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Erhan Deniz; Halit Orhan

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SOME PROPERTIES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS BY USING GENERALIZED RUSCHEWEYH DERIVATIVE OPERATOR

Erhan Deniz, Halit Orhan, Erzurum

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Abstract. By making use of the known concept of neighborhoods of analytic functions we prove several inclusions associated with the (j,δ)-neighborhoods of various subclasses of starlike and convex functions of complex order b which are defined by the generalized Ruscheweyh derivative operator. Further, partial sums and integral means inequalities for these function classes are studied. Relevant connections with some other recent investigations are also pointed out.

Keywords: neighborhoods, partial sums, integral means, generalized Ruscheweyh derivative

MSC 2010: 30C45

1. Introduction

Let $A(j)$ denote the class of functions f of the form

$$f(z)=z+\sum_{k=j+1}^{\infty}a_kz^k \quad (j\in\mathbb{N}:=\{1,2,\ldots\}) \tag{1.1}$$

which are analytic in the open unit disk $U:=\{z\colon z\in\mathbb{C} \text{ and } |z|<1\}$.

Denote by $T(j)$ the subclass of $A(j)$ consisting of functions f of the form:

$$f(z)=z-\sum_{k=j+1}^{\infty}a_kz^k \quad (a_k\geqslant 0;\ j\in\mathbb{N}:=\{1,2,\ldots\}). \tag{1.2}$$

Let Ω be the class of functions $w(z)$ analytic in U such that $w(0)=0$, $|w(z)|<1$. For the functions $f(z)$ and $g(z)$ in $A(j)$, $f(z)$ is said to be subordinate to $g(z)\in U$

if there exists an analytic function $w(z) \in \Omega$ such that $f(z) = g(w(z))$. This subordination is denoted by

$$f(z) \prec g(z).$$

Next, for the functions f_m $(m = 1, 2)$ given by

$$f_m(z) = z + \sum_{k=2}^{\infty} a_{k,m} z^k \quad (m = 1, 2),$$

let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2, defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{1.3}$$

Thus, the Ruscheweyh derivative operator $D^\lambda \colon T \to T$ is defined for a function $f \in T := T(1)$ by

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1)$$

or equivalently, by

$$D^\lambda f(z) = z - \sum_{k=2}^{\infty} \varpi(\lambda, k) a_k z^k$$

where

$$\varpi(\lambda, k) = \frac{(\lambda+1)_{k-1}}{(k-1)!} = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+k-1)}{(k-1)!} \quad (\lambda > -1).$$

Let $H^b_{j,\lambda}[A, B]$ denote the class of functions f in $A(j)$ satisfying the condition

$$1 + \frac{\lambda+1}{b}\Big(\frac{D^{\lambda+1} f(z)}{D^\lambda f(z)} - 1\Big) \prec \frac{1+Az}{1+Bz} \quad (-1 \leqslant B < A \leqslant 1,\ z \in U) \tag{1.4}$$

where $\lambda > -1$ and $b \neq 0$ is an arbitrary fixed complex number. We call $H^b_{j,\lambda}[A, B]$ the generalized Ruscheweyh class of analytic functions of complex order b. We note that $H^b_{j,\lambda}[A, B] = H^b_\lambda[A, B]$. The class $H^b_\lambda[A, B]$ was introduced and studied for Fekete-Szegö problem by Ahuja [1]. We also let $TH^b_{j,\lambda}[A, B] = H^b_{j,\lambda}[A, B] \cap T(j)$. It can be seen that, by specializing the parameters j, b, λ, A, B the subclass $TH^b_{j,\lambda}[A, B]$ reduces to several well-known subclasses of analytic functions. Some of these subclasses are listed below.

(1) $H^1_{1,0}[A, B] = S[A, B]$ (Silverman [23]);
(2) $H^1_{1,1}[A, B] \equiv K[A, B]$ (Silverman and Silva [24]);
(3) $H^1_{1,\lambda}[1 - \alpha, -1] \equiv R_\lambda(\alpha)$ (Ahuja [2] and [27]);

(4) $H_{1,\lambda}^{\lambda+\frac{1}{2}}[1,-1] = K_\lambda$ (Ruscheweyh [19]);
(5) $H_{1,0}^{b}[1,-1] = S_j^*(b) \equiv S_n^*(\gamma)$ (Nasr and Aouf [12]);
(6) $H_{1,1}^{1}[1,-1] = C_j(b) \equiv C_n(\gamma)$ (Wiatrowski [28]);
(7) $H_{1,0}^{1-\alpha}[1,-1] = ST(\alpha)$ (Robertson [17]);
(8) $H_{1,1}^{1-\alpha}[1,-1] = CV(\alpha)$ (Robertson [17]);
(9) $H_{1,0}^{b}[A,B]$ (N. S. Sohi and L. P. Singh [25]).

The main object of the present paper is to investigate the (j,δ)-neighborhoods of two subclasses of $T(j)$ of normalized analytic functions in U with negative and missing coefficients, which are introduced here by making use of the Ruscheweyh derivative operator defined in [19]. Further, we obtain partial sums and integral means inequalities for this class of functions.

1. Neighborhood for the class $TH_{j,\lambda}^{b}[A,B]$

Next, following the earlier investigations by Goodman [7], Ruscheweyh [18], and others including Srivastava et al. [26], Orhan ([13] and [14]), Altıntaş et al. [4] (see also [8], [11], [21], [5]), we define the (j,δ)-neighborhood of functions in the family $TH_{j,\lambda}^{b}[A,B]$.

Definition 2.1. For $f \in T(j)$ of the form (1.2) and $\delta \geqslant 0$ we define a (j,δ)-neighborhood of a function $f(z)$ by

$$N_{j,\delta}(f) := \left\{ g\colon\ g \in T(j),\ g(z) = z - \sum_{k=j+1}^{\infty} c_k z^k \ \text{and} \ \sum_{k=j+1}^{\infty} k|a_k - c_k| \leqslant \delta \right\}.$$

In particular, for the *identity* function

$$e(z) = z$$

we immediately have

$$N_{j,\delta}(e) := \left\{ g\colon\ g \in T(j),\ g(z) = z - \sum_{k=j+1}^{\infty} c_k z^k \ \text{and} \ \sum_{k=j+1}^{\infty} k|c_k| \leqslant \delta \right\}.$$

For the class $TH_{j,\lambda}^{b}[A,B]$ we prove the following lemma.

Lemma 2.2. *A function $f(z) \in T(j)$ is in the class $TH^b_{j,\lambda}[A,B]$ if and only if*

$$\sum_{k=j+1}^{\infty} \varphi_k(b,\lambda,A,B)a_k \leqslant 1, \tag{2.1}$$

where

$$\varphi_k(b,\lambda,A,B) = \frac{((k-1)+|(A-B)b-B(k-1)|)\varpi(\lambda,k)}{(A-B)|b|} \tag{2.2}$$

for $-1 \leqslant B < A \leqslant 1$.

P r o o f. Suppose that $f(z) \in TH^b_{j,\lambda}[A,B]$, then

$$1 + \frac{\lambda+1}{b}\Big(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1\Big) = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in U)$$

and

$$\frac{(\lambda+1)D^{\lambda+1}f(z) + (b-\lambda-1)D^{\lambda}f(z)}{bD^{\lambda}f(z)} = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in U).$$

Therefore

$$w(z) = \frac{(\lambda+1)(D^{\lambda}f(z) - D^{\lambda+1}f(z))}{B(\lambda+1)D^{\lambda+1}f(z) - ((A-B)b + B(\lambda+1))D^{\lambda}f(z)},$$

hence

$$|w(z)| = \left|\frac{(\lambda+1)(D^{\lambda}f(z) - D^{\lambda+1}f(z))}{B(\lambda+1)D^{\lambda+1}f(z) - ((A-B)b + B(\lambda+1))D^{\lambda}f(z)}\right| < 1,$$

this implies that

$$\Re\left\{\frac{\sum\limits_{k=j+1}^{\infty}(k-1)\varpi(\lambda,k)a_k z^k}{(A-B)bz - \sum\limits_{k=j+1}^{\infty}((A-B)b - B(k-1))\varpi(\lambda,k)a_k z^k}\right\} < 1.$$

If we take $z = r$ with $0 < r < 1$, we can write the inequality

$$\Re\left\{\frac{\sum\limits_{k=j+1}^{\infty}(k-1)\varpi(\lambda,k)a_k r^k}{(A-B)br - \sum\limits_{k=j+1}^{\infty}(A-B)b - B(k-1)\varpi(\lambda,k)a_k r^k}\right\} < 1.$$

Letting $r \to 1^-$, we have

$$\frac{\sum\limits_{k=j+1}^{\infty}(k-1)\varpi(\lambda,k)a_k}{(A-B)|b|-\sum\limits_{k=j+1}^{\infty}|(A-B)b-B(k-1)|\varpi(\lambda,k)a_k}<1$$

or

$$\sum_{k=j+1}^{\infty}((k-1)+|(A-B)b-B(k-1)|)\varpi(\lambda,k)a_k<(A-B)|b|, \tag{2.3}$$

then (2.3) gives

$$\sum_{k=j+1}^{\infty}\varphi_k(b,\lambda,A,B)a_k\leqslant 1$$

where

$$\varphi_k(b,\lambda,A,B)=\frac{((k-1)+|(A-B)b-B(k-1)|)\varpi(\lambda,k)}{(A-B)|b|}.$$

Conversely, suppose that the inequality (2.1) holds. Then we have for $z\in U$

$$\begin{aligned}
&(\lambda+1)|D^{\lambda+1}f(z)-D^{\lambda}f(z)|-|(A-B)bD^{\lambda}f(z)-B(\lambda+1)(D^{\lambda+1}f(z)-D^{\lambda}f(z))|\\
&\qquad=\Bigg|\sum_{k=j+1}^{\infty}(1-k)\varpi(\lambda,k)a_kz^k\Bigg|\\
&\qquad\quad-\Bigg|(A-B)bz\bigg(z-\sum_{k=j+1}^{\infty}\varpi(\lambda,k)a_kz^k\bigg)-B\sum_{k=j+1}^{\infty}(1-k)\varpi(\lambda,k)a_kz^k\Bigg|\\
&\qquad=\Bigg|\sum_{k=j+1}^{\infty}(1-k)\varpi(\lambda,k)a_kz^k\Bigg|\\
&\qquad\quad-\Bigg|(A-B)bz-\sum_{k=j+1}^{\infty}((A-B)b-B(k-1))\varpi(\lambda,k)a_kz^k\Bigg|\\
&\qquad\leqslant\sum_{k=j+1}^{\infty}(k-1)\varpi(\lambda,k)a_kr^k-(A-B)|b|r\\
&\qquad\quad+\sum_{k=j+1}^{\infty}|(A-B)b-B(k-1)|\varpi(\lambda,k)a_kr^k\\
&\qquad=\sum_{k=j+1}^{\infty}((k-1)+|(A-B)b-B(k-1)|)\varpi(\lambda,k)a_kr^k-(A-B)|b|r.
\end{aligned}$$

Letting $r \to 1^-$, we have

$$(\lambda+1)|D^{\lambda+1}f(z)-D^{\lambda}f(z)|-|(A-B)bD^{\lambda}f(z)-B(\lambda+1)(D^{\lambda+1}f(z)-D^{\lambda}f(z))| \leqslant 0.$$

Hence it follows that

$$\Big|\frac{(\lambda+1)(D^{\lambda+1}f(z)/D^{\lambda}f(z)-1)}{(A-B)b-B(\lambda+1)(D^{\lambda+1}f(z)/D^{\lambda}f(z)-1)}\Big| < 1 \quad (z \in U).$$

If we put

$$w(z)=\frac{(\lambda+1)(D^{\lambda+1}f(z)/D^{\lambda}f(z)-1)}{(A-B)b-B(\lambda+1)(D^{\lambda+1}f(z)/D^{\lambda}f(z)-1)},$$

then $w(0)=0$, $w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Hence

$$1+\frac{\lambda+1}{b}\Big(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\Big)=\frac{1+Aw(z)}{1+Bw(z)} \quad (-1 \leqslant B < A \leqslant 1,\ z \in U),$$

which shows that $f(z)$ belongs to $TH^b_{j,\lambda}[A,B]$ and the proof is complete. □

Remark 2.3. For $j=1$, $\lambda=0$, $b=1$, $B=0$ and $A=1-\alpha$, $0 \leqslant \alpha < 1$ we get the result obtained by Silverman [20].

Remark 2.4. For $j=1$, $b=1$, $B=0$ and $A=1-\alpha$, $0 \leqslant \alpha < 1$ we get the result obtained by Ahuja [3].

Remark 2.5. For $b=\gamma$, $B=0$ and $A=\beta$, $0<\beta \leqslant 1$ we find the result of Lemma 1 obtained by Mugurusundaramoorthy et. al. [11].

Remark 2.6. For $b=\gamma$, $B=0$, $A=\beta$, $0<\beta\leqslant 1$ and $\varpi(\lambda,k)=(k-1)!^{-1}\times \prod_{m=2}^{k}(m-2\alpha)$ we get the result of Lemma 1 obtained by Orhan [13]. Applying the above lemma, we prove the following result.

Theorem 2.7. $TH^b_{j,\lambda}[A,B] \subset N_{j,\delta}(e)$, *where*

$$\delta := \frac{j+1}{\varphi_{j+1}(b,\lambda,A,B)}.$$

P r o o f. For a function $f(z) \in TH^b_{j,\lambda}[A,B]$ of the form (1.2), Lemma 2.2 immediately yields

$$(j+|(A-B)b-Bj|)\varpi(\lambda,j+1)\sum_{k=j+1}^{\infty}a_k \leqslant (A-B)|b|,$$

$$\sum_{k=j+1}^{\infty}a_k \leqslant \frac{(A-B)|b|}{(j+|(A-B)b-Bj|)\varpi(\lambda,j+1)}=\frac{1}{\varphi_{j+1}(b,\lambda,A,B)}. \tag{2.4}$$

On the other hand, we also find from (2.1) and (2.4) that

$$\begin{aligned}\varpi(\lambda, j+1)\sum_{k=j+1}^{\infty} ka_k &\leqslant (A-B)|b| + (1-|(A-B)b-Bj|)\varpi(\lambda,j+1)\sum_{k=j+1}^{\infty} a_k\\ &\leqslant (A-B)|b| + (1-|(A-B)b-Bj|)\\ &\quad \times \varpi(\lambda,j+1)\frac{(A-B)|b|}{(j+|(A-B)b-Bj|)\varpi(\lambda,j+1)}\\ &= \frac{(j+1)(A-B)|b|}{(j+|(A-B)b-Bj|)},\end{aligned}$$

that is,

$$\sum_{k=j+1}^{\infty} ka_k \leqslant \frac{(j+1)(A-B)|b|}{(j+|(A-B)b-Bj|)\varpi(\lambda,j+1)} = \frac{j+1}{\varphi_{j+1}(b,\lambda,A,B)} := \delta, \tag{2.5}$$

which, in view of Definition 2.1, proves Theorem 2.7. □

Corollary 2.8. *$TH^b_{1,\lambda}[A,B] \subset N_{1,\delta}(e)$ where*

$$\delta := \frac{2}{\varphi_2(b,\lambda,A,B)} = \frac{2(A-B)|b|}{(1+|(A-B)b-B|)(\lambda+1)}.$$

Remark 2.9. For $b=\gamma$, $B=0$ and $A=\beta$, $0<\beta\leqslant 1$ we get the result of Theorem 1 obtained by Mugurusundaramoorthy et.al. [11].

Remark 2.10. For $b=\gamma$, $B=0$, $A=\beta$, $0<\beta\leqslant 1$ and $\varpi(\lambda,k)=(k-1)!^{-1}\times \prod_{m=2}^{k}(m-2\alpha)$ we get the result of Theorem 1 obtained by Orhan [13].

Remark 2.11. For $b=1$, $B=0$, $\lambda=0$ and $A=1-\alpha$, $0\leqslant\alpha<1$ we get the result of Theorem 2.1 obtained by Altıntaş et. al. [4].

3. Neighborhood for the class $K^b_{j,\lambda}[A,B,C,D]$

We define the following class.

Definition 3.1. A function $f(z)\in T(j)$ is said to be in the class $K^b_{j,\lambda}[A,B,C,D]$ if it satisfies

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{A-B}{1-B} \quad (z\in U) \tag{3.1}$$

for $-1\leqslant B<A\leqslant 1$, $-1\leqslant D<C\leqslant 1$ and $g(z)\in TH^b_{j,\lambda}[C,D]$.

Theorem 3.2. *$N_{j,\delta}(g) \subset K^b_{j,\lambda}[A,B,C,D]$ where $g(z) \in TH^b_{j,\lambda}[C,D]$ and*

$$\frac{1-A}{1-B} = 1 - \frac{\delta}{j+1}\frac{\varphi_{j+1}(b,\lambda,C,D)}{\varphi_{j+1}(b,\lambda,C,D)-1}. \tag{3.2}$$

Proof. Suppose that $f(z) \in N_{j,\delta}(g)$. Then Definition 2.1 yields

$$\sum_{k=j+1}^{\infty} k|a_k - c_k| \leqslant \delta,$$

which readily implies the coefficients inequality

$$\sum_{k=j+1}^{\infty} |a_k - c_k| \leqslant \frac{\delta}{j+1}.$$

Next, since $g(z) \in TH^b_{j,\lambda}[C,D]$, we have

$$\sum_{k=j+1}^{\infty} c_k \leqslant \frac{1}{\varphi_{j+1}(b,\lambda,C,D)}.$$

Further,

$$\begin{aligned}\Big|\frac{f(z)}{g(z)} - 1\Big| &\leqslant \frac{\sum\limits_{k=j+1}^{\infty}|a_k - c_k|}{1 - \sum\limits_{k=j+1}^{\infty} c_k} \leqslant \frac{\delta/(j+1)}{1 - 1/\varphi_{j+1}(b,\lambda,C,D)} \\ &= \frac{\delta}{j+1}\frac{\varphi_{j+1}(b,\lambda,C,D)}{\varphi_{j+1}(b,\lambda,C,D)-1} = \frac{A-B}{1-B}.\end{aligned}$$

This implies that $f(z) \in K^b_{j,\lambda}[A,B,C,D]$. □

Putting $j = 1$ in Theorem 3.2, we have

Corollary 3.3. *$N_{1,\delta}(g) \subset K^b_{1,\lambda}[A,B,C,D]$ where $g(z) \in TH^b_{1,\lambda}[C,D]$ and*

$$\frac{1-A}{1-B} = 1 - \frac{\delta}{2}\frac{\varphi_2(b,\lambda,C,D)}{\varphi_2(b,\lambda,C,D)-1}. \tag{3.3}$$

4. Partial sums

Following the earlier works by Silverman [22], N. C. Cho et al. [6] and others (see also [16], [9]), in this section we investigate the ratio of real parts of functions involving (1.2) and their sequence of partial sums defined by

$$f_1(z) = z; \qquad f_n(z) = z - \sum_{k=j+1}^{n} a_k z^k \quad (j \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{4.1}$$

and determine sharp lower bounds for

$$\Re\{f(z)/f_n(z)\}, \quad \Re\{f_n(z)/f(z)\}, \quad \Re\{f'(z)/f_n'(z)\} \quad \text{and} \quad \Re\{f_n'(z)/f'(z)\}.$$

Theorem 4.1. *If f of the form* (1.2) *satisfies condition* (2.1), *then*

$$\Re\Big\{\frac{f(z)}{f_n(z)}\Big\} \geqslant \frac{\varphi_{n+j+1}(b, \lambda, A, B) - 1}{\varphi_{n+j+1}(b, \lambda, A, B)} \tag{4.2}$$

and

$$\Re\Big\{\frac{f_n(z)}{f(z)}\Big\} \geqslant \frac{\varphi_{n+j+1}(b, \lambda, A, B)}{\varphi_{n+j+1}(b, \lambda, A, B) + 1} \tag{4.3}$$

where $\varphi_{n+j+1}(b, \lambda, A, B)$ is given by (2.2). *The results are sharp for every n, with the extremal function given by*

$$f(z) = z - \frac{1}{\varphi_{n+j+1}(b, \lambda, A, B)} z^{n+1}. \tag{4.4}$$

Proof. In order to prove (4.2), it is sufficient to show that

$$\varphi_{n+j+1}(b, \lambda, A, B)\Big[\frac{f(z)}{f_n(z)} - \Big(\frac{\varphi_{n+j+1}(b, \lambda, A, B) - 1}{\varphi_{n+j+1}(b, \lambda, A, B)}\Big)\Big] \prec \frac{1+z}{1-z} \quad (z \in U). \tag{4.5}$$

We can write

$$\varphi_{n+j+1}(b,\lambda,A,B)\Big[\frac{f(z)}{f_n(z)}-\Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B)-1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\Big]$$

$$=\varphi_{n+j+1}(b,\lambda,A,B)\left[\frac{1-\sum\limits_{k=j+1}^{\infty}a_k z^{k-1}}{1-\sum\limits_{k=j+1}^{n}a_k z^{k-1}}-\Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B)-1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\right]$$

$$=\varphi_{n+j+1}(b,\lambda,A,B)$$

$$\times\left[\frac{1-\sum\limits_{k=j+1}^{n}a_k z^{k-1}-\sum\limits_{k=n+j+1}^{\infty}a_k z^{k-1}}{1-\sum\limits_{k=j+1}^{n}a_k z^{k-1}}-\Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B)-1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\right]$$

$$=\frac{1+w(z)}{1-w(z)}.$$

Then

$$w(z)=\frac{-\varphi_{n+j+1}(b,\lambda,A,B)\sum\limits_{k=n+j+1}^{\infty}a_k z^{k-1}}{2-2\sum\limits_{k=j+1}^{n}a_k z^{k-1}-\varphi_{n+j+1}(b,\lambda,A,B)\sum\limits_{k=n+j+1}^{\infty}a_k z^{k-1}}.$$

Obviously $w(0)=0$ and

$$|w(z)|\leqslant\frac{\varphi_{n+j+1}(b,\lambda,A,B)\sum\limits_{k=n+j+1}^{\infty}a_k}{2-2\sum\limits_{k=j+1}^{n}a_k-\varphi_{n+j+1}(b,\lambda,A,B)\sum\limits_{k=n+j+1}^{\infty}a_k}.$$

Now, $|w(z)|\leqslant 1$ if and only if

$$2\varphi_{n+j+1}(b,\lambda,A,B)\sum_{k=n+j+1}^{\infty}a_k\leqslant 2-2\sum_{k=j+1}^{n}a_k,$$

which is equivalent to

$$\sum_{k=j+1}^{n}a_k+\varphi_{n+j+1}(b,\lambda,A,B)\sum_{k=n+j+1}^{\infty}a_k\leqslant 1. \tag{4.6}$$

In view of (2.1), this is equivalent to showing that

$$\sum_{k=j+1}^{n}(\varphi_k(b,\lambda,A,B)-1)a_k + \sum_{k=n+j+1}^{\infty}(\varphi_k(b,\lambda,A,B)-\varphi_{n+j+1}(b,\lambda,A,B))a_k\geqslant 0. \tag{4.7}$$

To see that the function f given by (4.4) gives the sharp result, we observe for $z = re^{2\pi i/n}$ that

$$\frac{f(z)}{f_n(z)} = 1 - \frac{1}{\varphi_{n+j+1}(b,\lambda,A,B)} z^n \to 1 - \frac{1}{\varphi_{n+j+1}(b,\lambda,A,B)}$$

where $r \to 1^-$.

Thus, we have completed the proof of (4.2). □

The proof of (4.3) is similar to (4.2) and will be omitted.

Similarly, we can establish the following theorem.

Theorem 4.2. *If $f(z)$ of the form* (1.2) *satisfies* (2.1), *then*

$$\Re\Big\{\frac{f'(z)}{f_n'(z)}\Big\} \geqslant \frac{\varphi_{n+j+1}(b,\lambda,A,B) - n - 1}{\varphi_{n+j+1}(b,\lambda,A,B)} \tag{4.8}$$

and

$$\Re\Big\{\frac{f_n'(z)}{f'(z)}\Big\} \geqslant \frac{\varphi_{n+j+1}(b,\lambda,A,B)}{\varphi_{n+j+1}(b,\lambda,A,B) + n + 1} \tag{4.9}$$

where $\varphi_{n+j+1}(b,\lambda,A,B)$ is given by (2.2). *The results are sharp for every n, with the extremal function given by* (4.4).

P r o o f. In order to prove (4.8), it is sufficient to show that

$$\frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1}\Big[\frac{f'(z)}{f_n'(z)} - \Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B) - n - 1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\Big] \prec \frac{1+z}{1-z} \quad (z \in U). \tag{4.10}$$

We can write

$$\begin{aligned}
&\frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1}\Big[\frac{f'(z)}{f_n'(z)} - \Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B) - n - 1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\Big] \\
&= \frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1}\left[\frac{1 - \sum\limits_{k=j+1}^{\infty} k a_k z^{k-1}}{1 - \sum\limits_{k=j+1}^{n} k a_k z^{k-1}} - \Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B) - n - 1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\right] \\
&= \frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1} \\
&\quad\times\left[\frac{1 - \sum\limits_{k=j+1}^{n} k a_k z^{k-1} - \sum\limits_{k=n+j+1}^{\infty} k a_k z^{k-1}}{1 - \sum\limits_{k=j+1}^{n} k a_k z^{k-1}} - \Big(\frac{\varphi_{n+j+1}(b,\lambda,A,B) - n - 1}{\varphi_{n+j+1}(b,\lambda,A,B)}\Big)\right] \\
&= \frac{1+w(z)}{1-w(z)}.
\end{aligned}$$

Then

$$w(z)=\frac{-\varphi_{n+j+1}(b,\lambda,A,B)(n+1)^{-1}\sum\limits_{k=n+j+1}^{\infty}ka_kz^{k-1}}{2-2\sum\limits_{k=j+1}^{n}ka_kz^{k-1}-\varphi_{n+j+1}(b,\lambda,A,B)(n+1)^{-1}\sum\limits_{k=n+j+1}^{\infty}ka_kz^{k-1}}.$$

Obviously $w(0)=0$ and

$$|w(z)|\leqslant\frac{\varphi_{n+j+1}(b,\lambda,A,B)(n+1)^{-1}\sum\limits_{k=n+j+1}^{\infty}ka_k}{2-2\sum\limits_{k=j+1}^{n}ka_k-\varphi_{n+j+1}(b,\lambda,A,B)(n+1)^{-1}\sum\limits_{k=n+j+1}^{\infty}ka_k}.$$

Now $|w(z)|\leqslant 1$ if and only if

$$2\frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1}\sum_{k=n+j+1}^{\infty}ka_k\leqslant 2-2\sum_{k=j+1}^{n}ka_k,$$

which is equivalent to

$$\sum_{k=j+1}^{n}ka_k+\frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1}\sum_{k=n+j+1}^{\infty}ka_k\leqslant 1. \tag{4.11}$$

In view of (2.1), this is equivalent to showing that

$$\begin{aligned}\sum_{k=j+1}^{n}&[\varphi_k(b,\lambda,A,B)-k]a_k\\&+\sum_{k=n+j+1}^{\infty}\Big[\varphi_k(b,\lambda,A,B)-\frac{\varphi_{n+j+1}(b,\lambda,A,B)}{n+1}k\Big]a_k\geqslant 0.\end{aligned} \tag{4.12}$$

Thus we have completed the proof of (4.8). □

The proof of (4.9) is similar to (4.8) and is omitted.

5. Integral means

The following subordination result due to Littlewood [10] will be required in our investigation. The integral means of analytic functions was studied in [16], [15].

Lemma 5.1. *If $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} \, d\theta \leqslant \int_0^{2\pi} |g(re^{i\theta})|^{\mu} \, d\theta,$$

where $\mu > 0$ $z = re^{i\theta}$ and $0 < r < 1$.

Application of Lemma 5.1 to functions $f(z)$ in the class $TH^b_{j,\lambda}[A, B]$ gives the following result using known procedures.

Theorem 5.2. *Let $\mu > 0$. If $f(z) \in TH^b_{j,\lambda}[A, B]$ is given by (1.2) and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1}{\varphi_{j+2}(b, \lambda, A, B)} z^2$$

where $\varphi_{n+j+1}(b, \lambda, A, B)$ is defined in (2.2), then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leqslant \int_0^{2\pi} |f_2(z)|^{\mu} \, d\theta. \tag{5.1}$$

Proof. For $f(z) = z - \sum\limits_{k=j+1}^{\infty} a_k z^k$, (5.1) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{k=j+1}^{\infty} a_k z^{k-1} \right|^{\mu} d\theta \leqslant \int_0^{2\pi} \left| 1 - \frac{1}{\varphi_{j+2}(b, \lambda, A, B)} z \right|^{\mu} d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{k=j+1}^{\infty} a_k z^{k-1} \prec 1 - \frac{1}{\varphi_{j+2}(b, \lambda, A, B)} z.$$

Setting

$$1 - \sum_{k=j+1}^{\infty} a_k z^{k-1} = 1 - \frac{1}{\varphi_{j+2}(b, \lambda, A, B)} w(z), \tag{5.2}$$

from (5.2) and (2.1) we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=j+1}^{\infty} \varphi_{j+2}(b, \lambda, A, B) a_k z^{k-1} \right| \leqslant |z| \sum_{k=j+1}^{\infty} \varphi_{j+2}(b, \lambda, A, B) a_k \\ &\leqslant |z| \sum_{k=j+1}^{\infty} \varphi_{n+j+1}(b, \lambda, A, B) a_k \leqslant |z|. \end{aligned}$$

This completes the proof of the theorem. □

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Authors' address: E. Deniz, H. Orhan, Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey, e-mail: `edeniz36@yahoo.com`, `horhan@atauni.edu.tr`.