Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 2, 437–459

Persistent URL: http://dml.cz/dmlcz/141545

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Czechoslovak Mathematical Journal, 61 (136) (2011), 437–459

A PETTIS-TYPE INTEGRAL AND APPLICATIONS TO TRANSITION SEMIGROUPS

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(Received January 27, 2010)

Abstract. Motivated by applications to transition semigroups, we introduce the notion of a norming dual pair and study a Pettis-type integral on such pairs. In particular, we establish a sufficient condition for integrability. We also introduce and study a class of semigroups on such dual pairs which are an abstract version of transition semigroups. Using our results, we give conditions ensuring that a semigroup consisting of kernel operators has a Laplace transform which also consists of kernel operators. We also provide conditions under which a semigroup is uniquely determined by its Laplace transform.

Keywords: Pettis-type integral, dual pairs, Laplace transform, transition semigroup

MSC 2010: 46G10, 47D06, 60J35

1. INTRODUCTION

In a certain way, the Bochner integral is the appropriate generalization of the Lebesgue integral to the Banach space setting. The criterion for Bochner integrability is fairly easy: a strongly measurable function is Bochner integrable if and only if its norm is integrable.

However, not for all applications this notion of integrability is suitable. In this case, one can sometimes resort to the Pettis integral, see [23] or Section II.3 of [8], which still yields a rich theory. But even the notion of weak measurability, which is a prerequisite for Pettis integrability, is often too strong. Indeed, already in the simple example of the shift semigroup on the space of bounded Borel measures on the real line, the orbits of the semigroup are not weakly measurable, cf. [10].

The author was supported partially by the Deutsche Forschungsgesellschaft in the framework of the DFG research training group 1100 at the University of Ulm and partially by VICI subsidy 639.033.604 in the 'Vernieuwingsimpuls' programme of the Netherlands Organization for Scientific Research (NWO).

It is thus natural to replace in the definition of the Pettis integral the dual X^* of a Banach space X with some subset Y of X^*. This leads to the notion of Y-integrability, see [22]. However, except for the special cases of the Pettis integral and the weak*-integral, this notion of integrability has not been the subject of broad investigation.

In Section 4, we will study Y-integrability in the case where Y is a norm-closed subspace of X^* which is norming for X. In particular, we prove a sufficient condition for Y-integrability (Theorem 4.4). Our main assumption in that theorem is the existence of a quasi-complete, consistent topology τ on X.

It should be noted that the Y-integral actually coincides with the Pettis integral on the locally convex space (X, τ), for any locally convex topology τ on X such that $(X, \tau)' = Y$. However, in contrast to conditions for Pettis integrability on locally convex spaces, see [13], our condition is more in the spirit of the characterization of Bochner integrability and the proof makes extensive use of the norm topologies on X and Y.

At first sight, the notion of Y-integrability seems quite technical and arbitrary since there is no canonical choice for the space Y. This might be the reason why this notion of integrability has not been studied in more detail so far. However, in applications to transition semigroups, it is quite clear which space Y should be chosen. This example serves both as a motivation and as an application of our theory.

We recall that associated to a Markov process $(X_t)_{t \geqslant 0}$, taking values in a measurable space (E, Σ), there are, in fact, *two* semigroups which both have a stochastic interpretation and which are connected to each other by duality. Namely, there is a semigroup $\mathbf{T}$ on the space $B_b(E)$ of all bounded measurable functions on E, used to compute conditional expectations, and another semigroup $\mathbf{T}'$ acting on $\mathscr{M}(E)$, the space of all bounded measures on E, which gives the distributions of the random variables X_t. This duality relation actually characterizes the operators in $\mathbf{T}$ as kernel operators, cf. [21] and Section 3. Furthermore, this duality relation suggests to replace the space $B_b(E)^*$, which is usually quite large, with $\mathscr{M}(E)$ for the purpose of integrating the orbits of $\mathbf{T}$ and, similarly, to replace $\mathscr{M}(E)^*$ with $B_b(E)$ to integrate the orbits of $\mathbf{T}'$.

In applications, it is also important to replace $B_b(E)$ by some closed subspace X of $B_b(E)$ which is invariant under the semigroup. For example, if E is a topological space, one wants to replace $B_b(E)$ with $X = C_b(E)$. If E is additionally locally compact, one wants to work on the space $X = C_0(E)$. This is the classical example of a *Feller semigroup*. In order to treat also these situations, we shall work on a general norming dual pair (X, Y), see Definition 2.1, and introduce in Section 5 the abstract notion of 'semigroups on norming dual pairs'.

We call such a semigroup 'integrable' if it is possible to compute the Laplace transform in an appropriate way and obtain again operators which respect the duality. Jefferies [15], [16] studied weakly integrable semigroups on locally convex spaces and made similar assumptions on the semigroup. However, he does not assume that the Laplace transform respects the duality. In Theorem 5.8, we will show that this assumption—actually it suffices to consider the Laplace integral at a single point—is equivalent to the requirement that all orbits of $\mathbf{T}$ are locally Y-integrable and all orbits of $\mathbf{T}'$ are locally X-integrable. Using the results of Section 4, we study integrable transition semigroups on the space $C_b(E)$ in Section 6.

In order to treat transition semigroups on $C_b(E)$, several approaches have been proposed in literature. We mention the theory of weakly continuous semigroups of Cerrai [6], the theory of bi-continuous semigroups of Kühnemund [18], see also [9], [19] for applications in the context of transition semigroups, and the theory of π-semigroups by Priola [25]. It should be noted that in these approaches additional assumptions, in particular continuity and equicontinuity assumptions, are made which ensure that a Riemann integral can be used to compute the Laplace transform. Even though integrability is not an issue in these approaches, the question remains in which sense the Laplace transform determines the semigroup. More precisely: is it possible that there exists another semigroup (not necessarily satisfying the continuity and equicontinuity assumptions) which yields the same Laplace transform? This is not the case as our uniqueness theorem (Theorem 5.4) shows.

It is also possible to interpret the continuity and equicontinuity assumptions in the articles mentioned above from our 'dual point of view'. This yields interesting new results for such semigroups which are presented elsewhere [20].

Notation. If (E, Σ) is a measurable space, then $B_b(E)$ denotes the space of all bounded, measurable functions $f\colon E \to \mathbb{C}$, endowed with the supremum norm. By $\mathscr{M}(E)$ we denote the space of all complex measures on (E, Σ). The *total variation* of a measure μ is defined by

$$|\mu|(A) = \sup_{\mathscr{Z}} \sum_{B \in \mathscr{Z}} |\mu(B)|,$$

where the supremum is taken over all partitions $\mathscr{Z}$ of A into finitely many, disjoint, measurable sets. Endowed with the total variation norm $\|\mu\| := |\mu|(E)$, the space $\mathscr{M}(E)$ is a Banach space.

Now suppose that E is a topological space. Then $C_b(E)$ denotes the Banach space of all bounded, continuous functions $f\colon E \to \mathbb{C}$. The Borel σ-algebra of E is denoted by $\mathscr{B}(E)$. If we speak about measures or measurable functions on a topological space, this is always to be understood with respect to the Borel σ-algebra. A positive measure $\mu \in \mathscr{M}(E)$ is a *Radon measure* if $\mu(A) = \sup\{\mu(K)\colon\ K \subset A,\ K \text{ compact}\}$ for all $A \in \mathscr{B}(E)$. An arbitrary $\mu \in \mathscr{M}(E)$ is called a Radon measure if $|\mu|$ is a Radon measure. We denote the space of all Radon measures on E by $\mathscr{M}_0(E)$. This is a closed subspace of $\mathscr{M}(E)$.

By $\mathbb{1}_A$ we denote the characteristic function of a set A. For a complex number z, $\operatorname{sgn} z$ denotes the signum of z, i.e. $\operatorname{sgn} z := |z|^{-1}\overline{z}$ if $z \neq 0$ and $\operatorname{sgn} 0 := 0$. The Dirac measure at a point x is denoted by δ_x. If E is a metric space, then $B(x,r)$ denotes the open ball of radius r centered at x and $\overline{B}(x,r)$ denotes the closure of that ball. If X is a Banach space, then X^* denotes the norm dual of X and $\langle\cdot,\cdot\rangle_*$ denotes the canonical duality between X and X^*.

2. Norming dual pairs

Definition 2.1. Let X and Y be nontrivial Banach spaces and let $\langle\cdot,\cdot\rangle$ be a duality pairing between X and Y. Then $(X, Y, \langle\cdot,\cdot\rangle)$ is called a *norming dual pair*, if

$$\|x\|_X = \sup\{|\langle x, y\rangle|\colon\ y \in Y,\ \|y\|_Y \leqslant 1\}$$

and

$$\|y\|_Y = \sup\{|\langle x, y\rangle|\colon\ x \in X,\ \|x\|_X \leqslant 1\}.$$

We will write (X, Y) for a norming dual pair if the duality pairing is understood. Note that if (X, Y) is a norming dual pair, then so is (Y, X).

As we have done already in the introduction, we will frequently consider Y as a closed subspace of X^* via $\langle x, y\rangle_* = \langle x, y\rangle$. With this interpretation, (X, Y) is a norming dual pair if and only if Y is a closed subspace of X^* which is norming for X in the sense of [3]. For $Y \subset X^*$ to be norming for X it is necessary that Y is weak*-dense in X^*. However, not every weak*-dense, closed subspace of X^* is norming, see [7].

Example 2.2. If X is a Banach space, then (X, X^*), and thus by symmetry also (X^*, X), is a norming dual pair with the canonical duality $\langle\cdot,\cdot\rangle_*$. If X is reflexive, then $Y = X^*$ is the only closed subspace of X^* such that (X, Y) is a norming dual pair. Indeed, if $Y \subset X^*$ is norm-closed, it follows from the Hahn-Banach theorem that Y is weakly closed and hence, by reflexivity, weak*-closed. Since Y is weak*-dense, it follows that $Y = X^*$.

Example 2.3. Let (E, Σ) be a measurable space. Then $(B_b(E), \mathscr{M}(E))$ is a norming dual pair with respect to the duality $\langle\cdot,\cdot\rangle$, given by

$$\langle f, \mu\rangle := \int_E f \,\mathrm{d}\mu. \tag{2.1}$$

P r o o f. We clearly have $|\int f\,\mathrm{d}\mu| \leqslant \|f\|_\infty \cdot \|\mu\|$. Considering Dirac measures, we obtain $\|f\|_\infty = \sup\{|\langle f,\mu\rangle|\colon \mu \in \mathscr{M}(E), \|\mu\| \leqslant 1\}$. Now let $\mu \in \mathscr{M}(E)$. If $\mathscr{X}$ is a partition of E into finitely many, pairwise disjoint, measurable sets, then $f_{\mathscr{X}} := \sum_{A\in\mathscr{X}} \operatorname{sgn}\mu(A)\cdot \mathbb{1}_A$ is a measurable function of norm at most 1. Furthermore, $|\langle f_{\mathscr{X}},\mu\rangle| = \sum_{A\in\mathscr{X}} |\mu(A)|$. Taking the supremum over all such partitions $\mathscr{X}$, it follows that $(B_b(E), \mathscr{M}(E))$ is a norming dual pair. $\square$

Example 2.4. Let E be a completely regular Hausdorff space. Then, endowed with the duality (2.1), $(C_b(E), \mathscr{M}_0(E))$ is a norming dual pair.

For a complete, separable metric space E, the proof of this statement is implicitly contained in the proof of Theorem 2.2 of [25]. We give a proof in the general case.

P r o o f. It suffices to show that $\|\mu\| \geqslant \sup\{|\langle f,\mu\rangle|\colon f \in C_b(E),\ \|f\|_\infty \leqslant 1\}$. Let $\mu \in \mathscr{M}_0(E)$ be fixed and let $\mathscr{X} = \{A_1,\dots,A_n\}$ be a finite partition of E into measurable sets. Since μ is a Radon measure, given $\varepsilon > 0$, we find compact sets $C_k \subset A_k$ for $k = 1,\dots,n$ such that $|\mu(A_k) - \mu(C_k)| \leqslant |\mu|(A_k\setminus C_k) \leqslant \varepsilon/n$. As E is completely regular, there exists a continuous function $f\colon E \to \mathbb{C}$ such that $\|f\|_\infty \leqslant 1$ and $f|_{C_k} \equiv \operatorname{sgn}\mu(C_k)$. Now

$$\left|\int f\,\mathrm{d}\mu\right| \geqslant \sum_{k=1}^n |\mu(C_k)| - \sum_{k=1}^n |\mu|(A_k\setminus C_k) \geqslant \sum_{k=1}^n |\mu(A_k)| - 2\varepsilon$$

follows from the reverse triangle inequality. As ε is arbitrary, $\sum_{k=1}^n |\mu(A_k)| \leqslant \sup\{|\langle f,\mu\rangle|\colon f \in C_b(E),\ \|f\|_\infty \leqslant 1\}$. Taking the supremum over all such partitions $\mathscr{X}$ of E, the claim follows. $\square$

In what follows, we will be interested in locally convex topologies τ on X which are *consistent (with the duality).* By this we mean that $(X,\tau)' = Y$, i.e. every τ-continuous linear functional φ on X is of the form $\varphi(x) = \langle x,y\rangle$ for some $y \in Y$. By the Mackey-Arens theorem, see [17, 21.4 (2)], a consistent topology is finer than the *weak topology* $\sigma(X,Y)$ and coarser than the *Mackey topology* $\mu(X,Y)$. To simplify notation, we will write σ for $\sigma(X,Y)$ and σ' for the $\sigma(Y,X)$ topology on Y. We will write $\rightharpoonup$ or $\rightharpoonup'$ to indicate convergence with respect to σ or σ', respectively. We will use the name of a topology as a label or prefix to topological notions to indicate that it is to be understood with respect to that topology. Without label or prefix, such notions are always understood with respect to the norm topology.

We now characterize bounded subsets in a norming dual pair.

Proposition 2.5. *Let (X,Y) be a norming dual pair and τ a consistent topology on X. For a subset $M \subset X$, the following conditions are equivalent.*

(i) *M is norm-bounded;*
(ii) *M is σ-bounded;*
(iii) *M is τ-bounded.*

Proof. (i) $\Rightarrow$ (ii). As M is σ-bounded iff $\sup_{x\in M}|\langle x,y\rangle|<\infty$ for all $y\in Y$, this implication is trivial.

(ii) $\Rightarrow$ (i). If M is σ-bounded, the uniform boundedness principle in Y^* implies that $\sup_{x\in M}\|x\|=\sup_{x\in M}\|x\|_{Y^*}$ is bounded.

(ii) $\Leftrightarrow$ (iii). See § 20.11 (7) in [17]. □

3. Operators on norming dual pairs

If τ is a locally convex topology on X, we denote the algebra of τ-continuous linear operators on X by $L(X,\tau)$. For $\tau=\tau_{\|\cdot\|}$, where $\tau_{\|\cdot\|}$ is the norm topology, we merely write $L(X)$ instead of $L(X,\tau_{\|\cdot\|})$. For $T\in L(X)$, we denote its norm-adjoint by T^*. If $T\in L(X,\sigma)$, then we denote its σ-adjoint by T'.

Proposition 3.1. *Let (X,Y) be a norming dual pair.*
(i) *$T\in L(X,\sigma)$ if and only if $T\in L(X)$ and $T^*Y\subset Y$. In this case, $T'=T^*|_Y$. Furthermore, $\|T\|_{L(X)}=\|T'\|_{L(Y)}$.*
(ii) *$L(X,\sigma)$ is closed in $L(X)$ with respect to the operator norm.*

Proof. (i) If T is σ-continuous, then T maps σ-bounded sets into σ-bounded sets. By Proposition 2.5, T is a bounded operator on X, hence $T\in L(X)$. Furthermore, as T is σ-continuous, it has a σ-adjoint S. But for $x\in X$ and $y\in Y$ we have $\langle Tx,y\rangle=\langle x,Sy\rangle=\langle x,T^*y\rangle_*$. It follows that $T^*y=Sy\in Y$, and thus T^* leaves Y invariant. Conversely, assume that $T\in L(X)$ and $T^*Y\subset Y$. Then we have $\langle Tx,y\rangle=\langle x,T^*y\rangle$ for all $x\in X$ and $y\in Y$. Since $T^*y\in Y$ by assumption, it follows that the map $x\mapsto\langle Tx,y\rangle$ is σ-continuous and thus, since y is arbitrary, $T\in L(X,\sigma)$. Finally, we have

$$\|T'\|_{L(Y)}=\sup_y\sup_x|\langle x,T'y\rangle|=\sup_x\sup_y|\langle Tx,y\rangle|=\|T\|_{L(X)},$$

where all suprema are taken over the elements of norm at most 1.

(ii) Let $(T_n)\subset L(X,\sigma)$ be given with $T_n\to T\in L(X)$ in the operator norm. By (i), it suffices to prove $T^*Y\subset Y$. Let $y\in Y$ be given. By assumption, $T_n'y\in Y$. Furthermore,

$$\|T_n'y-T_m'y\|=\sup_{\|x\|\leqslant 1}|\langle T_nx-T_mx,y\rangle|\leqslant\|T_n-T_m\|\cdot\|y\|.$$

Thus $T_n'y$ is a Cauchy sequence in Y and hence converges to some $\tilde{y} \in Y$. Now for arbitrary $x \in X$ we have $\langle Tx, y\rangle = \lim\langle x, T_n'y\rangle = \langle x, \tilde{y}\rangle$, proving that $T^*y = \tilde{y} \in X$. This completes the proof. □

In the study of transition semigroups, one is mainly interested in positive contraction operators which are kernel operators, as they give the transition probabilities for a Markov process. Let us recall the following definition.

Definition 3.2. Let (E, Σ) be a measurable space. A *bounded kernel* on E is a mapping $k\colon E \times \Sigma \to \mathbb{C}$ such that

(i) $k(x, \cdot)$ is a complex measure on (E, Σ) for all $x \in E$;
(ii) $k(\cdot, A)$ is measurable for all $A \in \Sigma$;
(iii) $\sup_{x \in E} |k|(x, E) < \infty$. Here, $|k|(x, \cdot)$ is the total variation of $k(x, \cdot)$.

A linear operator T on a closed subspace X of $B_b(E)$ is called a *kernel operator (on X)* if there exists a bounded kernel k on E such that

$$(3.1) \qquad (Tf)(x) = \int_E f(y)k(x, \mathrm{d}y), \quad \forall f \in X.$$

We now prove that for many spaces $X \subset B_b(E)$ a kernel operator on X is the same as a σ-continuous operator for the norming dual pair $(X, \mathscr{M})$. We need some preparation.

If S is any set of functions, we denote by $\sigma(S)$ the σ-algebra generated by S, i.e. the smallest σ-algebra such that every $f \in S$ is measurable with respect to this σ-algebra. If S is a Stonean vector lattice, i.e. a vector lattice of functions such that if $f \in S$ is real then also $\inf\{f, \mathbb{1}\} \in S$, then the system

$$\mathscr{E}(S) := \{A\colon \exists\, u_n \in S \text{ such that } 0 \leqslant u_n \uparrow \mathbb{1}_A \text{ pointwise}\}$$

generates $\sigma(S)$ and is closed under finite intersections, see [4, Lemma 39.4].

Definition 3.3. Let (E, Σ) be a measurable space and $X \subset B_b(E)$ a $\|\cdot\|_\infty$-closed subspace of $B_b(E)$ which is a Stonean vector lattice. Further, let $\mathscr{M}_{(0)}(E)$ denote either $\mathscr{M}(E)$ or $\mathscr{M}_0(E)$. In the latter case we additionally assume that E is a completely regular Hausdorff space. Then X is called an $\mathscr{M}_{(0)}(E)$-*transition space* for E if

(i) $(X, \mathscr{M}_{(0)}(E))$ is a norming dual pair (with the duality (2.1));
(ii) $\sigma(X) = \Sigma$;
(iii) there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset X$ such that $0 \leqslant f_n \uparrow \mathbb{1}$ pointwise.

Example 3.4. For every measurable space (E,Σ), the space $B_b(E)$ is an $\mathscr{M}(E)$-transition space for E. If E is a metric space, then $C_b(E)$ is an $\mathscr{M}_0(E)$-transition space for E. Indeed, $\mathscr{E}(C_b(E))$ contains every open F_σ-set and hence, since E is a metric space, every open set. Thus $\sigma(C_b(E)) = \mathscr{B}(E)$.

The following is a generalization of Theorem 4.8.1 in [12].

Proposition 3.5. *Let (E,Σ) be a measurable space and let X be an $\mathscr{M}_{(0)}(E)$-transition space for E. Denote by σ the $\sigma(X,\mathscr{M}_{(0)}(E))$-topology. Consider the following statements:*

(i) *$T \in L(X,\sigma)$;*

(ii) *T is a kernel operator on X.*

Then (i) $\Rightarrow$ (ii). *In this case, T has a unique extension to a kernel operator on $B_b(E)$. If $\mathscr{M}_{(0)}(E) = \mathscr{M}(E)$, then also* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). If $T \in L(X,\sigma)$, then $k(x,\cdot) := T'\delta_x \in \mathscr{M}_{(0)}(E)$. By definition, we have $(Tf)(x) = \langle Tf, \delta_x\rangle = \langle f, T'\delta_x\rangle = \int f(y)k(x,\mathrm{d}y)$. Furthermore, $\sup_x |k|(x,E) \leqslant \|T\| < \infty$. It remains to prove that $k(\cdot,A)$ is measurable for any $A \in \Sigma$. Denote the collection of the sets A for which this is true by $\mathscr{G}$. Then $\mathscr{E}(X) \subset \mathscr{G}$. Indeed, if $A \in \mathscr{E}(X)$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ with $0 \leqslant u_n \uparrow \mathbb{1}_A$. Now the dominated convergence theorem yields

$$k(x,A) = \langle \mathbb{1}_A, T'\delta_x\rangle = \lim_{n\to\infty} \langle u_n, T'\delta_x\rangle = \lim_{n\to\infty} (Tu_n)(x)$$

for all $x \in E$. Hence $k(\cdot,A)$ is measurable as the pointwise limit of measurable functions. Using the properties of a bounded kernel, it is easy to see that $\mathscr{G}$ is a Dynkin system. Now $\mathscr{G} = \Sigma$ follows from the Dynkin π-λ theorem since $\mathscr{E}(X)$ is closed under finite intersections.

(ii) $\Rightarrow$ (i). By hypothesis, there exists a kernel k such that (3.1) holds for all $f \in X$. However, the right hand side of (3.1) also defines a bounded linear operator on $B_b(E)$ (which we still denote by T). We may also define an operator S on $\mathscr{M}(E)$ by

$$(S\mu)(A) := \int_E k(x,A)\,\mathrm{d}\mu(x).$$

It is easy to see that $S \in L(\mathscr{M}(E))$. However, for $f = \mathbb{1}_A$ we have

$$\langle Tf, \mu\rangle = \int_E k(x,A)\,\mathrm{d}\mu = \langle f, S\mu\rangle \quad \forall\, \mu \in \mathscr{M}(E).$$

Using linearity and approximation, we see that the above equation holds for arbitrary $f \in B_b(E)$. This proves $T^*\mathscr{M} \subset \mathscr{M}$ and hence (i) by Proposition 3.1. □

4. A variant of the Pettis integral

Throughout this section we fix a norming dual pair (X, Y) and a σ-finite measure space $(\Omega, \mathscr{F}, m)$.

Definition 4.1. A function $f\colon \Omega \to X$ is called *scalarly Y-measurable* (*scalarly Y-integrable*), if the function $\omega \mapsto \langle f(\omega), y\rangle$ is measurable (integrable) for every $y \in Y$.

As in the proof of Lemma 1 in Section II.3 of [8], one sees that if f is scalarly Y-integrable, then for any $A \in \mathscr{F}$ the linear functional $\varphi_A := \big[y \mapsto \int_A \langle f(\omega), y\rangle \,\mathrm{d}m\big]$ is norm continuous and hence an element of Y^*.

Definition 4.2. If f is scalarly Y-integrable, then the element φ_A of Y^* is called *the Y-integral of f over A*. We write $\int_A f \,\mathrm{d}m := \varphi_A$. If $\varphi_A \in X \subset Y^*$ for every $A \in \mathscr{F}$, we say that f is *Y-integrable*.

If f is Y-integrable, then, by definition, we may interchange integration and application of linear functionals in Y. The following lemma shows that the same is true for linear operators in $L(X, \sigma)$. We omit its easy proof.

Lemma 4.3. *Let $f\colon \Omega \to X$ be scalarly Y-integrable such that $\int_\Omega f \,\mathrm{d}m \in X$. Then for $T \in L(X, \sigma)$, the function Tf is scalarly Y-integrable and we have $\int_\Omega Tf \,\mathrm{d}m = T\int_\Omega f \,\mathrm{d}m \in X$.*

Our main result about Y-integrability is the following.

Theorem 4.4. *Assume that there exists a consistent topology τ on X such that (X, τ) is quasi-complete, i.e., τ is complete on every bounded closed subset of (X, τ). Then every almost τ-separably valued, scalarly Y-integrable function $f\colon \Omega \to X$ such that $\|f\|$ is majorized by an integrable function, is Y-integrable. Here f is called almost τ-separably valued if there exists a null set N and a τ-separable subspace X_0 of X such that $f(\Omega \setminus N) \subset X_0$.*

Remark 4.5. As a consequence of [17, §18.4 (4)], there exists a quasi-complete consistent topology τ on X if and only if $\mu(X, Y)$ is quasi-complete.

We first prove some preliminary lemmas which will also be used independently of the theorem.

Lemma 4.6. *Assume that $f\colon \Omega \to X$ is scalarly Y-measurable and that $\|f\|$ is majorized by an integrable function g. Then f is scalarly Y-integrable and the Y-integral of f over any $A \in \mathscr{F}$ is sequentially σ'-continuous and satisfies the estimate*

$$\left\|\int_A f \,\mathrm{d}m\right\|_{Y^*} \leqslant \int_A g(\omega)\,\mathrm{d}m(\omega). \tag{4.1}$$

Proof. As f is scalarly Y-measurable and satisfies the estimate $|\langle f(\cdot), y\rangle| \leqslant g(\cdot)\|y\|$, it follows that f is scalarly Y-integrable. Integrating this inequality and taking the supremum over y with $\|y\| \leqslant 1$, the estimate (4.1) follows. Now, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y and assume $y_n \rightharpoonup' y \in Y$. Then $\langle f, y_n\rangle \to \langle f, y\rangle$ pointwise on Ω. By Proposition 2.5, $\|y_n\|$ is bounded, say by M. Hence $|\langle f, y_n\rangle| \leqslant M\cdot g$. Thus φ_A is sequentially σ'-continuous by the dominated convergence theorem. □

Lemma 4.7. *Let $f\colon \Omega \to X$ be a scalarly Y-measurable function such that $\|f\| \leqslant g$ a.e. for some integrable function g. Furthermore, let $(\alpha_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(m)$ converging pointwise a.e. to $\alpha \in L^\infty(m)$. Then $\int_\Omega \alpha_n f \,\mathrm{d}m$ converges to $\int_\Omega \alpha f \,\mathrm{d}m$ with respect to the norm in Y^*. In particular, $A \mapsto \int_A f \,\mathrm{d}m$ defines a countably additive vector measure with values in Y^*.*

Proof. By (4.1) we have

$$\left\| \int_\Omega \alpha_n f \,\mathrm{d}m - \int_\Omega \alpha f \,\mathrm{d}m \right\|_{Y^*} \leqslant \int_\Omega |\alpha_n - \alpha| g \,\mathrm{d}m \to 0,$$

by the dominated convergence. The addendum follows by applying this to $\alpha_n := \mathbb{1}_{\bigcup_1^n A_k}$ and $\alpha := \mathbb{1}_{\bigcup_1^\infty A_k}$ for some sequence $(A_k)_{k\in\mathbb{N}} \subset \mathscr{F}$ of pairwise disjoint sets.
□

Proof of Theorem 4.4. We first make some simplifying assumptions.

We assume without loss of generality that the set N is empty, otherwise changing f on a set of measure 0. We may furthermore assume that (X, τ) is separable. If this is not the case, we replace X by $X_1 := \overline{X_0}^{\tau}$ and Y by $Y/X_1^\perp$. Since the norm topology is finer than τ, the space X_1 is norm closed in X and hence a Banach space. Furthermore, $(X_1, \tau|_{X_1})$ is a quasi-complete locally convex space and, as a consequence of the Hahn-Banach theorem, we have $(X_1, \tau|_{X_1})' = Y/X_1^\perp$.

Last, we may assume that $\|f\|$ is bounded. Indeed, assuming that $\|f\| \leqslant g \in L^1(m)$, we may consider $f_n := \mathbb{1}_{A_n} f$, where $A_n := \{g \leqslant n\} \in \mathscr{F}$. If we know that the Y-integral of f_n over some set A belongs to X for every $n \in \mathbb{N}$, then so does the Y-integral of f over the set A by Lemma 4.7 and the closedness of X in Y^*.

Denote the completion of (X, τ) by $(\tilde{X}, \tilde{\tau})$. Then $(\tilde{X}, \tilde{\tau})$ is locally convex and separable. Furthermore, by [17, §21.4 (5)], $(\tilde{X}, \tilde{\tau})' = Y$.

Now let $A \in \mathscr{F}$ with (strictly) positive finite measure be given. By Lemma 4.6, the Y-integral $\varphi_A \in Y^*$ of f over A is sequentially σ'-continuous and hence in particular sequentially $\sigma(Y, \tilde{X})$-continuous. Since $(\tilde{X}, \tilde{\tau})$ is complete and separable, φ_A is $\sigma(Y, \tilde{X})$-continuous by [17, §21.9 (5)] and thus $\varphi_A \in (Y, \sigma(Y, \tilde{X}))' = \tilde{X}$.

Now consider $B_0 = \operatorname{co}\{f(\omega)\colon \omega \in \Omega\}$. Then B_0 is convex and bounded and hence so is its $\tilde{\tau}$-closure B. Since (X, τ) is quasi-complete, $B \subset X$.

We claim that $m(A)^{-1}\varphi_A \in B$. Indeed, if this was not the case, then, by the Hahn-Banach theorem, there would exist $\varepsilon > 0$ and $y \in Y = (\tilde{X}, \tilde{\tau})'$ such that $\operatorname{Re}\langle f(\omega), y\rangle + \varepsilon \leqslant m(A)^{-1} \operatorname{Re}\langle \varphi_A, y\rangle$ for every $\omega \in \Omega$. Integrating this equation yields $\operatorname{Re}\langle \varphi_A, y\rangle + \varepsilon m(A) \leqslant \operatorname{Re}\langle \varphi_A, y\rangle$ – a contradiction since $m(A) > 0$. It follows that $\varphi_A \in X$.

For a general set A of positive measure, approximate A by a sequence $(A_n)_{n\in\mathbb{N}}$ with $0 < m(A_n) < \infty$ and use Lemma 4.7. □

Let us briefly discuss the assumptions of Theorem 4.4.

The case $Y = X^*$. A function $f\colon \Omega \to X$ is X^*-integrable iff it is Pettis integrable in the classical sense. Note that the norm topology on X is a complete, consistent topology. Furthermore, the assumption that f is scalarly X^*-measurable and almost $\|\cdot\|$-separably valued implies that f is strongly measurable by the Pettis measurability theorem [8, II.1, Theorem 2]. Thus in this case, if f satisfies the hypothesis of Theorem 4.4, then f is Bochner integrable.

In the Pettis measurability theorem the assumption of scalar X^*-measurability can actually be weakened to scalar Y-measurability for any norming subset $Y \subset X^*$, see Corollary 4 in II.1 of [8]. We note that since we only require the range of f to be almost τ-separable in Theorem 4.4, in the case of an arbitrary Y we do not implicitly require that f is strongly measurable.

The case $X = Y^*$. In this case, the Y-integral coincides with the weak*-integral. Hence every scalarly Y-integrable function $f\colon \Omega \to Y^*$ is Y-integrable. We note that since closed, bounded balls in Y^* are weak*-compact, the weak*-topology is quasi-complete. We also note that in this case the separability assumption in Theorem 4.4 is not needed.

The above examples have been extensively studied in literature. The following is our basic example of a norming dual pair on which a complete, consistent topology exists.

Example 4.8. Let E be a completely regular Hausdorff space and consider the norming dual pair $(C_b(E), \mathscr{M}_0(E))$. Then, by Section 7.6 of [14], the *strict topology* is a consistent topology on X. It is complete if and only if $C(E)$, the space of *a*ll continuous functions on E, is complete with respect to the compact-open topology, see Section 3.6 of [14]. If E is metrizable or locally compact, this is certainly the case.

The question arises whether on *every* norming dual pair there exists a quasi-complete, consistent topology. This question was answered to the negative by Bonet and Cascales [5]. In Section 6 we will give a concrete example that the assertion of Theorem 4.4 may fail without the assumption that there exists a quasi-complete consistent topology.

The following result is useful in establishing Y-integrability.

Proposition 4.9. *Let $\mathscr{E}$ be a generator of $\mathscr{F}$ which is closed under finite intersections and let $f\colon \Omega \to X$ be a scalarly Y-measurable function with the following properties:*

(i) *there exists a measurable function g such that $\|f\| \leqslant g$;*

(ii) *there exists a sequence $(\Omega_n)_{n\in\mathbb{N}} \subset \mathscr{F}$ with $m(\Omega_n) < \infty$ for all $n \in \mathbb{N}$ and $\bigcup\limits_{n\in\mathbb{N}} \Omega_n = \Omega$ such that the function g from* (i) *satisfies $g\mathbb{1}_{\Omega_n} \in L^1(m)$ for all $n \in \mathbb{N}$;*

(iii) *$x_A := \int_A f\,\mathrm{d}m \in X$ for every $A \in \mathscr{E} \cup \{\Omega_n\colon n \in \mathbb{N}\}$.*

Then for every measurable function $\alpha\colon \Omega \to \mathbb{C}$ with $|\alpha|g \in L^1(m)$, the function αf is Y-integrable.

P r o o f. By Lemma 4.6, αf is scalarly Y-integrable on Ω. It suffices to prove that its Y-integral over Ω belongs to X, as we can clearly replace α by $\alpha \cdot \mathbb{1}_A$ for any $A \in \mathscr{F}$. We proceed in three steps.

Step 1. Let $n \in \mathbb{N}$ be arbitrary and let $\mathscr{D}_n$ denote the collection of all sets $A \in \mathscr{F}$ such that $\int_{A\cap\Omega_n} f\,\mathrm{d}m \in X$. By assumption (iii), $\mathscr{E} \subset \mathscr{D}_n$. Using Lemma 4.7, it is easy to see that $\mathscr{D}_n$ is a Dynkin system. Hence $\mathscr{D}_n = \mathscr{F}$ by Dynkin's π-λ theorem.

Step 2. Now we prove the assertion for a simple function α. By Step 1 and linearity, the Y-integral of αf over Ω_n is an element of X. By Lemma 4.7, $\int_{\Omega_n} \alpha f\,\mathrm{d}m \to \int_\Omega \alpha f\,\mathrm{d}m$ in Y^*, hence $\int_\Omega \alpha f\,\mathrm{d}m \in X$.

Step 3. Now let α be arbitrary. Then there exists a sequence of step functions $(\alpha_k)_{k\in\mathbb{N}}$ such that $|\alpha_k| \leqslant |\alpha|$ and $\alpha_k \to \alpha$ pointwise. By Step 2, $\int_\Omega \alpha_k f\,\mathrm{d}m \in X$ for every k. Again by Lemma 4.7 it follows that $\int_\Omega \alpha f\,\mathrm{d}m \in X$. □

5. Semigroups and their Laplace transforms

Definition 5.1. Let (X, Y) be a norming dual pair. A *semigroup* on (X, Y) is a family of operators $\mathbf{T} = (T(t))_{t\geqslant 0} \subset L(X, \sigma)$ such that $T(t + s) = T(t)T(s)$ for all $t, s \geqslant 0$ and $T(0) = \mathrm{id}_X$. A semigroup is called *exponentially bounded* if there exist $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$. In this case we say that $\mathbf{T}$ is of *type* (M, ω). A semigroup of some type (M, ω) is called *integrable* if $t \mapsto \langle T(t)x, y\rangle$ is measurable for all $x \in X$ and $y \in Y$ and there exist a complex number λ_0 with $\operatorname{Re}\lambda_0 > \omega$ and an operator $R_0 \in L(X, \sigma)$ such that

$$\langle R_0 x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda_0 t}\langle T(t)x, y\rangle\,\mathrm{d}t, \quad \forall\, x \in X,\ y \in Y. \tag{5.1}$$

Two remarks are in order. Let us first note that for a fixed λ_0 there is at most one operator $R_0 \in L(X,\sigma)$ such that (5.1) is satisfied. Secondly, note that the definition of an 'integrable semigroup' is symmetric, i.e., if $\mathbf{T}$ is an integrable semigroup on (X,Y), then the σ-adjoint semigroup $\mathbf{T}'$ is an integrable semigroup on (Y,X). To see this, note that if $R_0 \in L(X,\sigma)$, then $R_0' \in L(Y,\sigma')$. Furthermore, we have $\langle x, R_0' y\rangle = \int_0^\infty \mathrm{e}^{-\lambda_0 t}\langle x, T(t)' y\rangle \,\mathrm{d}t$ for all $x \in X$ and $y \in Y$.

We will see in a moment that if $\mathbf{T}$ is an integrable semigroup, then for *every* λ with $\operatorname{Re}\lambda > \omega$ there exists an operator $R(\lambda) \in L(X,\sigma)$ such that

$$\langle R(\lambda)x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda t}\langle T(t)x, y\rangle \,\mathrm{d}t, \quad \forall\, x \in X,\ y \in Y. \tag{5.2}$$

Clearly, $R(\lambda_0) = R_0$. The family $\mathbf{R} := (\mathbf{R}(\lambda))_{\operatorname{Re}\lambda > \omega}$ is called the *Laplace transform* of $\mathbf{T}$.

It is well known that the Laplace transform of a strongly continuous semigroup is the resolvent of its generator. Since we did not impose continuity assumptions, we cannot expect the Laplace transform to be injective. In particular, it need not be the resolvent of an operator. However, the following proposition shows that, similarly to [1], the Laplace transform of an integrable semigroup is a pseudo-resolvent.

We will use freely some results about pseudo-resolvents and multivalued (m.v. for short) operators. We refer the reader to [1] or Appendix A of [11] for more information.

Proposition 5.2. *Let $\mathbf{T}$ be an integrable semigroup of type (M,ω). Then there exists a pseudo-resolvent $(R(\lambda))_{\operatorname{Re}\lambda>\omega} \subset L(X,\sigma)$ such that (5.2) holds for every $\operatorname{Re}\lambda > \omega$. Furthermore, every $R(\lambda)$ commutes with every $T(t)$ and for $\operatorname{Re}\lambda > \omega$ and $k \in \mathbb{N}$ we have $\|(\operatorname{Re}\lambda - \omega)^k R(\lambda)^k\| \leqslant M$.*

Proof. By the definition of an 'integrable semigroup' there exist $\lambda_0 \in \{\operatorname{Re}\lambda > \omega\}$ and $R_0 \in L(X,\sigma)$ such that (5.1) holds. Define the m.v. operator $\mathscr{A}$ by $\mathscr{A} := \lambda_0 - R_0^{-1}$ and put $R(\lambda) := (\lambda - \mathscr{A})^{-1}$. Now define

$$\Omega_0 := \{\lambda\colon\ \operatorname{Re}\lambda > \omega,\ R(\lambda) \in L(X,\sigma) \text{ and (5.2) holds}\}.$$

Then we have $\lambda_0 \in \Omega_0 \subset \Omega := \{\lambda\colon\ \operatorname{Re}\lambda > \omega,\ R(\lambda) \in L(X)\}$. By [11, Proposition A.2.3], the $L(X)$-valued map $R\colon\ \Omega \to L(X)$ defines a pseudo-resolvent; in particular, Ω is open and R is holomorphic. More precisely, if $\lambda \in \Omega$ and $|\lambda - \mu| < \|R(\lambda)\|^{-1}$, then $\mu \in \Omega$ and

$$R(\mu) = \sum_{k=0}^{\infty}(\lambda - \mu)^k R(\lambda)^{k+1}. \tag{5.3}$$

Now fix $\lambda \in \Omega_0$ and $\mu \in B(\lambda, \|R(\lambda)\|^{-1})$. Equation (5.3) and Proposition 3.1 (ii) imply that $R(\mu) \in L(X, \sigma)$. Now note that for any $\nu \in \Omega_0$, $x \in X$ and $y \in Y$ we have

$$(5.4) \qquad \langle R(\nu)^k x, y\rangle = \int_{(0,\infty)^k} \mathrm{e}^{-\nu(t_1+\ldots+t_k)}\langle T(t_1+\ldots+t_k)x, y\rangle \,\mathrm{d}t_1 \ldots \mathrm{d}t_k = \int_0^\infty \frac{t^{k-1}}{(k-1)!}\mathrm{e}^{-\nu t}\langle T(t)x, y\rangle \,\mathrm{d}t.$$

Here the first equality follows from the semigroup law and the second equality is derived from the fact that the k-fold convolution of exponential distributions is a gamma distribution. Thus since $|\lambda - \mu| < \|R(\lambda)\|^{-1}$, we have

$$\langle R(\mu)x, y\rangle = \sum_{k=0}^\infty \langle (\lambda-\mu)^k R(\lambda)^{k+1}x, y\rangle = \int_0^\infty \sum_{k=0}^\infty \frac{((\lambda-\mu)t)^k}{k!}\mathrm{e}^{-\lambda t}\langle T(t)x, y\rangle \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\mu t}\langle T(t)x, y\rangle \,\mathrm{d}t$$

for all $x \in X$, $y \in Y$. Hence $\mu \in \Omega_0$. Since $\lambda \in \Omega_0$ is arbitrary, it follows that Ω_0 is an open subset of Ω.

Now assume that $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence in Ω_0 converging to some $\lambda \in \Omega$. Then $R(\lambda_n) \to R(\lambda)$ in the operator norm and hence $R(\lambda) \in L(X, \sigma)$ by Proposition 3.1 (ii). Fix $\gamma > \omega$ such that $\operatorname{Re}\lambda_n > \gamma$ for all $n \in \mathbb{N}$. Using the estimate $|\mathrm{e}^{-\lambda_n t}\langle T(t)x, y\rangle| \leqslant M\mathrm{e}^{(\omega-\gamma)t}\|x\| \cdot \|y\| \in L^1(0,\infty)$ for all $x \in X$ and $y \in Y$, we may infer from dominated convergence that $\langle R(\lambda)x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda t}\langle T(t)x, y\rangle \,\mathrm{d}t$ for all $x \in X$, $y \in Y$. This proves that Ω_0 is closed in Ω. It follows that Ω_0 contains the connected component of λ_0 in Ω.

Let us prove now that $\Omega_0 = \{\lambda\colon \operatorname{Re}\lambda > \omega\}$. To that end, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in the connected component of λ_0 in Ω converging to some λ in the boundary of that component. By [1, Proposition 3.5], $\|R(\lambda_n)\|$ must be unbounded. If this is the case, we infer from the uniform boundedness principle that we can find some $x \in X$ and some $y \in Y$ such that $\langle R(\lambda_n)x, y\rangle$ is unbounded. But this is impossible unless $\operatorname{Re}\lambda = \omega$. Indeed, if $\operatorname{Re}\lambda > \omega$, then, similarly to the above, we find

$$\limsup_{n\to\infty}|\langle R(\lambda_n)x, y\rangle| \leqslant \limsup_{n\to\infty} M\|x\| \cdot \|y\| \int_0^\infty \mathrm{e}^{(\omega - \operatorname{Re}\lambda_n)t}\,\mathrm{d}t < \infty.$$

Hence we must have $\operatorname{Re}\lambda = \omega$ and, thus, $\Omega_0 = \{\lambda\colon \operatorname{Re}\lambda > \omega\}$.

The fact that every $R(\lambda)$ commutes with every $T(t)$ is an easy consequence of Lemma 4.3 and the semigroup law. The estimate $\|(\operatorname{Re}\lambda - \omega)^k R(\lambda)^k\| \leqslant M$ may be deduced from (5.4) and the exponential boundedness of $\mathbf{T}$. □

The question arises whether an integrable semigroup is uniquely determined by its Laplace transform. Without further assumptions, this is not the case, not even if $Y = X^*$, see [24]. We need

Definition 5.3. Let X be a Banach space and M a subspace of X. A subset $W \subset X^*$ is said to *separate points in* M if for every $x \in M \setminus \{0\}$ there exists $w \in W$ with $\langle x, w\rangle \neq 0$. A norming dual pair (X, Y) is said to be *countably separated* if there exists a countable subset of X separating points in Y and there exists a countable subset of Y separating points in X.

Theorem 5.4. *Let* $\mathbf{T}, \mathbf{S}$ *be integrable semigroups on* (X, Y) *of type* (M_T, ω_T) *and* (M_S, ω_S) *respectively. Suppose that for the corresponding Laplace transforms we have* $R_{\mathbf{T}}(\lambda) = (\lambda - \mathscr{A})^{-1} = R_{\mathbf{S}}(\lambda)$ *for some* $\lambda > \max\{\omega_{\mathbf{T}}, \omega_{\mathbf{S}}\}$. *Then* $T(t) = S(t)$ *for all* $t \geqslant 0$, *provided one of the following conditions is satisfied:*

(i) $D(\mathscr{A})$ *is* σ*-dense in* X;
(ii) (X, Y) *is countably separated.*

The proof uses the following lemma which is taken from [2, Lemma 3.16.5].

Lemma 5.5. *Let* $M \subset (0, \infty)$ *be a set of Lebesgue measure* 0 *and assume that* $t, s \notin M$ *implies* $t + s \notin M$. *Then* $M = \emptyset$.

P r o o f of Theorem 5.4. As a consequence of Proposition 5.2, we have $(\lambda - \mathscr{A})^{-1} = R_{\mathbf{T}}(\lambda) = R_{\mathbf{S}}(\lambda) \in L(X, \sigma)$ for all $\lambda > \max\{\omega_T, \omega_S\}$. Hence, for such λ and any $x \in X$ and $y \in Y$ we have

$$\int_0^\infty \mathrm{e}^{-\lambda t} \langle T(t)x, y\rangle \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} \langle S(t)x, y\rangle \,\mathrm{d}t.$$

By the uniqueness theorem for Laplace transforms [2, Theorem 1.7.3], there exists a set $N(x, y)$ of Lebesgue measure zero such that $\langle T(t)x, y\rangle = \langle S(t)x, y\rangle$ for all $t \notin N(x, y)$.

First assume (i). Note that for every $\operatorname{Re}\lambda > \omega$, $u \in X$ and $y \in Y$ we have

$$\begin{aligned}\langle T(t)R_{\mathbf{T}}(\lambda)u, y\rangle &= \int_0^\infty \mathrm{e}^{-\lambda s} \langle T(t+s)u, y\rangle \,\mathrm{d}s = \mathrm{e}^{\lambda t} \int_t^\infty \mathrm{e}^{-\lambda r} \langle T(r)u, y\rangle \,\mathrm{d}r \\ &= \mathrm{e}^{\lambda t} \left(\langle R_{\mathbf{T}}(\lambda)u, y\rangle - \int_0^t \mathrm{e}^{-\lambda r} \langle T(r)u, y\rangle \,\mathrm{d}r \right),\end{aligned}$$

and thus

$$\int_0^t \mathrm{e}^{-\lambda r} \langle T(r)u, y\rangle \,\mathrm{d}r = \langle R_{\mathbf{T}}(\lambda)u, y\rangle - \mathrm{e}^{-\lambda t} \langle T(t)R_{\mathbf{T}}(\lambda)u, y\rangle. \tag{5.5}$$

Now let $x \in D(\mathscr{A}) = \operatorname{rg} R_{\mathbf{T}}(\lambda)$, say $x = R_{\mathbf{T}}(\lambda)z$. Then the above equation for $u = z$ and arbitrary $y \in Y$ yields

$$\langle T(t)x, y\rangle = \mathrm{e}^{\lambda t}\left(\langle x, y\rangle - \int_0^t \mathrm{e}^{-\lambda r}\langle T(r)z, y\rangle\,\mathrm{d}r\right),$$

implying that $t \mapsto \langle T(t)x, y\rangle$ is continuous. The same applies to the corresponding orbit of $\mathbf{S}$ and we find $N(x,y) = \emptyset$. Thus $T(t)x = S(t)x$ for every $t \geqslant 0$ and $x \in D(\mathscr{A})$. However, if the σ-continuous linear operators $T(t)$ and $S(t)$ coincide on the σ-dense subspace $D(\mathscr{A})$, then they are equal.

Now assume that (ii) is satisfied. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$ be countable subsets separating points in Y and X, respectively. Fix $x \in X$ and put $N(x) = \bigcup_{n\in\mathbb{N}} N(x, y_n)$. Then $N(x)$ is a null set and

$$\langle T(t)x, y_n\rangle = \langle S(t)x, y_n\rangle \quad \forall\, t \notin N(x),\ n \in \mathbb{N}.$$

Since $\{y_n\}$ separates points, $T(t)x = S(t)x$ for all $t \notin N(x)$. In particular, $\langle T(t)x, y\rangle = \langle S(t)x, y\rangle$ for all $t \notin N(x)$ and all $y \in Y$.

Now fix $y \in Y$ and put $N = \bigcup_{n\in\mathbb{N}} N(x_n)$. Then N has measure 0 and for $t \notin N$ and $n \in \mathbb{N}$ we have

$$\langle x_n, T(t)'y\rangle = \langle T(t)x_n, y\rangle = \langle S(t)x_n, y\rangle = \langle x_n, S(t)'y\rangle.$$

As $\{x_n\}$ separates points, it follows that $T(t)'y = S(t)'y$ for all $t \notin N$. Since y is arbitrary, $T(t) = S(t)$ for all $t \notin N$. Now let $M = \{t\colon T(t) \neq S(t)\}$. Then $M \subset N$, showing that M has measure 0. However, if $t, s \notin M$ then, by the semigroup law, $t + s \notin M$. Thus Lemma 5.5 implies $M = \emptyset$. □

Remark 5.6. It is proved in [20, Theorem 2.10] that if $\mathbf{T}$ is *σ-continuous at* 0, i.e. $T(t)x \rightharpoonup x$ as $t \downarrow 0$ for every $x \in X$, then $\operatorname{rg} R(\lambda) = D(\mathscr{A})$ is σ-dense in X. Hence very mild continuity assumptions ensure that condition (i) in Theorem 5.4 is satisfied.

We now generalize a result from the theory of strongly continuous semigroups, cf. [2, Proposition 3.1.9]. Note that in our situation the operator $\mathscr{A}$ may be multi-valued.

Proposition 5.7. *Let $\mathbf{T}$ be an integrable semigroup on (X, Y) with Laplace transform $\mathbf{R}$ and let $\mathscr{A}$ be the unique m.v. operator such that $R(\lambda) = (\lambda - \mathscr{A})^{-1}$.*

(i) *The following conditions are equivalent.*
 (a) *$x \in D(\mathscr{A})$ and $z \in \mathscr{A}x$;*
 (b) *for every $t > 0$ we have $\int_0^t T(s)z\,\mathrm{d}s = T(t)x - x$.*

(ii) *We have $\int_0^t T(s)x\,\mathrm{d}s \in D(\mathscr{A})$ and $T(t)x - x \in \mathscr{A}\int_0^t T(s)x\,\mathrm{d}s$ for every $x \in X$ and $t > 0$.*

Proof. We first note that (i) (a) is equivalent to $x = R(\lambda)(\lambda x - z)$.

(i) (a) $\Rightarrow$ (b): Fix $t > 0$ and $y \in Y$ and define analytic functions $f, g\colon \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) := \lambda \int_0^t \mathrm{e}^{-\lambda s} \langle T(s)x, y\rangle \,\mathrm{d}s - \int_0^t \mathrm{e}^{-\lambda s} \langle T(s)z, y\rangle \,\mathrm{d}s,$$
$$g(\lambda) := \langle x, y\rangle - \mathrm{e}^{-\lambda t}\langle T(t)x, y\rangle.$$

Setting $u = x = R(\lambda)(\lambda x - z)$ in (5.5), it follows that $f(\lambda) = g(\lambda)$ for all $\operatorname{Re}\lambda > \omega$. The uniqueness theorem for analytic functions yields $f(0) = g(0)$. As t and y are arbitrary, (b) is proved.

(b) $\Rightarrow$ (a): If $\int_0^t T(s)z \,\mathrm{d}s = T(t)x - x$, then

$$\begin{aligned}\lambda R(\lambda)x - x &= \int_0^\infty \lambda \mathrm{e}^{-\lambda t}(T(t)x - x)\,\mathrm{d}t = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} \int_0^t T(s)z \,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\infty \int_s^\infty \lambda \mathrm{e}^{-\lambda t} T(s)z \,\mathrm{d}t\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)z\,\mathrm{d}s = R(\lambda)z.\end{aligned}$$

It follows that $x = R(\lambda)(\lambda x - z)$.

(ii) Considering integrals as elements of Y^* at first, we have

$$\begin{aligned}\int_0^t T(s)x\,\mathrm{d}s &= \int_0^t T(s)(\lambda - \mathscr{A})R(\lambda)x\,\mathrm{d}s \\ &= \lambda \int_0^t T(s)R(\lambda)x\,\mathrm{d}s - \int_0^t T(s)\mathscr{A}R(\lambda)x\,\mathrm{d}s \\ &= \lambda \int_0^t T(s)R(\lambda)x\,\mathrm{d}s + R(\lambda)x - T(t)R(\lambda)x,\end{aligned}$$

where we have used $R(\lambda)x \in D(\mathscr{A})$ and part (i) in the previous step. Furthermore, in slight abuse of notation, we wrote $\mathscr{A}R(\lambda)x$ in place of an element in this set. Now note that $\int_0^t T(s)R(\lambda)x\,\mathrm{d}s \in X$ by part (i), hence also $\int_0^t T(s)x\,\mathrm{d}s \in X$ by the above equation. Now Lemma 4.3 yields

$$\int_0^t T(s)x\,\mathrm{d}s = R(\lambda)\left(\lambda \int_0^t T(s)x\,\mathrm{d}s + x - T(t)x\right),$$

which is equivalent to (ii). □

Theorem 5.8. *Let* $\mathbf{T}$ *be a semigroup of type* (M, ω) *on the norming dual pair* (X, Y). *The following conditions are equivalent:*

(i) $\mathbf{T}$ *is an integrable semigroup;*

(ii) *for every* $x \in X$ *the orbit* $T(\cdot)x$ *is locally* Y*-integrable and for every* $y \in Y$ *the orbit* $T(\cdot)'y$ *is locally* X*-integrable. Here 'local* X/Y*-integrability' means* X/Y*-integrability on every bounded interval in* $\mathbb{R}^+$.

P r o o f. (i) $\Rightarrow$ (ii). As a consequence of Proposition 5.7 (ii), $\int_a^b T(t)x\,\mathrm{d}t \in X$ for every $x \in X$ and $0 \leqslant a < b < \infty$. As such intervals generate the Borel σ-algebra on $(0,\infty)$ and are closed under finite intersections, it follows from Proposition 4.9 that $T(\cdot)x$ is locally Y-integrable. Applying the same argument to $T(\cdot)'y$ for every $y \in Y$, (ii) follows.

(ii) $\Rightarrow$ (i). Fix λ with $\operatorname{Re}\lambda > \omega$. It follows from (ii) and Proposition 4.9 that there exists an element $R(\lambda)x \in X$ such that $R(\lambda)x = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)x\,\mathrm{d}t$. It remains to prove that $R(\lambda) \in L(X,\sigma)$. It is easy to see that $R(\lambda)$ is linear. Furthermore, using the exponential boundedness of $\mathbf{T}$ and the dominated convergence theorem, it follows that $R(\lambda) \in L(X)$. However, arguing similarly, it follows that there exists $V(\lambda) \in L(Y)$ such that $V(\lambda)y = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)'y\,\mathrm{d}t$. It is easily seen that $\langle R(\lambda)x, y\rangle = \langle x, V(\lambda)y\rangle$, hence $V(\lambda) = R(\lambda)^*|_Y$. Proposition 3.1 implies $R(\lambda) \in L(X,\sigma)$. This proves (i). $\square$

We end this section with

Lemma 5.9. *Let* $\mathbf{T}$ *be a semigroup on the norming dual pair* (X,Y) *which is* σ*-continuous at* 0*. Then* T *is exponentially bounded.*

P r o o f. Let us first prove that σ-continuity at 0 implies $\sup\limits_{0\leqslant t\leqslant 1} \|T(t)\| < \infty$. To this end, observe that for any $x \in X$ there exists ε_x such that $A_x := \{\|T(t)x\| \colon 0 \leqslant t \leqslant \varepsilon_x\}$ is bounded. Indeed, if this were wrong, there would exist a sequence $t_n \downarrow 0$ such that $\|T(t_n)x\|$ is unbounded. However, as $T(t_n)x \rightharpoonup x$, the set $\{T(t_n)x\}$ has to be σ-bounded and hence, by Proposition 2.5, norm-bounded – a contradiction. Now the semigroup law implies that $\{T(t)x \colon 0 \leqslant t \leqslant 1\} \subset A_x \cup T(\varepsilon_x)A_x \cup \ldots \cup T(\varepsilon_x)^k A_x$ for some $k \in \mathbb{N}$. As all operators $T(t)$ are bounded, it follows that $\{T(t)x \colon 0 \leqslant t \leqslant 1\}$ is bounded. By the uniform boundedness principle, $\sup\limits_{0\leqslant t\leqslant 1} \|T(t)\| =: M < \infty$. Now let $\omega = \log M$. For $t \geqslant 0$ split $t = n + r$ for some $n \in \mathbb{N}_0$ and $r \in [0,1)$. Then $\|T(t)\| = \|T(r)T(1)^n\| \leqslant M\mathrm{e}^{\omega n} \leqslant M\mathrm{e}^{\omega t}$. $\square$

6. Integrable semigroups on $(C_b(E), \mathscr{M}_0(E))$

We now turn to the problem of integrability of transition semigroups. As we will not use positivity or contractivity, we will consider general semigroups of kernel operators. Taking Theorem 3.5 into account, this is exactly the same as a semigroup on the norming dual pair $(B_b(E), \mathscr{M}(E))$. Our first result states that measurability and integrability extends from $(X, \mathscr{M}_{(0)}(E))$ to $(B_b(E), \mathscr{M}_{(0)}(E))$ if X is an $\mathscr{M}_{(0)}(E)$-transition space for E.

Lemma 6.1. *Let $(\Omega, \mathscr{F}, m)$ be a σ-finite measure space, (E, Σ) a measurable space and let $\mathscr{M}_{(0)}(E)$ denote either $\mathscr{M}(E)$ or $\mathscr{M}_0(E)$ (in the latter case, assume additionally that E is a completely regular Hausdorff space). We write σ instead of $\sigma(B_b(E), \mathscr{M}_{(0)}(E))$. Let $T\colon \Omega \to L(B_b(E), \sigma)$ and let X be an $\mathscr{M}_{(0)}$-transition space for E.*

(i) *$T(\cdot)f$ is scalarly $\mathscr{M}_{(0)}(E)$-measurable for every $f \in B_b(E)$ if and only if $T(\cdot)f$ is scalarly $\mathscr{M}_{(0)}(E)$-measurable for every $f \in X$.*

(ii) *Assume additionally that $\|T\|$ is majorized by an integrable function. Then $T(\cdot)f$ is $\mathscr{M}_{(0)}$-integrable for every $f \in B_b(E)$ if and only if $T(\cdot)f$ is $\mathscr{M}_{(0)}$-integrable for every $f \in X$.*

P r o o f. (i) Assume that $T(\cdot)f$ is scalarly $\mathscr{M}_{(0)}$-measurable for every $f \in X$ and define

$$\mathscr{G} := \{A \in \Sigma\colon\ T(\cdot)\mathbb{1}_A \text{ is scalarly } \mathscr{M}_{(0)}\text{-measurable}\}.$$

If $\mathbb{1}_A = \sup f_n$ for a sequence $(f_n)_{n\in\mathbb{N}} \subset X$, then $T(\omega)f_n \rightharpoonup T(\omega)f$ for all $\omega \in \Omega$ by the σ-continuity of $T(\omega)$. Hence, for any $\mu \in \mathscr{M}_{(0)}(E)$ we have $\langle T(\cdot)\mathbb{1}_A, \mu\rangle = \lim\langle T(\cdot)f_n, \mu\rangle$. This proves that $\langle T(\cdot)\mathbb{1}_A, \mu\rangle$ is measurable. It follows that $\mathscr{E}(X) \subset \mathscr{G}$. It is easy to see that $\mathscr{G}$ is a Dynkin system and thus $\mathscr{G} = \Sigma$. By linearity, $T(\cdot)f$ is $\mathscr{M}_{(0)}$-measurable for every simple function f. Approximating an arbitrary function by a sequence of simple functions and using the σ-continuity of the operators $T(\cdot)$ again, the assertion follows.

(ii) Scalar $\mathscr{M}_{(0)}$-measurability of $T(\cdot)f$ for all $f \in B_b(E)$ follows from (i). To prove $\mathscr{M}_{(0)}$-integrability, we proceed as in (i). Define

$$\mathscr{G} := \{A \in \Sigma\colon\ T(\cdot)\mathbb{1}_A \text{ is } \mathscr{M}_{(0)}\text{-integrable}\}.$$

If $\mathbb{1}_A = \sup f_n$ for a sequence $(f_n)_{n\in\mathbb{N}} \subset X$, then it follows from Lemma 4.7 that $A \in \mathscr{G}$. Hence $\mathscr{E}(X) \subset \mathscr{G}$. The rest of the proof is similar to that in (i). □

We now consider semigroups of kernel operators on $C_b(E)$.

Theorem 6.2. *Let E be a completely regular Hausdorff space and $\mathbf{T}$ a semigroup on $(C_b(E), \mathscr{M}_0(E))$ which is σ-continuous at 0.*

(i) *If the strict topology on $C_b(E)$ is complete (cf. Example 4.8) then, for every $f \in C_b(E)$, the orbit $T(\cdot)f$ is locally $\mathscr{M}_0$-integrable.*

(ii) *If E is a complete metric space, then $\mathbf{T}$ is integrable.*

P r o o f. By Lemma 5.9, the semigroup $\mathbf{T}$ is exponentially bounded, say of type (M, ω). Furthermore, since every operator $T(t)$ is σ-continuous, the semigroup law and the σ-continuity at 0 imply that $t \mapsto \langle T(t)f, \mu\rangle$ is right continuous for every $\mu \in \mathscr{M}_0$ and $f \in C_b(E)$. In particular, for every $f \in C_b(E)$ the orbit $T(\cdot)f$ is $\mathscr{M}_0(E)$-measurable and the range of this function is σ-separable and hence, as a consequence of the Hahn-Banach theorem, separable with respect to any consistent topology. Now (i) follows from Theorem 4.4.

To prove (ii), note that if E is a complete metric space, then the strict topology on $C_b(E)$ is complete, hence (i) may be used. In view of Theorem 5.8, to prove (ii) it suffices to prove that $T(\cdot)'\mu$ is locally $C_b(E)$-integrable for every $\mu \in \mathscr{M}_0(E)$. Fix $\mu \in \mathscr{M}_0(E)$. Since E is a metric space, $C_b(E)$ is an $\mathscr{M}_0$-transition space for E, and hence every $T(t)$ is a kernel operator by Proposition 3.5. In particular, it has a unique extension to an operator $\tilde{T}(t) \in L(B_b(E), \sigma)$. We infer from Lemma 6.1 (i) that $t \mapsto \langle f, T(t)'\mu\rangle = \langle \tilde{T}(t)f, \mu\rangle$ is measurable for every $f \in B_b(E)$. Now let $S \subset [0, \infty)$ be a bounded, measurable set. By Lemma 4.6, the B_b-integral $\varphi := \int_S T(t)'\mu \, \mathrm{d}t$ is sequentially $\sigma(\mathscr{M}, B_b)$-continuous. If we put $\varrho(A) = \varphi(\mathbb{1}_A)$, then it follows from the sequential continuity that ϱ is a measure. Clearly $\varphi(f) = \int_S f \, \mathrm{d}\varrho$ for all $f \in B_b(E)$.

It remains to prove that $\varrho \in \mathscr{M}_0(E)$. Since E is a complete metric space, a measure on E is a Radon measure if and only if it has separable support. By assumption, the measure $T(t)'\mu$ is a Radon measure for every $t \in S$. Consequently, we find a separable set E_t such that $T(t)'\mu(A) = 0$ for all $A \subset E \setminus \overline{E_t}$. Define

$$E_0 := \overline{\bigcup_{r \in S \cap \mathbb{Q}} E_r}.$$

Then E_0 is a separable set. We claim that ϱ is supported in E_0. Let $A \subset E \setminus E_0$ be an open set. Then A is an F_σ-set, say $A = \bigcup F_n$ for an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets. By Tieze's extension theorem, there exist functions f_n such that $f|_{F_n} \equiv 1$ whereas $f|_{A^c} \equiv 0$. By the right continuity of the paths we have

$$\langle f_n, T(t)'\mu\rangle = \lim_{r \downarrow t,\, r \in \mathbb{Q}} \langle f_n, T(r)'\mu\rangle = 0$$

for all $n \in \mathbb{N}$ and $t \geqslant 0$. Integrating over S yields $\int_S \langle f_n, T(t)'\mu\rangle \, \mathrm{d}t = 0$. Now the dominated convergence theorem implies that $\varrho(A) = \lim_{n\to\infty} \int_S \langle f_n, T(t)'\mu\rangle \, \mathrm{d}t = 0$. This proves that ϱ is supported in E_0 and is hence a Radon measure. □

Remark 6.3. The assumption that $\mathbf{T}$ is σ-continuous at 0 is equivalent to $\mathbf{T}$ being 'stochastically continuous', cf. [21, Theorem 3.8].

Example 6.4. Let E denote the real line endowed with the Sorgenfrey topology τ_S which is generated by the collection of all intervals $[a, b)$ for $a < b$. Then the shift semigroup $\mathbf{T}$, given by $T(t)f(x) = f(x + t)$, defines a semigroup on $(C_b(E), \mathscr{M}_0(E))$ which has the following properties: (i) it is σ-continuous at 0; (ii) for every $f \in C_b(E)$ the orbit $T(\cdot)f$ is locally $\mathscr{M}_0(E)$-integrable; (iii) $\mathbf{T}$ is not integrable.

Proof. We note that $f \in C_b(E)$ if and only if f is bounded and right continuous as a function on $\mathbb{R}$ with its usual topology. Furthermore, the Borel σ-algebra of E is just the usual Borel σ-algebra of $\mathbb{R}$ when endowed with the usual topology. It is well known that every compact subset of E is necessarily countable (though not every countable subset of $\mathbb{R}$ is compact with respect to τ_S), and thus $\mathscr{M}_0(E) = \ell^1(\mathbb{R})$, the space of all discrete measures on $\mathbb{R}$.

From these observations it is easy to see that $T(t)'\mathscr{M}_0(E) \subset \mathscr{M}_0(E)$ – hence $\mathbf{T}$ is a semigroup on $(C_b(E), \mathscr{M}_0(E))$ – and that $\mathbf{T}$ is σ-continuous at 0. Let us show that $C(E)$ is complete with respect to the compact-open topology. As noted in Example 4.8 this implies that the strict topology on $C_b(E)$ is complete and thus assertion (ii) follows from Theorem 6.2.

So let $(f_\alpha) \subset C_b(E)$ be a net converging to a function f with respect to the compact-open topology. Fix $t \in \mathbb{R}$. To prove that $f \in C_b(E)$, it suffices to prove that $f(t_n) \to f(t)$ as $n \to \infty$ for every sequence $t_n \downarrow t$. However, given such a sequence, the set $K = \{t, t_n \colon n \in \mathbb{N}\}$ is τ_S-compact and thus $f_\alpha \to f$ uniformly on K. The convergence of $f(t_n) \to f(t)$ as $n \to \infty$ now follows from a standard $\frac{1}{3}\varepsilon$-argument.

Concerning assertion (iii), we note that for every $f \in C_b(E)$ we have

$$\int_0^1 \langle f, T(t)'\delta_0\rangle \,\mathrm{d}t = \langle f, \lambda_{(0,1)}\rangle,$$

where $\lambda_{(0,1)}$ denotes the restriction of the Lebesgue measure to $(0,1)$. Since this measure does not belong to $\mathscr{M}_0(E)$, the orbit of $T(\cdot)'\delta_0$ is not locally $C_b(E)$-integrable and (iii) follows from Theorem 5.8. □

We close this section by proving that if the topology of E is induced by a separable metric, in particular if E is a Polish space (i.e. the topology of E is induced by a complete, separable metric), then the norming dual pair $(C_b(E), \mathscr{M}_0(E))$ is countably separated. As a consequence of this, Theorem 6.5 may be applied, yielding that every integrable semigroup on $(C_b(E), \mathscr{M}_0(E))$ is uniquely determined by its Laplace transform. Furthermore, if E is a Polish space, then, given exponential boundedness, the σ-continuity assumption in Theorem 6.2 may be weakened to the requirement that $t \mapsto \langle T(t)f, \mu\rangle$ be measurable for every $f \in C_b(E)$ and $\mu \in \mathscr{M}_0(E)$. This is evident from the proof of that theorem.

Theorem 6.5. *Let $(E, \mathscr{B}(E))$ be a separable metric space endowed with its Borel σ-algebra. Then the norming dual pair $(C_b(E), \mathscr{M}_0(E))$ is countably separated.*

Proof. Let $D := \{x_m \colon m \in \mathbb{N}\}$ be a countable, dense subset of E. Then $\{\delta_{x_m} \colon m \in \mathbb{N}\} \subset \mathscr{M}_0(E)$ separates points in $C_b(E)$ as continuous functions which coincide on a dense subset are equal. To find a sequence in $C_b(E)$ which separates points in $\mathscr{M}_0(E)$, we proceed as follows. For $n, m \in \mathbb{N}$, choose $f_{n,m} \in C_b(E)$ such that

$$\mathbb{1}_{\overline{B}(x_m,1/n+1)} \leqslant f_{n,m} \leqslant \mathbb{1}_{B(x_m,1/n)^c}.$$

If $J \subset \mathbb{N}$ is a finite subset, we put $f_{n,J} := \max\{f_{n,m}\colon\ m \in J\}$ and define

$$M := \{f_{n,J}\colon\ n \in \mathbb{N},\ J \subset \mathbb{N} \text{ finite}\}.$$

Then M is a countable set. We claim that M separates points in $\mathscr{M}_0(E)$. To this end, let $\mu \in \mathscr{M}_0(E)$ satisfy $\int f\,\mathrm{d}\mu = 0$ for all $f \in M$. We have to prove that $\mu = 0$. Since μ is a Radon measure, it suffices to prove that $\mu(K) = 0$ for all compact sets K. So let a compact set $K \neq \emptyset$ be given. As D is dense in E, the set K is covered by $\{B(x_m, (n+1)^{-1})\colon\ m \in \mathbb{N}\}$ for every $n \in \mathbb{N}$. Since K is compact, there exist $m_1, \ldots, m_{k_n}$ such that K is already covered by $\mathscr{B}_n := \{B(x_{m_i}, (n+1)^{-1})\colon$ $i = 1, \ldots, k_n\}$. We may assume without loss of generality that every ball in $\mathscr{B}_n$ intersects K. Define $f_n := f_{n,\{m_1,\ldots,m_{k_n}\}} \in M$. Then $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence which converges pointwise to $\mathbb{1}_K$. As $\int f_n\,\mathrm{d}\mu \equiv 0$ by assumption, the dominated convergence theorem yields $\mu(K) = \lim f_n\,\mathrm{d}\mu = 0$. As K is arbitrary, $\mu = 0$. □

Acknowledgement. The author is grateful to M. Haase and J. van Neerven for some valuable comments on earlier versions of this paper. He also thanks J. Bonet and B. Cascales for providing an early preprint of [5].

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