \delta \leqslant 1/m$. Then there exists a constant $C > 0$ such that*

$$M_\delta^\sharp(\mathcal{I}_\alpha(\vec{f}))(x) \leqslant C\mathcal{M}_\alpha(\vec{f})(x)$$

for all $\vec{f}$ of bounded measurable functions with compact support.

Lemma 2.4 ([2]). *Let $0 < \alpha < mn$, $\vec{b} \in \mathrm{BMO}^m$, $0 < \delta \leqslant 1/m$, $\delta < \varepsilon$. Then there exists a constant $C > 0$ such that*

$$M_\delta^\sharp(\mathcal{I}_\alpha^{\vec{b}}(\vec{f}))(x) \leqslant C\|\vec{b}\|_{\mathrm{BMO}^m}(\mathcal{M}_{\alpha,L(\log L)}(\vec{f})(x) + M_\varepsilon(\mathcal{I}_\alpha(\vec{f}))(x)) \tag{2.2}$$

for all $\vec{f}$ of bounded measurable functions with compact support. We remark that the proof of Lemma 2.4 actually shows that we can replace $\mathcal{M}_{\alpha,L(\log L)}(\vec{f})$ on the right-hand side of (2.2) by the slightly smaller operator $\sum\limits_{i=1}^{m} \mathcal{M}_{\alpha,L(\log L)}^i(\vec{f})$.

Lemma 2.5 ([13]). *Let $0 < \alpha < mn$, $1 < p_1, \ldots, p_m < \infty$, $1/m < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Then*

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha(\vec{f})(x)v_{\vec{\omega}}(x))^q \,\mathrm{d}x\right)^{1/q} \leqslant C\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n} (|f_j(x)|\omega_j(x))^{p_j} \,\mathrm{d}x\right)^{1/p_j}$$

if and only if $\vec{\omega} \in A_{\vec{p},q}$.

3. Proof of Theorem 1.1

By linearity of the multilinear commutators and Lemma 2.4, it is enough to consider $\mathcal{I}_\alpha^b(\vec{f})(x) = b(x)\mathcal{I}_\alpha(f_1, \ldots, f_m)(x) - \mathcal{I}_\alpha(bf_1, \ldots, f_m)(x)$ with $b \in \mathrm{BMO}$.

We will need the following notation and facts, for further information see [4]. Given an increasing function $\varphi\colon [0, \infty) \to [0, \infty)$, define the function h_φ by

$$h_\varphi(s) = \sup \frac{\varphi(st)}{\varphi(t)}, \quad 0 \leqslant s < \infty.$$

If φ is submultiplicative, then $h_\varphi \approx \varphi$. Also, for all $s, t > 0$, $\varphi(st) \leqslant h_\varphi(s)\varphi(t)$. Since Φ is submultiplicative, $h_\Phi \approx \Phi$. Denote

$$\Psi(t) = \begin{cases} 0, & t = 0, \\ \dfrac{t^m}{\Phi(t^{\alpha/n})}, & t > 0. \end{cases}$$

So

$$\Psi(t) \approx t^{m-\alpha/n}(1+\log^+ t)^{-1}.$$

The function Ψ is invertible with

$$\Psi^{-1}(t) \approx \Gamma(t) = [t(1+\log^+ t)]^{n/(mn-\alpha)}.$$

We need the following inequality in the proof of the weighted endpoint estimate of the maximal operator $\mathcal{M}^i_{\alpha,L\log L}$.

Lemma 3.1 ([4]). *If $\varphi(t)/t$ is decreasing, then for any positive sequence $\{t_j\}$,*

$$\varphi\Bigl(\sum_j t_j\Bigr) \leqslant \sum_j \varphi(t_j).$$

The proof of Theorem 1.1 will be based on the following results.

Theorem 3.1. *Let $0 < \alpha < mn$, $\Gamma(t) = [t(1+\log^+ t)]^{n/(mn-\alpha)}$, $\Theta(t) = t(1+\log^+ t^{-1})$ and $q = n/(mn-\alpha)$. Then for each weight $\vec{\omega} \in A_{(1,\dots,1),q}$ there exists a constant $C > 0$ such that for any $t > 0$, $i = 1,\dots,m$, we have*

$$\begin{aligned} v^q_{\vec{\omega}}(\{x \in \mathbb{R}^n\colon\ |\mathcal{M}^i_{\alpha,L\log L}(\vec{f})(x)| > t^m\}) \\ \leqslant C\Gamma\Bigl(\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi\Bigl(\frac{|f_j(y)|}{t}\Bigr)\Theta(\omega_j(y))\,\mathrm{d}y\Bigr). \end{aligned}$$

P r o o f. Without loss of generality we may assume $i = 1$ and $\vec{f} \geqslant 0$. By homogeneity, we may assume that $t = 1$. Define the set

$$\Omega = \{x \in \mathbb{R}^n\colon\ \mathcal{M}^1_{\alpha,L(\log L)}(\vec{f})(x) > 1\}.$$

It is easy to see that Ω is open and we may assume that it is not empty. To estimate the size of Ω, it is enough to estimate the size of every compact set F contained in Ω. We can cover F by a finite family of cubes $\{Q_j\}$ for which

$$|Q_j|^{\alpha/mn}\|f_1\|_{L(\log L),Q_j}\prod_{j=2}^m |Q_j|^{\alpha/mn}(f_j)_{Q_j} > 1.$$

Using Vitali's covering lemma, we can extract a subfamily of disjoint cubes $\{Q_k\}$ such that

$$F \subset \bigcup_k CQ_k. \tag{3.1}$$

For each k, by homogeneity and the properties of the norm $\|\cdot\|_{\Phi,Q}$ we have

$$
\begin{aligned}
1 &< \frac{1}{|Q_k|}\int_{Q_k}\Phi\Big(f_1(y)|Q_k|^{\alpha/n}\prod_{j=2}^m (f_j)_{Q_k}\Big)\,\mathrm{d}y\\
&\leqslant C\frac{\Phi(|Q_k|^{\alpha/n})}{|Q_k|^m}\prod_{j=1}^m\int_{Q_k}\Phi(f_j(y))\,\mathrm{d}y\\
&\leqslant C\frac{C}{\Psi(|Q_k|)}\prod_{j=1}^m\int_{Q_k}\Phi(f_j(y))\,\mathrm{d}y.
\end{aligned}
$$

Notice that for $s_1,\dots,s_m>0$ we have

$$
\begin{aligned}
h_\Psi((s_1\dots s_m)^q) &= \sup_{t>0}\frac{\Psi((s_1\dots s_m)^q t)}{\Psi(t)}\\
&\approx (s_1\dots s_m)^{mq}\sup_{t>0}\frac{\Phi(t^{\alpha/n})}{\Phi(((s_1\dots s_m)^q t)^{\alpha/n})}\\
&\leqslant C(s_1\dots s_m)^{mq}\Phi((s_1\dots s_m)^{-q\alpha/n})\leqslant C\prod_{j=1}^m\Theta(s_j).
\end{aligned}
$$

Since $\vec{\omega}\in A_{(1,\dots,1),q}$, for each k we have

$$
\begin{aligned}
\Psi(v_{\vec{\omega}}^q(Q_k)) &\leqslant C\frac{\Psi(v_{\vec{\omega}}^q(Q_k))}{\Psi(|Q_k|)}\prod_{j=1}^m\int_{Q_k}\Phi(f_j(y))\,\mathrm{d}y\\
&\leqslant Ch_\Psi\Big(\frac{v_{\vec{\omega}}^q(Q_k)}{|Q_k|}\Big)\prod_{j=1}^m\int_{Q_k}\Phi(f_j(y))\,\mathrm{d}y\\
&\leqslant Ch_\Psi\bigg(\prod_{j=1}^m\Big(\inf_{Q_k}\omega_j\Big)^q\bigg)\prod_{j=1}^m\int_{Q_k}\Phi(f_j(y))\,\mathrm{d}y\\
&\leqslant C\prod_{j=1}^m\int_{Q_k}\Phi(f_j(y))\Theta(\omega_j(y))\,\mathrm{d}y.
\end{aligned}
$$

It is easy to see that $\Psi^{1/m}(t)/t$ is decreasing and by Lemma 3.1 and Hölder's inequality at discrete level we have

$$
\begin{aligned}
\Psi(v_{\vec{\omega}}^q(F)) &= [\Psi^{1/m}(v_{\vec{\omega}}^q(F))]^m \leqslant \bigg[\Psi^{1/m}\Big(\sum_k v_{\vec{\omega}}^q(Q_k)\Big)\bigg]^m\\
&\leqslant \bigg[\sum_k \Psi^{1/m}(v_{\vec{\omega}}^q(Q_k))\bigg]^m \leqslant \bigg[\sum_k (\Psi(v_{\vec{\omega}}^q(Q_k))^{1/m})\bigg]^m
\end{aligned}
$$

$$\leqslant C\Big(\sum_k \prod_{j=1}^m \Big(\int_{Q_k} \Phi(f_j(y))\Theta(\omega_j(y))\,\mathrm{d}y\Big)^{1/m}\Big)^m$$
$$\leqslant C\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi(f_j(y))\Theta(\omega_j(y))\,\mathrm{d}y.$$

□

Theorem 3.2. *Let* $0<\alpha<mn$, $\omega\in A_\infty$, $\varphi(t)=\Gamma(\Phi(t)^m)$ *and* $b\in\mathrm{BMO}$. *Then there exists a constant* $C>0$ *such that*

$$(3.2)\quad \sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon |\mathcal{I}_\alpha^b(\vec f)(x)|>t^m\}) \leqslant C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon \mathcal{M}^1_{\alpha,L(\log L)}(\vec f)(x)>t^m\})$$

for all $\vec f$ *of bounded measurable functions with compact support.*

P r o o f. We can assume that the right-hand side of (3.2) is finite. It is easy to see that $1/\varphi(1/t)$ is doubling. By Lebesgue differentiation theorem, Fefferman-Stein inequality (2.1), Lemma 2.3 and Lemma 2.4 with exponents $0<\delta<\varepsilon<1/m$, we have

$$\begin{aligned}
&\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon |\mathcal{I}_\alpha^b(\vec f)(x)|>t^m\})\\
&\quad\leqslant \sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon M_\delta(\mathcal{I}_\alpha^b(\vec f))(x)>t^m\})\\
&\quad\leqslant C\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon M^\sharp_\delta(\mathcal{I}_\alpha^b(\vec f))(x)>t^m\})\\
&\quad\leqslant C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon \mathcal{M}^1_{\alpha,L(\log L)}(\vec f)(x)>t^m\})\\
&\qquad+C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon M_\varepsilon(\mathcal{I}_\alpha(\vec f)(x))>t^m\})\\
&\quad\leqslant C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon \mathcal{M}^1_{\alpha,L(\log L)}(\vec f)(x)>t^m\})\\
&\qquad+C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon M^\sharp_\varepsilon(\mathcal{I}_\alpha(\vec f)(x))>t^m\})\\
&\quad\leqslant C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon \mathcal{M}^1_{\alpha,L(\log L)}(\vec f)(x)>t^m\})\\
&\qquad+C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon \mathcal{M}_\alpha(\vec f)(x)>t^m\})\\
&\quad\leqslant C\varphi(\|b\|_{\mathrm{BMO}}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}\,\omega(\{x\in\mathbb{R}^n\colon \mathcal{M}^1_{\alpha,L(\log L)}(\vec f)(x)>t^m\}).
\end{aligned}$$

We need to verify now that

$$\sup_{t>0} \frac{1}{\varphi(1/t)}\, \omega(\{x \in \mathbb{R}^n \colon\ M_\delta(\mathcal{I}_\alpha^b(\vec{f}))(x) > t^m\}) < \infty \tag{3.3}$$

and

$$\sup_{t>0} \frac{1}{\varphi(1/t)}\, \omega(\{x \in \mathbb{R}^n \colon\ M_\varepsilon(\mathcal{I}_\alpha(\vec{f})(x)) > t^m\}) < \infty. \tag{3.4}$$

We will only show (3.3) because the proof of (3.4) is very similar but easier. We may assume that ω is bounded. Note that $\omega_j = \min\{\omega, j\} \to \omega$ as $j \to \infty$ a.e. on $\mathbb{R}^n$, $\omega_j \in L^\infty$ and $[\omega_j]_{A_\infty} \leqslant 2[\omega]_{A_\infty}$. The result for general ω will follow then by applying the Monotone Convergence Theorem.

Notice that due to $t^{mn/(mn-\alpha)}\varphi(1/t) \geqslant 1$, $m\delta < 1$ and the fact

$$M_{m\delta} \colon\ L^{(mn/(mn-\alpha)),\infty}(\mathbb{R}^n) \to L^{(mn/(mn-\alpha)),\infty}(\mathbb{R}^n)$$

we have

$$\begin{aligned}\sup_{t>0} \frac{1}{\varphi(1/t)}\, &\omega(\{x \in \mathbb{R}^n \colon\ M_\delta(\mathcal{I}_\alpha^b(\vec{f}))(x) > t^m\}) \\ &\leqslant C \sup_{t>0} t^{mn/(mn-\alpha)} |\{x \in \mathbb{R}^n \colon\ M_{m\delta}(|\mathcal{I}_\alpha^b(\vec{f})|^{1/m})(x) > t\}| \\ &\leqslant C \sup_{t>0} t^{mn/(mn-\alpha)} |\{x \in \mathbb{R}^n \colon\ |\mathcal{I}_\alpha^b(\vec{f})|^{1/m}(x) > t\}|.\end{aligned}$$

If we assume that b is bounded, then

$$\begin{aligned}|\mathcal{I}_\alpha^b(\vec{f})(x)| &\leqslant \int_{(\mathbb{R}^n)^m} \frac{|b(x)-b(y_1)||f_1(y_1)|\dots|f_m(y_m)|}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}}\,\mathrm{d}\vec{y} \\ &\leqslant C\|b\|_\infty \mathcal{I}_\alpha(|f_1|,\dots,|f_m|)(x).\end{aligned}$$

Thus, by Lemma 2.1 (b), we have

$$\begin{aligned}\sup_{t>0} t^{mn/(mn-\alpha)} &|\{x \in \mathbb{R}^n \colon\ |\mathcal{I}_\alpha^b(\vec{f})|^{1/m}(x) > t\}| \\ &\leqslant C \sup_{t>0} t^{mn/(mn-\alpha)} |\{x \in \mathbb{R}^n \colon\ \mathcal{I}_\alpha(|f_1|,\dots,|f_m|)(x) > t^m\}| \\ &\leqslant C \prod_{j=1}^m \|f_j\|_{L^1}^{n/(mn-\alpha)} < \infty,\end{aligned}$$

since the family $\vec{f}$ is bounded with compact support.

This proves (3.2) provided b is bounded. To obtain the result for a general b in BMO, we consider the sequence of functions $\{b_j\}$ given by

$$b_j(x) = \begin{cases} j, & b(x) > j, \\ b(x), & |b(x)| \leqslant j, \\ -j, & b(x) < -j. \end{cases}$$

Note that the sequence converges pointwise to b and $\|b_j\|_{\mathrm{BMO}} \leqslant c\|b\|_{\mathrm{BMO}}$. Thus

$$\sup_{t>0} \frac{1}{\varphi(1/t)}\, \omega(\{x \in \mathbb{R}^n \colon\ |\mathcal{I}_\alpha^{b_j}(\vec{f}\,)(x)| > t^m\}) \\ \leqslant C \sup_{t>0} \frac{1}{\varphi(1/t)}\, \omega(\{x \in \mathbb{R}^n \colon\ \mathcal{M}^1_{\alpha, L(\log L)}(\vec{f}\,)(x) > t^m\}),$$

where the constant C depends on the BMO norm of b. Since the family $\vec{f}$ is bounded with compact support and $\mathcal{I}_\alpha \colon L^1 \times \ldots \times L^1 \to L^{(n/(mn-\alpha)),\infty}$, we have $\|\mathcal{I}_\alpha(b_j f_1, f_2, \ldots, f_m) - \mathcal{I}_\alpha(b f_1, f_2, \ldots, f_m)\|_{L^{(n/(mn-\alpha)),\infty}} \to 0$, $j \to \infty$. Thus for each compact set, an appropriate subsequence of $\{\mathcal{I}_\alpha^{b_j}(\vec{f}\,)\}$ converges to $\mathcal{I}_\alpha^b(\vec{f}\,)$ in measure. Hence for any $K > 0$,

$$\sup_{t>0} \frac{1}{\varphi(1/t)}\, \omega(\{x \in B(0,K) \colon\ |\mathcal{I}_\alpha^{b}(\vec{f}\,)(x)| > t^m\}) \\ \leqslant C \sup_{t>0} \frac{1}{\varphi(1/t)}\, \omega(\{x \in \mathbb{R}^n \colon\ \mathcal{M}^1_{\alpha, L(\log L)}(\vec{f}\,)(x) > t^m\}),$$

where the constant C is independent of K. Finally, taking the supremum in K completes the proof of the theorem. □

P r o o f of Theorem 1.1. By homogeneity it is enough to assume $t = 1$. Since Γ and Φ are submultiplicative, by Lemma 2.2, Theorem 3.1 and Theorem 3.2 we have

$$\begin{aligned} &v_{\vec{\omega}}^q(\{x \in \mathbb{R}^n \colon\ |\mathcal{I}_\alpha^b(\vec{f}\,)(x)| > 1\}) \\ &\quad \leqslant \Gamma(\Phi(1)^m) \sup_{t>0} \frac{1}{\Gamma(\Phi(1/t)^m)}\, v_{\vec{\omega}}^q(\{x \in \mathbb{R}^n \colon\ |\mathcal{I}_\alpha^b(\vec{f}\,)(x)| > t^m\}) \\ &\quad \leqslant C\Gamma(\Phi(\|b\|_{\mathrm{BMO}}^{1/m})^m) \sup_{t>0} \frac{1}{\Gamma(\Phi(1/t)^m)}\, v_{\vec{\omega}}^q(\{x \in \mathbb{R}^n : \mathcal{M}^1_{\alpha, L(\log L)}(\vec{f}\,)(x) > t^m\}) \\ &\quad \leqslant C\Gamma(\Phi(\|b\|_{\mathrm{BMO}}^{1/m})^m) \sup_{t>0} \frac{1}{\Gamma(\Phi(1/t)^m)}\, \Gamma\Big(\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi\Big(\frac{|f_j(y)|}{t}\Big)\Theta(\omega_j(y))\,\mathrm{d}y\Big) \\ &\quad \leqslant C\Gamma(\Phi(\|b\|_{\mathrm{BMO}}^{1/m})^m)\Gamma\Big(\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi(|f_j(y)|)\Theta(\omega_j(y))\,\mathrm{d}y\Big). \end{aligned}$$

□

4. Proof of Theorem 1.2

The proof of Theorem 1.2 will use the following Coifman type inequalities for the commutators of the multilinear fractional operators.

Lemma 4.1 ([1]). *Let $0 < \alpha < mn$, $0 < p < \infty$, $\vec{b} \in \mathrm{BMO}^m$ and $\omega \in A_\infty$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} |\mathcal{I}_\alpha^{\vec{b}}(\vec{f})(x)|^p \omega(x)\,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} \mathcal{M}_{\alpha,L(\log L)}(\vec{f})(x)^p \omega(x)\,\mathrm{d}x \tag{4.1}$$

for all $\vec{f}$ of bounded measurable functions with compact support.

Lemma 4.2. *Let $1 < p_1, \ldots, p_m < \infty$, $1/q = 1/p - \alpha/n$. Assume that $\vec{\omega} \in A_{\vec{p},q}$. Then there exists a finite constant $r_0 > 1$ such that for any $r\colon 1 < r \leqslant r_0$, $\vec{\omega}^r = (\omega_1^r, \ldots, \omega_m^r) \in A_{\vec{p}/r,q/r}$, and $[\vec{\omega}^r]_{A_{\vec{p}/r,q/r}} \leqslant C[\vec{\omega}]^r_{A_{\vec{p},q}}$.*

Proof. By Lemma 2.2, $\omega_j^{-p_j'} \in A_{mp_j'} \subset A_\infty$ for $j = 1, 2, \ldots, m$, hence there are constants c_j, $t_j > 1$, such that for any cube Q

$$\Big(\frac{1}{|Q|}\int_Q \omega_j^{-p_j' t_j}\Big)^{1/t_j} \leqslant \frac{c_j}{|Q|}\int_Q \omega_j^{-p_j'}. \tag{4.2}$$

Let $r_j = t_j p_j/(p_j + t_j - 1) > 1$, $j = 1, \ldots, m$, $r_0 = \min\{r_1, \ldots, r_m\}$ and $c = \max\{c_1, \ldots, c_m\}$. Then for any $r\colon 1 < r \leqslant r_0$, by Hölder's inequality and (4.2), we have

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q (v_{\vec{\omega}^r})^{q/r}\Big)^{r/q} \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q (\omega_j^r)^{-(p_j/r)'}\Big)^{1-r/p_j} \\
&\quad= \Big(\frac{1}{|Q|}\int_Q (v_{\vec{\omega}})^q\Big)^{r/q} \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q \omega_j^{-(r(p_j-1)/(p_j-r))\,p_j'}\Big)^{1-r/p_j} \\
&\quad\leqslant \Big(\frac{1}{|Q|}\int_Q (v_{\vec{\omega}})^q\Big)^{r/q} \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q \omega_j^{-t_j p_j'}\Big)^{r/t_j p_j'} \\
&\quad\leqslant c^m \Big(\frac{1}{|Q|}\int_Q (v_{\vec{\omega}})^q\Big)^{r/q} \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q \omega_j^{-p_j'}\Big)^{r/p_j'} \\
&\quad\leqslant c^m [\vec{\omega}]^r_{A_{\vec{p},q}}.
\end{aligned}$$

□

P r o o f of Theorem 1.2. By Lemma 4.1 and since $v_{\vec{w}}^q$ is in A_∞,

$$\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha^b(\vec{f})(x)| v_{\vec{w}}(x))^q \,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} (\mathcal{M}_{\alpha, L(\log L)}(\vec{f})(x) v_{\vec{w}}(x))^q \,\mathrm{d}x.$$

For $r > 1$, define the maximal operator

$$\mathcal{M}_{\alpha,r}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m |Q|^{\alpha/mn} \biggl(\frac{1}{|Q|} \int_Q |f_i(y)|^r \,\mathrm{d}y \biggr)^{1/r}.$$

Since $\Phi(t) = t(1 + \log^+ t) \leqslant t^r$, $t > 1$, we have

$$\mathcal{M}_{\alpha, L(\log L)}(\vec{f})(x) \leqslant C \mathcal{M}_{\alpha,r}(\vec{f})(x) = C(M_{\alpha r}(\vec{f^r})(x))^{1/r}.$$

Thus

$$\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha^b(\vec{f})(x)| v_{\vec{w}}(x))^q \,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} (\mathcal{M}_{\alpha,r}(\vec{f})(x) v_{\vec{w}}(x))^q \,\mathrm{d}x.$$

Proving

$$\biggl(\int_{\mathbb{R}^n} (\mathcal{M}_{\alpha,r}(\vec{f}) v_{\vec{w}})^q \,\mathrm{d}x \biggr)^{1/q} \leqslant C \prod_{j=1}^m \biggl(\int_{\mathbb{R}^n} (|f_j| \omega_j)^{p_j} \,\mathrm{d}x \biggr)^{1/p_j}$$

is equivalent to proving

$$\biggl(\int_{\mathbb{R}^n} \mathcal{M}_{\alpha r}(\vec{f})^{q/r} (v_{\vec{w}^r})^{q/r} \,\mathrm{d}x \biggr)^{r/q} \leqslant C \prod_{j=1}^m \biggl(\int_{\mathbb{R}^n} |f_j|^{p_j/r} (\omega_j^r)^{p_j/r} \,\mathrm{d}x \biggr)^{r/p_j}. \tag{4.3}$$

For $0 < \alpha < mn$, by Lemma 3.1, there exists $r > 1$ such that $0 < r\alpha < mn$ and $\vec{\omega}^r \in A_{\vec{p}/r, q/r}$. Noticing that $1/(q/r) = 1/(p/r) - \alpha r/n$, we conclude that (4.3) is true by Lemma 2.5. This completes the proof. $\square$

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