Thomas Apel; Dieter Sirch

L^2-error estimates for Dirichlet and Neumann problems on anisotropic finite element meshes

Applications of Mathematics, Vol. 56 (2011), No. 2, 177–206

Persistent URL: http://dml.cz/dmlcz/141438

56 (2011) APPLICATIONS OF MATHEMATICS No. 2, 177–206

L^2-ERROR ESTIMATES FOR DIRICHLET AND NEUMANN PROBLEMS ON ANISOTROPIC FINITE ELEMENT MESHES*

Thomas Apel, Dieter Sirch, München

(Received November 24, 2008)

Abstract. An L^2-estimate of the finite element error is proved for a Dirichlet and a Neumann boundary value problem on a three-dimensional, prismatic and non-convex domain that is discretized by an anisotropic tetrahedral mesh. To this end, an approximation error estimate for an interpolation operator that is preserving the Dirichlet boundary conditions is given. The challenge for the Neumann problem is the proof of a local interpolation error estimate for functions from a weighted Sobolev space.

Keywords: elliptic boundary value problem, a priori error estimates, interpolation of non-smooth functions, finite element error, non-convex domains, edge singularities, anisotropic mesh grading

MSC 2010: 65D05, 65N30

1. Introduction

The motivation for this paper comes from our investigation of a discretized version of the following optimal control problem. Minimize

$$F(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}$$

subject to an elliptic state equation $Ly = u$ in Ω with appropriate boundary conditions. Here, y_d is the desired state, the regularization parameter $\nu > 0$ is a fixed positive number, and the control variable u varies over a set $U_{\rm ad}$ which is the space $L^2(\Omega)$ or a convex subset of it. For the numerical solution, one usually considers the system of necessary and sufficient first order optimality conditions consisting of the state equation, an adjoint equation $L^*p = y - y_d$ (with boundary conditions) for the costate p and a projection $u = \Pi_{U_{\rm ad}}(-\nu^{-1}p)$ to the set of admissible controls.

* This work was supported by the DFG priority program 1253.

The classical discretization with piecewise linears for the state and costate as well as piecewise linears for the control leads to first order accuracy only, see [13], [14], [22], [8], [10]. In all those papers, a family of quasi-uniform meshes is discussed, and the solution is assumed to be sufficiently smooth. Second order approximation has been achieved by two different methods. In the variationally discrete approach, [18], only the state equation and its adjoint are discretized by piecewise linears; the control is obtained by a projection $u_h = \Pi_{U_{\rm ad}}(-\nu^{-1}p_h)$. In the superconvergence approach, [23], the control variable is discretized as well, but a postprocessing step generates the final approximation of u. This approach uses the fact that the piecewise constant approximate control is superclose to an interpolant of the exact control.

The error analysis of both approaches relies among other on an estimate of the finite element discretization error in the L^2-norm for the elliptic problem which is usually obtained by the Aubin-Nitsche method. In order to apply this method, a discretization error estimate in the energy norm is necessary for the situation where the right-hand side of the elliptic equation is only in $L^2(\Omega)$. This estimate is standard in many cases but not yet available for the discretization with anisotropic mesh grading as it is appropriate near edges of the computational domain. The aim of this paper is to derive such an estimate for two model problems. The application to the optimal control problems exceeds the scope of one paper and will be published elsewhere [7].

In order to introduce the reader more clearly into this topic let us fix some notation. In this paper two elliptic boundary value problems in a three-dimensional, non-convex domain Ω are treated. Since we discuss from now on the elliptic problem only, the standard notation with u being the solution of the partial differential equation is used, contrary to the optimal control problem above where the solution of the state equation is denoted by y. We consider the Dirichlet problem

$$-\Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1.1}$$

and the Neumann problem

$$-\Delta u + u = f \ \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial\Omega. \tag{1.2}$$

Robin or mixed boundary conditions are not discussed explicitly here since no further difficulties occur.

For both problems (1.1) and (1.2) it is well known that the solution has in general singularities near corners and edges, e.g. [19], [15]. Therefore, one can observe in this case that the convergence rate of the finite element method on quasiuniform meshes is smaller in comparison with that for problems with smooth solutions. To overcome this loss in accuracy, special adapted numerical methods have been developed. One approach is the *singular function method*, which is used for three-dimensional problems in [9], [21]. Another method is the *mesh refinement*. For the Dirichlet problem, refined *isotropic* meshes were considered in [3], [6]. However, it was observed that this technique leads to overrefinement near edges.

In order to avoid this overrefinement, *anisotropic meshes* in the neighborhood of the edges were used in [2], [5]. Anisotropic finite elements are more general than shape-regular elements; they are characterized by three size parameters $h_{i,T}$, $i = 1, 2, 3$, which may have different asymptotics. The anisotropic mesh grading is described by a relationship between the size parameters of each element and its distance from an edge. By estimating the approximation error of the standard nodal interpolation operator and using the projection property of the finite element method, it is shown that with u_h being the finite element solution using a linear ansatz space the estimate

$$|u - u_h|_{W^{1,2}(\Omega)} \lesssim h\|f\|_{L^p(\Omega)} \tag{1.3}$$

is valid for $p > 2$, $h = \max\limits_{T\in\mathcal{T}_h} \operatorname{diam} T$. The main drawback of this estimate is that the case $p = 2$ cannot be treated in this way and we do not obtain an L^2-estimate of the finite element error.

Let us discuss interpolation shortly. The standard Lagrangian (nodal) interpolation operator uses nodal values of the function for the definition of the interpolant. This implies the very useful property that Dirichlet boundary conditions are preserved. However, it was shown in [2] that the local interpolation error estimate

$$|u - I_h u|_{W^{1,p}(T)} \lesssim \sum_{i=1}^{3} h_{i,T} \Big|\frac{\partial u}{\partial x_i}\Big|_{W^{1,p}(T)} \tag{1.4}$$

and even its simplified version $|u-I_h u|_{W^{1,p}(T)} \lesssim \max\limits_i h_{i,T}\, |u|_{W^{2,p}(T)}$ is valid under the condition $p > 2$ only provided T is an anisotropic, three-dimensional finite element. This has led to the restriction $p > 2$ for (1.3) as mentioned above. A straightforward idea to overcome this problem is to use a quasi-interpolation operator as introduced for example in [11], [29]. The basic idea is to replace nodal values by suitable averaged values. Several variants of quasi-interpolation operators were investigated in [1] for anisotropic finite elements. It turned out that the classical operators according to [11], [29] are not uniformly $W^{1,p}$-stable in the aspect ratio and do not satisfy an estimate like (1.4) (with T replaced by a patch S_T on the right-hand side). The positive conclusion of [1] is, however, that three modifications of the Scott-Zhang interpolant are available for which such an estimate holds. The disadvantage of these modified operators is that they preserve Dirichlet boundary conditions on part of the boundary only. Moreover, the analysis is made for meshes with certain structure only; they are called *meshes of tensor product type* in [1].

The mixed boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_M, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_B \tag{1.5}$$

with $f \in L^2(\Omega)$ is considered in [1]. In view of the above mentioned difficulty with the boundary condition, the boundary parts are chosen such that one of these modified Scott-Zhang operators preserves the Dirichlet condition on Γ_M. In particular, we have $\Omega = G \times Z$, where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset \mathbb{R}$ is an interval. The different parts of the boundary are denoted by $\Gamma_B := \{x \in \partial\Omega \colon x_3 = 0 \text{ or } x_3 = z_0\}$ and $\Gamma_M := \partial\Omega \setminus \Gamma_B$. Besides it is assumed that the cross-section G has only one corner with interior angle $\omega > \pi$ at the origin; thus Ω has only one “singular edge” which is part of the x_3-axis. Since the edge singularities are of local nature, no additional difficulties are introduced by more than one reentrant corner in G. For appropriately graded anisotropic meshes the estimate

$$|u - u_h|_{W^{1,2}(\Omega)} \lesssim h\|f\|_{L^2(\Omega)} \tag{1.6}$$

is obtained where u_h is the finite element solution of (1.5) using linear elements, which leads easily to an L^2-estimate of the discretization error. One main ingredient of the proof is the description of the regularity of the solution in certain weighted Sobolev spaces.

Up to now there has been neither an L^2-estimate for the pure Dirichlet problem nor for the pure Neumann problem available. The specific difficulty with the Dirichlet problem is that the quasi-interpolant does not preserve the boundary conditions so that further modification is necessary and another error term has to be estimated. This was not elaborated ten years ago. The Neumann problem was not satisfactorily treated since its solution has to be described in other weighted Sobolev spaces than the Dirichlet and the mixed problems. It should be noted that the boundary condition on Γ_B is not important, only that the Neumann conditions are posed on both faces joining the “singular edge”. In view of this, Lemma 12 in [1] is wrong for the Neumann case, and in consequence also the proof of Theorem 14 in [1]. The aim of this paper is to fill these gaps.

The plan is to derive estimate (1.6) for problem (1.1) and (1.2), where $\Omega = G \times Z$ with G and Z defined as above. Then we are able to derive the optimal estimate of the $L^2(\Omega)$-error

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^2\|f\|_{L^2(\Omega)}.$$

Our further considerations begin with the introduction of some necessary notation. Then we recall regularity results for the Dirichlet problem and state one for the Neumann problem. Afterwards we show local error estimates for an interpolation operator for non-smooth functions that preserves Dirichlet boundary conditions. Furthermore, for an interpolation operator that is suitable for the Neumann problem we prove a local error estimate for functions in a special type of weighted Sobolev spaces. In the last section global estimates for the interpolation and the finite element error are shown.

From the previous paragraphs it may have become obvious that there is a close relationship between this paper and paper [1]. For brevity, we keep proofs short whenever they are applications or simple extensions of those in the former paper. Comprehensive proofs are given when new ideas have to be used.

2. Notation and analytical background

In this section we introduce the necessary notation. We further state an embedding result and prove a norm equivalence in a weighted Sobolev space.

For some positive constants C, C_1, and C_2, which are independent of the triangulation and the function under consideration, we write

$$x \lesssim y \Leftrightarrow x \leqslant Cy,$$
$$x \sim y \Leftrightarrow C_1 y \leqslant x \leqslant C_2 y.$$

For $x = (x_1, x_2, x_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_i non-negative integers, we use the multi-index notation

$$|\alpha| := \sum_{i=1}^{3} \alpha_i, \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \quad \text{and} \quad D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}}.$$

We denote the classical Sobolev spaces by $W^{k,p}(T)$, $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and use the norm and seminorm

$$\|v\|^p_{W^{k,p}(T)} := \sum_{|\alpha| \leqslant k} \int_T |D^\alpha v|^p, \quad |v|^p_{W^{k,p}(T)} := \sum_{|\alpha| = k} \int_T |D^\alpha v|^p$$

for $p < \infty$ with the usual modification for $p = \infty$. By introducing cylindrical coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, we define for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\beta \in \mathbb{R}$ the weighted Sobolev spaces

$$V^{k,p}_\beta(T) = \bigl\{v \in \mathcal{D}'(T)\colon\ \|v\|^p_{V^{k,p}_\beta(T)} < \infty\bigr\},$$
$$W^{k,p}_\beta(T) = \bigl\{v \in \mathcal{D}'(T)\colon\ \|v\|^p_{W^{k,p}_\beta(T)} < \infty\bigr\},$$

where

$$\|v\|^p_{V^{k,p}_\beta(T)} := \sum_{|\alpha|\leqslant k} \int_T |r^{\beta-k+|\alpha|} D^\alpha v|^p,$$
$$\|v\|^p_{W^{k,p}_\beta(T)} := \sum_{|\alpha|\leqslant k} \int_T |r^{\beta} D^\alpha v|^p$$

for $p < \infty$ with the usual modification for $p = \infty$. For the weighted space $W^{k,p}_\beta(\Omega)$ the following embedding result holds.

Lemma 2.1. *For $p \in (1,\infty)$, $\beta > 1 - 2/p$ and $k \geqslant 0$ one has the compact embedding*

$$W^{k+1,p}_\beta(\Omega) \overset{c}{\hookrightarrow} W^{k,p}_\beta(\Omega). \tag{2.1}$$

For $p \in (1,\infty)$, $k \geqslant 1$ and $\beta \in (1-2/p, 1]$ the continuous embeddings

$$W^{k,p}_\beta(\Omega) \hookrightarrow W^{k-1,p}_{\beta-1}(\Omega) \hookrightarrow W^{k-1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^1(\Omega) \tag{2.2}$$

are valid.

Proof. From Lemma 1.8 in [26] one has $W^{1,p}_\beta(\Omega) \hookrightarrow V^{1,p}_\beta(\Omega)$ for $\beta > 1 - 2/p$. Lemma 1.2 in [26] yields $V^{1,p}_\beta(\Omega) \overset{c}{\hookrightarrow} W^{0,p}_{\beta-1}(\Omega)$. Since $W^{0,p}_{\beta-1}(\Omega) \hookrightarrow W^{0,p}_\beta(\Omega)$ this shows the embedding $W^{1,p}_\beta(\Omega) \overset{c}{\hookrightarrow} W^{0,p}_\beta(\Omega)$. Applying this embedding to derivatives, one can conclude (2.1). The embedding $W^{k,p}_\beta(\Omega) \hookrightarrow W^{k-1,p}_{\beta-1}(\Omega)$ follows from Theorem 1.3 in [26]. The other embeddings in (2.2) can be concluded directly since $\beta \leqslant 1$ and $p > 1$. □

In the next lemma a norm equivalence is proved, that will be useful in the forthcoming derivation of a local interpolation error estimate for functions from the spaces $W^{k,p}_\beta(\Omega)$.

Lemma 2.2. *For $p > 1$, $\beta \in (1-2/p, 1]$ and a function $v \in W^{k+1,p}_\beta(\Omega)$ one has the norm equivalence*

$$\|v\|_{W^{k+1,p}_\beta(\Omega)} \sim |v|_{W^{k+1,p}_\beta(\Omega)} + \sum_{|\alpha|\leqslant k} \left| \int_\Omega D^\alpha v \right|.$$

Proof. Since one has for $p > 1$ and $\beta \in (1-2/p, 1]$, the embedding $W^{1,p}_\beta(\Omega) \hookrightarrow L^1(\Omega)$ (see (2.2)) the inequality

$$\|v\|_{W^{k+1,p}_\beta(\Omega)} \gtrsim |v|_{W^{k+1,p}_\beta(\Omega)} + \sum_{|\alpha|\leqslant k} \left| \int_\Omega D^\alpha v \right|$$

holds. In order to show the other direction,

$$\|v\|_{W_\beta^{k+1,p}(\Omega)} \lesssim |v|_{W_\beta^{k+1,p}(\Omega)} + \sum_{|\alpha|\leqslant k} \left| \int_\Omega D^\alpha v \right|, \tag{2.3}$$

we use proof by contradiction. If inequality (2.3) were not valid, then there would be a sequence (v_n) with $v_n \in W_\beta^{k+1,p}(\Omega)$ such that

$$\|v_n\|_{W_\beta^{k+1,p}(\Omega)} = 1, \tag{2.4}$$

$$|v_n|_{W_\beta^{k+1,p}(\Omega)} + \sum_{|\alpha|\leqslant k} \left| \int_\Omega D^\alpha v_n \right| \leqslant \frac{1}{n}. \tag{2.5}$$

Since (v_n) is a bounded sequence in $W_\beta^{k+1,p}(\Omega)$ and $W_\beta^{k+1,p}(\Omega) \overset{c}{\hookrightarrow} W_\beta^{k,p}(\Omega)$ (see (2.1)) there is a convergent subsequence $(v_{n_l}) \in W_\beta^{k,p}(\Omega)$. In the sequel we suppress the index l and write (v_n) for this subsequence. Because of the completeness of $W_\beta^{k,p}(\Omega)$ there is a function $v \in W_\beta^{k,p}(\Omega)$ such that

$$\|v - v_n\|_{W_\beta^{k,p}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0. \tag{2.6}$$

With (2.5) one can conclude $|v_n|_{W_\beta^{k+1,p}(\Omega)} \leqslant 1/n$, which results in

$$|v_n|_{W_\beta^{k+1,p}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0. \tag{2.7}$$

Next we show that (v_n) is a Cauchy sequence in $W_\beta^{k+1,p}(\Omega)$. For a fixed and arbitrarily small $\varepsilon > 0$ and numbers n, m large enough one obtains by virtue of (2.6) and (2.7)

$$\begin{aligned}
\|v_n - v_m\|^p_{W_\beta^{k+1,p}(\Omega)} &= \|v_n - v_m\|^p_{W_\beta^{k,p}(\Omega)} + |v_n - v_m|^p_{W_\beta^{k+1,p}(\Omega)} \\
&\leqslant \frac{\varepsilon}{3} + C|v_n|^p_{W_\beta^{k+1,p}(\Omega)} + C|v_m|^p_{W_\beta^{k+1,p}(\Omega)} \\
&\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since $W_\beta^{k+1,p}(\Omega)$ is complete, there is a function $v^* \in W_\beta^{k+1,p}(\Omega)$ with

$$\|v_n - v^*\|_{W_\beta^{k+1,p}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0$$

and using (2.4), one arrives at

$$\|v^*\|_{W_\beta^{k+1,p}(\Omega)} = 1. \tag{2.8}$$

Furthermore, one can conclude from (2.5)

$$|v^*|_{W^{k+1,p}_\beta(\Omega)} + \sum_{|\alpha|\leqslant k} \left| \int_\Omega D^\alpha v^* \right| = 0, \tag{2.9}$$

in particular $|v^*|_{W^{k+1,p}_\beta(\Omega)} = 0$, which means that $D^\alpha v^* = 0\ \forall\, \alpha\colon\ |\alpha| = k+1$, that is $v^* \in \mathcal{P}_{k,\Omega}$. The only function $v^* \in \mathcal{P}_{k,\Omega}$ with (2.9) is $v^* = 0$. This is a contradiction to (2.8), which proves (2.3). □

3. Regularity results

In this section we want to give some regularity results for our problems under consideration. Although the literature about elliptic boundary value problems in domains with edges is vast, there are only a few papers that include the Neumann problem. We start with the following well-known regularity result for the Dirichlet problem.

Lemma 3.1. *Let p and β be given real numbers with $p \in (1,\infty)$ and $\beta > 2 - \pi/\omega - 2/p$. Moreover, let f be a function in $V^{0,p}_\beta(\Omega)$. Then the weak solution of the boundary value problem* (1.1) *belongs to $H^1_0(\Omega) \cap V^{2,p}_\beta(\Omega)$. Moreover, the inequality*

$$\|u\|_{V^{2,p}_\beta(\Omega)} \lesssim \|f\|_{V^{0,p}_\beta(\Omega)}$$

is valid.

Proof. With $\operatorname{Im}\lambda_- = -\pi/\omega$ the assertion follows from Lemmas 1 and 2 of [28]. □

The drawback of describing the solution in the $V^{k,p}_\beta(\Omega)$-spaces is that the space $W^{1,2}(\Omega)$ does not belong to the scale of these weighted Sobolev spaces. A necessary condition for $u \in W^{1,2}(\Omega) \cap V^{1,2}_0(\Omega)$ is $u(r=0) = 0$. This condition is fulfilled for homogeneous Dirichlet boundary conditions, but cannot be guaranteed for a Neumann boundary. This is the reason why problem (1.2) is not included in the paper [6], where the authors demand $u \in W^{1,2}(\Omega) \cap V^{1,2}_0(\Omega)$. A way out is the description of the solution of (1.2) in the spaces $W^{k,p}_\beta(\Omega)$. Concerning the literature about elliptic boundary value problems with Neumann boundary in domains with edges, let us first mention the book of Grisvard [16], where estimates on the solution of the Neumann problem for the Laplace equation and the Lamé system in Sobolev and Sobolev-Slobodeckii spaces with $p = 2$ and without weight are given. Dauge [12] proved regularity results for linear elliptic Neumann problems in

L^p Sobolev spaces without weight. Maz'ya and Roßmann obtained regularity results in weighted Sobolev spaces in a cone for general p. Their result about the Neumann problem in a dihedron requires additional regularity on the solution, which cannot be guaranteed in our case. Zaionchkovskii and Solonnikov [30], Roßmann [27], and Nazarov and Plamenevsky [25] proved solvability theorems and regularity results for the Neumann problem in weighted Sobolev spaces for $p = 2$. Using the results of Zaionchkovskii and Solonnikov [30], we obtain the following theorem.

Theorem 3.2. *Let u be the solution of* (1.2). *If $f \in W^{0,2}_\beta(\Omega)$ with $\beta > 1 - \pi/\omega$, then u is contained in the space $W^{2,2}_\beta(\Omega)$ and satisfies the inequality*

$$\|u\|_{W^{2,2}_\beta(\Omega)} \lesssim \|f\|_{W^{0,2}_\beta(\Omega)}.$$

Proof. We first consider problem (1.2) in a dihedron $\mathcal{D}_\omega = \{x = (x', x_3)\colon\ x' \in K,\ x_3 \in \mathbb{R}\}$ where K denotes an infinite angle which has the form $\{x' = (x_1, x_2) \in \mathbb{R}^2\colon\ 0 < r < \infty,\ 0 < \varphi < \omega\}$ in polar coordinates r, φ. Setting $k = 0$ in Theorem 5.2 of [30], we can conclude

$$|u|_{W^{2,2}_\beta(\mathcal{D}_\omega)} + \|u\|_{W^{1,2}(\mathcal{D}_\omega)} \lesssim \|f\|_{W^{0,2}_\beta(\mathcal{D}_\omega)}. \tag{3.1}$$

Since $\beta > 0$, estimate (3.1) keeps valid if one replaces the left-hand side of the inequality by $\|u\|_{W^{2,2}_\beta(\mathcal{D}_\omega)}$. Problem (1.2) can be locally transformed near an edge point by a diffeomorphism into a boundary value problem in the dihedron $\mathcal{D}_\omega$. By the use of a partion of unity method one can fit together the local results to obtain the result for the domain Ω. Details on this technique can be found e.g. in the book of Kufner and Sändig [20, Section 8]. □

According to [17] the weak solution u of (1.1) or (1.2) can be written as a sum of a singular part u_s and a regular part u_r,

$$u = u_s + u_r, \tag{3.2}$$

where $u_r \in W^{2,2}(\Omega)$ and

$$u_s = \xi(r)\gamma(r, x_3) r^\lambda \Theta(\varphi) \quad \text{with } \lambda = \frac{\pi}{\omega}.$$

Here r and φ are polar coordinates in the plane perpendicular to the edge, $\xi(r)$ is a smooth cut-off function and $\Theta(\varphi) = \sin\lambda\varphi$ for the Dirichlet boundary conditions and $\Theta(\varphi) = \cos\lambda\varphi$ for the Neumann boundary conditions. The coefficient function γ can be written as a convolution integral,

$$\gamma(r, x_3) = \int_{\mathbb{R}} \frac{1}{\pi} \frac{r}{r^2 + s^2} q(x_3 - s)\,\mathrm{d}s,$$

where the smoothness of q can be characterized in Besov spaces depending on λ.

Lemma 3.3. *The singular part u_s of the weak solution u of* (1.1) *or* (1.2) *satisfies*

$$r^{\beta-1}\frac{\partial u_s}{\partial x_i} \in L^p(\Omega) \quad \text{and} \quad \Big\| r^{\beta-1}\frac{\partial u_s}{\partial x_i}\Big\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)} \quad \text{for } i = 1, 2, \tag{3.3}$$

$$r^{-1}\frac{\partial u_s}{\partial x_3} \in L^p(\Omega) \quad \text{and} \quad \Big\| r^{-1}\frac{\partial u_s}{\partial x_3}\Big\|_{L^p(\Omega)} \lesssim \|f\|L^p(\Omega), \tag{3.4}$$

$$\frac{\partial^2 u_s}{\partial x_i \partial x_3} \in L^p(\Omega)] \quad \text{and} \quad \Big\| \frac{\partial^2 u_s}{\partial x_i \partial x_3}\Big\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)} \quad \text{for } i = 1, 2, 3 \tag{3.5}$$

if $0 < \pi/\omega < 2 - 2/p$, $\pi/\omega > 1 - 2/p$ *and* $\beta > 2 - 2/p - \pi/\omega$.

P r o o f. For the Dirichlet problem this lemma is proved in [4, Section 2.2]. In order to get the result for the Neumann problem one just has to replace $\sin(j\pi\varphi/\omega)$ by $\cos(j\pi\varphi/\omega)$ in that proof. □

Corollary 3.4. *The weak solution u of* (1.1) *or (1.2) satisfies*

$$\frac{\partial u}{\partial x_3} \in W^{1,2}(\Omega) \quad \text{and} \quad \Big\| \frac{\partial u}{\partial x_3}\Big\|_{W^{1,2}(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{3.6}$$

P r o o f. Since $u \in W^{1,2}(\Omega)$ and $u_r \in W^{2,2}(\Omega)$ with $\|u_r\|_{W^{2,2}(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ the assertion follows from (3.2) and (3.5). □

R e m a r k 3.5. For the Dirichlet problem (1.1) one can replace u_s by u (see e.g. [20], [30]) in Lemma 3.3.

In the error analysis for a finite element discretization, local estimates of interpolation errors plays an important role. The interpolant has to be chosen such that the boundary conditions of the underlying partial differential equation are fulfilled by the interpolant. In this section we introduce two suitable interpolants for problems (1.1) and (1.2) and derive the corresponding local estimates. Before, we introduce a triangulation of Ω.

4.1. Triangulation of Ω

For a family of tetrahedral triangulations $\mathcal{T}_h = \{T_i\}_{i=1}^m$ we demand that

(A1) Ω is exactly triangulated by the tetrahedra, $\overline{\Omega} = \bigcup_{i=1}^{m} \overline{T}_i$,

(A2) the elements are disjoint, $T_i \cap T_j = \emptyset$ for $i \neq j$, and

(A3) any face of any element T_i is either a face of another element T_j or part of the boundary.

Notice that we do not demand the elements to be shape-regular. In contrast we are interested in *anisotropic elements*. If one denotes the diameter of the finite element T by h_T and the supremum of the diameters of all balls contained in T by ϱ_T, this type of element is characterized by huge values of the aspect ratio h_T/ϱ_T.

According to [4], we consider four reference elements

$$\begin{aligned}
\hat{T}_1 &:= \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 \colon\ 0 < \hat{x}_1 < 1,\, 0 < \hat{x}_2 < 1 - \hat{x}_1,\, 0 < \hat{x}_3 < 1 - \hat{x}_1 - \hat{x}_2\},\\
\hat{T}_2 &:= \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 \colon\ 0 < \hat{x}_1 < 1,\, 0 < \hat{x}_2 < 1 - \hat{x}_1,\, \hat{x}_1 < \hat{x}_3 < 1\},\\
\hat{T}_3 &:= \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 \colon\ 0 < \hat{x}_1 < 1,\, 0 < \hat{x}_2 < \hat{x}_1,\, 0 < \hat{x}_3 < \hat{x}_1 - \hat{x}_2\},\\
\hat{T}_4 &:= \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 \colon\ 0 < \hat{x}_1 < 1,\, 0 < \hat{x}_2 < \hat{x}_1,\, 1 - \hat{x}_1 < \hat{x}_3 < 1\}.
\end{aligned}$$

For elements with a face parallel to the plane $x_3 = 0$ we use $\hat{T}_1$ and $\hat{T}_3$, for elements without such a face $\hat{T}_2$ and $\hat{T}_4$ are considered. Elements with exactly one vertex with $r = 0$ are mapped to $\hat{T}_3$ or $\hat{T}_4$, in all other cases (zero or two vertices with $r = 0$) $\hat{T}_1$ and $\hat{T}_2$ are used. In the following we refer to the suitable reference element by $\hat{T}$. In order to be able to write down our proofs in a concise way, we restrict ourselves first to *tensor product meshes*. According to [1], an affine finite element is called a *tensor product element* when the transformation of a reference element $\hat{T}$ to the element T has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h_{1,T} & 0 & 0 \\ 0 & h_{2,T} & 0 \\ 0 & 0 & h_{3,T} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + b_T,$$

where $b_T \in \mathbb{R}^3$. Note that the vertices of a tensor element are located in the corners of a cuboid with edge lengths $h_{1,T}$, $h_{2,T}$ and $h_{3,T}$. We explain in Subsection 4.5 how the results extend to a more general mesh type. In addition we demand that there is no rapid change in the element sizes; this means that the relation

$$h_{i,T} \sim h_{i,T'} \quad \text{for all } T' \text{ with } \overline{T} \cap \overline{T'} \neq \emptyset$$

holds for $i = 1, 2, 3$. Furthermore, we define the set

$$M_T := \operatorname{int} \bigcup_{i \in I_T} \overline{T_i},$$

where the set I_T contains all indices i for which $\overline{T_i} \cap \overline{T} \neq \emptyset$ and the projection of T_i on the x_1x_2-plane is the same as that of T. By S_T we denote the smallest triangular prism that contains M_T. Notice that the height of S_T has the order of $h_{3,T}$. We further define

$$S_{\hat{T}} := \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 \colon\ 0 < \hat{x}_1 < 1,\ 0 < \hat{x}_2 < 1 - \hat{x}_1,\ 0 < \hat{x}_3 < 1\}.$$

With this definition one has $\hat{T} \subset S_{\hat{T}}$ for all reference elements mentioned above.

In this paper we consider the space of piecewise linear functions as the finite element space V_h,

$$V_h := \{v_h \in W^{1,2}(\Omega) \colon\ v_h|_T \in \mathcal{P}_{1,T} \text{ for all } T \in \mathcal{T}_h\}.$$

4.2. Interpolation operators

As already mentioned in Introduction the standard Lagrangian interpolant is not appropriate due to the fact that the estimate (1.4) is only true for $p > 2$. Therefore, we define the Scott-Zhang type interpolant E_h by

$$(E_h u)(x) := \sum_{i \in I} a_i \varphi_i(x),$$

which was originally introduced in [1]. Here the functions φ_i $(i \in I)$ are *nodal basis functions*, i.e. $\varphi_i(X_j) = \delta_{ij}$ for all $i, j \in I$, where $X_i = (X_{i,1}, X_{i,2}, X_{i,3}) \in \mathbb{R}^3$ are the nodes of the finite element mesh. In order to specify a_i, we first introduce the subset σ_i by the following properties.

(P1) σ_i is one-dimensional and parallel to the x_3-axis.
(P2) $X_i \in \overline{\sigma}_i$.
(P3) There exists an edge e of some element T such that the projection of e on the x_3-axis coincides with the projection of σ_i.
(P4) If the projections of any two points X_i and X_j on the x_3-axis coincide then so do the projections of σ_i and σ_j.

Note that the properties (P3) and (P4) make sense since we consider tensor product meshes. Now a_i is chosen as the value of the $L^2(\sigma_i)$-projection of u in the space of linear functions over $\sigma_i \subset \overline{\Omega}$ at the node X_i,

$$a_i := (\Pi_{\sigma_i} u)(X_i)$$

with

$$\Pi_{\sigma_i}\colon\ L^2(\sigma_i) \to \mathcal{P}_{1,\sigma_i},$$

where $\mathcal{P}_{1,\sigma_i}$ is the space of polynomials over σ_i with degree at most 1.

We denote by $\Phi_{0,i}$ and $\Phi_{1,i}$ the two one-dimensional linear nodal functions corresponding to $\sigma_i = \overrightarrow{X_i X_j}$, that is

$$\begin{aligned} \Phi_{0,i}(X_{i,3}) &= 1, \qquad & \Phi_{0,i}(X_{j,3}) &= 0,\\ \Phi_{1,i}(X_{i,3}) &= 0, \qquad & \Phi_{1,i}(X_{j,3}) &= 1. \end{aligned}$$

Besides, we define $\Psi_{0,i}$ and $\Psi_{1,i}$ as the two linear functions that are biorthogonal to $\{\Phi_{0,i}, \Phi_{1,i}\}$,

$$\int_{\sigma_i} \Phi_{k,i}\Psi_{l,i} = \delta_{k,l} \quad (k,l = 0,1). \tag{4.1}$$

Notice that $\Phi_{k,i}$ depends only on $X_{i,3}$, which means that $\Phi_{k,i} = \Phi_{k,m}$ if $X_{i,3} = X_{m,3}$ $(k = 0,1)$. The same is valid for $\Psi_{k,i}$. With this setting we can write the interpolation operator E_h as

$$\begin{aligned} E_h u(x) &= \sum_{i\in I} (\Pi_{\sigma_i} u)(X_i)\varphi_i(x) \\ &= \sum_{i\in I} \left[\Phi_{0,i}(X_{i,3}) \int_{\sigma_i} u\Psi_{0,i}\,\mathrm{d}s + \Phi_{1,i}(X_{i,3}) \int_{\sigma_i} u\Psi_{1,i}\,\mathrm{d}s\right]\varphi_i(x) \\ &= \sum_{i\in I} \left[\int_{\sigma_i} u\Psi_{0,i}\,\mathrm{d}s\right]\varphi_i(x). \end{aligned} \tag{4.2}$$

Remark 4.1. $E_h u$ is well-defined only for $u \in W^{l,p}(\Omega)$ with

$$l \geqslant 2 \ \text{ for } \ p = 1, \qquad l > \frac{2}{p} \ \text{ otherwise.}$$

This guarantees $u|_{\sigma_i} \in L^1(\Omega)$. In the special case that $u \in W^{2,2}_{1-\pi/\omega+\varepsilon}(\Omega)$ the interpolant $E_h u$ is also well-defined, since one has the imbedding $W^{2,2}_{1-\pi/\omega+\varepsilon}(\Omega) \hookrightarrow W_0^{1+\pi/\omega-\varepsilon,2}(\Omega)$ (see [26, Theorem 1.3]) and $1 + \pi/\omega - \varepsilon > 1$.

The disadvantage of E_h is that it preserves Dirichlet boundary conditions only on Γ_M, but not on Γ_B. But this is necessary in order to derive an estimate for the finite element error for problem (1.1). In order to be able to treat boundary value problems with Dirichlet boundary conditions on the whole boundary $\partial\Omega$, we introduce an operator E_{0h} as a modification of E_h.

Let J be the index set which includes the indices of all nodes not belonging to Γ_B and let

$$V_{0h} := \{v_h \in V_h\colon\ v_h|_{\partial\Omega} = 0\}.$$

We define $E_{0h}\colon\ W^{2,p}(\Omega) \to V_{0h}$ as

$$(E_{0h}u)(x) := \sum_{i\in J}(\Pi_{\sigma_i}u)(X_i)\varphi_i(x).$$

Since $\varphi_i(x) = 0$ for all $x \in \Gamma_B$ and $i \in J$, the operator E_{0h} is preserving homogeneous Dirichlet boundary conditions also on Γ_B.

In the following we assume

$$h_{1,T} \leqslant h_{2,T} \leqslant h_{3,T}. \tag{4.3}$$

4.3. Local estimates in classical Sobolev spaces

We first recall an approximation result from [1].

Theorem 4.2. *Consider an element T of a tensor product mesh and assume that* (4.3) *is fulfilled. Then the approximation error estimate*

$$|u - E_h u|_{W^{1,q}(T)} \lesssim |T|^{1/q-1/p} \sum_{|\alpha|=1} h_T^{\alpha}|D^{\alpha}u|_{W^{1,p}(S_T)} \tag{4.4}$$

holds for $p \in [1,\infty]$, q such that $W^{2,p}(T) \hookrightarrow W^{1,q}(T)$ and $u \in W^{2,p}(S_T)$.

P r o o f. If one sets $l = 2$, $m = 1$ formula (4.4) is exactly (6.6) in Theorem 10 of [1]. □

Our aim is now to estimate $|u - E_{0h}u|_{W^{1,q}(T)}$ for a function $u \in W^{2,p}(T)$, $p \in [1,\infty]$, q such that $W^{2,p}(T) \hookrightarrow W^{1,q}(T)$ and $u|_{\Gamma_B} = 0$. By the triangle inequality we get

$$|u - E_{0h}u|_{W^{1,q}(T)} \leqslant |u - E_h u|_{W^{1,q}(T)} + |E_h u - E_{0h}u|_{W^{1,q}(T)}. \tag{4.5}$$

The first term on the right-hand side is treated in Theorem 4.2. It remains to find an estimate for the second term. To this end, we first prove the following auxiliary result.

Lemma 4.3. *Let T be an element with $\overline{T}\cap\Gamma_B\neq\emptyset$, I the index set of the nodes in $\overline{T}\cap\Gamma_B$ and u a function in $W^{2,p}(S_T)$ with S_T as defined in Section 2, $p\in[1,\infty]$ and $u|_{\Gamma_B}=0$. Then for every $i\in I$ and every linear function $\tilde\Phi_{1,i}$ with $\tilde\Phi_{1,i}|_{\sigma_i}=\Phi_{1,i}$ and $\tilde\Phi_{1,i}|_{\Gamma_B}=0$ there exists a $c_i\in\mathbb{R}$ such that*

$$\sum_{|\alpha|\leqslant 2} h^\alpha\|D^\alpha(u-c_i\tilde\Phi_{1,i})\|_{L^p(S_T)}\lesssim\sum_{|\alpha|=2} h^\alpha\|D^\alpha u\|_{L^p(S_T)}. \tag{4.6}$$

Furthermore, one has

$$\sum_{|\alpha|\leqslant 1} h^\alpha|D^\alpha u|_{W^{1,p}(S_T)}\lesssim\sum_{|\alpha|=1} h^\alpha|D^\alpha u|_{W^{1,p}(S_T)}. \tag{4.7}$$

Proof. Let g be a continuous function with the properties of a norm, i.e.

$$\begin{aligned} g(t_1,\dots,t_n)\geqslant 0 \quad\text{and}\quad & g(t_1,\dots,t_n)=0\Leftrightarrow t_1=\dots=t_n=0,\\ g(\lambda t_1,\dots,\lambda t_n) &= |\lambda|g(t_1,\dots,t_n),\\ g(t_1+\tau_1,\dots,t_n+\tau_n) &\leqslant g(t_1,\dots,t_n)+g(\tau_1,\dots,\tau_n). \end{aligned}$$

In Theorem 4.5.1 of [24] it is shown that for such functions and for linear functionals $l_1,l_2,\dots,l_N$ that are bounded in $W^{k,p}(\Omega)$ and do not vanish simultaneously on a polynomial with degree less than k besides the zero polynomial, the inequality

$$\|u\|_{W^{k,p}(\Omega)}\lesssim g(l_1u,l_2u,\dots,l_Nu)+|u|_{W^{k,p}(\Omega)} \tag{4.8}$$

is valid. Here N is the number of independent monomials of degree $\leqslant k-1$.

Now we prove (4.6) and (4.7) for the reference patch $S_{\hat T}$. In our case we have $N=4$, which is the number of monomials of degree less than or equal to 1 in three dimensions. We denote by $\hat e_i$ $(i=1,2,3)$ the three edges of $S_{\hat T}$ in the x_1x_2-plane. Then we set $l_1v:=\int_{\hat e_1}v$, $l_2v:=\int_{\hat e_2}v$, $l_3v:=\int_{\hat e_3}v$ and $l_4v:=\int_{S_{\hat T}}v$. For g we choose $g(t_1,t_2,t_3,t_4)=\sum_{i=1}^4|t_i|$. Now we set c_i such that $\int_{S_{\hat T}}(\hat u-c_i\hat\Phi_{1,i})=0$ and get

$$\|\hat u-c_i\hat\Phi_{1,i}\|_{W^{2,p}(S_{\hat T})}\lesssim\sum_{i=1}^3\Bigl|\int_{\hat e_i}(\hat u-c_i\hat\Phi_{1,i})\Bigr|+\Bigl|\int_{S_{\hat T}}(\hat u-c_i\hat\Phi_{1,i})\Bigr|+|\hat u-c_i\hat\Phi_{1,i}|_{W^{2,p}(S_{\hat T})}.$$

Since $\hat\Phi_{1,i}$ is linear and $\hat\Phi_{1,i}|_{\hat e_j}=0$ $(j=1,2,3)$, we end up with

$$\|\hat u-c_i\hat\Phi_{1,i}\|_{W^{2,p}(S_{\hat T})}\lesssim|\hat u|_{W^{2,p}(S_{\hat T})}.$$

The transformation back to S_T yields (4.6).

In the case of (4.7) we have $k = 1$ and $N = 1$. We set $l_1 v = \int_{S_{\hat{T}} \cap \{z=0\}} v \, ds$. Since $\hat{u}$ vanishes on $S_{\hat{T}} \cap \{z = 0\}$, one has $l_1 \hat{u} = 0$ and by (4.8) this yields

$$\|\hat{u}\|_{W^{1,p}(S_{\hat{T}})} \lesssim |\hat{u}|_{W^{1,p}(S_{\hat{T}})}.$$

The transformation back to S_T results in (4.7). □

With this result at hand, we are now able to give an estimate of the second term of the right-hand side of inequality (4.5).

Theorem 4.4. *Consider an element T of a tensor product mesh and assume that* (4.3) *is fulfilled. Then the error estimate*

$$|E_{0h}u - E_h u|_{W^{1,q}(T)} \lesssim |T|^{1/q-1/p} \sum_{|\alpha|=1} h^\alpha |D^\alpha u|_{W^{1,p}(S_T)} \tag{4.9}$$

if $p \in [1, \infty]$, q is such that $W^{2,p}(T) \hookrightarrow W^{1,q}(T)$, $u \in W^{2,p}(S_T)$ and $u|_{\overline{T} \cap \Gamma_B} = 0$.

P r o o f. For an element T with $\overline{T} \cap \Gamma_B = \emptyset$ one has $E_{0h}u - E_h u = 0$ and (4.9) is valid. For an element T with $\overline{T} \cap \Gamma_B \neq \emptyset$ denote by B_T the index set of nodes belonging to Γ_B, $B_T := \{i\colon\ X_i \in \overline{T} \cap \Gamma_B\}$. We treat the derivatives in the different directions separately. For the estimate of the derivative in the x_3-direction we obtain together with (4.2) and (4.1)

$$\begin{aligned} \Big\|\frac{\partial}{\partial x_3}(E_h - E_{0h})u\Big\|_{L^q(T)} &= \Big\|\sum_{i \in B_T} (\Pi_{\sigma_i} u) \frac{\partial}{\partial x_3}\varphi_i\Big\|_{L^q(T)} \\ &= \Big\|\sum_{i \in B_T} \Big[\int_{\sigma_i} u \Psi_{0,i}\Big] \frac{\partial}{\partial x_3}\varphi_i\Big\|_{L^q(T)} \\ &= \Big\|\sum_{i \in B_T} \Big[\int_{\sigma_i} (u - c_i \Phi_{1,i}) \Psi_{0,i}\Big] \frac{\partial}{\partial x_3}\varphi_i\Big\|_{L^q(T)} \end{aligned} \tag{4.10}$$

for arbitrary $c_i \in \mathbb{R}$. We use

$$\|\Psi_{0,i}\|_{L^\infty(\sigma_i)} \lesssim |\sigma_i|^{-1}$$

and the trace theorem $W^{2,p}(S_T) \hookrightarrow L^1(\sigma_i)$, $p \geqslant 1$ in the form

$$\|v\|_{L^1(\sigma_i)} \lesssim |\sigma_i| |T|^{-1/p} \sum_{|\alpha| \leqslant 2} h^\alpha \|D^\alpha v\|_{L^p(S_T)} \tag{4.11}$$

to get the estimate

$$\left|\int_{\sigma_i}(u-c_i\Phi_{1,i})\Psi_{0,i}\,\mathrm{d}s\right| \leqslant \|\Psi_{0,i}\|_{L^\infty(\sigma_i)}\|u-c_i\Phi_{1,i}\|_{L^1(\sigma_i)}$$
$$\lesssim |T|^{-1/p}\sum_{|\alpha|\leqslant 2} h^\alpha\|D^\alpha(u-c_i\tilde{\Phi}_{i,1})\|_{L^p(S_T)},$$

where $\tilde{\Phi}_{i,1}$ is a linear function with $\tilde{\Phi}_{i,1}|_{\sigma_i}=\Phi_{i,1}$. By virtue of Lemma 4.3 we can conclude

$$\left|\int_{\sigma_i}(u-c_i\Phi_{1,i})\Psi_{0,i}\,\mathrm{d}s\right| \lesssim |T|^{-1/p}\sum_{|\alpha|=2} h^\alpha\|D^\alpha u\|_{L^p(S_T)}.$$

Taking into account that

$$\Big\|\frac{\partial}{\partial x_3}\varphi_i\Big\|_{L^q(T)} \lesssim |T|^{1/q}h_3^{-1} \quad \text{for } i\in B,$$

we can continue to obtain from (4.10)

$$\Big\|\frac{\partial}{\partial x_3}(E_h-E_{0h})u\Big\|_{L^q(T)} \lesssim \sum_{i\in B}\left|\int_{\sigma_i}(u-c_i\Phi_{1,i})\Psi_{0,i}\right|\Big\|\frac{\partial}{\partial x_3}\varphi_i\Big\|_{L^q(T)}$$
$$\lesssim |T|^{1/q-1/p}\,h_3^{-1}\sum_{|\alpha|=2} h^\alpha\|D^\alpha u\|_{L^p(S_T)}.$$

By (4.3) we finally conclude

$$\Big\|\frac{\partial}{\partial x_3}(E_h-E_{0h})u\Big\|_{L^q(T)} \lesssim |T|^{1/q-1/p}\sum_{|\alpha|=1} h^\alpha\|D^\alpha u\|_{W^{1,p}(S_T)}. \tag{4.12}$$

For the estimates concerning the derivatives in the x_2- and x_1-direction we use a different technique developed in [1]. Let us discuss the case of the x_2-derivative; the x_1-derivative can be proved by analogy.

First we consider the case that three nodes of T are contained in Γ_B, that is $|B_T|=3$. We denote these nodes by X_0, X_1, and X_2, where the edge spanned by X_0 and X_1 is parallel to the x_1-axis and the one spanned by X_0 and X_2 is parallel to the x_2-axis. Then one has

$$(E_{0h}u-E_hu)|_T=\sum_{i=0}^{2}a_i\varphi_i=(a_0-a_2)\varphi_0+a_2(\varphi_0+\varphi_2)+a_1\varphi_1,$$

where we have set $a_i:=\int_{\sigma_i}u\Psi_{0,i}$.

Taking into account that T is a tensor product element, we can conclude

$$\frac{\partial}{\partial x_2}\varphi_1 = 0 \quad \text{and} \quad \frac{\partial}{\partial x_2}(\varphi_0 + \varphi_2) = 0.$$

This yields

$$\Big\|\frac{\partial}{\partial x_2}(E_{0h} - E_h)u\Big\|_{L^q(T)} = |a_0 - a_2|\Big\|\frac{\partial}{\partial x_2}\varphi_0\Big\|_{L^q(T)}. \tag{4.13}$$

Since $\{x_3\colon (x_1,x_2,x_3) \in \sigma_0\} = \{x_3\colon (x_1,x_2,x_3) \in \sigma_2\}$, $\Psi_{0,0} = \Psi_{0,2}$ and $X_{0,1} = X_{2,1}$, we get for the first factor

$$\begin{aligned}
|a_0 - a_2| &= \Big|\int_{\sigma_0} u(X_{0,1}, X_{0,2}, z)\Psi_{0,0}(z)\,\mathrm{d}z - \int_{\sigma_2} u(X_{0,1}, X_{2,2}, z)\Psi_{0,2}(z)\,\mathrm{d}z\Big| \\
&= \Big|\int_{\sigma_0} \Psi_{0,0}(z)\int_{X_{0,2}}^{X_{2,2}} \frac{\partial}{\partial x_2}u(X_{0,1}, y, z)\,\mathrm{d}y\,\mathrm{d}z\Big| \\
&\lesssim \|\Psi_{0,0}\|_{L^\infty(\sigma_0)}\Big|\int_{\sigma_0}\int_{X_{0,2}}^{X_{2,2}} \frac{\partial}{\partial x_2}u(X_{0,1}, y, z)\,\mathrm{d}y\,\mathrm{d}z\Big| \\
&\lesssim h_1^{-1}h_3^{-1}\sum_{|\alpha|\leqslant 1} h^\alpha\Big\|D^\alpha\Big(\frac{\partial u}{\partial x_2}\Big)\Big\|_{L^1(S_T)}.
\end{aligned}$$

In the last estimate we have used the trace theorem $W^{1,1}(S_T) \hookrightarrow L^1(\Xi_1)$, where Ξ_1 is the two-dimensional manifold spanned by σ_0 and X_0X_2 in the form

$$\|u\|_{L^1(\Xi_1)} \leqslant |\Xi_1||T|^{-1}\sum_{|\alpha|\leqslant 1} h^\alpha\|D^\alpha u\|_{L^1(S_T)}.$$

From

$$\Big\|\frac{\partial}{\partial x_2}\varphi_0\Big\|_{L^q(T)} \lesssim h_2^{-1}|T|^{1/q},$$

obtained by using the inverse inequality, it follows from (4.13) together with the Hölder inequality that

$$\begin{aligned}
\Big\|\frac{\partial}{\partial x_2}(E_{0h} - E_h)u\Big\|_{L^q(T)}] &\lesssim (h_1h_2h_3)^{-1}|T|^{1/q}\sum_{|\alpha|\leqslant 1} h^\alpha\Big\|D^\alpha\Big(\frac{\partial u}{\partial x_2}\Big)\Big\|_{L^1(S_T)} \\
&\lesssim |T|^{1/q-1/p}\sum_{|\alpha|\leqslant 1} h^\alpha\Big\|D^\alpha\Big(\frac{\partial u}{\partial x_2}\Big)\Big\|_{L^p(S_T)}.
\end{aligned}$$

The application of Lemma 4.3 yields

$$\Big\|\frac{\partial}{\partial x_2}(E_{0h} - E_h)u\Big\|_{L^q(T)} \lesssim |T|^{1/q-1/p}\sum_{|\alpha|=1} h^\alpha\Big\|D^\alpha\Big(\frac{\partial u}{\partial x_2}\Big)\Big\|_{L^p(S_T)}, \tag{4.14}$$

since $\partial u/\partial x_2 = 0$ on Γ_B. Let us now consider the case where only two nodes X_0, X_1 of T are contained in Γ_B, which means $|B_T| = 2$. One has

$$(E_{0h}u - E_h u)|_T = a_0\varphi_0 - a_1\varphi_1. \tag{4.15}$$

We have to treat three different cases. First the case that the edge spanned by X_0 and X_1 is parallel to the x_2-axis, then the case that it is parallel to the x_1-axis and finally the case that is neither parallel to the x_1-axis nor to the x_2-axis. We first consider the case that the edge is parallel to the x_2-axis. One can rewrite (4.15) as

$$(E_{0h} - E_h u)|_T = (a_0 - a_1)\varphi_0 + a_1(\varphi_0 + \varphi_1).$$

Now one can proceed exactly as in the case with three nodes in Γ_B and obtain (4.14).

If the edge spanned by X_0 and X_1 is parallel to the x_1-axis one has

$$\frac{\partial}{\partial x_2}\varphi_0 = \frac{\partial}{\partial x_2}\varphi_1 = 0 \tag{4.16}$$

and from (4.15) one can conclude

$$\left\|\frac{\partial}{\partial x_2}(E_{0h} - E_h)u\right\|_{L^q(T)} = 0. \tag{4.17}$$

Consider now the case where the edge spanned by X_0 and X_1 is neither parallel to the x_1-axis nor to the x_2-axis. In the case that the remaining nodes X_2, X_3 of the tetrahedra span an edge that is parallel to the x_2-axis the nodal functions φ_0 and φ_1 do not depend on x_2 and equation (4.16) is valid. Equation (4.17) follows then from (4.15). If the edge spanned by X_2 and X_3 is parallel to the x_1-axis a more detailed analysis is necessary. Therefore, we rewrite (4.15) again as

$$(E_{0h}u - E_h u)|_T = (a_0 - a_1)\varphi_0 + a_1(\varphi_0 + \varphi_1).$$

A short computation shows that $\varphi_0 + \varphi_1 = 1 - x_3$ and, consequently,

$$\frac{\partial}{\partial x_2}(\varphi_0 + \varphi_1) = 0.$$

With $X_i = (X_{i,1}, X_{i,2}, X_{i,3})$ $(i = 0, 1)$ one can write

$$\begin{aligned} |a_0 - a_1| &= \left| \int_{\sigma_0} u(X_{0,1}, X_{0,2}, z)\Psi_{0,0}(z)\,\mathrm{d}z - \int_{\sigma_1} u(X_{1,1}, X_{1,2}, z)\Psi_{0,1}(z)\,\mathrm{d}z \right| \\ &= \left| \int_{\sigma_0} [u(X_{0,1}, X_{0,2}, z) - u(X_{1,1}, X_{1,2}, z)]\Psi_{0,0}(z)\,\mathrm{d}z \right|. \end{aligned}$$

The triangle inequality yields

$$
\begin{aligned}
|a_0 - a_1| \leqslant & \left| \int_{\sigma_0} [u(X_{0,1}, X_{0,2}, z) - u(X_{1,1}, X_{0,2}, z)] \Psi_{0,0}(z) \,\mathrm{d}z \right| \\
& + \left| \int_{\sigma_0} [u(X_{1,1}, X_{0,2}, z) - u(X_{1,1}, X_{1,2}, z)] \Psi_{0,0}(z) \,\mathrm{d}z \right| \\
= & \left| \int_{\sigma_0} \Psi_{0,0}(z) \int_{X_{1,1}}^{X_{0,1}} \frac{\partial}{\partial x_1} u(x, X_{0,2}, z) \,\mathrm{d}x \,\mathrm{d}z \right| \\
& + \left| \int_{\sigma_0} \Psi_{0,0}(z) \int_{X_{1,2}}^{X_{0,2}} \frac{\partial}{\partial x_2} u(X_{1,1}, y, z) \,\mathrm{d}y \,\mathrm{d}z \right|.
\end{aligned}
$$

Now one can proceed as in the case of three nodes in Γ_B arriving at

$$
|a_0 - a_1| \lesssim h_1^{-1} h_3^{-1} \sum_{\alpha \leqslant 1} h^\alpha \left[\left\| D^\alpha \Big(\frac{\partial u}{\partial x_1} \Big) \right\|_{L^1(S_T)} + \left\| D^\alpha \Big(\frac{\partial u}{\partial x_2} \Big) \right\|_{L^1(S_T)} \right].
$$

Since

$$
\left\| \frac{\partial}{\partial x_2} \varphi_0 \right\|_{L^q(T)} \lesssim h_2^{-1} |T|^{1/q} \quad \text{and} \quad h_1 \leqslant h_2,
$$

it follows as in the case of three nodes in Γ_B (comp. (4.14)) that

$$
\begin{aligned}
\left\| \frac{\partial}{\partial x_2} (E_{0h} - E_h) u \right\|_{L^q(T)} \lesssim & |T|^{1/q - 1/p} \sum_{|\alpha|=1} h^\alpha \left\| D^\alpha \Big(\frac{\partial u}{\partial x_1} \Big) \right\|_{L^p(S_T)} \\
& + |T|^{1/q - 1/p} \sum_{|\alpha|=1} h^\alpha \left\| D^\alpha \Big(\frac{\partial u}{\partial x_2} \Big) \right\|_{L^p(S_T)}.
\end{aligned}
$$

It remains to deal with the case where only one node X_0 of T is contained in Γ_B. The difference of E_{0h} and E_h in T reduces to

$$
(E_{0h} u - E_h u)_T = a_0 \varphi_0.
$$

Since T is a tensor product element, one has $\varphi_0 = \varphi_0(x_3)$ and consequently

$$
\left\| \frac{\partial}{\partial x_2} (E_{0h} - E_h) u \right\|_{L^q(T)} = |a_0|^{1/q} \left\| \frac{\partial}{\partial x_2} \varphi_0 \right\|_{L^q(T)} = 0.
$$

Summarizing all the cases, we obtain

$$
\left\| \frac{\partial}{\partial x_2} (E_{0h} - E_h) u \right\|_{L^q(T)} \lesssim |T|^{1/q - 1/p} \sum_{|\alpha|=1} h^\alpha \| D^\alpha u \|_{W^{1,p}(S_T)}. \tag{4.18}
$$

The proof for an estimate of the error in the x_1-derivative is analogous to the x_2-case and one gets

$$\left\|\frac{\partial}{\partial x_1}(E_{0h}-E_h)u\right\|_{L^q(T)} \lesssim |T|^{1/q-1/p}\sum_{|\alpha|=1} h^\alpha \|D^\alpha u\|_{W^{1,p}(S_T)}. \tag{4.19}$$

By virtue of (4.12), (4.18), and (4.19) the assertion is shown. □

Theorem 4.5. *Consider an element T of a tensor product mesh and assume that* (4.3) *is fulfilled. Then the error estimate*

$$|u-E_{0h}u|_{W^{1,q}(T)} \lesssim |T|^{1/q-1/p}\sum_{|\alpha|=1} h_T^\alpha |D^\alpha u|_{W^{1,p}(S_T)} \tag{4.20}$$

holds for $p\in[1,\infty]$, q such that $W^{2,p}(T)\hookrightarrow W^{1,q}(T)$, $u\in W^{2,p}(S_T)$ and $u|_{\overline{T}\cap\Gamma_B}=0$.

P r o o f. Inequality (4.20) follows by the triangle inequality from (4.4) and (4.9). □

4.4. Local estimates in weighted Sobolev spaces

In order to get a global estimate for the interpolation error, it is useful to have an estimate where certain first derivatives of the interpolant are estimated against the first derivatives of the solution u. This additional stability estimate is necessary, since $u\notin W^{2,2}(T)$ for elements T with $r_T=0$. Thus we prove the following estimate for functions from weighted Sobolev spaces.

Lemma 4.6. *Consider a tensor product element T and assume that $h_{1,T}\sim h_{2,T}\lesssim h_{3,T}$. Let $p,q\in[1,\infty]$, $1-2/p<\beta<2-2/p$ and $\beta\leqslant 1$. Then for $u\in W^{1,p}(S_T)\cap V^{2,p}_\beta(S_T)$ and $u|_{\overline{T}\cap\Gamma_B}=0$ one has the estimate*

$$|E_{0h}u|_{W^{1,q}(T)} \lesssim |T|^{1/q-1/p} h_{1,T}^{-\beta}\sum_{|\alpha|=1} h_T^\alpha \|D^\alpha v\|_{V^{1,p}_\beta(S_T)}. \tag{4.21}$$

For $u\in W^{1,p}(S_T)\cap W^{2,p}_\beta(S_T)$ the estimate

$$\|E_h u\|_{W^{1,q}(T)} \lesssim |T|^{1/q-1/p} h_{1,T}^{-\beta}\sum_{|\alpha|=1} h_T^\alpha \|D^\alpha v\|_{W^{1,p}_\beta(S_T)} \tag{4.22}$$

is valid.

P r o o f. By the triangle inequality, Lemma 11 in [1] with $m=1$ and (4.9) one has

$$|E_{0h}u|_{W^{1,q}(T)} \leqslant |E_h u|_{W^{1,q}(T)} + |E_{0h}u - E_h u|_{W^{1,q}(T)} \lesssim |T|^{1/q-1} \sum_{|\alpha|\leqslant 1} h^{\alpha} |D^{\alpha}u|_{W^{1,1}(S_T)}. \tag{4.23}$$

The step from $|\alpha| \leqslant 1$ to $|\alpha| = 1$ is analogous to the proof of Lemma 11 in [1]. One has just to replace (6.13) in that proof by (4.23), and (4.21) is shown.

The second inequality can be proved in the following way. Since $\beta < 2 - 2/p$ one has $r^{-\beta} \in L^{p'}(S_T)$ with $1/p' = 1 - 1/p$, and for $v \in W^{0,p}_{\beta}(S_T)$ one can write

$$\|v\|_{L^1(S_T)} \leqslant \|r^{-\beta}\|_{L^{p'}(S_T)} \|r^{\beta} v\|_{L^p(S_T)}. \tag{4.24}$$

Consider now two cylindrical sectors Z_1, Z_2 with radii $c_1 h_{1,T}$ and $c_2 h_{1,T}$ so that $Z_1 \subset S_T \subset Z_2$. Since $h_{1,T} \sim h_{2,T}$, we can conclude

$$\Bigl(\int_{Z_i} r^{-\beta p'}\Bigr)^{1/p'} \sim \Bigl(h_{3,T} \int_0^{c_i h_{1,T}} r^{-\beta p'+1}\Bigr)^{1/p'} \sim (h_{3,T} h_{1,T}^{2-\beta p'})^{1/p'} \sim (|S_T| \cdot h_{1,T}^{-\beta p'})^{1/p'}$$

for $i = 1, 2$. This results in the inequality

$$\|r^{-\beta}\|_{L^{p'}(S_T)} \leqslant |S_T|^{1/p'} h_{1,T}^{-\beta}. \tag{4.25}$$

The two inequalities (4.24) and (4.25) yield the embedding $W^{2,p}_{\beta}(S_T) \hookrightarrow W^{2,1}(S_T)$ and it follows that $u \in W^{2,1}(S_T)$. Therefore, one has from Theorem 10 in [1]

$$\|E_h u\|_{W^{1,q}(T)} \lesssim |T|^{1/q-1} \sum_{|\alpha|\leqslant 1} h_T^{\alpha} \|D^{\alpha} u\|_{W^{1,1}(S_T)}. \tag{4.26}$$

Notice that the patch S_T defined in [1] is a subset of S_T as defined in Section 2. Now we proceed from (4.26) to

$$\begin{aligned}\|E_h u\|_{W^{1,q}(T)} &\lesssim |T|^{1/q-1} \sum_{|\alpha|\leqslant 1} \sum_{|t|\leqslant 1} h_T^{\alpha} \|D^{\alpha+t} u\|_{L^1(S_T)} \\ &\lesssim |T|^{1/q-1} |S_T|^{1-1/p} h_{1,T}^{-\beta} \sum_{|\alpha|\leqslant 1} \sum_{|t|\leqslant 1} h_T^{\alpha} \|r^{\beta} D^{\alpha+t} u\|_{L^p(S_T)} \\ &\sim |T|^{1/q-1/p} h_{1,T}^{-\beta} \sum_{\alpha\leqslant 1} h_T^{\alpha} \|D^{\alpha} u\|_{W^{1,p}_{\beta}(S_T)}\end{aligned}$$

and the assertion (4.22) is shown. □

Next we prove an interpolation error estimate for functions in $W^{2,p}_{\beta}(T)$. This result is necessary for estimating the finite element error of the pure Neumann problem (1.2).

Theorem 4.7. *Consider an element T of a tensor product mesh and assume that $h_{1,T} \sim h_{2,T} \lesssim h_{3,T}$ is fulfilled. Then the error estimate*

$$\|u - E_h u\|_{W^{1,q}(T)} \lesssim |T|^{1/q-1/p} h_{1,T}^{-\beta} \sum_{|\alpha|=1} h_T^{\alpha} \|D^{\alpha} u\|_{W_{\beta}^{1,p}(S_T)} \tag{4.27}$$

holds for $p \in [1, \infty]$, $\beta \in (1-2/p, 1]$, q such that $W_{\beta}^{1,p}(T) \hookrightarrow L^q(T)$ and $u \in W_{\beta}^{2,p}(T)$.

Proof. From the triangle inequality we have for an arbitrary function $w \in W_{\beta}^{2,p}(S_T)$

$$\|u - E_h u\|_{W^{1,q}(T)} \leqslant \|u - w\|_{W^{1,q}(T)} + \|E_h(u - w)\|_{W^{1,q}(T)}. \tag{4.28}$$

For the first term in this inequality one can conclude from the embedding $W_{\beta}^{1,p}(S_T) \hookrightarrow L^q(S_T)$

$$\begin{aligned} \|u - w\|_{W^{1,q}(T)} &\leqslant \|u - w\|_{W^{1,q}(S_T)} \\ &\lesssim |T|^{1/q} \sum_{|t|\leqslant 1} h_T^{-t} \|D^t(\hat{u} - \hat{w})\|_{L^q(S_{\hat{T}})} \\ &\lesssim |T|^{1/q} \sum_{|t|\leqslant 1} h_T^{-t} \|D^t(\hat{u} - \hat{w})\|_{W_{\beta}^{1,p}(S_{\hat{T}})} \\ &= |T|^{1/q} \sum_{|t|\leqslant 1} \sum_{|\alpha|\leqslant 1} h_T^{-t} \|r^{\beta} D^{\alpha+t}(\hat{u} - \hat{w})\|_{L^p(S_{\hat{T}})}. \end{aligned} \tag{4.29}$$

The application of (4.22) to $u - w$ yields

$$\begin{aligned} \|E_h(u - w)\|_{W^{1,q}(T)} &\lesssim |T|^{1/q-1/p} h_{1,T}^{-\beta} \sum_{|\alpha|\leqslant 1} h_T^{\alpha} \|D^{\alpha}(u - w)\|_{W_{\beta}^{1,p}(S_T)} \\ &\lesssim |T|^{1/q} \sum_{|\alpha|\leqslant 1} \sum_{|t|\leqslant 1} h_T^{-t} \|r^{\beta} D^{\alpha+t}(\hat{u} - \hat{w})\|_{L^p(S_{\hat{T}})}. \end{aligned} \tag{4.30}$$

By virtue of (4.28), (4.29), and (4.30) one obtains

$$\begin{aligned} \|u - E_h u\|_{W^{1,q}(T)} &\lesssim |T|^{1/q} \sum_{|\alpha|\leqslant 1} \sum_{|t|\leqslant 1} h_T^{-t} \|r^{\beta} D^{\alpha+t}(\hat{u} - \hat{w})\|_{L^p(S_{\hat{T}})} \\ &= |T|^{1/q} \sum_{|t|\leqslant 1} h_T^{-t} \|D^t(\hat{u} - \hat{w})\|_{W_{\beta}^{1,p}(S_{\hat{T}})}. \end{aligned} \tag{4.31}$$

Now we specify w as the function of $\mathcal{P}_1(S_T)$ such that

$$\int_{S_{\hat{T}}} D^t(\hat{u} - \hat{w}) = 0 \quad \forall\, t\colon\ |t| \leqslant 1$$

and using Lemma 2.2, we can continue from (4.31) to

$$\begin{aligned}
\|u - E_h u\|_{W^{1,q}(T)} &\lesssim |T|^{1/q} \sum_{|t|\leqslant 1} h_T^{-t} |D^t \hat{u}|_{W^{1,p}_\beta(S_{\hat{T}})} \\
&= |T|^{1/q} \sum_{|t|\leqslant 1} \sum_{|\alpha|=1} h_T^{-t} \|r^\beta D^{\alpha+t} \hat{u}\|_{L^p(S_{\hat{T}})} \\
&\lesssim |T|^{1/q-1/p} h_{1,T}^{-\beta} \sum_{|t|\leqslant 1} \sum_{|\alpha|=1} h_T^{\alpha} \|r^\beta D^{\alpha+t} u\|_{L^p(S_T)} \\
&= |T|^{1/q-1/p} h_{1,T}^{-\beta} \sum_{|\alpha|=1} h_T^{\alpha} \|D^\alpha u\|_{W^{1,p}_\beta(S_T)}
\end{aligned}$$

and the assertion (4.27) is shown. □

4.5. Extension to more general meshes

If we consider the special case $h_1 \sim h_2$, we can extend our results to more general meshes. Instead of tensor product elements we introduce as in [1] *elements of tensor product type* that are defined by the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} B_T & 0 \\ 0 & \pm h_{3,T} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + b_T =: \hat{B} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + b_T,$$

where $b_T \in \mathbb{R}^3$ and $B_T \in \mathbb{R}^{2\times 2}$ with

$$|\det B_T| \sim h_{1,T}^2, \quad \|B_T\| \sim h_{1,T}, \quad \|B_T^{-1}\| \sim h_{1,T}^{-1}.$$

Further, we introduce a coordinate system $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ via the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h_1^{-1} B_T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} =: \tilde{B} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}.$$

This transformation maps T and S_T to $\tilde{T}$ and $\tilde{S_T}$. Since

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \tilde{B}^{-1}\hat{B} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \tilde{B}^{-1} b_T = \begin{bmatrix} h_{1,T} & 0 & 0 \\ 0 & h_{1,T} & 0 \\ 0 & 0 & h_{3,T} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \tilde{B}^{-1} b_T,$$

the mesh is a tensor product mesh in the coordinate system $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$. Since $\tilde{S_T} = S_{\tilde{T}}$, it follows from

$$\det \tilde{B} \sim 1, \quad \|\tilde{B}\| \sim 1, \quad \|\tilde{B}^{-1}\| \sim 1$$

that our results extend to meshes of tensor product type.

In this section the main results of this paper, namely the optimal L^2-error estimates for the finite element solution of problems (1.1) and (1.2), are given. In order to get an optimal error estimate it is necessary to introduce appropriate meshes. Therefore, we define according to [5] a family of graded triangulations $\mathcal{T}_h = \{T\}$ of Ω consisting of tensor product elements. With h being the global mesh parameter, $\mu \in (0,1]$ being the grading parameter and r_T being the distance of a tetrahedron T to the reentrant edge,

$$r_T := \min_{(x_1,x_2,x_3)\in\overline{T}} (x_1^2 + x_2^2)^{1/2},$$

we assume that the element sizes satisfy for some constant $R > 0$

$$h_{1,T} \sim h_{2,T} \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0, \\ h r_T^{1-\mu} & \text{for } 0 < r_T \leqslant R, \\ h & \text{for } r_T > R. \end{cases} \quad h_{3,T} \sim h, \tag{5.1}$$

In the following we show the optimal convergence rate for the finite element method on these meshes.

5.1. The Dirichlet problem

Let $V_0 = W_0^{1,2}(\Omega)$ be the space of all $W^{1,2}(\Omega)$-functions that vanish on $\partial\Omega$. With the bilinear form $a_D(\cdot,\cdot)\colon V_0 \times V_0 \to \mathbb{R}$ and the linear form $(f,\cdot)\colon V_0 \to \mathbb{R}$ defined by

$$a_D(u,v) := \int_\Omega \nabla u \cdot \nabla v, \quad (f,v) := \int_\Omega fv,$$

the variational form of (1.1) is given by

$$\text{find } u \in V_0 \text{ such that } a_D(u,v) = (f,v) \quad \text{for all } v \in V_0. \tag{5.2}$$

The finite element solution u_h is determined by

$$\text{find } u_h \in V_{0h} \text{ such that } a_D(u_h,v_h) = (f,v_h) \quad \text{for all } v_h \in V_{0h}, \tag{5.3}$$

where $V_{0h} := V_0 \cap V_h$. Notice that the Lax-Milgram lemma guarantees unique solutions u and u_h. We are now able to state the following global estimate.

Theorem 5.1. *Let u be the solution of* (5.2)*. Then the estimate*

$$|u - E_{0h}u|_{W^{1,2}(\Omega)} \lesssim h\|f\|_{L^2(\Omega)} \tag{5.4}$$

holds if $\mu < \pi/\omega$.

Proof. The theorem can be proved along the lines of the proof of Theorem 14 of [1]. The necessary prerequisites are provided here by Lemmas 3.1, 3.3, Remark 3.5, and estimates (4.20) and (4.21) for $p = q = 2$. □

Theorem 5.2. *Let u be the solution of* (5.2) *and let u_h be the finite element solution defined by* (5.3)*. Assume that the mesh is refined according to $\mu < \pi/\omega$. Then the finite element error can be estimated by*

$$|u - u_h|_{W^{1,2}(\Omega)} \lesssim h\|f\|_{L^2(\Omega)}, \tag{5.5}$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^2\|f\|_{L^2(\Omega)}. \tag{5.6}$$

Proof. Estimate (5.5) is a conclusion of (5.4) and the projection property of the finite element method. (5.6) follows by the Aubin-Nitsche trick. □

5.2. The Neumann problem

The variational formulation of (1.2) is given by

$$\text{find } u \in V \text{ such that } a_N(u, v) = (f, v) \quad \text{for all } v \in V, \tag{5.7}$$

where for $V := W^{1,2}(\Omega)$ the bilinear form $a_N(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ is defined by

$$a_N(u, v) := \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv.$$

The finite element solution u_h is defined by

$$\text{find } u_h \in V_h \text{ such that } a_N(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \tag{5.8}$$

As in the case of Dirichlet boundary conditions we can give a global estimate of the interpolation error. However, we cannot prove this estimate in the same way as in Theorem 14 of [1] as we did in the proof of Theorem 5.1. This is due to the fact that u admits a different regularity in case of Neumann boundary conditions and, in particular, does not vanish at the edge. Instead, the results of Theorem 4.7 play a key role.

Theorem 5.3. *Let u be the solution of* (5.7). *Then the estimate*

$$\|u - E_h u\|_{W^{1,2}(\Omega)} \lesssim h\|f\|_{L^2(\Omega)} \tag{5.9}$$

holds if $\mu < \pi/\omega$.

P r o o f. We use the estimations of the local error to get an estimate for the global error. Therefore, we distinguish between elements next to the edge M and elements away from M. We begin with the elements T with $\overline{T} \cap M = \emptyset$. Then $u \in W^{2,2}(T)$ and from (4.4) it follows for $p = 2$ that

$$\begin{aligned} \|u - E_h u\|_{W^{1,2}(T)} &\lesssim \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha u\|_{W^{1,2}(S_T)} \\ &\lesssim \sum_{i=1}^{2} h_{i,T} r_T^{-\beta} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{W_\beta^{1,2}(S_T)} + h_{3,T} \Big\| \frac{\partial u}{\partial x_3} \Big\|_{W^{1,2}(S_T)} \end{aligned}$$

for all $\beta < 1 - \pi/\omega$. For the last estimate we have used Lemma 3.3 and $r_T \lesssim h_{3,T}$. Since $\mu < \pi/\omega$, the choice $\beta = 1 - \mu$ is admissible and we obtain for $r_T \leqslant R$ from (5.1) the relation

$$h_{i,T} r_T^{-\beta} \sim h r_T^{1-\mu-\beta} = h \quad (i = 1, 2).$$

For $r_T > R$ we have

$$h_{i,T} r_T^{-\beta} \lesssim h R^{-\beta} \sim h.$$

Combining the last two estimates with the fact that $h_{3,T} \sim h$, one arrives at

$$\|u - E_h u\|_{W^{1,2}(T)} \lesssim h \sum_{i=1}^{2} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{W_\beta^{1,2}(S_T)} + h \Big\| \frac{\partial u}{\partial x_3} \Big\|_{W^{1,2}(S_T)}. \tag{5.10}$$

For an element T with $T \cap M \neq \emptyset$ we can estimate according to Theorem 4.7 for $p = q = 2$, since $W_\beta^{2,2}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ (see (2.2)):

$$\|u - E_h u\|_{W^{1,2}(T)} \lesssim \sum_{i=1}^{3} h_{i,T} h_{1,T}^{-\beta} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{W_\beta^{1,2}(T)}$$

$$\lesssim \sum_{i=1}^{2} h^{(1-\beta)/\mu} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{W_\beta^{1,2}(T)} + h_{3,T} \Big\| \frac{\partial u}{\partial x_3} \Big\|_{W^{1,2}(T)} \tag{5.11}$$

$$\lesssim \sum_{i=1}^{2} h \Big\| \frac{\partial u}{\partial x_i} \Big\|_{W_\beta^{1,2}(T)} + h \Big\| \frac{\partial u}{\partial x_3} \Big\|_{W^{1,2}(T)}, \tag{5.12}$$

where we have used the additional regularity of u in the x_3-direction (see Corollary 3.4), $r^\beta \lesssim h_{1,T}^\beta$ in (5.11) and $\beta = 1 - \mu$.

The estimates (5.10) and (5.12) yield together with the fact that the number of elements in S_T is bounded by a constant the inequality

$$\begin{aligned}\|u - E_h u\|^2_{W^{1,2}(\Omega)} &= \sum_{T\in\mathcal{T}_h} \|u - E_h u\|^2_{W^{1,2}(T)} \\ &\lesssim h^2\Big(\sum_{i=1}^{2}\Big\|\frac{\partial u}{\partial x_i}\Big\|^2_{W^{1,2}_\beta(T)} + \Big\|\frac{\partial u}{\partial x_3}\Big\|^2_{W^{1,2}(T)}\Big).\end{aligned}$$

Together with the regularity results from Theorem 3.2 and Corollary 3.4 this proves the desired estimate (5.9). □

This global estimate of the interpolation error yields an estimate for the finite element error.

Theorem 5.4. *Let u be the solution of* (5.7) *and let u_h be the finite element solution defined by* (5.8). *Assume that the mesh is refined according to $\mu < \pi/\omega$. Then the finite element error can be estimated by*

$$\begin{aligned}|u - u_h|_{W^{1,2}(\Omega)} &\lesssim h\|f\|_{L^2(\Omega)},\\ \|u - u_h\|_{L^2(\Omega)} &\lesssim h^2\|f\|_{L^2(\Omega)}.\end{aligned}$$

Proof. The assertion follows from inequality (5.9) like the assertion of Theorem 5.2 from (5.4). □

References

[1] *Th. Apel*: Interpolation of non-smooth functions on anisotropic finite element meshes. M2AN, Math. Model. Numer. Anal. *33* (1999), 1149–1185.

[2] *Th. Apel, M. Dobrowolski*: Anisotropic interpolation with applications to the finite element method. Computing *47* (1992), 277–293.

[3] *Th. Apel, B. Heinrich*: Mesh refinement and windowing near edges for some elliptic problem. SIAM J. Numer. Anal. *31* (1994), 695–708.

[4] *Th. Apel, S. Nicaise*: Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes. Preprint. TU Chemnitz-Zwickau SPC94_16, 1994.

[5] *Th. Apel, S. Nicaise*: Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes. Partial Differential Equations and Functional Analysis. In Memory of Pierre Grisvard (J. Cea, D. Chenais, G. Geymonat, J.-L. Lions, eds.). Birkhäuser, Boston, 1996, pp. 18–34, shortened version of Preprint SPC94_16, TU Chemnitz-Zwickau, 1994.

[6] *Th. Apel, A.-M. Sändig, J. R. Whiteman*: Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. Math. Methods Appl. Sci. *19* (1996), 63–85.

[7] *Th. Apel, D. Sirch, G. Winkler*: Error estimates for control contstrained optimal control problems: Discretization with anisotropic finite element meshes. Preprint SPP1253-02-06 publ DFG Priority Program 1253. Erlangen, 2008.
[8] *N. Arada, E. Casas, F. Tröltzsch*: Error estimates for the numerical approximation of a semilinear elliptic control problem. Comput. Optim. Appl. *23* (2002), 201–229.
[9] *A. E. Beagles, J. R. Whiteman*: Finite element treatment of boundary singularities by augmentation with non-exact singular functions. Numer. Methods Partial Differ. Equations *2* (1986), 113–121.
[10] *E. Casas, M. Mateos, F. Tröltzsch*: Error estimates for the numerical approximation of boundary semilinear elliptic control problems. Comput. Optim. Appl. *31* (2005), 193–219.
[11] *P. Clément*: Approximation by finite element functions using local regularization. Rev. Franc. Automat. Inform. Rech. Operat. *2* (1975), 77–84.
[12] *M. Dauge*: Neumann and mixed problems on curvilinear polyhedra. Integral Equations Oper. Theory *15* (1992), 227–261.
[13] *R. S. Falk*: Approximation of a class of optimal control problems with order of convergence estimates. J. Math. Anal. Appl. *44* (1973), 28–47.
[14] *T. Geveci*: On the approximation of the solution of an optimal control problem governed by an elliptic equation. RAIRO, Anal. Numér. *13* (1979), 313–328.
[15] *P. Grisvard*: Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, Vol. 24. Pitman, Boston, 1985.
[16] *P. Grisvard*: Singularities in boundary value problems. Research Notes in Applied Mathematics, Vol. 22. Masson/Springer, Paris/Berlin, 1992.
[17] *P. Grisvard*: Singular behavior of elliptic problems in non Hilbertian Sobolev spaces. J. Math. Pures Appl., IX. Sér. *74* (1995), 3–33.
[18] *M. Hinze*: A variational discretization concept in control constrained optimization: The linear-quadratic case. Comput. Optim. Appl. *30* (2005), 45–61.
[19] *V. A. Kondrat'ev*: Singularities of the solution of Dirichlet problem for a second-order elliptic equation in the neighborhood of an edge. Differ. Uravn. *13* (1977), 2026–2032. (In Russian.)
[20] *A. Kufner, A.-M. Sändig*: Some Applications of Weighted Sobolev Spaces. Teubner, Leipzig, 1987.
[21] *J. M.-S. Lubuma, S. Nicaise*: Finite element method for elliptic problems with edge singularities. J. Comput. Appl. Math. *106* (1999), 145–168.
[22] *K. Malanowski*: Convergence of approximations vs. regularity of solutions for convex, control-constrained optimal-control problems. Appl. Math. Optimization *8* (1982), 69–95.
[23] *C. Meyer, A. Rösch*: Superconvergence properties of optimal control problems. SIAM J. Control Optim. *43* (2004), 970–985.
[24] *S. G. Michlin*: Partielle Differentialgleichungen in der mathematischen Physik. Akademie-Verlag, Berlin, 1978. (In German.)
[25] *S. A. Nazarov, B. A. Plamenevsky*: Elliptic Problems in Domains with Piecewise Smooth Boundaries. Walther de Gruyter & Co., Berlin, 1994.
[26] *J. Roßmann*: Gewichtete Sobolev-Slobodetskiĭ-Räume und Anwendungen auf elliptische Randwertaufgaben in Gebieten mit Kanten. Habilitationsschrift. Universität Rostock, Rostock, 1988. (In German.)
[27] *J. Roßmann*: The asymptotics of the solutions of linear elliptic variational problems in domains with edges. Z. Anal. Anwend. *9* (1990), 565–578.
[28] *A.-M. Sändig*: Error estimates for finite-element solutions of elliptic boundary value problems in non-smooth domains. Z. Anal. Anwend. *9* (1990), 133–153.

[29] *L. R. Scott, S. Zhang*: Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comput. *54* (1990), 483–493.
[30] *V. Zaionchkovskii, V. A. Solonnikov*: Neumann problem for second-order elliptic equations in domains with edges on the boundary. J. Sov. Math. *27* (1984), 2561–2586.

Authors' addresses: *T. Apel*, Universität der Bundeswehr München, Institut für Mathematik und Bauinformatik, 85577 Neubiberg, Germany, e-mail: thomas.apel@unibw.de; *D. Sirch*, Universität der Bundeswehr München, Institut für Mathematik und Bauinformatik, 85577 Neubiberg, Germany (until Oct. 2010), e-mail: dieter.sirch@gmx.de.