Czechoslovak Mathematical Journal

Raffaella Cilia; Joaquín M. Gutiérrez

Integral polynomials on Banach spaces not containing ℓ_1

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 221–231

Persistent URL: http://dml.cz/dmlcz/140564

Czechoslovak Mathematical Journal, 60 (135) (2010), 221–231

INTEGRAL POLYNOMIALS ON BANACH SPACES NOT CONTAINING ℓ_1

Raffaella Cilia, Catania, Joaquín M. Gutiérrez, Madrid

(Received June 20, 2008)

Abstract. We give new characterizations of Banach spaces not containing ℓ_1 in terms of integral and p-dominated polynomials, extending to the polynomial setting a result of Cardassi and more recent results of Rosenthal.

Keywords: (Pietsch) integral polynomial, Banach space not containing ℓ_1, p-dominated polynomial

MSC 2010: 46G25, 46B20, 47H60

1. Introduction

Cardassi [8] and Rosenthal [29] have given characterizations of Banach spaces not containing ℓ_1 in terms of integral and p-summing (linear) operators. Using these results and the extendibility of integral polynomials due to Carando and Lassalle [7], we obtain polynomial characterizations of Banach spaces not containing ℓ_1.

Throughout, E, F, and G denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers, and by $\mathbb{K}$ the scalar field (real or complex). We use the symbol $\mathcal{L}(E,F)$ for the space of all (linear bounded) operators from E into F endowed with the operator norm. Given a space F, we shall denote by k_F the natural isometric embedding of F into its bidual F^{**}.

For $m \in \mathbb{N}$, we denote by $\mathcal{P}(^mE,F)$ the space of all m-homogeneous (continuous) polynomials from E into F endowed with the supremum norm. Recall that with each

The first named author was supported in part by G.N.A.M.P.A. (Italy).
Both authors were supported in part by Dirección General de Investigación, MTM 2006-03531 (Spain).

$P \in \mathcal{P}(^mE, F)$ we can associate a unique symmetric m-linear (continuous) mapping $\hat{P}\colon E \times \overset{(m)}{\cdots} \times E \to F$ so that

$$P(x) = \hat{P}(x, \overset{(m)}{\ldots}, x) \qquad (x \in E).$$

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer the reader to [16] and [26]. The notion of the ideal of multilinear mappings or polynomials may be seen, for instance, in [20], [21], or [12].

We use the notation $\bigotimes^m E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E. The symbol $E \otimes_\pi F$ (or $E \otimes_\varepsilon F$) denotes the completed projective (respectively, injective) tensor product of E and F (see [15] or [13] for the theory of tensor products). By $\bigotimes_s^m E := E \otimes_s \overset{(m)}{\cdots} \otimes_s E$ we denote the m-fold symmetric tensor product of E, that is, the set of all elements $u \in \bigotimes^m E$ of the form

$$u = \sum_{j=1}^n \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \qquad (n \in \mathbb{N}, \lambda_j \in \mathbb{K}, x_j \in E, 1 \leqslant j \leqslant n).$$

By $\bigotimes_{\pi,s}^m E$ (or $\bigotimes_{\varepsilon,s}^m E$) we represent the space $\bigotimes_s^m E$ endowed with the topology induced by that of $\bigotimes_\pi^m E$ (respectively, $\bigotimes_\varepsilon^m E$).

By

$$\delta_m\colon E \longrightarrow \bigotimes_{\pi,s}^m E$$

we denote the canonical polynomial given by $\delta_m(x) := x \otimes \overset{(m)}{\cdots} \otimes x$ for all $x \in E$.

For symmetric tensor products, the reader is referred to [19].

For a polynomial $P \in \mathcal{P}(^mE, F)$, its *linearization*

$$\overline{P}\colon \bigotimes_{\pi,s}^m E \longrightarrow F$$

is the operator given by

$$\overline{P}\left(\sum_{j=1}^n \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j\right) = \sum_{j=1}^n \lambda_j P(x_j)$$

for all $x_j \in E$ and $\lambda_j \in \mathbb{K}$ $(1 \leqslant j \leqslant n)$.

Given $1 \leqslant r < \infty$, a polynomial $P \in \mathcal{P}(^mE, F)$ is r-*dominated* (see, e.g., [24], [25]) if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^n \|P(x_i)\|^{r/m}\right)^{m/r} \leqslant k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^r\right)^{m/r}.$$

The space of all r-dominated polynomials in $\mathcal{P}(^mE,F)$ is denoted by $\mathcal{P}_{r\text{-d}}(^mE,F)$. Note that, for $m=1$, we obtain the (absolutely) r-summing operators.

A polynomial $P\in\mathcal{P}(^mE,F)$ is *Pietsch integral* (or *(Grothendieck) integral*) [1] if there exists a regular countably additive, F-valued (or F^{**}-valued) Borel measure $\mathscr{G}$ of bounded variation on B_{E^*} such that

$$P(x)=\int_{B_{E^*}}[x^*(x)]^m\,\mathrm{d}\mathscr{G}(x^*)\qquad(x\in E).$$

The symbol $\mathcal{P}_{\mathrm{PI}}(^mE,F)$ ($\mathcal{P}_{\mathrm{I}}(^mE,F)$) denotes the space of all Pietsch integral (resp., integral) m-homogeneous polynomials from E into F, endowed with the *Pietsch integral norm* (*integral norm*)

$$\|P\|_{\mathrm{PI}}:=\inf|\mathscr{G}|(B_{E^*})\qquad(\text{respectively,}\quad\|P\|_{\mathrm{I}}:=\inf|\mathscr{G}|(B_{E^*})),$$

where $|\mathscr{G}|$ denotes the variation of $\mathscr{G}$ and the infimum is taken over all vector measures $\mathscr{G}$ satisfying the definition.

A polynomial $P\in\mathcal{P}(^mE,F)$ is *compact* if $P(B_E)$ is relatively compact in F. We use $\mathcal{P}_{\mathrm{K}}(^mE,F)$ for the space of all compact polynomials in $\mathcal{P}(^mE,F)$. A polynomial $P\in\mathcal{P}(^mE,F)$ is *weakly continuous on bounded subsets* if, for each bounded net $(x_\alpha)\subset E$ weakly converging to x, $(P(x_\alpha))$ converges to $P(x)$ in norm. We denote by $\mathcal{P}_{\mathrm{wb}}(^mE,F)$ the space of all polynomials in $\mathcal{P}(^mE,F)$ which are weakly continuous on bounded sets. Every polynomial in $\mathcal{P}_{\mathrm{wb}}(^mE,F)$ is compact ([5, Lemma 2.2] and [4, Theorem 2.9]). An operator is weakly continuous on bounded sets if and only if it is compact [5, Proposition 2.5].

When the range space is omitted, it is understood to be the scalar field, for instance:

$$\mathcal{P}(^mE):=\mathcal{P}(^mE,\mathbb{K}),\qquad\mathcal{P}_{\mathrm{wb}}(^mE):=\mathcal{P}_{\mathrm{wb}}(^mE,\mathbb{K}),\qquad\mathcal{P}_{\mathrm{I}}(^mE):=\mathcal{P}_{\mathrm{I}}(^mE,\mathbb{K}).$$

Every polynomial $P\in\mathcal{P}(^mE,F)$ admits an *Aron-Berner extension*

$$\tilde{P}\in\mathcal{P}(^mE^{**},F^{**})$$

[3] (see also [23]).

2. The results

The following result is given in [29].

Theorem 2.1. *Let E be a Banach space. The following assertions are equivalent:*

(a) *E contains no copy of ℓ_1;*
(b) *for every F, every integral operator from F into E^* is compact;*
(c) *every integral operator from ℓ_1 into E^* is compact;*
(d) *every integral operator from E^* into E^* is compact;*
(e) *for every F, every integral operator from E into F is compact;*
(f) *every integral operator from E into E^* is compact;*
(g) *for all F and $1 \leqslant r < \infty$, every absolutely r-summing operator from E into F is compact;*
(h) *every absolutely 2-summing operator from E into ℓ_2 is compact;*
(i) *every absolutely 2-summing operator from E into E^* is compact.*

The equivalence (a) $\Leftrightarrow$ (e) was proved in [8]. The implication (a) $\Rightarrow$ (g) is given in [28, Corollary 1.7].

Here we extend Theorem 2.1 to the polynomial setting.

Recall that a Banach space F is said to have the *compact range property* (*CRP*, for short) if every F-valued countably additive measure of bounded variation has compact range [27]. Every Banach space with the weak Radon-Nikodým property has the CRP. A dual Banach space has the CRP if and only if its predual contains no copy of ℓ_1 [18, Corollary 5]. We refer the reader to [17], [18], [27], [31] for more about the CRP.

Theorem 2.2. *Let E be a Banach space. The following assertions are equivalent:*

(a) *E contains no copy of ℓ_1;*
(b) *for all F and $m \in \mathbb{N}$ ($m \geqslant 2$), we have $\mathcal{P}_{\mathrm{I}}({}^mF, E^*) \subseteq \mathcal{P}_{\mathrm{K}}({}^mF, E^*)$;*
(c) *for every $m \in \mathbb{N}$ ($m \geqslant 2$), we have $\mathcal{P}_{\mathrm{I}}({}^m\ell_1, E^*) \subseteq \mathcal{P}_{\mathrm{K}}({}^m\ell_1, E^*)$;*
(d) *there exists $m \in \mathbb{N}$ ($m \geqslant 2$) such that, for every polynomial $P \in \mathcal{P}({}^m\ell_1, E^*)$ of the form $P = Q \circ T$ where T is an integral operator and Q is a polynomial, P is compact;*
(e) *for every $m \in \mathbb{N}$ ($m \geqslant 2$), we have $\mathcal{P}_{\mathrm{I}}({}^mE^*, E^*) \subseteq \mathcal{P}_{\mathrm{K}}({}^mE^*, E^*)$;*
(f) *there exist $m \in \mathbb{N}$ ($m \geqslant 2$) and a Banach space F containing ℓ_1 such that $\mathcal{P}_{\mathrm{I}}({}^mF, E^*) \subseteq \mathcal{P}_{\mathrm{K}}({}^mF, E^*)$;*
(g) *there exist $m \in \mathbb{N}$ ($m \geqslant 2$) and a Banach space F containing ℓ_1 such that, for every polynomial $P \in \mathcal{P}({}^mF, E^*)$ of the form $P = Q \circ T$ where T is an integral operator and Q is a polynomial, P is compact.*

Proof. (a) $\Rightarrow$ (b). Let $P \in \mathcal{P}_{\mathrm{I}}(^m F, E^*)$. Since its range is a dual space, by the same easy argument as in the linear case (see the proof of [15, Corollary VIII.2.10]), P is Pietsch integral. Since E contains no copy of ℓ_1, E^* has the CRP, and P is compact by [11, Theorem 4.10].

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (d) is clear by virtue of [9, Corollary 2.7].

(d) $\Rightarrow$ (a). If E contains a copy of ℓ_1, then E^* does not have the CRP. By [11, Proposition 4.8], there exists a polynomial $P \in \mathcal{P}(^m \ell_1, E^*)$ of the form $P = Q \circ T$ where T is an integral operator and Q is a polynomial, but P is not compact.

(b) $\Rightarrow$ (e) is obvious.

(e) $\Rightarrow$ (a). Suppose that E contains a copy of ℓ_1. Then, by Theorem 2.1, there exists an integral operator $T \in \mathcal{L}(E^*, E^*)$ which is not compact. Consider the polynomial $P \in \mathcal{P}(^m E^*, E^*)$ given by

$$P := T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ \delta_m$$

where $\pi_p \colon \bigotimes_{\pi,s}^{p+1} E^* \to \bigotimes_{\pi,s}^{p} E^*$ $(1 \leqslant p \leqslant m-1)$ are the projections introduced in [6, page 168]. These operators are also continuous with respect to the injective tensor norm [2, 3.5]. So the linearization of P,

$$T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \in \mathcal{L}\Bigl(\bigotimes_{\varepsilon,s}^{m} E^*, E^*\Bigr),$$

is well defined. Moreover, it is integral since T is integral. By [32, page 62], P is an integral polynomial. Hence it is compact by (e). It follows that

$$T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \in \mathcal{L}\Bigl(\bigotimes_{\pi,s}^{m} E^*, E^*\Bigr)$$

is compact [30, Lemma 4.1]. Let $j_p \colon \bigotimes_{\pi,s}^{p} E^* \to \bigotimes_{\pi,s}^{p+1} E^*$ $(1 \leqslant p \leqslant m-1)$ be the operator [6, page 168] such that $\pi_p \circ j_p$ is the identity map on $\bigotimes_{\pi,s}^{p} E^*$. Then

$$T = T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ j_{m-1} \circ \ldots \circ j_1$$

is a compact operator and we get a contradiction.

(b) $\Rightarrow$ (f) and (f) $\Rightarrow$ (g) are obvious.

(g) $\Rightarrow$ (d). Let $m \in \mathbb{N}$ $(m \geqslant 2)$ and F be, respectively, the index and the Banach space provided by (g). Suppose that the polynomial $P \in \mathcal{P}(^m \ell_1, E^*)$ is of the form $P = Q \circ T$ where $T \in \mathcal{L}(\ell_1, G)$ is an integral operator and $Q \in \mathcal{P}(^m G, E^*)$. Since F contains a copy of ℓ_1, we can extend T to an integral operator $\tilde{T} \in \mathcal{L}(F, G^{**})$

[14, Proposition 6.12]. Let $\tilde{Q}$ be the Aron-Berner extension of Q and let H be the canonical projection of E^{***} onto E^* (see the diagram below).

$$\begin{tikzcd}
\ell_1 \arrow[r, "P"] \arrow[d, "i"'] \arrow[dr, "T"] & E^* \arrow[r, "k_{E^*}"] & E^{***} \arrow[r, "H"] & E^* \\
F \arrow[dr, "\tilde{T}"'] & G \arrow[u, "Q"'] \arrow[d, "k_G"'] & & \\
 & G^{**} \arrow[uur, "\tilde{Q}"'] & &
\end{tikzcd}$$

By (g), the polynomial $H \circ \tilde{Q} \circ \tilde{T} \in \mathcal{P}({}^mF, E^*)$ is compact. Since its restriction to ℓ_1 coincides with P, P is also compact. $\square$

Theorem 2.3. *Let E be a Banach space. The following assertions are equivalent:*

(a) *E contains no copy of ℓ_1;*

(b) *for all F and $m \in \mathbb{N}$ $(m \geqslant 2)$, we have $\mathcal{P}_{\mathrm{I}}({}^mE, F) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mE, F)$;*

(c) *for every $m \in \mathbb{N}$ $(m \geqslant 2)$, we have $\mathcal{P}_{\mathrm{I}}({}^mE, E^*) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mE, E^*)$;*

(d) *there exists $m \in \mathbb{N}$ $(m \geqslant 2)$ such that for every polynomial $P \in \mathcal{P}({}^mE, E^*)$ of the form $P = Q \circ T$ where T is an integral operator and Q is a polynomial, we have $P \in \mathcal{P}_{\mathrm{wb}}({}^mE, E^*)$;*

(e) *for each $m \in \mathbb{N}$ $(m \geqslant 2)$, there is a Banach space F such that*

$$\mathcal{P}_{\mathrm{I}}({}^mE, F) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mE, F);$$

(f) *for every $m \in \mathbb{N}$ $(m \geqslant 2)$, we have $\mathcal{P}_{\mathrm{I}}({}^mE) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mE)$;*

(g) *there exists $m \in \mathbb{N}$ $(m \geqslant 2)$ such that $\mathcal{P}_{\mathrm{I}}({}^mE) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mE)$;*

(h) *there are $m \in \mathbb{N}$ $(m \geqslant 2)$ and a Banach space F such that*

$$\mathcal{P}_{\mathrm{I}}({}^mE, F) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mE, F).$$

P r o o f. (a) $\Rightarrow$ (b). Let $P \in \mathcal{P}_{\mathrm{I}}({}^mE, F)$. Then $k_F \circ P \in \mathcal{P}_{\mathrm{PI}}({}^mE, F^{**})$. By [11, Theorem 4.9], $k_F \circ P$ is weakly continuous on bounded sets. Hence P is also weakly continuous on bounded sets.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (d) is clear by virtue of [9, Corollary 2.7].

(d) $\Rightarrow$ (a). Suppose that E contains a copy of ℓ_1. Then, by Theorem 2.2,(d), there exists a noncompact polynomial $P \in \mathcal{P}({}^m\ell_1, E^*)$ of the form $P = Q \circ T$ where T is an integral operator from ℓ_1 into some Banach space G. We can extend T to an integral operator $\tilde{T} \in \mathcal{L}(E, G^{**})$ [14, Proposition 6.12]. Let $\tilde{Q}$ be the Aron-Berner extension of Q and let H be the canonical projection from E^{***} onto E^*. Then the polynomial $\tilde{P} := H \circ \tilde{Q} \circ \tilde{T} \in \mathcal{P}(^mE, E^*)$ cannot be weakly continuous on bounded sets since its restriction to ℓ_1 coincides with P. This contradicts (d) and completes the proof.

(b) $\Rightarrow$ (e) $\Rightarrow$ (h) and (f) $\Rightarrow$ (g) are obvious.

(e) $\Rightarrow$ (f). Assume (f) fails. Then, for some $m \in \mathbb{N}$ ($m \geqslant 2$), there is a polynomial

$$P \in \mathcal{P}_{\mathrm{I}}(^mE) \setminus \mathcal{P}_{\mathrm{wb}}(^mE).$$

Let F be the Banach space provided by (e). Choose $y_0 \in F$, $y_0 \neq 0$. Let $j\colon \mathbb{K} \to F$ be given by $j(\lambda) = \lambda y_0$. By the ideal property, the polynomial $j \circ P \in \mathcal{P}(^mE, F)$ is integral. However, $j \circ P$ is not weakly continuous on bounded sets.

(g) $\Rightarrow$ (a). Suppose that E contains a copy of ℓ_1. Consider the polynomial $P \in \mathcal{P}(^m\ell_1)$ given by

$$P(x) := \sum_{n=1}^{\infty} x_n^m \qquad \text{for } x = (x_n)_{n=1}^{\infty} \in \ell_1.$$

By [16, Example 2.25], P is integral. By [7, Theorem 5], we can extend P to a polynomial $\tilde{P} \in \mathcal{P}_{\mathrm{I}}(^mE)$.

Denoting by $\mathcal{L}_{\mathrm{s}}(^{m-1}\ell_1)$ the space of all $(m-1)$-linear symmetric continuous forms on ℓ_1, let

$$T_P\colon \ell_1 \longrightarrow \mathcal{L}_{\mathrm{s}}(^{m-1}\ell_1)$$

be the operator given by

$$T_P(x)(y_1, \ldots, y_{m-1}) := \hat{P}(x, y_1, \ldots, y_{m-1}) \qquad \text{for } x, y_1, \ldots, y_{m-1} \in \ell_1.$$

Then, for $n \neq k$, we have

$$\begin{aligned} \|T_P(e_n) - T_P(e_k)\| &\geqslant |T_P(e_n)(e_n, \overset{(m-1)}{\ldots}, e_n) - T_P(e_k)(e_n, \overset{(m-1)}{\ldots}, e_n)| \\ &= |P(e_n) - \hat{P}(e_k, e_n, \overset{(m-1)}{\ldots}, e_n)| = 1, \end{aligned}$$

so T_P is not compact, which implies that $P \notin \mathcal{P}_{\mathrm{wb}}(^m\ell_1)$ [4, Theorem 2.9]. Hence, $\tilde{P} \notin \mathcal{P}_{\mathrm{wb}}(^mE)$.

(h) $\Rightarrow$ (g) is proved as in (e) $\Rightarrow$ (f). $\square$

Using [12, Theorem 2.3], it is easy to see that every integral polynomial takes weakly convergent sequences into norm convergent sequences. Then, the implication (g) $\Rightarrow$ (a) of Theorem 2.3 improves [22, Theorem 4, (e) $\Rightarrow$ (a)].

We may ask if the following assertion could be included in Theorem 2.3:

(d′) *for some (or for all)* $m \in \mathbb{N}$ $(m \geqslant 2)$ *and for every polynomial* $P \in \mathcal{P}(^m E)$ *of the form* $P = Q \circ T$ *where* $T \in \mathcal{L}(E, G)$ *is integral and* $Q \in \mathcal{P}(^m G)$*, we have* $P \in \mathcal{P}_{\mathrm{wb}}(^m E)$*.*

The answer is negative, as a consequence of the following result.

Proposition 2.4. *Let* E *and* F *be Banach spaces such that* F^* *contains no copy of* ℓ_1*. Let* $P \in \mathcal{P}(^m E, F)$ *be a polynomial of the form* $P = Q \circ T$ *where* $T \in \mathcal{L}(E, G)$ *is an integral operator and* $Q \in \mathcal{P}(^m G, F)$ *for some Banach space* G*. Then* $P \in \mathcal{P}_{\mathrm{wb}}(^m E, F)$*.*

P r o o f. Let $\tilde{Q} \in \mathcal{P}(^m G^{**}, F^{**})$ be the Aron-Berner extension of Q.

$$\begin{tikzcd} E \arrow[r, "P"] \arrow[dr, "T"'] & F \arrow[r, "k_F"] & F^{**} \\ & G \arrow[u, "Q"'] \arrow[r, "k_G"] & G^{**} \arrow[u, "\tilde{Q}"'] \end{tikzcd}$$

The operator $k_G \circ T$ is Pietsch integral. Since F^{**} has the CRP, we have

$$k_F \circ P = \tilde{Q} \circ k_G \circ T \in \mathcal{P}_{\mathrm{I}}(^m E, F^{**}) = \mathcal{P}_{\mathrm{PI}}(^m E, F^{**}) \subseteq \mathcal{P}_{\mathrm{wb}}(^m E, F^{**})$$

(see [9, Corollary 2.7] and [11, Theorem 4.10]). Therefore, $P \in \mathcal{P}_{\mathrm{wb}}(^m E, F)$. $\square$

With the same argument as that used in the proof of [9, Proposition 3.1], we can prove the following lemma where

$$\pi_p \colon \bigotimes_{\pi,s}^{p+1} E \longrightarrow \bigotimes_{\pi,s}^{p} E \qquad (1 \leqslant p \leqslant m-1)$$

are the projections introduced in [6, page 168].

Lemma 2.5. *Let* $T \in \mathcal{L}(E, F)$ *be an* r*-summing operator for* $1 \leqslant r < \infty$*. Then the polynomial*

$$P := T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ \delta_m \in \mathcal{P}(^m E, F)$$

is r*-dominated.*

Theorem 2.6. *Let E be a Banach space. The following assertions are equivalent:*

(a) *E contains no copy of ℓ_1;*

(b) *for all $m \in \mathbb{N}$ ($m \geqslant 2$) and $1 \leqslant r < \infty$ and for every Banach space F, we have $\mathcal{P}_{r\text{-d}}(^mE, F) \subseteq \mathcal{P}_{\mathrm{wb}}(^mE, F)$;*

(c) *for all $m \in \mathbb{N}$ ($m \geqslant 2$), we have $\mathcal{P}_{2\text{-d}}(^mE, \ell_2) \subseteq \mathcal{P}_{\mathrm{wb}}(^mE, \ell_2)$;*

(d) *for all $m \in \mathbb{N}$ ($m \geqslant 2$), we have $\mathcal{P}_{2\text{-d}}(^mE, E^*) \subseteq \mathcal{P}_{\mathrm{wb}}(^mE, E^*)$;*

(e) *for all $m \in \mathbb{N}$ ($m \geqslant 2$), there is a Banach space F such that, for all $1 \leqslant r < \infty$, we have $\mathcal{P}_{r\text{-d}}(^mE, F) \subseteq \mathcal{P}_{\mathrm{wb}}(^mE, F)$;*

(f) *for all $m \in \mathbb{N}$ ($m \geqslant 2$) and $1 \leqslant r < \infty$, we have $\mathcal{P}_{r\text{-d}}(^mE) \subseteq \mathcal{P}_{\mathrm{wb}}(^mE)$;*

(g) *there are $m \in \mathbb{N}$ ($m \geqslant 2$) and $1 < r < \infty$ such that $\mathcal{P}_{r\text{-d}}(^mE) \subseteq \mathcal{P}_{\mathrm{wb}}(^mE)$.*

Proof. (a) $\Rightarrow$ (b). Let $P \in \mathcal{P}_{r\text{-d}}(^mE, F)$. Then there are a Banach space G, an r-summing operator $T \in \mathcal{L}(E, G)$, and a polynomial $Q \in \mathcal{P}(^mG, F)$ such that $P = Q \circ T$ [10, Theorem 5]. By Theorem 2.1, T is compact. Hence, P is weakly continuous on bounded sets.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Suppose that E contains a copy of ℓ_1. By Theorem 2.1, there exists a 2-summing operator $T \in \mathcal{L}(E, \ell_2)$ which is not compact. We shall use the operators π_p and j_p as in the proof of Theorem 2.2. Consider the polynomial $P \in \mathcal{P}(^mE, \ell_2)$ given by

$$P := T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ \delta_m.$$

By Lemma 2.5, P is 2-dominated and, by (c), P is weakly continuous on bounded sets. It follows that its linearization $T \circ \pi_1 \circ \ldots \circ \pi_{m-1}$ is compact [30, Lemma 4.1] and then

$$T = T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ j_{m-1} \circ \ldots \circ j_1$$

is compact, which is false.

(b) $\Rightarrow$ (d), (b) $\Rightarrow$ (e), and (f) $\Rightarrow$ (g) are obvious.

(d) $\Rightarrow$ (a) by the same argument as in the proof of (c) $\Rightarrow$ (a).

(e) $\Rightarrow$ (f). Assume (f) fails. Then for some $m \in \mathbb{N}$ ($m \geqslant 2$) and some $1 \leqslant r < \infty$, there is a polynomial

$$P \in \mathcal{P}_{r\text{-d}}(^mE) \setminus \mathcal{P}_{\mathrm{wb}}(^mE).$$

Let F be the Banach space provided by (e). Choose $y_0 \in F$, $y_0 \neq 0$. Let $j\colon \mathbb{K} \to F$ be given by $j(\lambda) = \lambda y_0$. Then the polynomial $j \circ P \in \mathcal{P}(^mE, F)$ is r-dominated [25, Theorem 9], but it is not weakly continuous on bounded sets.

(g) $\Rightarrow$ (a). Suppose that E contains a copy of ℓ_1. Let $m \in \mathbb{N}$ ($m \geqslant 2$) and $1 < r < \infty$ be given by (g). Let $P := Q \circ i \in \mathcal{P}(^m\ell_1)$, where $i \in \mathcal{L}(\ell_1, \ell_2)$ is the natural inclusion and $Q \in \mathcal{P}(^m\ell_2)$ is defined by

$$Q(y) = \sum_{n=1}^{\infty} y_n^m \qquad \text{for } y = (y_n)_{n=1}^{\infty} \in \ell_2.$$

The operator i is r-integral [14, Proposition 5.12], so it can be extended to an r-integral operator $V \in \mathcal{L}(E, \ell_2)$ [14, Proposition 6.12]. Then V is r-summing [14, Proposition 5.5] and $Q \circ V \in \mathcal{P}_{r\text{-d}}(^m E)$ [10, Theorem 5]. By (g), $Q \circ V \in \mathcal{P}_{\mathrm{wb}}(^m E)$ and so $P \in \mathcal{P}_{\mathrm{wb}}(^m \ell_1)$, which contradicts the proof of (g) $\Rightarrow$ (a) in Theorem 2.3. $\square$

In assertion (g) of Theorem 2.6 we cannot replace $r > 1$ by $r \geqslant 1$, as a consequence of the following result. The definition of the $\mathscr{L}_\infty$-space may be found in [14, Chapter 3].

Proposition 2.7. *Let E be an $\mathscr{L}_\infty$-space and let F be a Banach space such that F^* contains no copy of ℓ_1. Choose $m \in \mathbb{N}$ $(m \geqslant 2)$. Then*

$$\mathcal{P}_{1\text{-d}}(^m E, F) \subseteq \mathcal{P}_{\mathrm{wb}}(^m E, F).$$

P r o o f. Let $P \in \mathcal{P}_{1\text{-d}}(^m E, F)$. By [11, Theorem 4.5], we may factor P in the form $P = Q \circ T$, where T is an integral operator and Q is a polynomial. It only remains to apply Proposition 2.4. $\square$

References

[1] *R. Alencar*: On reflexivity and basis for $\mathcal{P}(^m E)$. Proc. Roy. Irish Acad. Sect. A *85* (1985), 131–138.

[2] *J. M. Ansemil and K. Floret*: The symmetric tensor product of a direct sum of locally convex spaces. Studia Math. *129* (1998), 285–295.

[3] *R. M. Aron and P. D. Berner*: A Hahn-Banach extension theorem for analytic mappings. Bull. Soc. Math. France *106* (1978), 3–24.

[4] *R. M. Aron, C. Hervés and M. Valdivia*: Weakly continuous mappings on Banach spaces. J. Funct. Anal. *52* (1983), 189–204.

[5] *R. M. Aron and J. B. Prolla*: Polynomial approximation of differentiable functions on Banach spaces. J. Reine Angew. Math. *313* (1980), 195–216.

[6] *F. Blasco*: Complementation in spaces of symmetric tensor products and polynomials. Studia Math. *123* (1997), 165–173.

[7] *D. Carando and S. Lassalle*: Extension of vector-valued integral polynomials. J. Math. Anal. Appl. *307* (2005), 77–85.

[8] *C. S. Cardassi*: Strictly p-integral and p-nuclear operators, in: Analyse harmonique: Groupe de travail sur les espaces de Banach invariants par translation, Exp. II. Publ. Math. Orsay (1989).

[9] *R. Cilia, M. D'Anna and J. M. Gutiérrez*: Polynomial characterization of $\mathscr{L}_\infty$-spaces. J. Math. Anal. Appl. *275* (2002), 900–912.

[10] *R. Cilia and J. M. Gutiérrez*: Polynomial characterization of Asplund spaces. Bull. Belgian Math. Soc. Simon Stevin *12* (2005), 393–400.

[11] *R. Cilia and J. M. Gutiérrez*: Integral and S-factorizable multilinear mappings. Proc. Roy. Soc. Edinburgh *136A* (2006), 115–137.

[12] *R. Cilia and J. M. Gutiérrez*: Ideals of integral and r-factorable polynomials. Bol. Soc. Mat. Mexicana *14, 3a Serie* (2008).

[13] *A. Defant and K. Floret*: Tensor Norms and Operator Ideals. Math. Studies, vol. 176, North-Holland, Amsterdam, 1993.
[14] *J. Diestel, H. Jarchow and A. Tonge*: Absolutely Summing Operators. Cambridge Stud. Adv. Math. *43*; , Cambridge University Press, Cambridge, 1995.
[15] *J. Diestel and J. J. Uhl, Jr.*: Vector Measures. Math. Surveys Monographs 15, American Mathematical Society, Providence RI, 1977.
[16] *S. Dineen*: Complex Analysis on Infinite Dimensional Spaces. Springer Monogr. Math., Springer, Berlin, 1999.
[17] *G. Emmanuele*: Dominated operators on $C[0,1]$ and the (CRP). Collect. Math. *41* (1990), 21–25.
[18] *G. Emmanuele*: Banach spaces with the (CRP) and dominated operators on $C(K)$. Ann. Acad. Sci. Fenn. Math. *16* (1991), 243–248.
[19] *K. Floret*: Natural norms on symmetric tensor products of normed spaces. Note Mat. *17* (1997), 153–188.
[20] *K. Floret*: On ideals of n-homogeneous polynomials on Banach spaces. Topological algebras with applications to differential geometry and mathematical physics (Athens, 1999), 19–38, Univ. Athens, Athens, 2002.
[21] *K. Floret and S. Hunfeld*: Ultrastability of ideals of homogeneous polynomials and multilinear mappings on Banach spaces. Proc. Amer. Math. Soc. *130* (2002), 1425–1435.
[22] *J. M. Gutiérrez*: Weakly continuous functions on Banach spaces not containing ℓ_1. Proc. Amer. Math. Soc. *119* (1993), 147–152.
[23] *J. M. Gutiérrez and I. Villanueva*: Extensions of multilinear operators and Banach space properties. Proc. Roy. Soc. Edinburgh *133A* (2003), 549–566.
[24] *M. C. Matos*: Absolutely summing holomorphic mappings. An. Acad. Bras. Ci. *68* (1996), 1–13.
[25] *Y. Meléndez and A. Tonge*: Polynomials and the Pietsch domination theorem. Proc. Roy. Irish Acad. Sect. A *99* (1999), 195–212.
[26] *J. Mujica*: Complex Analysis in Banach Spaces. Math. Studies, vol. 120, North-Holland, Amsterdam, 1986.
[27] *K. Musial*: Martingales of Pettis integrable functions (D. Kölzow, ed.). Measure Theory. Oberwolfach 1979, Lecture Notes in Math. 794, Springer, Berlin, 1980, pp. 324–339.
[28] *G. Pisier*: Factorization of Linear Operators and Geometry of Banach Spaces. CBMS Reg. Conf. Ser. Math. 60, American Mathematical Society, Providence RI, 1987.
[29] *H. P. Rosenthal*: Some new characterizations of Banach spaces containing ℓ_1, preprint.
[30] *R. A. Ryan*: Applications of topological tensor products to infinite dimensional holomorphy. Ph. D. Thesis, Trinity College, Dublin (1980).
[31] *M. Talagrand*: Pettis Integral and Measure Theory. Mem. Amer. Math. Soc. 307, American Mathematical Society, Providence RI, 1984.
[32] *I. Villanueva*: Integral mappings between Banach spaces. J. Math. Anal. Appl. *279* (2003), 56–70.

Authors' addresses: Raffaella Cilia, Dipartimento di Matematica, Facoltà di Scienze, Università di Catania, Viale Andrea Doria 6, 95125 Catania, Italy, e-mail: `cilia@dmi.unict.it`; Joaquín M. Gutiérrez, Departamento de Matemática Aplicada, ETS de Ingenieros Industriales, Universidad Politécnica de Madrid, C. José Gutiérrez Abascal 2, 28006 Madrid, Spain, e- mail: `jgutierrez@etsii.upm.es`.