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A METHOD CONSTRUCTING DENSITY FUNCTIONS: THE CASE OF A GENERALIZED RAYLEIGH VARIABLE

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Abstract. In this paper we propose a new generalized Rayleigh distribution different from that introduced in Apl. Mat. *47* (1976), pp. 395–412. The construction makes use of the so-called "conservability approach" (see Kybernetika *25* (1989), pp. 209–215) namely, if X is a positive continuous random variable with a finite mean-value $E(X)$, then a new density is set to be $f_1(x) = xf(x)/E(X)$, where $f(x)$ is the probability density function of X. The new generalized Rayleigh variable is obtained using a generalized form of the exponential distribution introduced by Isaic-Maniu and the present author as $f(x)$.

Keywords: generalized Rayleigh variable (GRV), generalized exponential (GE), generating differential equation (GDE), conservability, probability density function (p.d.f.), pseudo-Weibull variable

MSC 2010: 60E10, 60K40, 62P30

1. Introduction

A famous guru of SQC (Statistical Quality Control), I refer here to William Edwards Deming (1900–1993), wrote in one of his books [13, p. 1] that the object of taking data is *to provide a basis for action.* These data must be analyzed in the frame of a *statistical model*, otherwise we deal only with a "pure raw material" whose generating mechanisms are unknown to us. In a later work [14, p. 148], he wrote that such a model has to be *a statistical distribution* personalized by a specific function which describes the behavior of the considered characteristic of interest. Most of these characteristics are measurable ones: "static" product quality characteristics

* The editors learnt with great sadness that Professor Viorel Vodă passed away on May 8, 2009. The galleys of this paper were therefore not proofread by the author, and the responsibility for any typesetting inaccuracies lies solely with the editors.

such as hardness, strength, geometric features, weights etc. or "dynamic" ones as for instance durability or time to failure of a technical component or system.

Nowadays, the specific literature devoted to statistical distributions, especially to those modeling various measurable variables, is extremely large. We have now the second expanded edition of a four volume collection of the most used distributions, see [31], and also some specialized monographs such as that of Patel and Read [35] about the classical normal variable and the pioneering works of Pollard and Rivoire [37] and Isaic-Maniu [25] on the Weibull one. From a theoretical point of view, a specific distribution function (d.f.), or its associated probability density function (p.d.f.), can be derived from what is called "a system of frequency curves".

Rodriguez [40] discussed five such classical systems, namely those of Karl Pearson, Gram-Charlier-Edgeworth, Burr-Hatke, Johnson, and Tukey's lambda which are in fact: a differential equation, a series expansion, again a differential equation, a transformation to normality and a special transformation (Rodriguez's words, [40, p. 218]). It is interesting to notice that there exists also an original system proposed in 1939 by Rafaele D'Addario (1899–1974) [11] reconsidered by himself in 1969 [12] which is in fact a functional equation involving some partially expected values of the underlying variable (details are given by Guerrieri [21, pp. 56–57]).

Let us mention also the so-called "suprasystem of probability distributions" proposed by Savageau [42] which is a set of simultaneous ordinary differential equations, and the S-system of distributions presented by Voit [53] and defined as a four parameter ordinary differential equations which "appears to be a good candidate for representing and analyzing failure data" ([54, p. 596]). A generalization of the Burr-Hatke and S-system of distributions has been recently done by the present author and posted on Gnedenko forum [52].

Another way to obtain p.d.f.'s is the one currently used in the reliability theory via the formula $f(x) = h(x)\exp[-\int h(u)\,\mathrm{d}u]$, where integration is taken over $[0, x]$ and $h(u)$ is the hazard (or failure) rate associated to the X variable representing the time-to-failure of the given entity.

Various choices for $h(u)$ provide a wide range of p.d.f.'s, as Blischke and Murthy presented [6, pp. 128–129].

A linear choice, like $h(u) = 2au$ furnishes just the Rayleigh p.d.f.

$$f(x) = 2ax\exp(-ax^2), \quad a > 0, \ x \geqslant 0,$$

which will be one of our concerns.

A third way to derive peculiar p.d.f.'s is to consider a general multiparameter p.d.f., as for instance the so-called generalized Gamma (GG); see Section 3 of this paper. Then by punctual particularizations of its parameters, one may obtain some usual (known) or quite "exotic" (less known) p.d.f.'s.

In one monograph on statistical tolerances, I quote here that of Miloš Jílek [30, pp. 18–27 and pp. 132–136], the Czech speaking readers (for instance) can find a sound synthesis of some peculiar cases covered by this GG p.d.f.

In the present paper we will construct and analyze statistical properties of a new density function via the following procedure: if X is a positive continuous random variable with a finite mean value $E(X)$ and $f(x)$ is its p.d.f. then $f_1(x) = xf(x)/E(X)$ is also a p.d.f. As a working material for $f_1(x)$ we will choose $f(x)$ to be a generalized variant of the exponential distribution proposed in [27].

Details will be given in the subsequent sections. We will now recall briefly some elements which may be not widely known.

2. The classical Rayleigh distribution

The classical Rayleigh distribution was proposed in 1880 by John William Strutt (1842–1919), better known as Lord Rayleigh (Nobel Prize in Physics, 1904), as the distribution of the amplitude resulting from the harmonic oscillations. Its density is $f(x;a) = (x/a^2)\exp(-x^2/2a^2)$, $x \geqslant 0$, $a > 0$, which provides a d.f. $F(x;a) = 1 - \exp(-x^2/2a^2)$ and a reliability function $R(x;a) = 1 - F(x;a) = \exp(-x^2/2a^2)$. The main characteristics are: mean $= E(X) = a(\pi/2)^{1/2} \approx 1.253314\,a$; variance $= \mathrm{var}(X) = a^2(4-\pi)/2 \approx 0.429204\,a^2$; mode $= a$; skewness ≈ 0.631111 and kurtosis ≈ 3.245089.[1]

The usual way to generate a Rayleigh distributed variable is the following: let $M(X,Y)$ be a point in a rectangular coordinate system, where X and Y are normally distributed random variables with zero mean and the same variance a^2. Then the distance $d = (X^2+Y^2)^{1/2}$ is a random element which obeys the Rayleigh law with parameter $a > 0$. It is worth to notice that in a less known but still interesting paper, Sergio Bruscantini [7, p. 103] enlists some of instances where the Rayleigh law occurs: 1) in naval research to simulate the sea waves, 2) in telecommunications to describe the signal fluctuations due to multipath effects in the line-of-sight links, 3) as a model for wind speed, 4) in bombing problems to describe the distributions of distances from target to the actual impact points.

As a curiosity, the last field of applications (bombing problems) has been largely debated in the so-called "hottest cold war period" (approximately after 1946) when the hysteria as regards "a probable nuclear attack on an urban center" became quite a passion for some scholars. Hunter [23] quotes a study concerning a comparison of estimates of nuclear bomb casualties from two different urban models: the cities under such an imaginary attack in 1947 are Sydney and Brisbane (Australia) and

[1] `http://www.brighton-webs.co.uk/distributions/rayleigh.asp`

using a two dimensional Rayleigh variable the estimated number of casualities should be 539,000 and 260,000 respectively (scaring!).

Let us focus on more peaceful illustrations. One of the main fields of applications being that of cutting and grinding tool durability since the hazard rate function of a Rayleigh variable is linearly increasing: this property is suitable to describe the *irreversible wear-out process* of such items used in metalworking. Actual case studies/examples may be found in [16, pp. 70–72 and 145–147] or in [18, pp. 44–59].

At the same time, linearity has obvious restrictions in describing the functional behavior of more complex systems which exhibit often a bath tube shape of the hazard rate. This is one of the reasons, we believe, which raised the interest in constructing various forms which generalize the classical Rayleigh p.d.f. A short review of some of these p.d.f.'s is presented in the next section.

3. A little bit of (quite a personal) statistical history

Some decades ago, I did publish in the Prague journal Apl. Mat. two papers (see [48], [49]) regarding a generalized variant of the Rayleigh density function

$$f(x;\theta,k)=\frac{2\theta^{k+1}}{\Gamma(k+1)}\,x^{2k+1}\exp(-\theta x^2),\quad x\geqslant 0,\ \ \theta>0,\ \ k\geqslant 0, \tag{1}$$

where $\Gamma(u)$ is the well-known Gamma function

$$\Gamma(u)=\int_0^\infty t^{u-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{2}$$

The form (1) includes, apart from the classical Rayleigh density (for $k=0$), some others, such as Maxwell (for $k=\frac{1}{2}$) and Chi (χ) (for $k=\frac{1}{2}a-1$, $a\in\mathbb{N}$, $a>2$, and $\theta=1/2b^2$, $b>0$).

Also, if we quit the positivity request for k and take $k=-\frac{1}{2}$ and $\theta=1/2\sigma^2$, $\sigma>0$, we obtain the "half-normal" density.

This form (1) has found its place in the well-known book of Johnson, Kotz, and Balakrishnan [31, p. 479].

There exist a lot of density functions which may be considered generalizations of the Rayleigh one. For instance, in a monograph by E. S. Pereverzev [36] I detected two forms, namely

$$F(x;k,a)=1-\exp(-x^{2k}/2a^2),\quad x\geqslant 0,\ \ k,a>0, \tag{3}$$

this distribution becoming for $k=\frac{1}{2}$ the exponential one and for $k=1$ the Rayleigh one

$$F(x)=1-\exp\Big[-\frac{a(x-x_1)^n}{(x_2-x_1)^m}\Big],\quad x\in[x_1,x_2]\subset\mathbb{R} \tag{4}$$

with $a > 0$ and $n, m \in \mathbb{N}$, which generalizes Rayleigh ($m = 0$, $x_1 = 0$, $n = 2$, and $x \in [0, \infty)$).

The best known generalization is probably that of Waloddi Weibull (1887–1979):

$$f(x; \theta, k) = \theta k x^{k-1} \exp(-\theta x^k), \quad x \geqslant 0, \ \ k, \theta > 0, \tag{5}$$

which gives Rayleigh for $k = 2$.

Blischke and Murthy [6] consider that this distribution was a response to the overused exponential model (which has a constant hazard rate $h(x) = \theta$ for $k = 1$) inapplicable to strength of materials or cutting-tool durability studies, for instance.

Weibull distribution had a tremendous career amongst the practitioners. Weibull himself collected in 1977 in a technical report of Förvarets Teletekniska Laboratorium (in Stockholm, Sweden) a number of 1019 references (papers) and 36 titles of books in which his model is mentioned (all these titles are only in English). Facing the enthusiasm related to this distribution, A. C. Giorski [20] draws the attention of what he calls "Weibull euphoria" arguing that the model is very useful but not "universal" (one year later, Ravenis [39] just proclaimed Weibull's model a *"potentially universal p.d.f. for scientists and engineers"*...).

A more general model (which includes the Weibull one) is the one called the generalized Gamma:

$$f(x; b, k, p) = \frac{k}{b\Gamma(p)} \cdot \left(\frac{x}{b}\right)^{pk-1} \exp\Big\{\Big(-\frac{x}{b}\Big)^k\Big\}, \tag{6}$$

where $x \geqslant 0$, $b, p, k > 0$, proposed by Stacy [45], which for $p = 1$ becomes the Weibull one, with b^k as a scale parameter and k as a shape parameter. It is interesting to notice that a similar form of (6) was used in 1925 by an Italian economist, Luigi Amoroso (1886–1965), in "Annali di Matematica Pura ed Applicata" in a long paper (No. 421, pp. 123–159) entitled "Ricerche intorno alla curva dei redditi" (Researches on the curve of incomes). Since it was published in a journal not very widespread, Amoroso's work remained unknown for a long time (until late sixties) when Henrick J. Malik discovered it and seized the merits of this generalized Gamma (see [5]). Amoroso-Stacy's distribution has been intensively studied by two Polish engineers: K. Ciechanowicz [9] and S. Firkowicz [19]. U. Hjorth [33] proposed another generalization of Rayleigh distribution:

$$F(x; \theta, \beta, \delta) = 1 - (1 + \beta x)^{-\theta/\beta} \cdot \exp\{-\delta x^2/2\}, \tag{7}$$

where $x \geqslant 0$, $\theta \geqslant 0$, $\beta, \delta > 0$. If we take $\theta = 0$, we obtain the classical Rayleigh form. It is interesting to notice that exponential distribution is obtained if $\delta = 0$ and $\beta \to 0$.

Recently, Soleha and Sewilam [44] introduced what they called “an entropy-like transformation”, namely

$$g(x) = F(x) + R(x) \ln R(x), \quad x \geqslant 0, \tag{8}$$

where $F(x)$ and $R(x)$ are respectively the distribution and the reliability function of a positive continuous random variable X.

If X is Rayleigh distributed, that is $F(x) = 1 - \exp(-ax^2)$, $x \geqslant 0$, $a > 0$, they found that the first derivative of g, namely

$$g'(x) = 2a^3x^3 \exp(-ax^2), \quad x \geqslant 0, \ \ a > 0, \tag{9}$$

is just a peculiar case of our GRV given by (1). These authors stated that (1) is “one of the earliest forms of generalized Rayleigh” [44, § 2, p. 8].

In our opinion, the form (8) is convenient from the analytical treatment point of view if $R(x)$ is of an exponential type: $R(x) = \exp[-A(x)]$ (with suitable assumptions on A) since (9) may be written as $g'(x) = -f(x) \ln R(x) = R'(x) \ln R(x)$ and finally $g'(x) = A(x)A'(x) \exp[-A(x)]$, $x \geqslant 0$ (this could be regarded as another possibility constructing density functions!).

Raqab and Kundu [38] have studied a general d.f. of the form

$$F_{\mathrm{EW}}(x; a, b, c) = [1 - \exp(-a^b x^b)]^c, \quad x \geqslant 0, \ \ a, b, c > 0, \tag{10}$$

which was baptized “exponentiated Weibull” (EW) and in a later paper, the same authors (Kundu and Raqab, [32]) taking $b = 2$ in (10) obtained the following form of a generalized Rayleigh:

$$F_{\mathrm{EW}}(x; a, c) = [1 - \exp(-a^2 x^2)]^c, \quad x \geqslant 0, \ \ a, c > 0 \tag{11}$$

(if $c = 1$, one has the classical Rayleigh d.f.).

Babus et al. [3, pp. 324–326] suggest that (11) may be used as a model in SPC (Statistical Process Control): they constructed and studied the behavior of a new control chart of Shewhart type but only for the case $c = 1$ (for details of Shewhartian methodology, see Caulcutt [8, pp. 122–129 and pp. 220–224]. Other results on control charts in the Rayleigh case belong to Drane et al. [17, pp. 237–241] where a comparison with Shewhart type charts is performed.

In the next section we will examine some generalized forms of the exponential p.d.f., choosing one of them as working material.

4. Generalized exponentials

Various ways how to generalize exponential distribution have been proposed. We mention two of them:

Dobó's form ([15])**:**

$$F(x;\lambda,a,\theta)=1-\Big(\frac{\lambda+\theta}{\lambda+\theta\mathrm{e}^{ax}}\Big)^{1/a\theta}, \tag{12}$$

where $x\geqslant 0$, $a>0$, $\lambda,\theta\geqslant 0$, which gives the classical $F(x;\theta)=1-\exp(-x/\theta)$ for $\lambda=0$.

Khan's form ([34] or [5, pp. 44–46])**:**

$$f(x;r,\theta)=\frac{\mathrm{e}^{-x/\theta}\cdot x^{rx}}{\theta^{1+rx}\cdot\Gamma(2+rx)}, \tag{13}$$

with $x\geqslant 0$, $r\geqslant 0$ but $r\theta<1$, which for $r=0$ gives $f(x;\theta)=\theta^{-1}\exp(-x/\theta)$.

It is worth noticing that if in (10) one takes $b=1$, then the so-called generalized exponential (GE) distribution is obtained which has been studied in detail by Gupta and Kundu [22].

Consider now the differential equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}=a(x)\cdot\varphi^{\alpha}+b(x)\cdot\varphi^{\beta}, \tag{14}$$

where φ is a positive real function, $x\in[a,b]\subseteq\mathbb{R}$ and for a reliability context one may choose $[a,b]\equiv[0,+\infty)$.

We will call (14) a generating differential equation (GDE) since for various choices of real continuous functions $a(x)$, $b(x)$ on $\mathbb{R}$ and two real numbers α, β one could obtain a wide range of densities φ.

Now, in (14) let us take $\alpha=-1$, $b(x)=0$, and β is an arbitrary real number. We have

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}=a(x)\cdot\frac{1}{\varphi}\quad\text{or}\quad\frac{1}{2}\varphi^2=\int a(x)\,\mathrm{d}x. \tag{15}$$

If $x\geqslant 0$ and $a(x)=-A\cdot(k/\lambda)x^{k-1}\exp(-x^k/\lambda)$, where A is a norming factor, we obtain

$$\frac{1}{2}\varphi^2=A\exp\Big(-\frac{x^k}{\lambda}\Big)\quad\text{or}\quad\varphi=\sqrt{2A}\exp\Big(-\frac{x^k}{2\lambda}\Big). \tag{16}$$

If we denote $2\lambda = \theta > 0$ and take A as $\frac{1}{2}k^2/\theta^{2/k} \cdot \Gamma^2(1/k)$ where $k > 0$, we get $\varphi(x)$ as a density function, namely

$$\varphi(x;\theta,k) = \frac{k}{\theta^{1/k}\Gamma(1/k)} \cdot \exp\Big(-\frac{k^k}{\theta}\Big), \tag{17}$$

which for $k = 1$ gives the classical exponential p.d.f.

It is interesting to draw the attention to the so-called "generalized error distribution" studied by T. Taguchi [47] who states that it was introduced by a Russian mathematician M. T. Subbotin in [46]. Its density is

$$f(x;p) = \frac{1}{2p^{\frac{1}{p}-1}\Gamma(1/p)} \cdot \exp\Big(-\frac{|x|^2}{p}\Big), \tag{18}$$

where $x \in \mathbb{R}$ and $p > 0$. This form resembles (17) but it does not include the exponential, since for $p = 1$ one obtains the Laplace density function. Our form (17) has two parameters (scale and shape ones) and x is restricted to $[0, \infty)$.

Now, if we compute the theoretical mean value for (17) we will obtain easily

$$E(X) = \theta^{1/k} \cdot \frac{\Gamma(2/k)}{\Gamma(1/k)} \tag{19}$$

(see for other details [26]).

5. The new GRV (Generalized Rayleigh Variable)

Our variable will be obtained in a more general framework which was presented in [51]. It is known that in the reliability theory some classes of time-to-failure distributions are obtained using a so-called "generator", which is also a p.d.f. One of the problems of interest is for instance the following: if we have an IFR (increasing failure rate) distribution function (d.f.) $F(x)$, and if we construct a new d.f.

$$F_1(x) = \mu \cdot \int_0^x R(u)\,\mathrm{d}u, \quad \text{where } R(x) = 1 - F(x), \quad \mu = \int_0^\infty x \cdot \mathrm{d}F(x), \tag{20}$$

does this $F_1(x)$ preserve the IFR property? According to Barlow and Proschan [4], the answer is *yes*. Let us recall that a failure distribution $F(x)$ has an increasing failure rate (IFR) if its associated failure (or hazard) rate $h(x) = f(x)/[1 - F(x)]$ where $F'(x) = f(x)$ is increasing in x [6, p. 121].

Now, if we define a p.d.f. as

$$f_1(x) = \frac{x}{E(X)} \cdot f(x), \tag{21}$$

where $0 < E(X) < \infty$ is the mean value of X and where $f(x)$ is known (that is, it has a well-defined form and specified parameters), which means it belongs to a certain class, say Weibull, is the new $f_1(x)$ also a Weibull type density? In this case, the answer is *no* (see [51, p. 211]). The new p.d.f. was nicknamed a "pseudo-Weibull distribution".

The property to preserve the initial class of belongness has been called conservativeness. Therefore, in this approach, Weibull (and also the classical exponential) are not conservative.

Now, if we take as $f(x)$ the form (17), that is, our generalized exponential, we obtain

$$f_1(x) = \frac{x}{E(X)} \cdot f(x) = \frac{kx}{\theta^{2/k}\Gamma(2/k)} \cdot \exp\Bigl(-\frac{x^k}{\theta}\Bigr) \tag{22}$$

with $x \geqslant 0$, $k, \theta > 0$, which for $k = 2$ yields the usual Rayleigh p.d.f. $f_1(x) = (2/\theta) \cdot x \cdot \exp(-x^2/\theta)$.

Hence, this new generalization of Rayleigh p.d.f. is not conservative as regards the transformation given by (21).

One interesting fact is the following: if $k = 1$, then we obtain (curiously) the pseudo-Weibull p.d.f., $PW(x; \theta, 1)$, that is $f_1(x) = \theta^{-2} \cdot x \cdot \exp(-x/\theta)$, since the general $PW(x; \theta, k)$ is

$$f_1(x; \theta, k) = kx^k \bigl[\theta^{1+1/k} \cdot \Gamma(1 + 1/k)\bigr]^{-1} \cdot \exp(-x^k/\theta). \tag{23}$$

The p.d.f. $PW(x; \theta, 1)$ has been studied in [51]. In our case (22) we have, for instance (for $m \in \mathbb{N}$)

$$E(X^m) = \frac{k}{\theta^{2/k}\Gamma(2/k)} \int_0^\infty x^{m+1} \cdot \mathrm{e}^{-x^k/\theta} \,\mathrm{d}x = \theta^{m/k} \cdot \frac{\Gamma\bigl((m+2)/k\bigr)}{\Gamma(2/k)}. \tag{24}$$

If $m = 1$ and $m = 2$ we have the first two noncentral moments

$$E(X) = \theta^{1/k}\Gamma(3/k)/\Gamma(2/k) \quad \text{and} \quad E(X^2) = \theta^{2/k}\Gamma(4/k)/\Gamma(2/k), \tag{25}$$

which give the variance of the variable

$$\operatorname{var}(X) = \theta^{2/k} \biggl[\frac{\Gamma(4/k)}{\Gamma(2/k)} - \frac{\Gamma^2(3/k)}{\Gamma^2(2/k)}\biggr]. \tag{26}$$

We will now prove the following lemma:

Lemma. *If X is a GRV with k known, then the variable $Y = X^k$ is a Gamma type random variable with parameters θ and $2/k$.*

P r o o f. We write the distribution function of Y, namely

$$f(y) = \mathrm{Prob}\{X^k < y\} = \mathrm{Prob}\{X < y^{1/k}\} = \int_0^{y^{1/k}} f(x;\theta,k)\,\mathrm{d}x, \tag{27}$$

where $f(x;\theta,k)$ is the p.d.f. of X. Hence, we have

$$F(y) = \frac{k}{\theta^{2/k}\Gamma(2/k)} \int_0^{y^{1/k}} x\cdot\exp(-x^k/\theta)\,\mathrm{d}x. \tag{28}$$

Taking into account the general formula

$$\frac{\mathrm{d}}{\mathrm{d}x}\biggl(\int_{a(x)}^{b(x)} f(u)\,\mathrm{d}u\biggr) = b'(x)\cdot f[b(x)] - a'(x)f[(a(x)], \tag{29}$$

we obtain from (24)

$$F(y) = f(y) = \frac{y^{2/k-1}}{\theta^{2/k}\cdot\Gamma(2/k)}\exp(-y/\theta), \quad y \geqslant 0,\ \ \theta,k > 0, \tag{30}$$

which is just the p.d.f. of a Gamma random variable (one may denote $(2/k = a)$ to have the usual form). □

If k is known, then the estimation of θ is easy to find by the maximum likelihood method. Indeed, if we have a sample $\{x_1, x_2, \ldots, x_n\}$ on X, the likelihood function is

$$L = \frac{k^n\cdot\prod_1^n x_i}{\theta^{2n/k}\cdot\Gamma^n(2/k)}\cdot\exp\biggl(-\frac{1}{\theta}\sum_1^n x_i^k\biggr) \tag{31}$$

and taking the logarithms and the derivative with respect to θ, we find

$$\ln L = n\ln k + \sum_1^n \ln x_i - \frac{2n}{k}\cdot\ln\theta - n\ln\Gamma(2/k) - \frac{1}{\theta}\sum_1^n x_i^k, \tag{32}$$

$$\frac{\partial\ln L}{\partial\hat\theta} = -\frac{2n}{k}\cdot\frac{1}{\hat\theta} + \frac{1}{\hat\theta^2}\cdot\sum_1^n x_i^k = 0, \tag{33}$$

which provides the solution

$$\hat\theta_{ML} = \Bigl(\frac{k}{2n}\Bigr)\cdot\sum_1^n x_i^k. \tag{34}$$

The distribution of $\hat{\theta}_{ML}$ is now almost obvious: since X_i^k has a Gamma distribution, the sum of i.i.d. (independent and identically distributed) Gamma variables is also Gamma (the stability property of Gamma distribution; see [4]).

If both parameters are unknown then we obtain the system

$$
(35) \qquad \begin{cases} \dfrac{\partial \ln L}{\partial \hat{k}} = \dfrac{n}{\hat{k}} + 2n \cdot \dfrac{1}{\hat{k}^2} \cdot \ln \hat{\theta} - n \cdot \dfrac{\Gamma'(2/k)}{\Gamma(2/k)} - \dfrac{1}{\hat{\theta}} \displaystyle\sum_1^n x_i^{\hat{k}} \ln x_i = 0, \\ \dfrac{\partial \ln L}{\partial \hat{k}} = \dfrac{2n}{\hat{k}} + \dfrac{1}{\hat{\theta}^2} \cdot \displaystyle\sum_1^n x_i^{\hat{k}} = 0. \end{cases}
$$

As one can see, the so-called DiGamma function is involved (that is, the derivative of the Gamma function $\Gamma(u)$), and consequently, some numerical methods are needed. Using some formulas from Ryshyk-Gradstein tables [41], namely

$$
(36) \qquad \frac{\mathrm{d} \ln \Gamma(u)}{\mathrm{d}u} = \psi(u) = \ln u + \int_0^\infty \mathrm{e}^{-ut} \Big(\frac{1}{t} - \frac{1}{1 - \mathrm{e}^{-t}} \Big)\, \mathrm{d}t,
$$

one may approximate $\psi(u+1)$ as

$$
(37) \qquad \psi(u+1) \approx -0.577215664 + 1.644934067 \cdot u - 1.202056904 \cdot u^2.
$$

As regards $\ln \theta$, if $0 < \theta < 1$, we can use the well-known Taylor series, and if $\theta > 1$, we may approximate $\ln \theta$ with the upper bound $(\theta - 1)/\sqrt{\theta}$ since we have the elementary inequality

$$
(38) \qquad \frac{\ln x}{x - 1} \leqslant \frac{1}{\sqrt{x}} \quad \text{if } x \in (0, +\infty) \setminus \{1\}.
$$

Let us remark that in (33), the first constant is just Euler's $C = \lim\limits_{n \to \infty} (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n) \approx 0.57721$.

6. Some comments

a) One can detect in literature some p.d.f.'s more general than the generalized Gamma, as for instance the so-called generalized Gompertz-Verhulst one (see [1], [2] or [50]). Such p.d.f.'s contain two or more special functions, which are difficult to be analytically treated.

b) The generating differential equation (14) may provide not only p.d.f.'s *but also distribution functions*. For instance, if we take $a(x) = \theta$, $b(x) = -\theta$, $\theta > 0$, $\alpha = 1$, and $\beta = 2$, we obtain

$$
(39) \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}x} = \theta \cdot \varphi(1 - \varphi) \quad \text{or} \quad \frac{\mathrm{d}\varphi}{\varphi(1 - \varphi)} = \theta\, \mathrm{d}x,
$$

where by straightforward integration one gets

$$\varphi(x;\theta)=\frac{1}{1+\mathrm{e}^{-\theta x}},\quad x\in\mathbb{R},\ \theta>0,$$

which is just a reduced Verhulst distribution function, proposed by Pierre François Verhulst (1804–1849) (see [24]).

c) An interesting form of a GRV can be obtained from the reliability function proposed in 1998 by Silvia Spătaru and Angela Galupa [43]. Their construction was the following: let $Y\geqslant 0$ be a random variable and let $G(x)=\mathrm{Prob}\{Y\leqslant z\}$, $x\geqslant 0$, its distribution function. Starting from a generalized form of the power distribution (see [26]) the following reliability function is obtained:

$$R(x)=[1-G(x)]^{\lambda}\cdot\mathrm{e}^{-\theta x},\quad x\geqslant 0,\ \theta\geqslant 0, \tag{40}$$

which provides a hazard rate

$$h(x)=\lambda\cdot\frac{g(x)}{1-G(x)}+\theta,\quad g(x)=G'(x) \tag{41}$$

having the following interpretation: if $\lambda=1$, the first element on the right-hand side of $h(x)$ represents the hazard rate of Y component, and θ is the failure rate of the second element having an exponential distribution. If we take now $G(x)$ to be the classical Rayleigh, namely $G(x)=1-\exp(-ax^2)$, then the distribution

$$F(x)=1-R(x)=1-\exp(-\lambda ax^2+\theta x) \tag{42}$$

is obtained with $x\geqslant 0$, $a>0$, $\lambda,\theta\geqslant 0$ (we have $F(x)=1-\exp(-ax^2)$ if $\lambda=1$, $\theta=0$).

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