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DIV-CURL LEMMA REVISITED: APPLICATIONS IN ELECTROMAGNETISM

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Two new time-dependent versions of div-curl results in a bounded domain $\Omega \subset \mathbb{R}^3$ are presented. We study a limit of the product $\boldsymbol{v}_k\boldsymbol{w}_k$, where the sequences $\boldsymbol{v}_k$ and $\boldsymbol{w}_k$ belong to $\mathbf{L}_2(\Omega)$. In Theorem 2.1 we assume that $\nabla \times \boldsymbol{v}_k$ is bounded in the L_p-norm and $\nabla \cdot \boldsymbol{w}_k$ is controlled in the L_r-norm. In Theorem 2.2 we suppose that $\nabla \times \boldsymbol{w}_k$ is bounded in the L_p-norm and $\nabla \cdot \boldsymbol{w}_k$ is controlled in the L_r-norm. The time derivative of $\boldsymbol{w}_k$ is bounded in both cases in the norm of $\mathbf{H}^{-1}(\Omega)$. The convergence (in the sense of distributions) of $\boldsymbol{v}_k\boldsymbol{w}_k$ to the product $\boldsymbol{v}\boldsymbol{w}$ of weak limits of $\boldsymbol{v}_k$ and $\boldsymbol{w}_k$ is shown.

Keywords: compensated compactness, convergence, vector fields

Classification: 35B05, 65M99

1. INTRODUCTION

The structure of this paper is as follows. Section 1.1 shows the main differences in compactness arguments for diffusion processes and for electromagnetic fields. Section 1.2 collects some known important versions of the div-curl lemma. New compactness results are derived in Section 2, namely in Theorems 2.1 and 2.2. A numerical experiment from electromagnetic fields is presented in Section 3.

1.1. Different ways of compactness

Modeling the process of real physical events is a complex procedure including the following steps:

- Physical model and its analysis.
- Mathematical description including partial differential equations (PDEs) and boundary conditions (BCs), which reflect the situation outside the region of interest.
- Qualitative and quantitative mathematical analysis of the present model.
- Numerical study including discretization, (a priori and/or a posteriori) estimates, convergence analysis, (possible) error estimates.

- Computations and visualization of results.
- Calibration of model parameters.

This scheme is usually repeated until the coincidence between the computed results and the measurements becomes reasonable. Only very few problems admit exact solutions. In most cases one can find only an approximation of a solution in an appropriate function space. The choice of such a space can be crucial for the well posedness of the problem. Variational framework has been developed in order to decrease impositions put on a solution. Using integration by parts (Green formula) one can put the half of the derivatives from the differential operator onto a suitable test function. A solution obtained in this way is called variational or weak. The appropriate test spaces (usually Hilbert or reflexive Banach) have to reflect the BCs in a natural way. Using the Galerkin approximation the test space can be approximated by finite dimensional subspaces containing approximate solutions. If the problem setting in considerations is linear, then a weak convergence is sufficient to prove convergence of approximations towards an exact solution. In a case of nonlinear settings strong convergence of approximations is necessary.

Considering diffusion processes, the variational framework is usually based on Sobolev spaces $H^{k,p}(\Omega)$ or their suitable subspaces, whose properties are familiar. Convergence of approximations for steady-state elliptic problems is frequently based on the compact embedding $H^{k,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)$ – see [16]. For monotone operators (cf. [23, 29]) one can apply the well known Minty–Browder trick based on a monotone behavior of the nonlinearity, cf. [9, 10]. Time dependent problems have to contain some information about the time derivative. The proof of convergence is then based on the Arzela–Ascoli theorem or Kolmogorov's argument ([16]). Possible (nonlinear) BCs can be handled using trace theorems and the Nečas inequality – see [22]

$$\|z\|_\Gamma^2 \le \varepsilon \|\nabla z\|^2 + C_\varepsilon \|z\|^2, \qquad \forall\, z \in H^1(\Omega), \qquad 0 < \varepsilon < \varepsilon_0.$$

The situation in electromagnetism is more delicate, which is given by the nature of differential operator appearing in the modeling of electromagnetic fields. Let us consider an open bounded domain $\Omega \subset \mathbb{R}^3$ (with a Lipschitz boundary Γ), which is occupied by a ferromagnetic material. The electromagnetic field in Ω can be described by several vector fields $\boldsymbol{B}$ – magnetic induction, $\boldsymbol{H}$ – magnetic field, $\boldsymbol{E}$ – electric field, $\boldsymbol{D}$ – electric displacement field, and $\boldsymbol{J}$ – free current density. We consider the Maxwell equations of the form

$$\begin{array}{rcll} \partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} + \boldsymbol{J} &=& \mathbf{0}, & \text{Ampere's law} \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &=& \mathbf{0}, & \text{Faraday's law} \\ \boldsymbol{J} &=& \boldsymbol{J}_0 + \sigma \boldsymbol{E}, & \text{Ohm's law} \end{array} \qquad (1)$$

where $\boldsymbol{J}_0$ is a given vector field and σ denotes the conductivity. Variational framework usually uses subspaces of $\mathbf{H}(\mathbf{curl};\Omega)$, $\mathbf{H}(div;\Omega)$ as natural test spaces. The main disdvantage is that their embedding into $L_p(\Omega)$ is generally not compact. The traces of elements of $\mathbf{H}(\mathbf{curl};\Omega)$ or $\mathbf{H}(div;\Omega)$ do not generally belong to $L_p(\Gamma)$ spaces, cf. [6, 20]. Nevertheless suitable subspaces of $\mathbf{H}(\mathbf{curl};\Omega)$ are compactly embedded into $L_2(\Omega)$ (cf. [1, 20, 30]), namely

$$\{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(div;\Omega)\ \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0}\} \hookrightarrow\hookrightarrow \mathbf{L}_2(\Omega),$$

$$\{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(div;\Omega);\ \boldsymbol{u} \cdot \boldsymbol{\nu} = 0\} \hookrightarrow\hookrightarrow \mathbf{L}_2(\Omega),$$

$$\{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(div;\Omega);\ \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0},\ \boldsymbol{u} \cdot \boldsymbol{\nu} = 0\} = H_0^1(\Omega).$$

Inhomogeneous BCs can be handled using the following inequalities (cf. [8]) in a bounded Lipschitz domain in $\mathbb{R}^3$

$$\|\boldsymbol{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \le C\left(\|\boldsymbol{u}\|_{\mathbf{L}_2(\Omega)} + \|\nabla \times \boldsymbol{u}\|_{\mathbf{L}_2(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|_{\mathbf{L}_2(\Omega)} + \|\boldsymbol{u} \times \boldsymbol{\nu}\|_{\mathbf{L}_2(\Gamma)}\right)$$

and

$$\|\boldsymbol{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \le C\left(\|\boldsymbol{u}\|_{\mathbf{L}_2(\Omega)} + \|\nabla \times \boldsymbol{u}\|_{\mathbf{L}_2(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|_{\mathbf{L}_2(\Omega)} + \|\boldsymbol{u} \cdot \boldsymbol{\nu}\|_{L_2(\Gamma)}\right)$$

Moreover, for a convex or smooth domain with $\boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0}$ we obtain $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$.

1.2. Compensated compactness

The famous div-curl lemma represents a basic result of the compensated compactness theory in Sobolev spaces (see [10, 21, 28]). It was introduced by Murat and Tartar.

Lemma 1.1. (Murat [21]) Assume that $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$, $\{\boldsymbol{w}_k\}_{k=1}^{\infty}$ are bounded sequences in $L_2(\Omega)$ such that

(i) $\nabla \times \boldsymbol{w}_k$ lies in a compact subset of $W^{-1,2}$,

(ii) $\nabla \cdot \boldsymbol{v}_k$ lies in a compact subset of $W^{-1,2}$.

Suppose further

$$\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}, \qquad \boldsymbol{w}_k \rightharpoonup \boldsymbol{w} \qquad \text{in } L_2(\Omega).$$

Then

$$\lim_{k\to\infty} \boldsymbol{v}_k \boldsymbol{w}_k = \boldsymbol{v}\boldsymbol{w}$$

in the sense of distributions.

Here, the missing information about the gradient is compensated by some regularity of the divergence and of the curl operators. Then the convergence (but not up to the boundary) of the product $\boldsymbol{v}_k\boldsymbol{w}_k$ to $\boldsymbol{v}\boldsymbol{w}$ can be proved. Specially, if $\boldsymbol{v}_k = \boldsymbol{w}_k$ for all indices, then a suitable information about the divergence and of the curl implies the strong convergence of $\boldsymbol{v}_k$ in the $L_2(\Omega')$ for any compact subset $\Omega' \subset \Omega$. The proof of Murat's lemma is essentially based on the identity $-\Delta \boldsymbol{M} = \nabla\times(\nabla\times\boldsymbol{M}) - \nabla(\nabla\cdot\boldsymbol{M})$. Suitable information over $\boldsymbol{M}, \nabla\times\boldsymbol{M}$ and $\nabla\cdot\boldsymbol{M}$ can yield a compactness argument. First results were determined in $L_2(\Omega)$ spaces and later generalized to $L_p(\Omega)$ case, cf. [15]. Generalization to a setting, where every component of the vectors $\boldsymbol{v}^k$ and $\boldsymbol{w}^k$ can lie in different $L_{p_i}(\Omega)$. This is of special interest in problems arising from limiting procedures in the hydrodynamic equations for plasmas and semiconductors.

Lemma 1.2. (Gasser & Marcati [12]) Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, smooth domain. Let $1 < p_i < \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ for $i = 1, \ldots, n$. Denote $p_{\min} = \min_{1 \le i \le n} p_i$, $p_{\max} = \max_{1 \le i \le n} p_i$. Assume

- $v_k^i \in L_{p_i'}(\Omega)$, $w_k^i \in L_{p_i}(\Omega)$ are uniformly bounded with respect to k in corresponding norms, and $\frac{1}{p_{\min}} - \frac{1}{n} < \frac{1}{p_{\max}}$;
- $\nabla \cdot \boldsymbol{v}_k$ lies in a compact set of $W^{-1,t}(\Omega)$ with $t \ge \max_{1 \le i \le n} p_i' = (p_{\min})'$;
- $\nabla \times \boldsymbol{w}_k$ lies in a compact set of $W^{-1,s_{ij}}(\Omega)$ with $\min_{1 \le j \le n} s_{ji} \ge p_i$ for $i = 1, \ldots, n$.

Then

$$\lim_{k \to \infty} \boldsymbol{v}_k \boldsymbol{w}_k = \boldsymbol{v}\boldsymbol{w}$$

in the sense of distributions, where $\boldsymbol{v}, \boldsymbol{w}$ are the corresponding weak limits.

In a case when the vectorial fields $\boldsymbol{v}_k$ and $\boldsymbol{w}_k$ are time dependent, one also needs some information about their time derivatives. More exactly, it is enough to keep the time derivative of one of the vectorial fields under control, e. g.,

Lemma 1.3. (Slodička [26]) Assume that $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$, $\{\boldsymbol{w}_k\}_{k=1}^{\infty}$ are bounded sequences in $L_2\left((0,T), \mathbf{L}_2(\Omega)\right)$ such that

(i) $\displaystyle\int_0^T \|\nabla \times \boldsymbol{w}_k\|^2 \le C,$

(ii) $\displaystyle\int_0^T \left[\|\nabla \cdot \boldsymbol{v}_k\|^2 + \|\partial_t \boldsymbol{v}_k\|_{-1}^2\right] \le C.$

Suppose further

$$\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}, \qquad \boldsymbol{w}_k \rightharpoonup \boldsymbol{w} \qquad \text{in } L_2\left((0,T), \mathbf{L}_2(\Omega)\right).$$

Then

$$\lim_{k \to \infty} \int_0^T (\Phi \boldsymbol{v}_k, \boldsymbol{w}_k) = \int_0^T (\Phi \boldsymbol{v}, \boldsymbol{w})$$

for any $\Phi \in C_0^\infty(\overline{\Omega})$

or

Lemma 1.4. (Slodička [27]) Assume that $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$, $\{\boldsymbol{w}_k\}_{k=1}^{\infty}$ are bounded sequences in $L_2\left((0,T), \mathbf{L}_2(\Omega)\right)$ such that

(i) $\displaystyle\int_0^T \left[\|\partial_t \boldsymbol{w}_k\|^2 + \|\nabla \times \boldsymbol{w}_k\|^2\right] \le C,$

(ii) $\displaystyle\int_0^T \|\nabla \cdot \boldsymbol{v}_k\|^2 \le C.$

Suppose further $\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}$, $\boldsymbol{w}_k \rightharpoonup \boldsymbol{w}$ in $L_2\left((0,T), \mathbf{L}_2(\Omega)\right)$. Then

$$\lim_{k \to \infty} \int_0^T (\Phi \boldsymbol{v}_k, \boldsymbol{w}_k) = \int_0^T (\Phi \boldsymbol{v}, \boldsymbol{w}) \quad \text{for any} \quad \Phi \in C_0^\infty(\overline{\Omega}).$$

2. MAIN RESULTS

Now, we prove our first main result. Here, we keep $\nabla \times \boldsymbol{v}_k$ under control in the L_p-norm and $\nabla \cdot \boldsymbol{w}_k$ in the L_r-norm.

Theorem 2.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain with a Lipschitz continuous boundary. Assume that $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$, $\{\boldsymbol{w}_k\}_{k=1}^{\infty}$ are sequences in $L_2\left((0,T),\mathbf{L}_2(\Omega)\right)$ such that

(i) $\displaystyle\int_0^T \left[\|\boldsymbol{w}_k\|^2 + \|\nabla\cdot\boldsymbol{w}_k\|_{\mathbf{L}_r(\Omega)}^r + \|\partial_t \boldsymbol{w}_k\|_{\mathbf{H}^{-1}(\Omega)}^2\right] \le C, \qquad r > \tfrac{6}{5},$

(ii) $\displaystyle\int_0^T \left[\|\boldsymbol{v}_k\|^2 + \|\nabla\times\boldsymbol{v}_k\|_{\mathbf{L}_p(\Omega)}^p\right] \le C\ , \qquad p > \tfrac{6}{5}.$

Suppose further $\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}$, $\boldsymbol{w}_k \rightharpoonup \boldsymbol{w}$ in $L_2\left((0,T),\mathbf{L}_2(\Omega)\right)$. Then

$$\lim_{k\to\infty}\int_0^T (\Phi\boldsymbol{v}_k,\boldsymbol{w}_k) = \int_0^T (\Phi\boldsymbol{v},\boldsymbol{w})$$

for any $\Phi \in C_0^\infty(\overline{\Omega})$.

P r o o f. Consider for each $k = 1,2,\dots$ the vector field $\boldsymbol{u}_k$ solving

$$\begin{aligned} -\Delta \boldsymbol{u}_k &= \boldsymbol{w}_k && \text{in } \Omega,\\ \boldsymbol{u}_k &= \mathbf{0} && \text{on } \Gamma. \end{aligned} \tag{2}$$

Applying [13, Theorem 8.13] we see that $\boldsymbol{u}_k$ belongs to $L_2\left((0,T),\mathbf{H}^2(\Omega)\right)$. We differentiate (2) with respect to the time variable and we get

$$\begin{aligned} -\Delta \partial_t\boldsymbol{u}_k &= \partial_t\boldsymbol{w}_k && \text{in } \Omega,\\ \partial_t\boldsymbol{u}_k &= \mathbf{0} && \text{on } \Gamma. \end{aligned} \tag{3}$$

Due to the fact that $\displaystyle\int_0^T \|\partial_t\boldsymbol{w}_k\|_{\mathbf{H}^{-1}(\Omega)}^2 \le C$, we easily deduce that

$$\|\nabla\partial_t\boldsymbol{u}_k\|^2 = (\partial_t\boldsymbol{w}_k,\partial_t\boldsymbol{u}_k) \le \varepsilon\,\|\partial_t\boldsymbol{u}_k\|_{\mathbf{H}^1(\Omega)}^2 + C_\varepsilon\,\|\partial_t\boldsymbol{w}_k\|_{\mathbf{H}^{-1}(\Omega)}^2 .$$

Choosing a sufficiently small positive ε we conclude

$$\partial_t\boldsymbol{u}_k \in L_2\left((0,T),\mathbf{H}^1(\Omega)\right). \tag{4}$$

Taking into account that $\Omega \subset \mathbb{R}^3$ we have $\mathbf{H}^2(\Omega) \hookrightarrow\hookrightarrow \mathbf{W}^{1,s}(\Omega)$ for any $1 \le s < 6$ (see [16, Thm.5.8.2]), which implies the compactness of $\boldsymbol{u}_k$ in $L_2\left((0,T),\mathbf{W}^{1,s}(\Omega)\right)$.

One can easily see that $\boldsymbol{z} = \nabla\cdot\boldsymbol{u}_k$ solves

$$\begin{aligned} -\Delta \boldsymbol{z} &= \nabla\cdot\boldsymbol{w}_k && \text{in } \Omega,\\ \boldsymbol{z} &= \nabla\cdot\boldsymbol{u}_k && \text{on } \Gamma. \end{aligned} \tag{5}$$

Using [14, Thm. 9.2.2] we see that $\nabla\cdot\boldsymbol{u}_k \in L_2\left((0,T),\mathbf{W}^{2,r}(\Omega')\right)$ for any sub-domain $\Omega' \subset\subset \Omega$. The relation (4) yields $\partial_t\nabla\cdot\boldsymbol{u}_k \in L_2\left((0,T),\mathbf{L}_2(\Omega)\right)$, which according [16, Thm. 5.8.2] we have $\mathbf{W}^{2,r}(\Omega') \hookrightarrow\hookrightarrow \mathbf{W}^{1,2}(\Omega')$, which gives the compactness of $\nabla\cdot\boldsymbol{u}_k$ in $L_2\left((0,T),\mathbf{W}^{1,2}(\Omega')\right)$.

Therefore, upon passing to subsequences as necessary, we have

$$\begin{array}{rcll} \boldsymbol{u}_k & \to & \boldsymbol{u} & \text{in } L_2\left((0,T),\mathbf{W}^{1,s}(\Omega)\right), \\ \nabla\cdot\boldsymbol{u}_k & \to & \nabla\cdot\boldsymbol{u} & \text{in } L_2\left((0,T),\mathbf{W}^{1,2}(\Omega')\right), \end{array} \tag{6}$$

where $\boldsymbol{u}$ solves

$$\begin{array}{rcll} -\Delta\boldsymbol{u} & = & \boldsymbol{w} & \text{in } \Omega, \\ \boldsymbol{u} & = & \mathbf{0} & \text{on } \Gamma, \end{array} \tag{7}$$

which has been obtained from (2) passing to the limit as $k\to\infty$.

Now, using the identity $-\Delta\boldsymbol{M} = \nabla\times(\nabla\times\boldsymbol{M}) - \nabla(\nabla\cdot\boldsymbol{M})$, which is valid for any vector $\boldsymbol{M}$, we can write for arbitrary $\Phi\in C_0^\infty(\overline{\Omega})$

$$\begin{aligned} \int_0^T (\Phi\boldsymbol{v}_k,\boldsymbol{w}_k) &= \int_0^T (\Phi\boldsymbol{v}_k,-\Delta\boldsymbol{u}_k) \\ &= \int_0^T (\Phi\boldsymbol{v}_k,\nabla\times\nabla\times\boldsymbol{u}_k) - \int_0^T (\Phi\boldsymbol{v}_k,\nabla(\nabla\cdot\boldsymbol{u}_k)) \\ &= \int_0^T (\nabla\times(\Phi\boldsymbol{v}_k),\nabla\times\boldsymbol{u}_k) - \int_0^T (\Phi\boldsymbol{v}_k,\nabla(\nabla\cdot\boldsymbol{u}_k)) \\ &= \int_0^T (\Phi\nabla\times\boldsymbol{v}_k,\nabla\times\boldsymbol{u}_k) + \int_0^T (\nabla\Phi\times\boldsymbol{v}_k,\nabla\times\boldsymbol{u}_k) \\ &\quad - \int_0^T (\Phi\boldsymbol{v}_k,\nabla(\nabla\cdot\boldsymbol{u}_k)). \end{aligned}$$

According to (6) we obtain

$$\begin{aligned} \int_0^T (\Phi\boldsymbol{v}_k,\boldsymbol{w}_k) &\to \int_0^T (\Phi\nabla\times\boldsymbol{v},\nabla\times\boldsymbol{u}) + \int_0^T (\nabla\Phi\times\boldsymbol{v},\nabla\times\boldsymbol{u}) \\ &\quad - \int_0^T (\Phi\boldsymbol{v},\nabla(\nabla\cdot\boldsymbol{u})) \\ &= \int_0^T (\nabla\times(\Phi\boldsymbol{v}),\nabla\times\boldsymbol{u}) - \int_0^T (\Phi\boldsymbol{v},\nabla(\nabla\cdot\boldsymbol{u})) \\ &= \int_0^T (\Phi\boldsymbol{v},\nabla\times\nabla\times\boldsymbol{u}) - \int_0^T (\Phi\boldsymbol{v},\nabla(\nabla\cdot\boldsymbol{u})) \\ &= \int_0^T (\Phi\boldsymbol{v},-\Delta\boldsymbol{u}) \\ &= \int_0^T (\Phi\boldsymbol{v},\boldsymbol{w}), \end{aligned}$$

which concludes the proof. □

Now, we prove our second main result. Here, we keep $\nabla\times\boldsymbol{w}_k$ under control in the L_p-norm and $\nabla\cdot\boldsymbol{w}_k$ in the L_r-norm.

Theorem 2.2. Assume that $\{\boldsymbol{v}_k\}_{k=1}^{\infty}$, $\{\boldsymbol{w}_k\}_{k=1}^{\infty}$ are sequences in $L_2\left((0,T),\mathbf{L}_2(\Omega)\right)$ such that

(i) $\displaystyle\int_0^T \left[\|\boldsymbol{w}_k\|^2 + \|\nabla\cdot\boldsymbol{w}_k\|^r_{\mathbf{L}_r(\Omega)} + \|\partial_t\boldsymbol{w}_k\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\nabla\times\boldsymbol{w}_k\|^p_{\mathbf{L}_p(\Omega)}\right] \le C$ for some $p, r > \frac{6}{5}$,

(ii) $\displaystyle\int_0^T \|\boldsymbol{v}_k\|^2 \le C$.

Suppose further $\boldsymbol{v}_k \rightharpoonup \boldsymbol{v},\ \boldsymbol{w}_k \rightharpoonup \boldsymbol{w}$ in $L_2\left((0,T),\mathbf{L}_2(\Omega)\right)$. Then

$$\lim_{k\to\infty}\int_0^T (\Phi\boldsymbol{v}_k,\boldsymbol{w}_k) = \int_0^T (\Phi\boldsymbol{v},\boldsymbol{w})$$

for any $\Phi\in C_0^\infty(\overline{\Omega})$.

Proof. The proof is similar to the one of Theorem 2.1. Therefore we point out the main steps only.

Let $\boldsymbol{u}_k$ solves (2). The theory of linear elliptic equations yields that $\boldsymbol{u}_k$ belongs to $L_2\left((0,T),\mathbf{H}^2(\Omega)\right)$ and (4) holds true. Moreover we also get the compactness of $\nabla\cdot\boldsymbol{u}_k$ in $L_2\left((0,T),\mathbf{W}^{1,2}(\Omega')\right)$. Further $\boldsymbol{r}=\nabla\times\boldsymbol{u}_k$ solves

$$\begin{array}{rcll} -\Delta\boldsymbol{r} &=& \nabla\times\boldsymbol{w}_k & \text{in } \Omega,\\ \boldsymbol{r} &=& \nabla\times\boldsymbol{u}_k & \text{on } \Gamma. \end{array} \tag{8}$$

Applying [14, Thm. 9.2.2] we see that $\nabla\times\boldsymbol{u}_k\in L_2\left((0,T),\mathbf{W}^{2,p}(\Omega')\right)$ for any sub-domain $\Omega'\subset\subset\Omega$. The relation (4) yields $\partial_t\nabla\cdot\boldsymbol{u}_k\in L_2\left((0,T),\mathbf{L}_2(\Omega)\right)$, which according [16, Thm. 5.8.2] we have $\mathbf{W}^{2,p}(\Omega')\hookrightarrow\hookrightarrow\mathbf{W}^{1,2}(\Omega')$, which gives the compactness of $\nabla\times\boldsymbol{u}_k$ in $L_2\left((0,T),\mathbf{W}^{1,2}(\Omega')\right)$.

Now, we can write

$$\begin{aligned} \int_0^T (\Phi\boldsymbol{v}_k,\boldsymbol{w}_k) &= \int_0^T (\Phi\boldsymbol{v}_k,-\Delta\boldsymbol{u}_k)\\ &= \int_0^T (\Phi\boldsymbol{v}_k,\nabla\times\nabla\times\boldsymbol{u}_k) - \int_0^T (\Phi\boldsymbol{v}_k,\nabla(\nabla\cdot\boldsymbol{u}_k)). \end{aligned}$$

Passing to the limit for $k\to\infty$ we obtain

$$\begin{aligned} \int_0^T (\Phi\boldsymbol{v}_k,\boldsymbol{w}_k) &\to \int_0^T (\Phi\boldsymbol{v},\nabla\times\nabla\times\boldsymbol{u}) - \int_0^T (\Phi\boldsymbol{v},\nabla(\nabla\cdot\boldsymbol{u}))\\ &= \int_0^T (\Phi\boldsymbol{v},-\Delta\boldsymbol{u})\\ &= \int_0^T (\Phi\boldsymbol{v},\boldsymbol{w}), \end{aligned}$$

which concludes the proof.

□

3. NUMERICAL EXPERIMENT

We recall the London model of the nonlinear diffusion in superconductors, see London [17, 18]. The "London" name comes from brothers F. London and H. London, who in 1935 created a theoretical model of superconductivity. It is well known that high-field (hard) type-II superconductors are not ideal conductors of electric current. From the point of view of phenomenological electrodynamics, type-II superconductors can be treated as electrically nonlinear conductors. The process of electromagnetic field penetration in such devices is the process of nonlinear diffusion. Understanding this process is of practical and theoretical importance and it helps by evaluation of magnetic hysteresis and the study of creep phenomena. For a very nice overview of models with some hierarchy structure we refer the reader to Chapman [7], Fabrizio and Morro [11].
The idea of using nonlinear diffusion equations for the description of flux creep can be traced back to the landmark papers of Anderson and Kim [2] and Beasley et al. [4].

By the modelling we start from the quasi-static Maxwell equations

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \boldsymbol{J}, \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= \boldsymbol{0}. \end{aligned}$$

We assume linear materials

$$\boldsymbol{B} = \mu_0 \boldsymbol{H},$$

where μ_0 denotes the magnetic permeability of free space. Actual resistive transitions are gradual and it is customary to describe them by the following power law (cf. Mayergoyz [19])

$$\boldsymbol{J} = \sigma_c |\boldsymbol{E}|^{\frac{1}{p}-1} \boldsymbol{E}, \qquad p > 1, \tag{9}$$

where σ_c is some parameter that coordinates the dimensions of both sides in expression. Let us note that in the case as $p \to 1$ we obtain the linear Ohm law, and for $p \to \infty$ we obtain the Bean critical-state model – see, e. g., Bean [3], Prigozhin [24, 25].

Elimination of $\boldsymbol{H}$ gives

$$\mu_0 \sigma_c \partial_t \left(|\boldsymbol{E}|^{\frac{1}{p}-1} \boldsymbol{E} \right) + \nabla \times \nabla \times \boldsymbol{E} = \boldsymbol{0}. \tag{10}$$

We recall that one can also use the power law (9) in the form

$$\boldsymbol{E} = \sigma_c^{-p} |\boldsymbol{J}|^{p-1} \boldsymbol{J}, \qquad p > 1$$

instead of (9). Elimination of $\boldsymbol{E}$ gives

$$\mu_0 \partial_t \boldsymbol{H} + \nabla \times \left(\sigma_c^{-p} |\nabla \times \boldsymbol{H}|^{p-1} \nabla \times \boldsymbol{H} \right) = \boldsymbol{0}. \tag{11}$$

Of course, both equations (10) and (11) have to be accompanied by appropriate BCs and some reasonable physical information about the divergence of $\boldsymbol{E}$ or $\boldsymbol{H}$. Testing

(10) with $\boldsymbol{E}$, integrating over Ω and the time interval $(0,T)$, using integration by parts one can readily derive the following stability of $\boldsymbol{E}$

$$\max_{t\in[0,T]} \|\boldsymbol{E}(t)\|^{\frac{p+1}{p}}_{\mathbf{L}_{\frac{p+1}{p}}(\Omega)} + \int_0^T \|\nabla\times\boldsymbol{E}\|^2 \leq C.$$

Testing (11) with $\boldsymbol{H}$ similarly yields

$$\max_{t\in[0,T]} \|\boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla\times\boldsymbol{H}\|^{p+1}_{\mathbf{L}_{p+1}(\Omega)} \leq C.$$

These stability results indicate the need of div-curl lemma in some L_r-norm with $r \neq 2$.

Applying the time discretization (Rothe's method) to (10) or (11) one has to solve a recurrent system of nonlinear steady-state problems starting from initial data. The existence of a weak solution at each time step follows from the theory of monotone operators. The more and less standard analysis gives rise to suitable stability results for the approximate solution. Passing to the limit in the variational formulation for a time step approaching 0 and applying the generalized div-curl lemma one can get the existence of a weak solution to the original transient problem. Now, we present an illustrative numerical example.

3.1. Study case

We consider the following variational problem

$$(\partial_t(\sigma(|\boldsymbol{E}|)\boldsymbol{E}),\boldsymbol{\varphi}) + (\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{\varphi}) = (\boldsymbol{F},\boldsymbol{\varphi}) \tag{12}$$

with

$$\sigma(s) = s^{-\alpha}, \qquad \alpha = 0.5,$$

Dirichlet boundary condition $\boldsymbol{E}_\Gamma$, initial condition

$$\boldsymbol{E}(\boldsymbol{x},0) = 2\begin{pmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{pmatrix} \quad \text{for } \boldsymbol{x}\in\Omega,$$

and the right-hand side $\boldsymbol{F}$ chosen in such a way that

$$\boldsymbol{E}(\boldsymbol{x},t) = \begin{pmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{pmatrix}(\sin(2\pi t)+1)$$

is the exact solution.

To solve problem (12) we first discretize it using Rothe's method and afterwards we solve linearized problem obtained in this way using Newton's method. The discretization of (12) reads for $i = 1, 2, \ldots$

$$((\sigma(|\boldsymbol{e}_i|)\boldsymbol{e}_i),\boldsymbol{\varphi}) + \tau(\nabla\times\boldsymbol{e}_i,\nabla\times\boldsymbol{\varphi}) = (\overline{\boldsymbol{F}}_i,\boldsymbol{\varphi}), \tag{13}$$

where

$$\overline{\boldsymbol{F}}_i = \tau \boldsymbol{F}_i + \sigma(|\boldsymbol{e}_{i-1}|)\boldsymbol{e}_{i-1},$$

τ is the length of the time step, and $\boldsymbol{e}_0$ represents the initial condition.

To solve equation (13) using Newton's method we define a functional Φ as

$$\Phi(\boldsymbol{v}) = |\boldsymbol{v}|^{-\alpha}\boldsymbol{v} + \tau\nabla\times\nabla\times\boldsymbol{v} - \overline{\boldsymbol{F}}_i.$$

For the weak formulation and its Fréchet derivative $D\Phi(\boldsymbol{v})$ we can write

$$(\Phi(\boldsymbol{v}), \boldsymbol{\varphi}_j) = (|\boldsymbol{v}|^{-\alpha}\boldsymbol{v}, \boldsymbol{\varphi}_j) + \tau(\nabla\times\boldsymbol{v}, \nabla\times\boldsymbol{\varphi}_j) - (\overline{\boldsymbol{F}}_i, \boldsymbol{\varphi}_j),$$
$$(D\Phi(\boldsymbol{v})\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = (-\alpha|\boldsymbol{v}|^{-\alpha-2}[\boldsymbol{v}\cdot\boldsymbol{\varphi}_i]\boldsymbol{v} + |\boldsymbol{v}|^{-\alpha}\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) + \tau(\nabla\times\boldsymbol{\varphi}_i, \nabla\times\boldsymbol{\varphi}_j),$$

where $\boldsymbol{\varphi_i}, \boldsymbol{\varphi_j} \in \mathbf{H}_0(\mathbf{curl};\Omega)$.

On each time layer we now solve

$$\Phi(\boldsymbol{E}_i) = 0$$

by iteratively solving

$$D\Phi(\boldsymbol{E_{i_m}})\boldsymbol{d_m} = \Phi(\boldsymbol{E_{i_m}})$$

and setting

$$\boldsymbol{E_{i_{m+1}}} = \boldsymbol{E_{i_m}} - \boldsymbol{d_m},$$

until some stopping criterion is met. Index i stands for the time layer and the index m denotes iterations at singular time layer. Stopping criterion used in considered problem was

$$\|\boldsymbol{d_m}\|_{\mathbf{H}(\mathbf{curl};\Omega)} < 1.0\cdot 10^{-6},$$

and linearized problem in Newton's algorithm was solved using GMRes solver.

We have calculated our testing problem on a unit cube, which was split "quasi uniformly" into 384 tetrahedrons (leading to 4184 degrees of freedom). The mesh diameter was 0.026. For the approximation of the field $\boldsymbol{E}$ we have used the lowest order Whitney's edge elements, cf. [5, 6]. The computational error was calculated in the L_2-norm. Proposed numerical scheme has been computed using "The Finite Element Toolbox ALBERT"[1]. This toolbox was modified at our workgroup for the use of Whitney's elements.

Table shows the dependence of the error of obtained numerical approximation on the length of the time step τ. The same data are presented in Figure. Here the linear dependency of the error on the length of the time step is clearly visible. The obtained convergence rate ξ is 0.9613.

[1]ALBERT can be downloaded from `http://www.alberta-fem.de/` and the name stands for **A**daptive multi-**L**evel finite element toolbox using **B**isectioning refinement and **E**rror control by **R**esidual **T**echniques. Its successor ALBERTA can be downloaded from the same site.

Table. Dependence of the errors for the time discretization of the test problem on the length of the time step.

τ	Absolute Err	log(τ)	log(err)
0.50000000	0.37024160	-0.30103000	-0.43151479
0.25000000	0.19737710	-0.60205999	-0.70470324
0.12500000	0.10340850	-0.90308999	-0.98544376
0.06250000	0.05326687	-1.20411998	-1.27354282
0.03125000	0.02734697	-1.50514998	-1.56309079
0.01562500	0.01383989	-1.80617997	-1.85886736
0.00781250	0.00696084	-2.10720997	-2.15733854
0.00390625	0.00349059	-2.40823997	-2.45710066

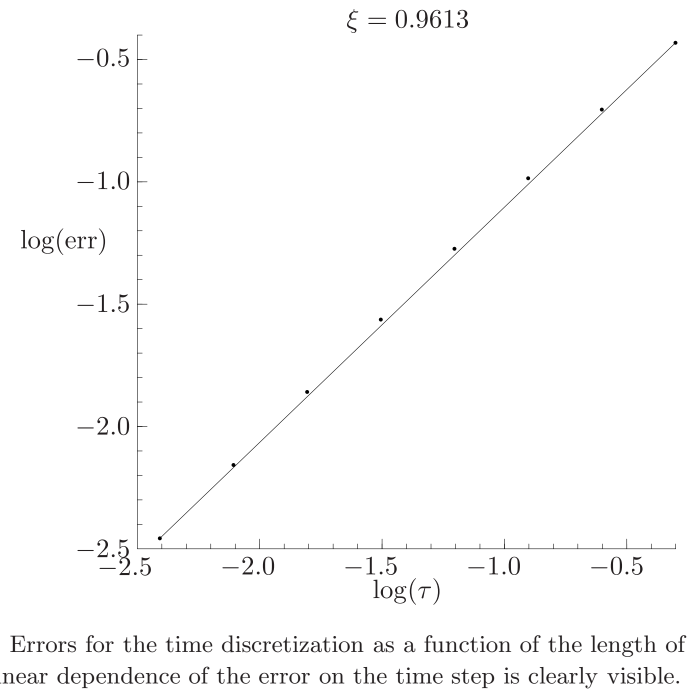

Fig. Errors for the time discretization as a function of the length of the time step τ. Linear dependence of the error on the time step is clearly visible. ξ denotes the numerically obtained convergence rate.

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