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Kybernetika, Vol. 48 (2012), No. 1, 144--164

Persistent URL: http://dml.cz/dmlcz/142068

KYBERNETIKA — VOLUME *48* (2012), NUMBER 1, PAGES 144–164

ON THE L-VALUED CATEGORIES OF L-E-ORDERED SETS

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The aim of this paper is to construct an L-valued category whose objects are L-E-ordered sets. To reach the goal, first, we construct a category whose objects are L-E-ordered sets and morphisms are order-preserving mappings (in a fuzzy sense). For the morphisms of the category we define the degree to which each morphism is an order-preserving mapping and as a result we obtain an L-valued category. Further we investigate the properties of this category, namely, we observe some special objects, special morphisms and special constructions.

Keywords: category, L-valued category, fuzzy order relation

Classification: 03E72, 18A05, 18B35

1. INTRODUCTION

The concept of order relation plays an important role in theoretical mathematics and its applications. The first definition of a fuzzy order was introduced by L. A. Zadeh in his paper [24] under the name of *fuzzy partial ordering* . Afterwards the interest of many researchers was focused on the study and development of the theory of L-valued, in particular fuzzy, orders (see e. g. [3, 4, 6, 7, 13, 21]). Recently the theoretical results of the theory of fuzzy relations were involved for solving some applied problems (see e. g. [5, 8]).

The aim of this work is to study L-E-ordered sets, where L is a cl-monoid, from L-valued category theory point of view. This means that starting from the small categories we already use the concept of L-valued category, defining the degree to which each morphism is indeed the morphism in the L-valued category. The alternative approach is to study generalized ordered structures in the frame of categories enriched over a quantale Ω (or simply, Ω-categories), see e. g. [17, 18, 19].

In our work we use the notion of L-E-order relation and construct an analogue of **POS** category (Partially Ordered Sets). In our category objects are L-E-ordered sets and morphisms are order-preserving mappings (in a fuzzy sense). By constructing the category from small categories, we show that the category is constructed in a natural way from the point of view of L-valued category theory. To realize this goal we follow the construction of the crisp **POS** category. We continue by studying the structure of the constructed category. Namely, we consider some special morphisms, objects and standard constructions such as product and coproduct.

In order to fuzzify the constructed category, we introduce an L-valued subclass of the class of all its morphisms as a mapping from the class of morphisms to a cl-monoid L and thus we obtain an L-valued category. The intuitive meaning of the value of this mapping for a given morphism is the degree to which this morphism is an order-preserving mapping. Therefore we obtain the L-valued category whose objects are L-E-ordered sets and morphisms are "potential" order-preserving mappings. Further we study also the structure of this L-valued category and consider some operations such as products and coproducts.

2. PRELIMINARIES

In this section we provide basic definitions which will be used in our consideration.

2.1. Commutative cl-monoid

Definition 2.1. (cf. [15])A commutative cl-monoid is a complete lattice $(L, \leq)$ enriched with a further binary commutative operation $*$, satisfying the isotonicity condition:

$$\forall \alpha, \beta, \gamma \in L \;\; \alpha \leq \beta \Rightarrow \alpha * \gamma \leq \beta * \gamma$$

and the infinite distributive law:

$$\forall \alpha \in L \;\; and \;\; \forall \{ \beta_i : i \in I\} \;\subset L \quad \alpha * \Big(\bigvee_i \beta_i \Big) = \bigvee_i (\alpha * \beta_i).$$

A commutative cl-monoid is integral if and only if the unit element 1 is also the universal upper bound in $(L, \leq)$. It is known that for the integral commutative cl-monoid with zero element 0 (i. e. $\forall \alpha \in L \;\; \alpha * 0 = 0$) the universal lower bound is zero element 0.

Every integral commutative cl-monoid with zero is residuated, i. e., there exists a binary operation " $\mapsto$ " (implication) on L satisfying the following condition:

$$\forall \alpha, \beta, \gamma \in L \quad \alpha * \beta \leq \gamma \Leftrightarrow \alpha \leq (\beta \mapsto \gamma).$$

Explicitly the implication is given by

$$\alpha \mapsto \beta = \bigvee \{\lambda \in L : \alpha * \lambda \leq \beta\}.$$

For the given implication " $\mapsto$ " a biimplication " $\leftrightarrow$ " is defined by:

$$\alpha \leftrightarrow \beta = (\alpha \mapsto \beta) * (\beta \mapsto \alpha).$$

Let us mention some properties of integral commutative cl-monoid which will be useful in the sequel:

(i) $\alpha \leq \beta \;\; and \;\; \gamma \leq \delta \Rightarrow \alpha * \gamma \leq \beta * \delta$;

(ii) $\alpha * \beta \leq \alpha \wedge \beta$;

(iii) $(\alpha \mapsto \gamma) * (\gamma \mapsto \beta) \leq \alpha \mapsto \beta$.

In our work we use an integral commutative cl-monoid L with zero element but for simplicity in the sequel we will write simply cl-monoid.

2.2. L-valued categories

In this section, we recall the basic notions for L-valued categories. Main concepts and results on the classical (crisp) category theory can be found in [1, 12].

Definition 2.2. [23]An L-valued category $\mathbb{C}$ consists of:

1. A class $Ob(\mathbb{C})$ of potential objects.
2. An L-valued subclass ω of $Ob(\mathbb{C})$:
$$\omega : Ob(\mathbb{C}) \to L.$$
3. A class $Mor(\mathbb{C}) = \bigcup\{Mor_{\mathbb{C}}(X,Y) : X, Y \in Ob(\mathbb{C})\}$ of pairwise disjoint sets $Mor_{\mathbb{C}}(X,Y)$. For each pair of potential objects $X, Y \in Ob(\mathbb{C})$ the members of $Mor_{\mathbb{C}}(X,Y)$ are called potential morphisms from X to Y and the members of $Mor(\mathbb{C})$ are called potential morphisms of the category $\mathbb{C}$.
4. An L-valued subclass μ of $Mor(\mathbb{C})$:
$$\mu : Mor(\mathbb{C}) \to L,$$
such that if $f \in Mor_{\mathbb{C}}(X,Y)$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$.
5. A composition $\circ$ of morphisms, i. e. for each triple $X, Y, Z \in Ob(\mathbb{C})$ there exists a map
$$\circ : Mor_{\mathbb{C}}(X,Y) \times Mor_{\mathbb{C}}(Y,Z) \to Mor_{\mathbb{C}}(X,Z)\ ((f,g) \to g \circ f),$$
such that the following axioms are satisfied:
    - preservation of morphisms:
    $\mu(g \circ f) \geq \mu(g) * \mu(f)$;
    - associativity:
    if $f \in Mor_{\mathbb{C}}(X,Y)$, $g \in Mor_{\mathbb{C}}(Y,Z)$ and $h \in Mor_{\mathbb{C}}(Z,U)$,
    then $h \circ (g \circ f) = (h \circ g) \circ f$;
    - existence of identities:
    for each $X \in Ob(\mathbb{C})$ there exists an identity $id_X \in Mor_{\mathbb{C}}(X,X)$ such that $\mu(id_X) = \omega(X)$ and for all $X, Y, Z \in Ob(\mathbb{C})$, all $f \in Mor_{\mathbb{C}}(X,Y)$ and all $g \in Mor_{\mathbb{C}}(Z,X)$ it holds $f \circ id_X = f$ and $id_X \circ g = g$.

Remark 2.3. Let $Ob_\alpha(\mathbb{C}) = \{X \in Ob(\mathbb{C}) : \omega(X) \geq \alpha\}$.
The elements of $Ob_\alpha(\mathbb{C})$ will be referred to α-objects of the L-valued category $\mathbb{C}$, while the elements of $Ob(\mathbb{C})$ will be called potential objects of $\mathbb{C}$ or $\mathbb{C}$-objects.

Similarly, the elements of $Mor_\alpha(\mathbb{C})$ ($Mor_\alpha(\mathbb{C}) = \{f \in Mor(\mathbb{C}) : \mu(f) \geq \alpha\}$) will be referred to α-morphisms of the L-valued category $\mathbb{C}$, while the elements of $Mor(\mathbb{C})$ will be called potential morphisms of $\mathbb{C}$ or $\mathbb{C}$-morphisms.

Given an L-valued category $\mathbb{C} = (Ob(\mathbb{C}), \omega, Mor(\mathbb{C}), \mu, \circ)$ and $X \in Ob(\mathbb{C})$, intuitively we understand the value $\omega(X)$ as the degree to which a potential object X of the L-valued category $\mathbb{C}$ is indeed its object; similarly, for $f \in Mor(\mathbb{C})$ the intuitive meaning of $\mu(f)$ is the degree to which a potential morphism f of $\mathbb{C}$ is indeed its morphism.

Definition 2.4. (Sostak [22]) Given an L-valued category $\mathbb{C} = (Ob(\mathbb{C}), \omega, Mor(\mathbb{C}), \mu, \circ)$ one can construct a crisp category $\mathbb{C}_0 = (Ob(\mathbb{C}), Mor(\mathbb{C}), \circ)$ by taking all potential objects and potential morphisms as objects and morphisms of L-valued category $\mathbb{C}$ and leaving the composition law unchanged. $\mathbb{C}_0$ is called the bottom frame of L-valued category $\mathbb{C}$. We also denote the top frame $\mathbb{C}_1 = (Ob_1(\mathbb{C}), Mor_1(\mathbb{C}), \circ)$, where $Ob_1(\mathbb{C}) = \{X \in Ob(\mathbb{C}) : \omega(X) = 1\}$; $Mor_1(\mathbb{C}) = \{f \in Mor(\mathbb{C}) : \mu(f) = 1\}$ and where 1 is the universal upper bound in $(L, \leq)$.

A general scheme for fuzzification of classical categories [23]

Let $\mathbb{C} = (Ob(\mathbb{C}), Mor(\mathbb{C}), \circ)$ and $\mathbb{D} = (Ob(\mathbb{D}), Mor(\mathbb{D}), \circ)$ be two ordinary categories and let $\Phi : \mathbb{C} \to \mathbb{D}$ be a functor. We define a new ordinary category Cat by setting $Ob(Cat) = Ob(\mathbb{C})$ and $Mor_{Cat}(X, Y) = Mor_{\mathbb{D}}(\Phi(X), \Phi(Y))$. Thus the morphisms from X to Y in Cat are the same as the morphisms from $\Phi(X)$ to $\Phi(Y)$ in $\mathbb{D}$. The composition law in Cat is naturally induced by the composition law in $\mathbb{D}$. Now, defining in a certain way an L-subclass of objects: $\omega : Ob(Cat) \to L$ and an L-subclass of morphisms $\mu : Mor(Cat) \to L$ satisfying Definition 2.2 we come to an L-valued category $(Ob(Cat), \omega, Mor(Cat), \mu)$. The category Cat could be also denoted as $\mathbb{C}_{\mathbb{D}\Phi}$ or $\mathbb{C}_{\mathbb{D}}$.

We continue with the definition of a functor between two L-valued categories.

Definition 2.5. [23] Let $\mathbb{C} = (Ob(\mathbb{C}), \omega_{\mathbb{C}}, Mor(\mathbb{C}), \mu_{\mathbb{C}}, \circ)$ and $\mathbb{D} = (Ob(\mathbb{D}), \omega_{\mathbb{D}}, Mor(\mathbb{D}), \mu_{\mathbb{D}}, \circ)$ be L-valued categories, then a functor F from $\mathbb{C}$ to $\mathbb{D}$ is a function that assigns to each $\mathbb{C}$-object A a $\mathbb{D}$-object $F(A)$, and to each $\mathbb{C}$-morphism $f : A \to A'$ a $\mathbb{D}$-morphism $F(f) : F(A) \to F(A')$, in such way that the following properties are satisfied:

1. F preserves composition, i. e. $F(g \circ f) = F(g) \circ F(f)$ provided the composition $g \circ f$ is defined;
2. F preserves identities, i. e. $F(id_X) = id_{F(X)}$ for any $X \in Ob(\mathbb{C})$;
3. $\mu_{\mathbb{C}}(f) \leq \mu_{\mathbb{D}}(F(f))$ for any $f \in Mor(\mathbb{C})$.

We now proceed with the definitions of some special objects and special morphisms for an L-valued category.

Let $X, Y \in Ob(\mathbb{C})$, where $\mathbb{C}$ is an L-valued category and let $\alpha, \beta \in L$.

Definition 2.6. (Sostak [22]) An object I in the L-valued category $\mathbb{C}$ is called α-initial if for every α-object X there exists a unique α-morphism $f : I \to X$. An object I is called initial if it is α-initial for every α.

Definition 2.7. (Sostak [22]) An object T in the L-valued category $\mathbb{C}$ is called α-terminal if for every α-object X there exists a unique α-morphism $f : X \to T$. An object T is called terminal if it is α-terminal for every α.

Definition 2.8. (Sostak [22]) An object Z in the L-valued category $\mathbb{C}$ is called α-zero if it is both α-initial and α-terminal. An object Z is called zero-object if it is α-zero for every α.

Definition 2.9. (Sostak [22]) An α-morphism $f : X \to Y$ is called a β-mono-α-morphism, (or just a β-monomorphism for short) provided for all β-morphisms $h : Z \to X$ and $k : Z \to X$ such that $f \circ h = f \circ k$ it follows that $h = k$.

Definition 2.10. (Sostak [22]) An α-morphism $f : X \to Y$ is called a β-epi-α-morphism, (or just a β-epimorphism for short) provided for all β-morphisms $h : Y \to Z$ and $k : Y \to Z$ such that $h \circ f = k \circ f$ it follows that $h = k$.

Definition 2.11. (Sostak [22]) An α-morphism $f : X \to Y$ is called a β-bi-α-morphism, (or just a β-bimorphism for short) if it is both β-monomorphisms and β-epimorphism.

2.3. L-valued relations

First time the definition of fuzzy order relation was introduced by L. A. Zadeh in 1971 under the name of fuzzy partial ordering. Slightly modifying Zadeh's definition we demonstrate the following concept of L-valued order relation (we use the term "fuzzy" when $L = [0, 1]$ and "L-valued" for an arbitrary cl-monoid L).

Let L be a fixed cl-monoid.

Definition 2.12. (cf. e. g. [24]) Let X be a set. By an L-valued order relation we call an L-valued relation $P : X \times X \to L$ such that the following three axioms are fulfilled for all $x, y, z \in X$:

1. $P(x, x) = 1$ - reflexivity;
2. $P(x, y) * P(y, z) \leq P(x, z)$ - transitivity;
3. $x \neq y \Rightarrow P(x, y) * P(y, x) = 0$ - antisymmetry.

Fifteen years later U. Höhle and N. Blanchard in their paper [13] proposed to involve L-valued equality[1] for the definition of L-valued order (partial ordering). For more resent results about fuzzy order defined with respect to the fuzzy equality see [3].

Let us first define an L-valued set and related category:

Definition 2.13. (cf. e. g. [13]) By an L-valued set we call a pair (X, E) where X is a set and E is an L-valued equality, i. e. a mapping $E : X \times X \to L$ such that the following three axioms are fulfilled for all $x, y, z \in X$:

1. $E(x, x) = 1$ - reflexivity;
2. $E(x, y) * E(y, z) \leq E(x, z)$ - transitivity;
3. $E(x, y) = E(y, x)$ - symmetry.

A mapping $f : (X, E_X) \to (Y, E_Y)$ is called extensional if

$$E_X(x_1, x_2) \leq E_Y(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$. L-valued sets and extensional mappings between them obviously form a category which is denoted L-**SET**.

[1]in the original work the term L-equality was used

Remark 2.14. In the paper [13] an L-valued set is called as an L-underdeterminate set and an L-valued equality as an L-equality relation. For the particular choices of cl-monoid an L-valued equality could be also called an M-equivalence relation (where $M = (L, \leq, *)$) [7], an L-equivalence [2], an M-valued similarity relation or an M-similarity [14], a fuzzy equivalence relation w.r.t $*$ [3, 4],[6] and also a global M-valued equality, $*$-equality and $*$-equivalence.

We continue with the definition of an L-E-order relation and an L-E-ordered set:

Definition 2.15. (cf. e.g. [13]) Let (X, E) be an L-valued set. By an L-E-order relation on the L-valued set (X, E) we call an L-valued relation $P : X \times X \to L$ such that the following three axioms are fulfilled for all $x, y, z \in X$:

1. $E(x, y) \leq P(x, y)$ - E-reflexivity;
2. $P(x, y) * P(y, z) \leq P(x, z)$ - transitivity;
3. $P(x, y) * P(y, x) \leq E(x, y)$ - E-antisymmetry.

A pair (X, P, E) is called L-E-ordered set.

In the sequel we use Definition 2.15 of L-E-order relation and we also give some comments how concrete results depend on the definition of reflexivity and antisymmetry.

Remark 2.16. For the particular choices of cl-monoid an L-E-order relation could be also called an M-E-partial ordering (where $M = (L, \leq, *)$) [7] , a fuzzy ordering w.r.t $*$ or $*$-E-ordering [3, 4].

3. CATEGORY WHOSE OBJECTS ARE L-E-ORDERED SETS

3.1. Construction of the category

Our aim is to construct an analogue of the category of partially ordered sets (**POS** or **POSET**). As we know in classical mathematics **POS** category can be constructed from small categories (categories, where the class of objects is a set and the class of morphisms is also a set). Actually we can observe each ordered set $(X, \leq)$ as a small category where elements of a set are objects and morphism from an object x to an object y exists if and only if $x \leq y$. Then the functors between these small categories must be order-preserving mappings of the corresponding ordered sets. Thus category **POS** can be observed as a category whose objects are small categories and morphisms are functors between them.

Our aim now is to construct a category whose objects are L-E-ordered sets. To realize this construction we follow the construction of the crisp **POS** category. Thus, first we observe a small category whose objects are elements of an L-E-ordered set. Further we involve the degree to which each morphism is indeed the morphism of the category and the degree to which each object is indeed the object of the category, thus we get an L-valued category (a small L-valued category). Then we construct functors between small categories. Finally we observe the category whose objects are small categories and morphisms are functors between them. By the construction of category from small categories we show that this category is constructed in a natural way from the point of view of L-valued categories.

For construction of the following small category we have to use a cl-monoid without zero divisors.
Let L be a cl-monoid without zero divisors ($\alpha * \beta = 0 \Rightarrow \alpha = 0 \quad or \quad \beta = 0$), P be an L-E-order relation and (X, P, E) be an L-E-ordered set. Let us construct a small category $pos_{(X,P,E)}$ which has as objects elements of L-E-ordered set:

- Objects: $Ob(pos_{(X,P,E)}) = \{x : x \in X\}$;
- Morphisms: $f : x \to y,\ f \in Mor(pos_{(X,P,E)}) \Leftrightarrow P(x, y) > 0$.

Now we should check that all properties of a category are fulfilled:

1. We first prove that if $f \in Mor(pos_{(X,P,E)})$ and $g \in Mor(pos_{(X,P,E)})$ then $f \circ g \in Mor(pos_{(X,P,E)})$.
   Let $g : x \to y,\ g \in Mor(pos_{(X,P,E)})$ and $f : y \to z,\ f \in Mor(pos_{(X,P,E)})$.
   Then $P(x, y) > 0$ and $P(y, z) > 0$.
   Thus $P(x, z) \geq P(x, y) * P(y, z) > 0$. Hence we obtain the existence of morphism $f \circ g \in Mor(pos_{(X,P,E)})$. Here we used the property that the cl-monoid L is without zero divisors to establish that $P(x, y) * P(y, z) > 0$ since $P(x, y) > 0$ and $P(y, z) > 0$.

2. Let us prove the associativity condition for morphisms:
   $f, g, h \in Mor(pos_{(X,P,E)}) \Rightarrow (f \circ g) \circ h = f \circ (g \circ h)$.

   Assume that $h : t \to x,\ g : x \to y,\ f : y \to z$. From the previous consideration we can conclude that compositions $(f \circ h) \circ h$ and $f \circ (g \circ h)$ exist. We know that from t to z exists only one morphism, hence $(f \circ g) \circ h = f \circ (g \circ h)$.

3. $P(x, x) = 1$, this means that there exists an identity morphism $id_x : x \to x$. Obviously $f \circ id_x = f$ for all $f : x \to y$ and $id_x \circ h = h$ for all $h : z \to x$.

Remark 3.1. Since the antisymmetry is defined by the means of L-valued equality, there could exist morphisms in the both ways between objects x and y: from x to y and from y to x. But in the case of antisymmetry from Definition 2.12, since L is without zero divisors we conclude that for all $x \neq y$ $P(x, y) = 0$ or $P(y, x) = 0$. This means that only one morphism between objects x and y could exist: either from x to y or from y to x.

Now let us involve an L-valued subclass of the class of morphisms as a mapping from the class of morphisms to the cl-monoid L

$$\mu : Mor(pos_{(X,P,E)}) \to L$$

in the following way:

$$f : x \to y \Rightarrow \mu(f) = P(x, y)$$

and an L-valued subclass of the class of objects as a mapping from the class of objects to the cl-monoid L

$$\omega : Ob(pos_{(X,P,E)}) \to L$$

$$x \in X \Rightarrow \omega(x) = 1.$$

We have constructed a small L-valued category

$$L\text{-}pos_{(X,P,E)} = (Ob(pos_{(X,P,E)}), \omega, Mor(pos_{(X,P,E)}), \mu, \circ).$$

Remark 3.2. Intuitively the value $\mu(f)$ is the degree to which a potential morphism $f : x \to y$ of the category is indeed its morphism, this means the degree to which x is less or equal to y, what is actually characterized by the value $P(x, y)$.

Let us verify all necessary properties for the L-valued category:

- The condition that $\mu(f) \leq \omega(x) \wedge \omega(y)$ for all $x, y \in Ob(L\text{-}pos_{(X,P,E)})$ and for all $f : x \to y$ is obvious since $\omega(x) = \omega(y) = 1$.
- The next property we have to prove is $\mu(g \circ f) \geq \mu(g) * \mu(f)$, if the morphism $g \circ f$ exists:

  Let $f : x \to y$ and $g : y \to z$.

  Then by transitivity of the L-E-order P we have
  $\mu(g \circ f) = P(x, z) \geq P(x, y) * P(y, z) = \mu(f) * \mu(g)$.
- Obviously $\mu(id_x) = \omega(x)$, where id_x is the identity morphism.

Thus we have proven that the category $L\text{-}pos_{(X,P,E)}$ is constructed correctly.

Let us construct a functor F from one small category to another. Let $L\text{-}pos_{(X,P,E)}$ and $L\text{-}pos_{(Y,P',E')}$ be L-valued categories. We construct a functor F from the category $L\text{-}pos_{(X,P,E)}$ to the category $L\text{-}pos_{(Y,P',E')}$ such that $E(x_1, x_2) \leq E'(F(x_1), F(x_2))$.

The necessary condition for F to be a functor is:

$$\forall f \in Mor(L\text{-}pos_{(X,P,E)}) \quad \mu_{L\text{-}pos_{(X,P,E)}}(f) \leq \mu_{L\text{-}pos_{(Y,P',E')}}(F(f)).$$

This means that $P(x_1, x_2) \leq P'(F(x_1), F(x_2))$.

$$\begin{array}{ccc} x_1 & \xrightarrow{F} & F(x_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ x_2 & \xrightarrow{F} & F(x_2) \end{array}$$

$$P(x_1, x_2) \leq P'(F(x_1), F(x_2))$$

Let us prove that this is also a sufficient condition for F to be a functor:

1. $f \in Mor_{L\text{-}pos_{(X,P,E)}}(x_1, x_2) \Rightarrow P(x_1, x_2) > 0 \Rightarrow$
   $\Rightarrow P'(F(x_1), F(x_2)) > 0 \Rightarrow F(f) \in Mor_{L\text{-}pos_{(Y,P',E')}}(F(x_1), F(x_2))$.
2. If the composition $g \circ f$ is defined, then $F(g \circ f) = F(g) \circ F(f)$
   $(f \in Mor_{L\text{-}pos_{(X,P,E)}}(x_1, x_2), g \in Mor_{L\text{-}pos_{(X,P,E)}}(x_2, x_3))$, because of the existence of only one morphism from $F(x_1)$ to $F(x_3)$.

3. $F(id_x) = id_{F(x)}$ for all $x \in Ob(L\text{-}pos_{(X,P,E)})$, because of the existence of only one morphism from $F(x)$ to $F(x)$.

We have proven that the functor was constructed correctly.

Let us define the category $\mathbf{POS}(L)$ (the analogue of crisp **POS** category) as a category of small categories. The objects of $\mathbf{POS}(L)$ will be L-E-ordered sets (categories of the type *L-pos*) and the morphisms will be functors between them.

$$\mathbf{POS}(L) = (Ob(\mathbf{POS}(L)), Mor(\mathbf{POS}(L)), \circ)$$

- Objects: $Ob(\mathbf{POS}(L)) = \{(X, P, E)\}$ where (X, P, E) are L-E-ordered sets;
- Morphisms: $Mor(\mathbf{POS}(L)) = \{f : (X, P, E) \to (Y, P', E') \mid \forall x_1, x_2 \in X$ $E(x_1, x_2) \leq E'(f(x_1), f(x_2));\ P(x_1, x_2) \leq P'(f(x_1), f(x_2))\}$.

Remark 3.3. If we use Definition 2.12 the constructions of the category *L-pos* and the category $\mathbf{POS}(L)$ are the same. We only have to skip the property $E(x_1, x_2) \leq E'(f(x_1), f(x_2))$ for the morphisms of the category $\mathbf{POS}(L)$.

Remark 3.4. In the sequel we say that a mapping $f : (X, P, E) \to (Y, P', E')$ is order-preserving if for all $x_1, x_2 \in X$ $P(x_1, x_2) \leq P'(f(x_1), f(x_2))$.

Remark 3.5. The property that a cl-monoid L is without zero divisors was useful only for the construction of the category of the type *pos* and, namely, for the existence of composition. If we are not interested in the construction we can define the category $\mathbf{POS}(L)$ straightway and skip the above mentioned condition for a cl-monoid.

3.2. Properties of the category

Our next aim is to consider some properties of the category $\mathbf{POS}(L)$. We are going to study some special objects, morphisms and some standard constructions of $\mathbf{POS}(L)$ category.

We begin by studying some properties of $\mathbf{POS}(L)$ category.

Proposition 3.6. The category **POS** is a full subcategory of the category $\mathbf{POS}(L)$.

Proof. We have to prove that for all $(X, \leq)$ and $(Y, \leq')$, $Mor_{\mathbf{POS}}((X, \leq), (Y, \leq')) = Mor_{\mathbf{POS}(L)}((X, \chi_{\leq}, \chi_{=}), (Y, \chi_{\leq'}, \chi_{='}))$, where $\chi_{\leq} : X \times X \to L$ such that $\chi_{\leq}(x, y) = 1$ if $x \leq y$ and $\chi_{\leq}(x, y) = 0$ otherwise; $\chi_{=}(x, y) = 1$ if $x = y$ and $\chi_{=}(x, y) = 0$ otherwise. This is obvious since for crisp ordered sets $(X, \leq)$ and $(Y, \leq')$ the condition of preserving order for the function f is equivalent to the condition $\chi_{\leq}(x_1, x_2) \leq \chi_{\leq'}(f(x_1), f(x_2))$. □

Proposition 3.7. Let L_1 and L_2 be two isomorphic cl-monoids (φ is an isomorphism) and $\mathbf{POS}(L_1)$, $\mathbf{POS}(L_2)$ correspondent categories. Then we can define the functor $F : \mathbf{POS}(L_1) \to \mathbf{POS}(L_2)$ such that $F((X, P_1, E_1)) = (X, P_2, E_2)$, where $E_2(x_1, x_2) = \varphi(E_1(x_1, x_2))$, $P_2(x_1, x_2) = \varphi(P_1(x_1, x_2))$ and $F(f) = f$. Thus defined functor F is an isomorphism and categories $\mathbf{POS}(L_1)$ and $\mathbf{POS}(L_2)$ are isomorphic.

Proposition 3.8. If L_1 and L_2 are cl-monoids and $\varphi : L_1 \hookrightarrow L_2$ is an order-embedding and operation-preserving mapping then $\mathbf{POS}(L_1)$ is isomorphic to $\mathbf{POS}(\varphi(L_1))$, which is a full subcategory of the category $\mathbf{POS}(L_2)$.

P r o o f. We define the functor $F : \mathbf{POS}(L_1) \to \mathbf{POS}(\varphi(L_1))$ as in previous proposition. Thus defined functor F is an isomorphism and categories $\mathbf{POS}(L_1)$ and $\mathbf{POS}(\varphi(L_1))$ are isomorphic. It is easy to see that $Ob(\mathbf{POS}(\varphi(L_1))) \subseteq Ob(\mathbf{POS}(L_2))$ and that $Mor_{\mathbf{POS}(\varphi(L_1))}((X,P),(Y,P')) = Mor_{\mathbf{POS}(L_2)}((X,P),(Y,P'))$, where $(X,P),(Y,P') \in Ob(\mathbf{POS}(\varphi(L_1)))$. □

Proposition 3.9. The category $\mathbf{POS}(L)$ is a connected category.

P r o o f. To prove that the category $\mathbf{POS}(L)$ is a connected category we should show that for every two objects (X,P,E) and (Y,P',E') (X and Y are nonempty sets) $Mor_{\mathbf{POS}(L)}((X,P,E),(Y,P',E')) \neq \emptyset$, that means there exists a morphism $f : (X,P,E) \to (Y,P',E')$.
Let us fix objects (X,P,E) and (Y,P',E'). Further we build a morphism $f : (X,P,E) \to (Y,P',E')$ in the following way: $f(x) = y_0$ for all $x \in X$ where y_0 is a fixed element from the set Y. Obviously $E(x_1,x_2) \leq E'(f(x_1),f(x_2))$ since $E'(f(x_1),f(x_2)) = E'(y_0,y_0) = 1$ and $P(x_1,x_2) \leq P'(f(x_1),f(x_2))$ since $P'(f(x_1),f(x_2)) = P'(y_0,y_0) = 1$. □

We continue by considering special objects in the category $\mathbf{POS}(L)$.

Proposition 3.10. Empty set is the unique initial object in $\mathbf{POS}(L)$.

Proposition 3.11. The singleton set with the uniquely constructed L-valued equality E_1 and L_1-E_1-order is the terminal object in $\mathbf{POS}(L)$.

Proposition 3.12. There are no zero objects in $\mathbf{POS}(L)$.

We continue by considering special morphisms in the category $\mathbf{POS}(L)$.

Proposition 3.13. A morphism $f : (X,P,E) \to (Y,P',E')$ is a monomorphism if and only if f is an injective mapping.

P r o o f. The sufficiency is obvious. We continue by proving the necessity.
If the mapping $f : (X,P,E) \to (Y,P',E')$ is not an injection then there exist two elements x_1 and x_2 in the set X such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.
Let (Z,P_Z,E_Z) be an L-E_Z-ordered set such that $Z = \{z\}$, $P_Z(z,z) = 1$, $E_Z(z,z) = 1$ and let $u,v : Z \to X$ be functions such that $u(z) = x_1$ but $v(z) = x_2$. Thus, obviously $f \circ u = f \circ v$ but $u \neq v$. Hence f is not a monomorphism. □

Proposition 3.14. A morphism $f : (X,P,E) \to (Y,P',E')$ is an epimorphism if and only if f is a surjection.

P r o o f. The sufficiency is obvious. We continue by proving the necessity.
If the mapping $f : (X, P, E) \to (Y, P', E')$ is not a surjection then there exists an element y_0 in the set Y such that $\forall x \in X \; f(x) \neq y_0$.
Let $Z = Y \bigcup \{z\}$, the L-valued equality E'' and L-E''-order P'' on the set Z we define in the following way:

$P''(y_1, y_2) = P'(y_1, y_2)$, $E''(y_1, y_2) = E'(y_1, y_2)$ if $y_1, y_2 \in Y$;

$P''(z, y) = P'(y_0, y)$, $E''(z, y) = E'(y_0, y)$ if $y \in Y$ and $y \neq y_0$;

$P''(y, z) = P'(y, y_0)$, $E''(y, z) = E'(y, y_0)$ if $y \in Y$ and $y \neq y_0$;

$P''(y_0, z) = 1$, $E''(y_0, z) = 1$;

$P''(z, y_0) = 1$, $E''(z, y_0) = 1$;

$P''(z, z) = 1$, $E''(z, z) = 1$.

It is easy to verify that E'' is an L-valued equality and P'' fulfills all necessary condition: E''-reflexivity, transitivity and E''-antisymmetry. Let us define now the functions $u : (Y, P', E') \to (Z, P'', E'')$ and $v : (Y, P', E') \to (Z, P'', E'')$ in the following way: $u(y) = y$ for all $y \in Y$; $v(y) = 0$ if $y \neq y_0$, $v(y) = z$ otherwise. Obviously, functions u and v are extensional, order-preserving mappings and $u \circ f = v \circ f$ but $u \neq v$. Hence f is not an epimorphism. □

Remark 3.15. In the case of definition of L-valued order 2.11 the relation P'' for the elements (z, y_0) should be defined as $P''(z, y_0) = 0$. The other part of the proof is similar.

From the previous two propositions and the definition of bimorphism we get the following proposition:

Proposition 3.16. A morphism $f : (X, P, E) \to (Y, P', E')$ is a bimorphism if and only if f is a bijection.

We know that in the category **POS** not every injection is a section, not every surjection is a retraction and not every bijection is an isomorphism. Provided that the category **POS** is a full subcategory of the category **POS**(L), in the category **POS**(L) we can find injections which are not sections, surjections which are not retractions and bijections which are not isomorphisms. Hence the category **POS**(L) is not balanced.

A morphism $f : (X, P, E) \to (Y, P', E')$ is an isomorphism in the category **POS**(L) if and only if it is a bijection, $E(x_1, x_2) = E'(f(x_1), f(x_2))$ and
$P(x_1, x_2) = P'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

We now turn to the study of special constructions in the category **POS**(L).

Theorem 3.17. The product of a family $((X_i, P_i, E_i))_{i \in I}$ of **POS**(L) objects is a pair $((\prod_i X_i, P_\wedge, E_\wedge), (\pi_i)_{i \in I})$, where $\prod_i X_i = \{f : I \to \bigcup_i X_i \mid \forall i \; f(i) \in X_i\}$,
$E_\wedge(f, h) = \bigwedge_{i \in I} E_i(f(i), h(i))$, $P_\wedge(f, h) = \bigwedge_{i \in I} P_i(f(i), h(i))$ and
$\pi_i : (\prod_i X_i, P_\wedge, E_\wedge) \to (X_i, P_i, E_i)$ is defined by $\pi_i(f) = f(i)$.

If I is a finite set then the Theorem 3.17 reduces to the following (more lucid) form:

Theorem 3.17'. The product of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of **POS**(L) objects is a pair $((X_1 \times X_2 \times \cdots \times X_n, P_\wedge, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$, where
$E_\wedge((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \bigwedge_i E_i(a_i, b_i)$,
$P_\wedge((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \bigwedge_i P_i(a_i, b_i)$ and
$\pi_i : (\prod_i X_i, P_\wedge, E_\wedge) \to (X_i, P_i, E_i)$ is defined by $\pi_i((a_1, a_2, \ldots, a_n)) = a_i$.

Here we prove the general Theorem 3.17.

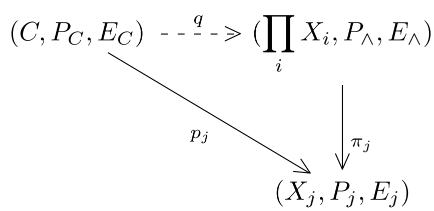
P r o o f.

- Let us prove that $(\prod_i X_i, P_\wedge, E_\wedge)$ is an object in the category **POS**(L). This means we should prove that $E_\wedge$ is an L-valued equality and $P_\wedge$ is an L-$E_\wedge$-order relation. We skip the proof that $E_\wedge$ is an L-valued equality since it is similar to the proof that $P_\wedge$ is an L-$E_\wedge$-order relation. We now turn to prove $E_\wedge$-reflexivity, transitivity and $E_\wedge$-antisymmetry of the L-$E_\wedge$-order relation $P_\wedge$:
  - Since all relations P_i are E_i-reflexive $E_i(f(i), g(i)) \leq P_i(f(i), g(i))$ for all $i \in I$. Therefore $\bigwedge_i E_i(f(i), g(i)) \leq \bigwedge_i P_i(f(i), g(i))$ for all f, g .
    Thus $E_\wedge(f, g) \leq P_\wedge(f, g)$. We have proven the $E_\wedge$-reflexivity of L-$E_\wedge$-order relation $P_\wedge$.
  - $P_\wedge(f, g) * P_\wedge(g, h) =$
    $= \bigwedge_i P_i(f(i), g(i)) * \bigwedge_i P_i(g(i), h(i)) \leq \bigwedge_i (P_i(f(i), g(i)) * P_i(g(i), h(i))) \leq$
    $\leq \bigwedge_i P_i(f(i), h(i)) = P_\wedge(f, h)$. We have proven the transitivity of L-$E_\wedge$-order relation $P_\wedge$.
  - We have to prove that for all f, g $P_\wedge(f, g) * P_\wedge(g, f) \leq E_\wedge(f, g)$:
    $P_\wedge(f, g) * P_\wedge(g, f) =$
    $= \bigwedge_i P_i(f(i), g(i)) * \bigwedge_i P_i(g(i), f(i)) \leq \bigwedge_i (P_i(f(i), g(i)) * P_i(g(i), f(i))) \leq$
    $\leq \bigwedge_i E_i(f(i), g(i)) = E_\wedge(f, g)$. We have proven the $E_\wedge$-antisymmetry of L-$E_\wedge$-order relation $P_\wedge$.

- We proceed to show that π_j are morphisms for all $j \in I$:

  $E_\wedge(f, h) = \bigwedge_{i\in I} E_i(f(i), h(i)) \leq E_j(f(j), h(j)) = E_j(\pi_j(f), \pi_j(h))$ for all $j \in I$;
  $P_\wedge(f, h) = \bigwedge_{i\in I} P_i(f(i), h(i)) \leq P_j(f(j), h(j)) = P_j(\pi_j(f), \pi_j(h))$ for all $j \in I$.

- The task is now to prove that for each pair $((C, P_C, E_C), (p_i)_{i\in I})$, where (C, P_C, E_C) is a **POS**(L) object and for each $j \in I$,
  $p_j : (C, P_C, E_C) \to (X_j, P_j, E_j)$ is a morphism there exists a unique **POS**(L) morphism $q : (C, P_C, E_C) \to (\prod_i X_i, P_\wedge, E_\wedge)$ such that for each $j \in I$, the triangle



  commutes.
  Let us first prove the existence of the morphism q.

We define $q : (C, P_C, E_C) \to (\prod_i X_i, P_\wedge, E_\wedge)$ in the following way:

$$\forall c \in C \; q(c) = f_c : \; f_c(j) = p_j(c) \; \forall j \in I.$$

We have to prove that q is an extensional, order-preserving mapping:
We know that p_j is an extensional, order-preserving mapping for all $j \in I$. This means that $E_C(c_1, c_2) \leq E_j(p_j(c_1), p_j(c_2))$ and $P_C(c_1, c_2) \leq P_j(p_j(c_1), p_j(c_2))$ for all $j \in I$.
Thus $E_C(c_1, c_2) \leq \bigwedge_i E_i(p_i(c_1), p_i(c_2)) = E_\wedge(f_{c_1}, f_{c_2}) = E_\wedge(q(c_1), q(c_2))$;
$P_C(c_1, c_2) \leq \bigwedge_i P_i(p_i(c_1), p_i(c_2)) = P_\wedge(f_{c_1}, f_{c_2}) = P_\wedge(q(c_1), q(c_2))$.

Now it is sufficient to prove that the above diagram commutes and that q is the unique morphism for which this diagram commutes. The proof is similar as in the case of product in the category **SET**.

□

The scheme of the proof for the following construction is similar, so we leave the proposition without proof.

Theorem 3.18. A coproduct of a family $((X_i, P_i, E_i))_{i \in I}$ of **POS**(L) objects is a pair $((\mu_i)_{i \in I}, (\bigcup_i X_i, P_\cup, E_\cup))$ where $\bigcup_i X_i$ is disjoint union,

$$E_\cup(a, b) = \begin{cases} E_i(a, b), & \text{if } a, b \in X_i \\ 0, & \text{otherwise} \end{cases}; \; P_\cup(a, b) = \begin{cases} P_i(a, b), & \text{if } a, b \in X_i \\ 0, & \text{otherwise} \end{cases}$$

and $\mu_i : (X_i, P_i, E_i) \to (\bigcup_i X_i, P_\cup, E_\cup)$ such that $\mu_i(a) = a$.

4. L-VALUED ANALOGUE OF **POS**(L) CATEGORY

We describe here three different L-valued categories whose objects are L-E-ordered sets:

1. Let us introduce the mapping μ for the category **POS**(L) described in the previous section in the following way:

$$\mu(f) = \inf_{x_1, x_2 \in X} (P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2))$$

where $f : (X, P) \to (Y, P')$, $f \in Mor(\mathbf{POS}(L))$ and the mapping ω:
$\omega((X, P, E)) = 1$ where (X, P, E) is the objects of the category **POS**(L).
In this case we obtain the category, where the intuitive meaning of the value $\mu(f)$ is the degree to which a morphism f is an order-reflecting mapping. Thus the obtained L-valued category is something more than just an L-valued analogue of **POS** category, because all morphisms are order-preserving mappings, but additionally we introduce the "order-reflecting" degrees for the morphisms. It is worth to mention that the bottom frame of this L-valued category is exactly the category **POS**(L). This approach was investigated in [20] and we do not discuss it here.

2. The idea of the second approach is not to use the following order-preserving property: $P(x_1, x_2) \leq P'(f(x_1), f(x_2))$ for the morphism $f : (X, P) \to (Y, P')$, but to use the graded order-preserving property described by the mapping μ:

$$\mu(f) = \inf_{x_1, x_2 \in X} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))).$$

In this case we obtain an L-valued analogue of the $\mathbf{POS}(L)$ category. To be more formal we describe this case by applying the scheme proposed in paper [23]. In our case the scheme can be described as follows.

Let $\phi : \mathbf{POS}(L) \to L\text{-}\mathbf{SET}$ be the functor assigning to each $\mathbf{POS}(L)$ object (X, P, E) the support set (X, E) and leaving morphisms unchanged. Then according to the scheme we come to the category $\mathbf{POS}(L)_{L\text{-}\mathbf{SET}}$. Its objects are the same as in $\mathbf{POS}(L)$, but its morphisms are all mappings between the corresponding support sets. Starting from this category as the crisp bottom frame we define the L-valued category $L\text{-}\mathbf{POS}(L)$ by setting $\omega((X, P, E)) = 1$ for every $L\text{-}\mathbf{POS}(L)$ object (X, P, E) and the mapping μ which is introduced above.

3. The idea of the third approach is not to use the following order-preserving property: $P(x_1, x_2) \leq P'(f(x_1), f(x_2))$ for the morphism $f : (X, P) \to (Y, P')$, but use the graded order-preserving-and-reflecting property described by the mapping μ:

$$\mu(f) = \inf_{x_1, x_2 \in X} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))).$$

In this case we obtain something more than just an L-valued analogue of the $\mathbf{POS}(L)$ category.

In the section below we consider only constructions 2 and 3. But we study the properties of the category of the form 2 only, because it is a direct L-valued analogue of $\mathbf{POS}(L)$ category.

4.1. Construction of the category

Let us observe the category $L\text{-}\mathbf{POS}(L)$.

$L\text{-}\mathbf{POS}(L)$-objects are L-E-ordered sets and $L\text{-}\mathbf{POS}(L)$-morphisms are extensional mappings between them.

$$L\text{-}\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), \omega, Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ),$$

where

$$\mu(f) = \inf_{x_1, x_2 \in X} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2)))$$

for $f : (X, P, E) \to (Y, P', E')$ and

$$\forall (X, P, E) \in Ob(L\text{-}\mathbf{POS}(L)) \quad \omega((X, P, E)) = 1.$$

Theorem 4.1.

$$L\text{-}\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), \omega, Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ)$$

is an L-valued category.

P r o o f. It is obvious that $Ob(L\text{-}\mathbf{POS}(L))$ and $Mor(L\text{-}\mathbf{POS}(L))$ form a crisp category, thus we have to prove the conditions for the mappings μ and ω, the part which characterizes the L-valued case.

1. $\mu(f) \leq \omega((X, P, E)) \wedge \omega((Y, P', E'))$ for all $(X, P, E), (Y, P', E') \in Ob(L\text{-}\mathbf{POS}(L))$ and for all $f \in Mor_{L\text{-}\mathbf{POS}(L)}((X, P, E), (Y, P', E'))$, since $\omega((X, P, E)) = 1$ and $\omega((Y, P', E')) = 1$.

2. Let us prove that $\mu(g \circ f) \geq \mu(g) * \mu(f)$ where $f : (X, P, E) \to (Y, P', E')$, $g : (Y, P', E') \to (Z, P'', E'')$ and $x_1, x_2 \in X$:

$$\begin{aligned}
\mu(g \circ f) &= \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P''(g(f(x_1)), g(f(x_2)))) \geq \\
&\geq \inf_{x_1,x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
&* (P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2))))) \geq \\
&\geq \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
&* \inf_{x_1,x_2} (P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) \geq \\
&\geq \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * \\
&* \inf_{y_1,y_2} (P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2))) = \\
&= \mu(f) * \mu(g).
\end{aligned}$$

We obtain that $\mu(g \circ f) \geq \mu(g) * \mu(f)$.

We have used the properties of the cl-monoid and the following inequality in the proof :

$$\begin{aligned}
&\inf_{x_1,x_2} (P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) \geq \\
\geq &\inf_{y_1,y_2} (P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2))).
\end{aligned}$$

This follows from the fact that:

$$\begin{aligned}
&\{\ P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2))) : x_1, x_2 \in X\} \subseteq \\
\subseteq &\{\ P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2)) : y_1, y_2 \in Y\}.
\end{aligned}$$

3. $\mu(id_{(X,P,E)}) = \inf_{x_1,x_2} (P(x_1, x_2) \mapsto P(x_1, x_2)) = 1 = \omega((X, P, E))$.

□

Now let us define the category $FL\text{-}\mathbf{POS}(L)$ which we have discussed in the third clause of the previous section. It is worth to mention that the only difference between L-valued category $FL\text{-}\mathbf{POS}(L)$ and L-valued category $L\text{-}\mathbf{POS}(L)$ is in the choice of mapping μ.

FL-$\mathbf{POS}(L)$-objects are L-E-ordered sets and FL-$\mathbf{POS}(L)$-morphisms are extensional mappings.

$$FL\text{-}\mathbf{POS}(L) = (Ob(FL\text{-}\mathbf{POS}(L)), \omega, Mor(FL\text{-}\mathbf{POS}(L)), \mu, \circ), \text{ where}$$
$$Ob(FL\text{-}\mathbf{POS}(L)) = \{(X, P, E) \mid (X, P, E) \text{ is an } L\text{-}E\text{-ordered set}\};$$
$$Mor(FL\text{-}\mathbf{POS}(L)) = \{f : (X, P, E) \to (Y, P', E') \mid$$
$$\forall x_1, x_2 \in X \; E(x_1, x_2) \leq E'(f(x_1), f(x_2))\};$$
$$\mu(f) = \inf_{x_1, x_2 \in X} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))), \text{ where}$$
$$f : (X, P, E) \to (Y, P', E');$$
$$\omega((X, P, E)) = 1 \quad \forall (X, P, E) \in Ob(FL\text{-}\mathbf{POS}(L)).$$

Theorem 4.2.

$$FL\text{-}\mathbf{POS}(L) = (Ob(FL\text{-}\mathbf{POS}(L)), \omega, Mor(FL\text{-}\mathbf{POS}(L)), \mu, \circ)$$

is an L-valued category.

Proof. All properties of an L-valued category (except the property $\mu(g \circ f) \geq \mu(g) * \mu(f)$) are straightforward. It is only necessary to prove that

$$\inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P''(g(f(x_1)), g(f(x_2)))) \geq$$
$$\geq \inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))) * \inf_{y_1, y_2} (P'(y_1, y_2) \leftrightarrow P''(g(y_1), g(y_2)))$$

where $f : (X, P, E) \to (Y, P', E')$, $g : (Y, P', E') \to (Z, P'', E'')$, $x_1, x_2 \in X$ and $y_1, y_2 \in Y$:

$$\inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P''(g(f(x_1)), g(f(x_2)))) =$$
$$= \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P''(g(f(x_1)), g(f(x_2)))) *$$
$$*(P''(g(f(x_1)), g(f(x_2))) \mapsto P(x_1, x_2))) \geq$$
$$\geq \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) *$$
$$*(P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) *$$
$$*(P''(g(f(x_1)), g(f(x_2))) \mapsto P'(f(x_1), f(x_2))) *$$
$$*(P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2))) \geq$$
$$\geq \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * (P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2))) *$$
$$* \inf_{x_1, x_2} ((P'(f(x_1), f(x_2)) \mapsto P''(g(f(x_1)), g(f(x_2)))) *$$
$$*(P''(g(f(x_1)), g(f(x_2))) \mapsto P'(f(x_1), f(x_2)))) \geq$$
$$\geq \inf_{x_1, x_2} ((P(x_1, x_2) \mapsto P'(f(x_1), f(x_2))) * (P'(f(x_1), f(x_2)) \mapsto P(x_1, x_2)))) *$$
$$* \inf_{y_1, y_2} ((P'(y_1, y_2) \mapsto P''(g(y_1), g(y_2))) * (P''(g(y_1), g(y_2)) \mapsto P'(y_1, y_2)))) =$$
$$= \inf_{x_1, x_2} (P(x_1, x_2) \leftrightarrow P'(f(x_1), f(x_2))) * \inf_{y_1, y_2} (P'(y_1, y_2) \leftrightarrow P''(g(y_1), g(y_2))).$$

□

Remark 4.3. If the value of a mapping ω is equal to 1 for all objects of an L-valued category we do not write it. For instance we will write

$$L\text{-}\mathbf{POS}(L) = (Ob(L\text{-}\mathbf{POS}(L)), Mor(L\text{-}\mathbf{POS}(L)), \mu, \circ).$$

4.2. Properties of the category

In this section we study properties of the category L-$\mathbf{POS}(L)$.

Proposition 4.4. If we consider the crisp category **POS** as an L-valued category $\mathbf{POS} = (Ob(\mathbf{POS}), Mor(\mathbf{POS}), \mu_{\mathbf{POS}}, \circ)$, where $\mu_{\mathbf{POS}}(f)$ is equal to 1 if and only if f is an order-preserving mapping and 0 otherwise, then the category **POS** is an L-valued subcategory of the category L-$\mathbf{POS}(L)$.

We continue by considering special objects and special morphisms in the category L-$\mathbf{POS}(L)$.

Proposition 4.5. The empty set is the unique initial (1-initial) object in L-$\mathbf{POS}(L)$.

Proposition 4.6. The singleton set with the uniquely constructed L_1-E_1-order on it is the terminal (1-terminal) object in L-$\mathbf{POS}(L)$.

Proposition 4.7. There are no α-zero objects in L-$\mathbf{POS}(L)$.

Proposition 4.8. An α-morphism $f : (X, P) \to (Y, P')$ is a β-monomorphism (for any $\beta \in L$) if and only if f is an injective mapping.

Proposition 4.9. An α-morphism $f : (X, P) \to (Y, P')$ is a β-epimorphism (for any $\beta \in L$) if and only if f is a surjection.

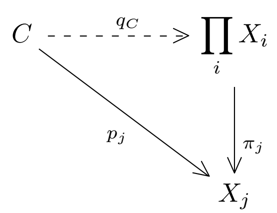
Proposition 4.10. An α-morphism $f : (X, P) \to (Y, P')$ is a β-bimorphism (for any $\beta \in L$) if and only if f is a bijection.

We continue by describing product in the category L-$\mathbf{POS}(L)$. To do this we define the product in the context of L-valued categories.

Let $\mathbb{C} = (Ob(\mathbb{C}), Mor(\mathbb{C}), \mu, \circ)$ be an L-valued category.

Definition 4.11. A pair $(\prod_i X_i, (\pi_i)_{i\in I})$ is an α-product of a family $(X_i)_{i\in I}$ of $\mathbb{C}$-objects if and only if:

- $\prod_i X_i$ is a $\mathbb{C}$-object;
- π_i are $\mathbb{C}$-morphisms such that $\inf_i \mu(\pi_i) \geq \alpha$;
- for each pair $(C, (p_i)_{i\in I})$, where C is a $\mathbb{C}$-object and for each $j \in I$, $p_j : C \to X_j$ is a $\mathbb{C}$-morphism and $\mu(p_j) \geq \mu(\pi_j)$ there exists a unique $\mathbb{C}$-morphism $q_C : C \to \prod_i X_i$ such that $\mu(q_C) \geq \alpha$ and for each $j \in I$, the triangle



commutes.

Now we propose an alternative definition where we try to separate the "crisp" and the "fuzzy parts", but first we should define the notion of an α-source.

Definition 4.12. An α-source in an L-valued category $\mathbb{C}$ is a pair $(X, (f_i)_{i\in I})$, where X is a $\mathbb{C}$-object and $(f_i : X \to X_i)_{i\in I}$ is a family of $\mathbb{C}$-morphisms each with domain X and $\inf_i \mu(f_i) \geq \alpha$.

Definition 4.13. An α-source $(\prod_i X_i, (\pi_i)_{i\in I})$ in an L-valued category $\mathbb{C}$ is an α-product of a family $(X_i)_{i\in I}$ of $\mathbb{C}$-objects if and only if it is a product in a crisp category $\mathbb{C} = (Ob(\mathbb{C}), Mor(\mathbb{C}), \circ)$ and for each α-source $(C, (p_i)_{i\in I})$ (for all $j \in I$, $p_j : C \to X_j$ and $\mu(p_j) \geq \mu(\pi_j)$) such that q_C is a unique morphism $q_C : C \to \prod_i X_i$

$$\mu(q_C) \geq \alpha.$$

We observe some α-products of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-**POS**(L) objects, where L is a concrete cl-monoid in the next examples.

Example 4.14. Let L be a cl-monoid. A pair $((X_1 \times \cdots \times X_n, P_\wedge, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$ is a 1-product of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-**POS**(L) objects, where
$E_\wedge((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \bigwedge_i E_i(a_i, b_i)$,
$P_\wedge((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \bigwedge_i P_i(a_i, b_i)$ and
$\pi_i : (\prod_i X_i, P_\wedge, E_\wedge) \to (X_i, P_i, E_i)$ are defined by $\pi_i((a_1, a_2, \ldots, a_n)) = a_i$.

In the next two examples for the brevity of explanation the relations E_i for all i are crisp equivalence relations: $E_i(a, b) = 1$ if $a = b$, $E_i(a, b) = 0$ otherwise and the relation $E_\wedge$ will be defined as follows:
$E_\wedge((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \bigwedge_i E_i(a_i, b_i)$.

Example 4.15. Let $L = ([0, 1], \leq, \wedge, \vee, T, \mapsto_T)$ be a cl-monoid, where T is a t-norm without zero divisors, $\mapsto_T$ is a corresponding residuum and let A_ω be the weakest aggregation function defined by:

$$A_\omega(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & x_1 = x_2 = \ldots = x_n = 1 \\ 0, & \text{otherwise} \end{cases}.$$

The requirement on a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-**POS**(L)-objects is that sets X_i are not empty sets and there exists an index j such that $\exists\, a_j, b_j \in X_j$:
$P_j(a_j, b_j) \in (0, 1)$.

Then a pair $((X_1 \times \cdots \times X_n, P_{A_\omega}, E_\wedge), (\pi_i)_{i\in\{\overline{1,n}\}})$ is a 0-product of a family $((X_i, P_i, E_i))_{i\in\{\overline{1,n}\}}$ of L-**POS**(L) objects, where
$P_{A_\omega}((a_1, \cdots, a_n), (b_1, \ldots, b_n)) = A_\omega(P_1(a_1, b_1), \ldots, P_n(a_n, b_n))$ and
$\pi_i : (\prod_i X_i, P_{A_\omega}, E_\wedge) \to (X_i, P_i, E_i)$ are defined by $\pi_i((a_1, a_2, \ldots, a_n)) = a_i$.

For example, for L-**POS**(L)-morphism
$h : (X_1 \times \cdots \times X_n, P_\wedge, E_\wedge) \to (X_1 \times \cdots \times X_n, P_{A_\omega}, E_\wedge)$ $\mu(h)$ is equal to 0 if the above conditions are fulfilled.

Example 4.16. Let $L = ([0,1], \leq, \wedge, \vee, T_L, \mapsto_{T_L})$ be a cl-monoid, where T_L is Łukasiewicz t-norm and $\mapsto_{T_L}$ is the corresponding residuum.
Then a pair $((X_1 \times X_2, P_{T_L}, E_\wedge), (\pi_i)_{i\in\{1,2\}})$ is a 0.5-product of a family $((X_i, P_i, E_i))_{i\in\{1,2\}}$ of L-**POS**(L) objects, where
$P_{T_L}((a_1,a_2),(b_1,b_2)) = T_L(P_1(a_1,b_1), P_2(a_2,b_2))$ and
$\pi_i : (\prod_i X_i, P_{T_L}, E_\wedge) \to (X_i, P_i, E_i)$ are defined by $\pi_i((a_1,a_2)) = a_i$.

Proof. We know that for the Łukasiewicz t-norm T_L (as it is a left-continuous t-norm) $x \mapsto_{T_L} y = 1 \Leftrightarrow x \leq y$. Obviously $T_L(P_1(a_1,b_1), P_2(a_2,b_2)) \leq P_i(a_i,b_i)$ for all i and for all $a_i, b_i \in X_i$. Thus $\mu(\pi_i) = 1$ for all i. So for every other source $((C, P_C, E_C), (f_i)_{i\in\{1,2\}})$, where $\mu(\pi_i) \leq \mu(f_i)$ we know that $\mu(f_i) = 1$ and then $P_C(c_1,c_2) \leq P_i(f_i(c_1), f_i(c_2))$ for all $c_1, c_2 \in C$.

Thus there exists unique morphism $h_C : (C, P_C, E_C) \to (X_1 \times X_2, P_\wedge, E_\wedge)$ such that $P_C(c_1,c_2) \leq P_\wedge(h_C(c_1), h_C(c_2))$ (Proposition 3.15.), where
$P_\wedge((a_1,a_2),(b_1,b_2)) = P_1(a_1,b_1) \wedge P_2(a_2,b_2)$. This gives $\mu(h_C) = 1$. Thus for every morphism $q_C : (C, P_C, E_C) \to (X_1 \times X_2, P_{T_L}, E_\wedge)$ the following inequality holds: $\mu(q_C) \geq \mu(e_{X_1\times X_2}) * \mu(h_C)$, where $e_{X_1\times X_2} : (X_1 \times X_2, P_\wedge, E_\wedge) \to (X_1 \times X_2, P_{T_L}, E_\wedge)$ such that $e_{X_1\times X_2}((x_1,x_2)) = (x_1,x_2)$. It is easy to calculate that $\mu(e_{X_1\times X_2}) \geq 0.5$, thus $\mu(q_C) \geq 0.5$. □

In the same way we could define and investigate other special constructions, but for the sake of brevity we do not do it here.

5. CONCLUSION

In the paper we have constructed the L-valued category whose objects are L-E-ordered sets and morphisms are "potential" order-preserving mappings. To realize the construction we have introduced the degree (for the class of potential morphisms) to which each morphism is an order-preserving mapping, or, in other words, monotone mapping. In the natural way we can use this idea for some practical applications. For example to construct an aggregation process in the context of L-valued **POS** categories and by this define the graded property of monotonicity for the aggregation function (see e. g. [10]).

It is also possible to apply the above idea to the aggregation of fuzzy relations, namely, to the aggregation of fuzzy orders. Our proposition is to involve the degree to which aggregation operator preserves properties of fuzzy relations. So we will be able to calculate this degree for any aggregation function (not only for aggregation operators which preserve properties of fuzzy relations), see e. g. [11].

In the future we are going to apply the properties of constructed categories for the above mentioned practical applications.

ACKNOWLEDGEMENT

This work was partially supported by ESF research project 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008.

(Received July 7, 2010)

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