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The purpose of these lectures is to present a rather elementary and selfcontained approach to the weak first and second boundary value problem for the Laplacian in L^q where $1 < q < \infty$. These problems are basic for a lot of applications in mathematical physics, like as e.g. Stokes' problem. From the viewpoint of applications it is necessary to consider as well bounded as exterior domains. Our approach rests on two variational inequalities in L^q and a type of regularity argument. The results presented here are part of a joint work with H. Sohr (Paderborn/FRG) [9].

1. Notations. Throughout this paper $G \subseteq \mathbb{R}^n$ $(n \geq 2)$ denotes a domain, i.e. G is open and connected. G is called an exterior domain if G is a domain and if there exists a bounded open set $\emptyset \neq K \subset \mathbb{R}^n$ such that $G = \mathbb{R}^n \setminus \bar{K}$. Without loss of generality we may assume $0 \in K$. If $G \subseteq \mathbb{R}^n$ is a domain we write $\partial G \in C^1$ if the boundary is of class C^1. If A,B are subsets of $\mathbb{R}^n$ we write $A \subset\subset B$ if A and B are open, $\bar{A}$ is compact and $\bar{A} \subset B$. For $x \in \mathbb{R}^n$ and $r > 0$ by $B_r(x)$ we denote the open ball with radius r centered a x. If $x = 0$ we use the abbreviation $B_r := B_r(0)$. If $M \subset \mathbb{R}^n$ is a Lebesgue measurable subset of $\mathbb{R}^n$ by $|M|$ we denote its Lebesgue measure. Let $1 < q < \infty$ and let q' be defined by $\frac{1}{q} + \frac{1}{q'} = 1$, that is $q' = \frac{q}{q-1}$. Observe $(q')' = q$. For a domain $G \subset \mathbb{R}^n$ by $L^q(G)$ we denote the usual (real) Lebesgue space equipped with norm $\|u\|_{L^q(G)} := \|u\|_{q,G} := (\int_G |u(x)|^q dx)^{1/q}$. For $f \in L^q(G)$ and $g \in L^{q'}(G)$ we write $\langle f,g\rangle := \int_G f(x)g(x)dx$. If $f \in (L^q(G))^n$, $g \in (L^{q'}(G))^n$ are vector fields we use the same notation $\langle f,g\rangle := \sum_{i=1}^n \langle f_i,g_i\rangle$. Beside the usual space

$L^q_{loc}(G) := \{f : G \longrightarrow \mathbb{R} : f \text{ measurable in } G \text{ and } f|_K \in L^q(K) \text{ for each } K \subset\subset G\}$

we use the convenient abbreviation

$L^q_{loc}(\bar{G}) := \{f: G \to \mathbb{R} : f \text{ measurable in } G \text{ and } f|_{G\cap B_R} \in L^q(G\cap B_R) \text{ for each } R > 0\}$.

We write in the sequel $G_R := G \cap B_R$. Observe that for G bounded $L^q_{loc}(\bar{G}) = L^q(G)$.

So the notation $L^q_{loc}(\bar{G})$ is interesting only in connection with unbounded domains. In the same sense we use the notation $C^\infty_o(\bar{G}) := \{\phi|_{\bar{G}} : \phi \in C^\infty_o(\mathbb{R}^n)\}$. Observe again for G bounded that $C^\infty_o(\bar{G}) = C^\infty(\bar{G})$. For $i = 1,..,n$ by $\partial_i := \frac{\partial}{\partial x_i}$ we denote the partial derivates, $\nabla := \nabla_n := (\partial_1,..,\partial_n)$ denotes the gradient and $\Delta := \Delta_n := \partial_1^2 + .. + \partial_n^2$ the Laplacian. If X is a Banach space by X^* we denote the dual space equipped with norm

$$\|x^*\|_{X^*} := \sup_{0 \neq x \in X} \frac{|x^*(x)|}{\|x\|_X} \quad \text{for } x^* \in X^*.$$

2. Sobolev spaces. For $1 \leq q < \infty$ and a domain $G \subset \mathbb{R}^n$ by $H^{1,q}(G) := \{p \in L^q(G): \partial_i p \in L^q(G),\ i = 1,..,n\}$ we denote the usual Sobolev space equipped with norm $\|p\|_{1,q} := (\|p\|_q^q + \|\nabla p\|_q^q)^{1/q}$ where $\|\nabla p\|_q := (\sum_{i=1}^n \|\partial_i p\|_q^q)^{1/q}$. Observe that this norm is equivalent to the norm $(\int_G |\nabla p(x)|^q dx)^{1/q}$ where $|\nabla p| = (\sum_i (\partial_i p)^2)^{1/2}$. Here $\partial_i p$ denotes the weak (= distributional) derivative of p. For the well known properties of these spaces we refer e.g. to Ne as [6] or Kufner, John, Fu ik [5]. As usual $H^{1,q}_o(G) := \overline{C^\infty_o(G)}^{\|\cdot\|_{1,q}}$. Considering for G bounded the space $H^{1,q}_o(G)$, because of the elementary Poincaré-inequality

$$(2.1) \qquad \|p\|_q \leq C(G)\ \|\nabla p\|_q \quad \text{for } p \in C^\infty_o(G)$$

a norm being equivalent to $\|p\|_{1,q}$ is defined by $\|\nabla p\|_q$. If G is a bounded domain say with boundary $\partial G \in C^1$ (or if G is convex, see e.g. [4]) then the general Poincaré-inequality

$$(2.2) \qquad \|p\|_q \leq C(G)\|\nabla p\|_q \quad \text{for } p \in H^{1,q}(G) \text{ with } \int_G p\,dy = 0$$

holds true. Considering the quotient space $H^{1,q}(G)/\mathbb{R}$ (identifying elements whose difference is constant) then again by $\|\nabla p\|_q$ an equivalent norm on $H^{1,q}(G)/\mathbb{R}$ is defined. This procedure is no longer possible for G unbounded. But from the viewpoint of applications we have to use Sobolev spaces equipped with the (order homogeneous) norm $\|\nabla p\|_q$. For this purpose we define for a domain $G \subset \mathbb{R}^n$ with boundary $\partial G \in C^1$ and for $1 \leq q < \infty$

$$(2.3) \quad E^q(G) := \{\nabla p : p \in L^q_{loc}(\bar{G}),\ \nabla p \in L^q(G)^n\}.$$

This space is equipped with norm $\|\nabla p\|_q$. Observe that for G bounded we have $E^q(G) = \{\nabla p : p \in H^{1,q}(G)\}$. For technical reasons we need the following lemma (compare [8], Lemma 2.2) admitting an elementary proof using solely the Poincaré-inequality (2.2) for balls:

Lemma 2.1. Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be a domain. Let $1 \leq q < \infty$ and let a sequence $(p_i) \subset H^{1,q}_{loc}(G)$ be given such that (∇p_i) is a Cauchy sequence in $L^q_{loc}(G)^n$. Then there exists a sequence $(c_i) \subset \mathbb{R}$ and some $p \in H^{1,q}_{loc}(G)$ such that $(p_i - c_i)$ converges in $H^{1,q}_{loc}(G)$ to p. The sequence (c_i) may be chosen independently of q.

Using this lemma and the fact that for $\partial G \in C^1$ we may conclude from $p \in L^1_{loc}(G)$ with $\nabla p \in L^q(G)$ that $p \in L^q_{loc}(\bar{G})$ (see e.g. Necas [6], p. 114), it is not too hard to see that $E^q(G)$ is complete. Since $E^q(G)$ for $1 < q < \infty$ may be regarded as a closed subspace of the reflexive space $L^q(G)^n$ it is reflexive too. So we end with

Theorem 2.2. Let $G \subset \mathbb{R}^n$ be a domain with boundary $\partial G \in C^1$ and let $1 < q < \infty$. Then $E^q(G)$ is a reflexive Banach space. For $q = 2$ $E^2(G)$ is a Hilbert space with inner product $\langle \nabla p, \nabla \phi \rangle$ for $\nabla p, \nabla \phi \in E^2(G)$.

Next we study approximation properties. If G is bounded with $\partial G \in C^1$ from the fact $E^q(G) = \{\nabla p : p \in H^{1,q}(G)\}$ and the classical density result for Sobolev spaces $H^{1,q}(G) = \overline{C^\infty(\bar{G})}^{\|\cdot\|_{1,q}}$ (compare e.g. Ne as [6], p. 67) we immediately derive

Theorem 2.3. Let $G \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial G \in C^1$ and let $1 < q < \infty$. Then

(2.4) $E^q(G) = \overline{\{\nabla p : p \in C^\infty(\bar{G})\}}^{\|\cdot\|_q}$

For exterior domains we get

Theorem 2.4. Let $G \subset \mathbb{R}^n$ be an exterior domain with boundary $\partial G \in C^1$ and let $1 < q < \infty$. Then

(2.5) $E^q(G) = \overline{\{\nabla p : p \in C^\infty_0(\bar{G})\}}^{\|\cdot\|_q}$

Proof. i) For $k \in \mathbb{N}$ let $R_k := \{x \in \mathbb{R}^n : k < |x| < 2k\}$. Then there is $k_0 \in \mathbb{N}$ such that $R_k \subset\subset G$ for $k \geq k_0$. For $k = 1$, R_1 is a C^1-domain and the Poincaré-inequality (2.2) holds for q with $1 < q < \infty$ and for R_1 with a certain constant $C_1 = C_1(q) > 0$. If $k \in \mathbb{N}$ and $u \in H^{1,q}(R_k)$ with $\int_{R_1} u dy = 0$ then with $p(x) := u(kx)$ for $x \in R_1$ we have $p \in H^{1,q}(R_1)$. Further

$$\int_{R_1} p(x)dx = k^{-n} \int_{R_k} u(y)dy = 0, \qquad \|p\|_{q,R_1} = k^{-n/q}\|u\|_{q,R_k},$$

$\|\nabla p\|_{q,R_1} = k^{1-n/q}\|\nabla u\|_{q,R_k}$. From (2.2) valid for p and R_1 we derive

$$(2.6) \qquad \|u\|_{q,R_k} \leq k \cdot C_1 \|\nabla u\|_{q,R_k} \quad \text{for } u \in H^{1,q}(R_k) \text{ with } \int_{R_k} u\, dy = 0.$$

ii) Choose $\rho \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \rho \leq 1$ with $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = 0$ for $|x| \geq 2$. For $k \in \mathbb{N}$ put $\rho_k(x) := \rho(k^{-1}x)$. Then supp $\nabla\rho_k \subset R_k$ and $|\nabla\rho_k(x)| \leq M \cdot k^{-1}$ where $M = \max_{z \in \mathbb{R}^n} |\nabla\rho(z)|$. Take now some $p : G \to \mathbb{R}$ such that $\nabla p \in E^q(G)$. For $k \geq k_o$ put $h_k(x) := \rho_k(x)(p(x)-c_k)$ where $c_k := |R_k|^{-1} \int_{R_k} p dy$. By the properties of p and ρ_k we see $h_k|_{G_{2R}} \in H^{1,q}(G_{2R})$ where $G_{2R} := G \cap B_{2R}$.
Further $\nabla h_k = \rho_k \nabla p + \nabla\rho_k(p-c_k)$. Clearly $\|\nabla p - \rho_k \nabla p\|_q \to 0$. Since supp $\nabla\rho_k \subset R_k$ we derive from (2.6)

$$\|\nabla\rho_k(p-c_k)\|_{q,G} \leq M \cdot k^{-1}\|p-c_k\|_{q,R_k} \leq M \cdot C_1\|\nabla p\|_{q,R_k} \to 0$$

for $k \to \infty$. Therefore $\|\nabla p - \nabla h_k\|_{q,G} \to 0$. Since

$p|_{G_{2k}} \in H^{1,q}(G_{2k}) = \overline{C^\infty(\bar{G}_{2k})}^{\|\cdot\|_{1,q}}$ there exists $\phi_k \in C^\infty(\bar{G}_{2k})$ such that

$$\|\nabla p - \nabla\phi_k\|_{q,G_{2k}} \leq \|p - \phi_k\|_{1,q,G_{2k}} \leq k^{-1}.$$

Put $p_k := \rho_k(\phi_k - d_k)$ where $d_k := |R_k|^{-1} \int_{R_k} \phi_k dy$.

Then $p_k \in C_0^\infty(\bar{G})$ is vanishing outside G_{2k} and again by (2.6)

$$\|\nabla h_k - \nabla p_k\|_q \leq \|\rho_k(\nabla p - \nabla\phi_k)\|_{q,G_{2k}} + M \cdot k^{-1}\|(p-c_k)-(\phi_k-d_k)\|_{q,R_k}$$

$$\leq \|\nabla p - \nabla\phi_k\|_{q,G_{2k}} + MC_1\|\nabla p - \nabla\phi_k\|_{q,R_k} \leq (1+MC_1) \cdot \frac{1}{k}$$

Altogether we get $\|\nabla p - \nabla p_k\|_{q,G} \to 0$. ∎

As an interesting corollary we derive

<u>Corollary 2.5.</u> Let $G \subset \mathbb{R}^n$ be an exterior domain with boundary $\partial G \in C^1$ and let $1 < q < \infty$.

Then

(2.7) $\quad E^q(G) = \overline{\{\nabla p : p \in H^{1,q}(G)\}}^{\|\cdot\|_q}$

Proof. Clearly $\{\nabla p : p \in C_0^\infty(\bar{G})\} \subset \{\nabla p : p \in H^{1,q}(G)\} \subset E^q(G)$ and (2.7) follows from (2.5). ■

As we will see later for exterior domains we have $\{\nabla p : p \in H^{1,q}(G)\} \subsetneq E^q(G)$. In connection with the Dirichlet problem with homogeneous boundary data we consider for $1 < q < \infty$ and $G \subset \mathbb{R}^n$ an open set the spaces

(2.8) $\quad E_0^q(G) := \overline{\{\nabla \phi : \phi \in C_0^\infty(G)\}}^{\|\cdot\|_q}$

Clearly $E_0^q(G) \subset E^q(G)$. If G is bounded we immediately see by means of (2.1) that $E_0^q(G) = \{\nabla p : p \in H_0^{1,q}(G)\}$. This representation no longer holds for exterior domains. A useful partial substitute for (2.1) is given by

Theorem 2.6. Let $G \subset \mathbb{R}^n$ be an exterior domain, $G = \mathbb{R}^n \backslash \bar{K}$ where $\phi \neq K \subset\subset \mathbb{R}^n$. Suppose $0 \in K$. Let $1 < q < \infty$. Then there is a constant $C = C(q,G,n) > 0$ such that for each $R > 0$ with $K \subset\subset B_R$

(2.9) $\quad \|p\|_{p,G_R} \leq CR^{n/q + 1/q'}\|\nabla p\|_{q,G}$

holds for $p \in C_0^\infty(G)$, where $q' = \frac{q}{q-1}$.

Proof. Since $0 \in K$ and K is open there is $\delta > 0$ such that $B_\delta \subset\subset K$. Let $p \in C_0^\infty(G)$. Then p vanishes in a neighborhood of B_δ. Let $S := \{\zeta \in \mathbb{R}^n : |\zeta| = 1\}$ denote the unit sphere. For $0 \neq x \in \mathbb{R}^n$ write $x = r\zeta$, $r = |x|$, $\zeta = \frac{x}{|x|} \in S$. Then

$$p(x) = p(r\zeta) = p(r\zeta) - p(0) = \int_0^r \sum_{i=1}^n (\partial_i p)(t\zeta)\zeta_i dt$$

and by Hölder's inequality

$$|p(r\zeta)|^q \leq r^{q-1} \int_0^r |\nabla p(t\zeta)|^q dt$$

Since p vanishes in B_δ we get after integrating with respect to $\zeta \in S$ for $0 < \delta < r \leq R$

$$\int_S |p(r\zeta)|^q d\omega_\zeta \le R^{q-1} \int_\delta^R \frac{t^{n-1}}{t^{n-1}} \int_S |\nabla p(t\zeta)|^q d\omega_\zeta dt \le \delta^{1-n} R^{q-1} \|\nabla p\|^q_{q,G_R}$$

Multiplying by r^{n-1} and integrating with respect to $r \in [0,R]$ yields (2.9). ∎

As an immediate consequence we get

<u>Theorem 2.7.</u> Let $G \subset \mathbb{R}^n$ be an exterior domain and let $1 < q < \infty$. Then

$$(2.10) \quad E^q_0(G) = \{\nabla p \in E^q(G)\text{: there exists a sequence } (p_i) \subset C^\infty_0(G) \text{ such that } \|\nabla p - \nabla p_i\|_q \to 0 \text{ and } \|p - p_i\|_{q,G_R} \to 0 \text{ for each } R > 0\}$$

<u>Proof. i)</u> If $\nabla p \in E^q_0(G)$ then by (2.8) there is a sequence $(p_i) \subset C^\infty_0(G)$ such that $\|\nabla p_i - \nabla p_j\|_q \to 0$. By (2.9) (p_i) is a Cauchy sequence in G_k for each fixed $k \in \mathbb{N}$. Denote the $L^q(G_k)$-limit of (p_i) by $p^{(k)}$. Then after eventually changing $p^{(k+1)}$ on a subset $N_k \subset G_k$ of measure zero we may assume $p^{(k+1)}|_{G_k} = p^{(k)}$. So we get a measurable $p : G \to \mathbb{R}$ such that $p|_{G_R} \in L^q(G_R)$ for $R > 0$ and for $\phi \in C^\infty_0(G)$ we conclude

$$\langle p, \partial_j \phi \rangle = \lim_{i\to\infty} \langle p_i, \partial_j \phi \rangle = - \lim_{i\to\infty} \langle \partial_j p_i, \phi \rangle$$

telling us that the distributional gradient of p is given by the L^q-limit of the sequence (∇p_i). That is $\nabla p \in E^q(G)$ and the above approximation property holds.

ii) If conversely ∇p belongs to the set at the right hand side of (2.10), by (2.8) we see $\nabla p \in E^q_0(G)$. ∎

By the Sobolev embedding theorem ([6], p.69, [5], p.282) we see that for any domain $G \subset \mathbb{R}^n$ (bounded or unbounded) and $1 < q < n$ holds

$$(2.11) \quad \nabla p \in E^q_0(G) \Rightarrow p \in L^{q^*}(G) \text{ where } q^* = \frac{nq}{n-q} .$$

To study conversely the case $q > n$ we first consider the Morrey-estimate (compare e.g. [1], p.242): Let $G \subset \mathbb{R}^n$ be an open set and $0 < \alpha \le 1$ and let $p \in C^\infty_0(G)$ with the property that there is a constant $M \ge 0$ such that

$$(2.12) \quad \int_{G\cap B_r(x_0)} |\nabla p| dx \le M r^{n-1+\alpha}$$

holds for all $x_0 \in G$ and $r > 0$. Then there is a constant $C = C(n,\alpha) > 0$

independent of p such that for $x_1,x_2 \in G$

(2.13) $|p(x_1)-p(x_2)| \le C\cdot M\ |x_1-x_2|^{\alpha}$.

For last estimate compare in addition e.g. [6], p. 73 or [5], p. 289.
If now $\nabla p \in E_0^q(G)$ with $q > n$ then

$$\int_{G\cap B_r(x_o)} |\nabla p|dx \le \|\nabla p\|_{q;B_r(x_o)} |B_r(x_o)|^{1/q'} \le C_1\|\nabla p\|_q\ r^{n/q'}$$

where $C_1 = C_1(n,q) > 0$. Since $\frac{n}{q'} = n - \frac{n}{q} = n-1 + (1-\frac{n}{q})$

(2.12) holds with $0 < \alpha := 1 - \frac{n}{q} < 1$. By definition there is a sequence $(p_i) \subset C_0^\infty(G)$ such that $\|\nabla p-\nabla p_i\|_q \to 0$. By (2.9) $\|p-p_i\|_{q,G_k} \to 0$ for each $k \in \mathbb{N}$. So we may select a subsequence again denoted by (p_i) such that $p_i \to p$ a.e. in G. With $M := C_1 \sup_{i\in\mathbb{N}} \|\nabla p_i\|_q < \infty$ (2.13) holds for the p_i and at the end for p and almost all $x_1,x_2 \in G$. After changing p on a set of measure zero (2.13) holds for all $x \in G$. So p is Hölder-continuous with Hölder exponent $\alpha = 1 -\frac{n}{q}$.
Suppose now that $G \subset \mathbb{R}^n$ is an exterior domain, $G = \mathbb{R}^n\backslash K$ where $0 \in K \subset\subset \mathbb{R}^n$. Given $\nabla p \in E_0^q(G)$ we may extend p by zero to the whole $\mathbb{R}^n$ leading to $\nabla p \in E^q(\mathbb{R}^n)$. (2.13) holds for this extension too and because of $p(0) = 0$ we get

(2.14) $|p(x)| \le CM\ |x|^{1-n/q}$.

By no means p need to vanish near $x = \infty$, neither pointwise nor in any L^s-mean. Conversely let $\varphi \in C^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, $\varphi(x) = 0$ for $|x| \le R$ and $\varphi(x) = 1$ for $|x| \ge 2R$ where $R > 0$ is such that $K \subset B_R$. Let $q > n$ and $0 < \lambda < 1 -\frac{n}{q}$ and put $p(x) := \varphi(x)|x|^\lambda$. Then p vanishes in a neighborhood of ∂G. Let $\rho \in C_0^\infty(\mathbb{R}^n)$, $0 \le \rho \le 1$, $\rho(x) = 1$ for $|x| \le 1$, $\rho(x) = 0$ for $|x| \ge 2$ and for $k \in \mathbb{N}$ let $\rho_k(x) := \rho(k^{-1}x)$ and put $p_k := \rho_k\ p$. Then $p_k \in C_0^\infty(\mathbb{R}^n)$.
Since $|\nabla p(x)| \le c|x|^{\lambda-1}$ for $|x| \ge 2R$ we see $|\nabla p| \in L^q(G)$ for $q > n$ and $\|\nabla p(1-\rho_k)\|_q \to 0(k \to \infty)$. Since $|\nabla\rho_k(x)| \le \frac{c}{|x|}$ for $k \le |x| \le 2k$ we see $\|p\cdot\nabla\rho_k\|_q \to 0$ and therefore $\|\nabla p-\nabla p_k\|_q \to 0$, that is $p \in E_0^q(G)$.

3. Some auxiliary tools. First we need some facts on harmonic functions. If $G \subset \mathbb{R}^n$ is an open set and $u \in C^\infty(G)$, $\Delta u = 0$, then we have the two mean value properties: If $x \in G$, $R > 0$ such that $B_R(x) \subset\subset G$, then

(3.1) $u(x) = \frac{1}{\omega_n} \int_S u(x+r\zeta)d\omega_\zeta$ for $0 < r \leq \mathbb{R}$ where $S = \{\zeta \in \mathbb{R}^n : |\zeta| = 1\}$ and

(3.2) $u(x) = |B_r(x)|^{-1} \int_{B_r(x)} u(y)dy$ for $0 < r \leq R$

We consider Friedrichs' mollifier with a r a d i a l depending kernel: $j \in C_o^\infty(\mathbb{R}^n)$, $j(z) = j(|z|)$, $0 \leq j(z)$, $j(z) = 0$ for $|z| \geq 1$ and $\int_{\mathbb{R}^n} j(z)dz = 1$ (with suitable $c \geq 0$ choose e.g. $j(z) := c \exp[(1-|z|^2)^{-1}\sim$ for $|z| < 1$ and $j(z) = 0$ for $|z| \geq 1$). For $\epsilon > 0$ put $j_\epsilon(z) := \epsilon^{-n}j(\frac{z}{\epsilon})$ and for $f \in L^1(G)$ put

(3.3) $f_\epsilon(x) := \int_G j_\epsilon(x-y)\ f(y)dy = (j_\epsilon * f)(x).$

As is well known (see e.g. [6], p.58 or [5],p.72)

$$f_\epsilon \in C^\infty(\mathbb{R}^n) \text{ and } \|f-f_\epsilon\|_{L^1(G)} \to 0$$

Suppose now that u is harmonic in G and let $x \in G, \epsilon_o > 0$ such that $B_{\epsilon_o}(x) \subset\subset G$. Then introducing polar coordinates $y = x+r\zeta$, $\zeta \in S$, we get for $0 < \epsilon \leq \epsilon_o$

$$u_\epsilon(x) = \int j_\epsilon(x-y)u(y)dy = \int_0^\epsilon r^{n-1}\ j_\epsilon(r) \int_S u(x+r\zeta)d\omega_\zeta dr$$

By (3.1) $\int u(x+r\zeta)d\omega_\zeta = \omega_n u(x)$ and

$$\int_0^\epsilon r^{n-1}j_\epsilon(r)dr\omega_n = \int j_\epsilon(z)dz = 1. \text{ Therefore}$$

(3.4) $u_\epsilon(x) = u(x)$ for harmonic u, $B_\epsilon(x) \subset\subset G$.

Last observation admits a rather simple proof of

Theorem 3.1 (Weyl's lemma). Let $G \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}(G)$ such that

(3.5) $\int_G u(x)\Delta\phi(x)dx = 0$ for $\phi \in C_o^\infty(G)$

Then u coincides a. e. in G with a harmonic C^∞-function.

Proof. Because there exists a sequence (G_k) with $G_k \subset\subset G$, $G_k \subset\subset G_{k+1}$, $G = \bigcup_{k=1}^\infty G_k$ it suffices to proof the theorem for any $G' \subset\subset G$. Choose a set G" such that $G' \subset\subset G'' \subset\subset G$. Let $\epsilon_o := \frac{1}{2}\min(\text{dist}(G',\partial G''), \text{dist}(G'',\partial G)) > 0$. Then for $y \in G''$ and $0 < \epsilon \leq \epsilon_o$ with $\phi(x) := j_\epsilon(y-x)$ we see $\phi \in C_o^\infty(G)$ and therefore by (3.5)

$$0 = \int u(x)\Delta_x j_\epsilon(x-y)dx = \int u(x)\Delta_y j_\epsilon(x-y)dx = \Delta u_\epsilon(y).$$

Therefore u_ϵ is harmonic in G". Let $0 < \delta \le \epsilon_0$ and $x \in G'$. Then by (3.4)

$$u_\epsilon(x) = u_{\epsilon\delta}(x) = u_{\delta\epsilon}(x) = u_\delta(x)$$

since the convolutions commute. But then for $x \in G'$ $u_\epsilon(x)$ does not depend on ϵ. Since $u \in L^1(G')$ and $\|u-u_\epsilon\|_{L^1(G')} \to 0$ we conclude $u = u_\epsilon$ a.e. in G', proving the theorem since $u_\epsilon \in C^\infty(G')$ and $\Delta u_\epsilon = 0$. ■

An easy consequence is now

Theorem 3.2. Let $1 < q < \infty$. Then

$$(3.6) \qquad L^q(\mathbb{R}^n) = \overline{\{\Delta\phi : \phi \in C_0^\infty(\mathbb{R}^n)\}}^{\|\cdot\|_q}$$

Proof. Denote by M_q the right hand side of (3.6) and suppose $M_q \subsetneq L^q(\mathbb{R}^n)$. By the Hahn-Banach theorem there exists $F^* \in L^q(\mathbb{R}^n)^*$ with $\|F^*\|_* > 0$ and $F^*|_{M_q} = 0$. Since $L^q(\mathbb{R}^n)^* \cong L^{q'}(\mathbb{R}^n)$ isometrically isomorphic ($q' = \frac{q}{q-1}$) there is $f \in L^{q'}(\mathbb{R}^n)$, $\|f\|_{q'} = \|F\|_* > 0$ such that $F(g) = \langle f,g \rangle$ for $g \in L^q(\mathbb{R}^n)$. Since $F|_{M_q} = 0$ we conclude $\langle f,\Delta\phi \rangle = 0$ for $\phi \in C_0^\infty(\mathbb{R}^n)$. By Theorem 3.1 f is harmonic (eventually after change on a set of measure zero) and for $x \in \mathbb{R}^n$ and arbitrary $r > 0$ by (3.2)

$$|f(x)| \le |B_r(x)|^{-1} \int_{B_r(x)} |f(y)|dy \le |B_r(x)|^{-1/q'} \|f\|_{q',\mathbb{R}^n} \to 0$$

for $r \to \infty$. Therefore $f \equiv 0$ contradicting $\|f\|_{q'} > 0$. ■

With the fundamental solution

$$S(z) := \begin{cases} \frac{2}{(n-2)\omega_n} |z|^{2-n} & \text{for } n \ge 3 \\ -\frac{1}{2\pi} \ln|z| & \text{for } n = 2 \end{cases}$$

we have for $u \in C_0^\infty(\mathbb{R}^n)$ the representation

$$(3.7) \qquad u(x) = - \int S(x-y)\Delta u(y)dy$$

This formula is basic to derive L^q-estimates for second derivatives of u via

Theorem 3.3 (Calderon-Zygmund estimate). Let $n \geq 2$, $S_n := \{z \in \mathbb{R}^n : |z| = 1\}$ and let $K : S_n \to \mathbb{R}$ be a continuous function with the property $\int_{S_n} K(z)d\omega_z = 0$. Let $1 < q < \infty$, $f \in L^q(\mathbb{R}^n)$ and define for $\epsilon > 0$

$$(T_\epsilon f)(x) := \int_{\{y \in \mathbb{R}^n \,:\, |y-x| \geq \epsilon\}} \frac{K(\frac{x-y}{|x-y|})}{|x-y|^n} f(y)dy$$

Then $Tf := \lim_{\epsilon\to 0} T_\epsilon f$ exists in $L^q(\mathbb{R}^n)$ and there is a constant $C = C(n,q,K) > 0$ such that

(3.8) $\qquad \|Tf\|_q \leq C\|f\|_q.$

For a proof see e.g. [1], p.277, [10], p.39

Theorem 3.4. Let $1 < q < \infty$. Then there exists a constant $C = C(n,q) > 0$ such that for $u \in C_o^\infty(\mathbb{R}^n)$

(3.9) $\qquad \left(\sum_{j,k=1}^{n} \|\partial_j\partial_k u\|_q^q\right)^{1/q} \leq C\|\Delta u\|_q$

Proof. By partial integration we derive from (3.7)

$$\partial_j u(x) = - \int S(x-y)\, \Delta\partial_j u(y)dy \text{ and therefore}$$

$$\partial_k\partial_j u(x) = - \int \partial_{x_k} S(x-y)\, \partial_j\Delta u(y)dy =$$

$$= - \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \int_{\{y:|y-x|>\epsilon\}} \partial_{x_k} S(x-y)\, \partial_j\Delta u(y)dy$$

Let $\epsilon > 0$ be fixed. Integrating by parts leads to

$$I_\epsilon(x) := -\int_{\{y:|y-x|>\epsilon\}} \partial_{x_k} S(x-y)\, \partial_j\Delta u(y)dy = \int_{\{y:|y-x|=\epsilon\}} \partial_{x_k} S(x-y) \frac{y_j-x_j}{|y-x|} \Delta u(y)dw_y$$

$$+ \int_{\{y:|y-x|>\epsilon\}} \partial_{x_k}\partial_{y_j} S(x-y)\, \Delta u(y)dy =: D_\epsilon(x) + T_\epsilon(x)$$

For $n \geq 2$ and $x \neq y$ we have $\partial_{x_k} S(x-y) = \frac{1}{\omega_n} \frac{y_k-x_k}{|y-x|^n}$

and therefore writing $y = x + \epsilon\zeta$, $\zeta \in S_n$,

$$D_\epsilon(x) = \frac{1}{\omega_n} \int_{S_n} \zeta_k \zeta_j (\Delta u)(x+\epsilon\zeta) d\omega_\zeta$$

Since $u \in C_0^\infty(\mathbb{R}^n)$, D_ϵ has compact support too and is bounded. Since $\lim_{\epsilon\to 0} D_\epsilon(x) = \frac{\Delta u(x)}{n} \delta_{jk}$ we conclude by Lebesgue's theorem

$$\text{(3.10)} \qquad \|\Delta u \cdot \frac{\delta_{jk}}{n} - D_\epsilon\|_q \to 0.$$

Further $\partial_{x_k} \partial_{y_j} S(x-y) = \frac{1}{\omega_n} \left[\frac{\delta_{jk}}{|y-x|^n} - n \frac{(y_k-x_k)(y_j-x_j)}{|y-x|^{n+2}} \right]$

Writing $K(z) := \frac{1}{\omega_n} (\delta_{jk} - n z_j z_k)$ for $z \in S_n$ we get

$$T_\epsilon(x) = \int_{\{y\in\mathbb{R}^n : |y-x|>\epsilon\}} \frac{K(\frac{x-y}{|x-y|})}{|x-y|^n} \Delta u(y) dy$$

If $k = j$, then $\omega_n \int_{S_n} K(z) d\omega_z = \int_{S_n} d\omega_z - n \int_{S_n} z_j^2 d\omega_z = 0$

If $k \neq j$, then $\omega_n \int_{S_n} K(z) d\omega_z = -n \int_{S_n} z_j z_k d\omega_z = 0$

By Theorem 3.3 we then derive the existence of the L^q-limit T of T_ϵ and by (3.8) $\|T\|_q \le C\|\Delta u\|_q$. Combining this with (3.10) we are finished. ∎

4. Main Theorems. A consequence of the following theorems is the weak solvability of the Dirichlet and of the Neumann problem in L^q for the Laplacian under the assumptions given there.

Theorem 4.1 ("Neumann problem"). Let $n \ge 2$ and let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with boundary $\partial G \in C^1$ and let $1 < q < \infty$, $q' := \frac{q}{q-1}$. Then:

a) There exists a constant $C = C(G,q) > 0$ such that

$$\text{(4.1)} \qquad \|\nabla p\|_q \le C \sup_{0\neq\nabla\phi\in E^{q'}(G)} \frac{|\langle\nabla p, \nabla\phi\rangle|}{\|\nabla\phi\|_{q'}} \quad \text{for all } \nabla p \in E^q(G)$$

b) For $F^* \in (E^{q'}(G))^*$, $\|F^*\|_{(E^{q'})^*} := \sup_{0\neq\nabla\phi\in E^{q'}(G)} \frac{|F^*(\nabla\phi)|}{\|\nabla\phi\|_{q'}}$

there exists a unique $\nabla p \in E^q(G)$ such that

(4.2) $F^*(\nabla\phi) = \langle\nabla p,\nabla\phi\rangle$ for all $\nabla\phi \in E^{q'}(G)$

and

(4.3) $\|\nabla p\|_q \le C\ \|F^*\|_{(E^{q'})^*} \le C\ \|\nabla p\|_q$

with the same constant $C > 0$ as in (4.1).

<u>Theorem 4.2</u> ("Dirichlet problem"). Let $n \ge 2$ and $G \subset \mathbb{R}^n$ be either bounded or an exterior domain and assume $\partial G \in C^1$.

a) If G is bounded, let $1 < q < \infty$.
If G is an exterior domain and if $n \ge 3$ let $1 < q < n$ and if $n = 2$ let $1 < q \le 2$. Then there exists a constant $C = C(G,q) > 0$ such that

(4.4) $$\|\nabla p\|_q \le C \sup_{0\neq\nabla\phi\in E_o^{q'}(G)} \frac{|\langle\nabla p,\nabla\phi\rangle|}{\|\nabla\phi\|_{q'}}$$

holds for all $\nabla p \in E_o^q(G)$.

b) If G is bounded, let $1 < q < \infty$.
If G is an exterior domain and if $n \ge 3$, let $\frac{n}{n-1} < q < n$ and if $n = 2$ let $q = 2$. Then for $F^* \in (E_o^{q'}(G))^*$, $q' = \frac{q}{q-1}$, $\|F^*\|_{(E^{q'})^*} := \sup_{0\neq\nabla\phi\in E_o^{q'}(G)} \frac{|F^*(\nabla\phi)|}{\|\nabla\phi\|_{q'}}$
there exists a unique $\nabla p \in E_o^q(G)$ such that

(4.5) $F^*(\nabla\phi) = \langle\nabla p,\nabla\phi\rangle$ for all $\nabla\phi \in E_o^{q'}(G)$

and

(4.6) $\|\nabla p\|_q \le C\|F^*\|_{(E_o^{q'})^*} \le C\|\nabla p\|_q$

with the constant $C > 0$ from (4.4).

In case $n = q = 2$ for the exterior domain too a) and b) are trivially satisfied by the Frechet-Riesz theorem. If e.g. $n \ge 3$ and $q \ge n$ in case of an exterior domain there is a one-dimensional exceptional space such that (4.4) don't hold. This case has to be treated separately and demands a more detailed analysis.

5. <u>A priori estimates.</u> Roughly spoken the proof of (4.1) resp. (4.4) is based on local estimates of the same type and at the end is performed by a partition of unity. The local estimates are derived from estimates in the whole space (interior estimates) and in the half-space (estimates up to the boundary). The case of the half-space is reduced in both cases to that of the whole space by means of reflection arguments. It turns out that the uniqueness results of Theorem 5.18 are decisive. These in turn are based on certain "regularity" properties, that is, e.g. under the assumptions of Theorem 4.1 we may conclude if $\nabla p \in E^q(G)$ for a q with $1 < q < \infty$ and $<\nabla p, \nabla\phi> = 0$ for all $\nabla\phi \in E^{q'}(G)$, then $\nabla p \in L^2(G)$, analogously for the Dirichlet problem. For this reason we proof in Lemma 5.2 the estimates as well as the regularity property. The idea how part b) in Theorems 4.1 and 4.2 is derived from part a) by purely functional analytic considerations may be read off from Lemma 5.1.

In the following let $G \subset \mathbb{R}^n$ be a domain and $1 < s < \infty$. For i = 0 we write $E^s_o(G)$ (compare 2.8) and for i = 1 let $E^s_1(G) := E^s(G)$.

We say that G has <u>property $P^i_a(s)$</u> for i = 0 or 1

if there exists a constant $C_s = C(s,G) > 0$ such that

(5.1.s.i) $$\|\nabla p\|_s \leq C_s \sup_{0 \neq \nabla\phi \in E^{s'}_i(G)} \frac{<\nabla p, \nabla\phi>}{\|\nabla\phi\|_{s'}}$$

holds for all $\nabla p \in E^s_i(G)$, w here $s' = \frac{s}{s-1}$.

We say that G has the <u>property $P^i_b(s)$</u> for i = 0 or 1 if

the map $\sigma^i_s : E^s_i(G) \to (E^{s'}_i(G))^*$ defined by $\sigma^i_s : \nabla p \to <\nabla p, .>$ (that is $(\sigma^i_s(\nabla p))(\nabla\phi) = <\nabla p, \nabla\phi>$ for $\nabla p \in E^s_i(G)$ and $\nabla\phi \in E^{s'}_i(G)$) is a bijection and there is a constant $\tilde{C}_s = \tilde{C}(S,G) > 0$ such that for $\nabla p \in E^s_i(G)$

(5.2.s.i) $$\tilde{C}_s \|\nabla p\|_s \leq \|\sigma^i_s(\nabla p)\|_{(E^{s'}_i(G))^*} \leq \|\nabla p\|_s$$

<u>Lemma 5.1</u> Let $G \subset \mathbb{R}^n$ be a domain, $1 < q < \infty$ and $q' := \frac{q}{q-1}$. For i = 0 or 1 holds: G has the property $P^i_a(s)$ for s = q <u>and</u> s = q' if and only if G has the property $P^i_b(s)$ for s = q <u>and</u> s = q'.

Proof. We abbreviate $E_i^s := E_i^s(G)$. Observe $(s')' = s$.

i) Suppose G has the property $P_a^i(s)$ for $s = q$ and $s = q'$.

By (5.2.s.i) we conclude for $\nabla p \in E_i^s$

$$\text{(5.3.s.i)}\quad C_s^{-1}\,\|\nabla p\|_s \le \sup_{0\ne\nabla\phi\in E_i^{s'}} \frac{|<\nabla p,\nabla\phi>|}{\|\nabla\phi\|_{s'}} = \|\sigma_s^i(\nabla p)\|_{(E^{s'})^*} \le \|\nabla p\|_s$$

Therefore $\sigma_s^i(E_i^s)$ is a closed linear subspace of $(E^{s'})^*$. Suppose $\sigma_s^i(E_i^s) \subsetneqq (E_i^{s'})^*$. By the Hahn-Banach theorem there exists $F^{**} \in (E_i^{s'})^{**}$ such that $F^{**} \ne 0$ but $F^{**}|_{\sigma_s^i(E_i^s)} = 0$. Since $E_i^{s'}$ may be regarded as a closed subspace of the reflexive space $L^{s'}(G)^n$, it is reflexive too and we may identify $(E_i^{s'})^{**}$ with $E_i^{s'}$. Then there exists a unique $\nabla\phi \in E_i^{s'}$ such that $F^{**}(F^*) = F^*(\nabla\phi)$ for all $F^* \in (E_i^{s'})^*$ and $\|\nabla\phi\|_{s'} = \|F^{**}\|_{(E_i^{s'})^{**}} > 0$.

But for each $\nabla p \in E_i^s$ we then have $0 = (\sigma_s^i(\nabla p))(\nabla\phi) = <\nabla p,\nabla\phi>$ and therefore by (5.1.s'.i) we conclude $\|\nabla\phi\|_{s'} = 0$ what is a contradiction.

ii) Suppose conversely that $P_b^i(s)$ holds for $s = q$ and $s = q'$. Then because of $\sigma^i(E^{s'}) = (E^s)^*$ and (5.2.s'.i)

$$\|\nabla p\|_s = \sup_{0\ne F^*\in(E_i^s)^*} \frac{F^*(\nabla p)}{\|F^*\|_{(E_i^s)^*}} \le \sup_{0\ne\nabla\phi\in E_i^{s'}} \frac{|\sigma_{s'}^i(\nabla\phi))(\nabla p)|}{\tilde{C}_{s'}\|\nabla\phi\|_{s'}} = \tilde{C}_{s'}^{-1} \sup_{0\ne\nabla\phi\in E_i^{s'}} \frac{<\nabla p,\nabla\phi>}{\|\nabla\phi\|_{s'}}$$

Therefore (5.1.s.i) holds with $C_s = \tilde{C}_{s'}^{-1}$. ■

In the terminology used above e.g. Theorem 4.1 tells that if G is bounded or an exterior domain with $\partial G \in C^1$ then G has property $P_a^2(q)$ and $P_b^2(q)$ for all $1 < q < \infty$. Analogously we may understand Theorem 4.2. The proof of Theorems 4.1 and 4.2 is given via a number of steps. In fact we will prove more than section 4 says. First we show that the whole space and the half-space have property $P_a^i(q)$ for $1 < q < \infty$ and $i = 0$ and $i = 1$ (and by Lemma 5.1 they have property $P_b^i(q)$ too). Then we will prove by a perturbation argument that a sufficiently small "bended" half-space (the "smalness" depends on q) has still property $P_a^i(q)$ for $i = 0, 1$. The following lemma constitutes the basis for all subsequent estimates. Solely in the proof of Lemma 5.2 we need estimate (3.9), a consequence of the Calderon-Zygmund-Theorem. Conversely, Remark 5.3 tells us that (3.9) is equivalent to the assertion of Lemma 5.2. According to (2.3) we have for $1 < s < \infty$ that $E^s(\mathbb{R}^n) = \{\nabla p : p \in L^s_{loc}(\mathbb{R}^n),\ \nabla p \in L^s(\mathbb{R}^n)\}$.

Lemma 5.2. Let $1 < q < \infty$, $1 < r < \infty$ and suppose $\nabla p \in E^r(\mathbb{R}^n)$ and

$$(5.4) \qquad \sup_{0 \neq v \in C_o^\infty(\mathbb{R}^n)} \frac{|\langle \nabla p, \nabla v\rangle|}{\|\nabla v\|_{q'}} < \infty.$$

Then $\nabla p \in E^q(\mathbb{R}^n)$ and there is a constant $C_1 = C_1(n,q) > 0$ such that

$$(5.5) \qquad \|\nabla p\|_q \leq C_1 \sup_{0 \neq v \in C_o^\infty(\mathbb{R}^n)} \frac{|\langle \nabla p, \nabla v\rangle|}{\|\nabla v\|_{q'}}.$$

Proof. For $i = 1,\dots,n$ we conclude with $C_1 := C(n,q')^{-1}$ and $C(n,q')$ via Theorem 3.4 by means of (3.9)

$$(5.6) \qquad \infty > \sup_{0 \neq v \in C_0^\infty(\mathbb{R}^n)} \frac{|\langle \nabla p, \nabla v\rangle|}{\|\nabla v\|_{q'}} \geq \sup_{0 \neq u \in C_o^\infty(\mathbb{R}^n)} \frac{|\langle \nabla p, \nabla \partial_i u\rangle|}{\|\nabla \partial_i u\|_{q'}} =$$

$$= \sup_{0 \neq u \in C_o^\infty(\mathbb{R}^n)} \frac{|\langle \partial_i p, \Delta u\rangle|}{\|\nabla \partial_i u\|_{q'}} \geq C_1 \sup_{0 \neq u \in C_o^\infty(\mathbb{R}^n)} \frac{|\langle \partial_i p, \Delta u\rangle|}{\|\Delta u\|_{q'}}.$$

From this we conclude that the linear functional $F^*(f) := \langle \partial_i p, f\rangle$ for $f \in M := \Delta u : u \in C_o^\infty(\mathbb{R}^n) \subset L^{q'}(\mathbb{R}^n)$ is well defined and continuous. By Theorem 3.2 M is dense in $L^{q'}(\mathbb{R}^n)$ with respect to $L^{q'}$-norm. Therefore this functional may be uniquely and norm-preserving extended to a continuous linear functional on the whole space $L^{q'}(\mathbb{R}^n)$. Therefore there is a unique $g \in L^q(\mathbb{R}^n)$ such that $\langle \partial_i p, \Delta u\rangle = \langle g, \Delta u\rangle$ for all $u \in C_o^\infty(\mathbb{R}^n)$. From Weyl's lemma (Theorem 3.1) follows that $W := \partial_i p - g$ is harmonic on $\mathbb{R}^n$. For fixed $x \in \mathbb{R}^n$ and $R > 0$ by (3.2)

$$W(x) = |B_R(x)|^{-1}\Big(\int_{B_R(x)} \partial_i p(y)dy - \int_{B_R(x)} g(y)dy\Big)$$

and by Hölder's inequality

$$|W(x)| \leq |B_R(x)|^{-\frac{1}{r}} \|\partial_i p\|_r + |B_R(x)|^{-\frac{1}{q}} \|g\|_q \to 0 \ (R \to \infty).$$

Therefore $\partial_i p = g \in L^q(\mathbb{R}^n)$ and again by Theorem 3.4

$$(5.7)\qquad \sup_{0\neq u\in C_o^\infty(\mathbb{R}^n)} \frac{|<\partial_i p,\Delta u>|}{\|\Delta u\|_{q'}} = \sup_{0\neq f\in L^{q'}(\mathbb{R}^n)} \frac{|<\partial_i p,f>|}{\|f\|_{q'}} = \|\partial_i p\|_{q'}.$$

Combining (5.6) and (5.7) yields (5.5).
It remains to show $p \in L^q_{loc}(\mathbb{R}^n)$. Since $p \in L^r_{loc}(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ and given any ball $B_R \subset \mathbb{R}^n$ we see for the mollified functions
$c_\epsilon := |B_R|^{-1} \int_{B_R} p_\epsilon(y)dy \to c := |B_R|^{-1} \int_{B_R} p(y)dy$. Put $\tilde{p}_\epsilon := p_\epsilon - c_\epsilon$. By (2.2) with a constant $\gamma = \gamma(R,q) > 0$

$\|\tilde{p}_{\epsilon'} - \tilde{p}_{\epsilon''}\|_{q,B_R} \le \gamma\|\nabla(\tilde{p}_{\epsilon'} - \tilde{p}_{\epsilon''})\|_{q,B_R} \le \gamma\|\nabla p_{\epsilon'} - \nabla p_{\epsilon''}\|_q \to 0.$

Since (c_ϵ) converges in $\mathbb{R}$ we conclude that $(p_{\epsilon'})$ forms a Cauchy-sequence in $L^q(B_R)$ and has the limit $p_1 \in L^q(B_R) \subset L^1(B_R)$ and therefore $\|p_1 - p\|_{L^1(B_R)} = 0$. So $p = p_1 \in L^q(B_R)$. ∎

<u>Remark 5.3:</u> Suppose $1 < q < \infty$ and (5.5) holds for all $p \in C_o^\infty(\mathbb{R}^n)$. Then (3.9) holds for all $p \in C_o^\infty(\mathbb{R}^n)$ too: Let $1 \le i \le n$. Then by (5.5)

$$\|\nabla\partial_i p\|_q \le C \sup_{0\neq v\in C_o^\infty(\mathbb{R}^n)} \frac{|<\nabla\partial_i p,\nabla v>|}{\|\nabla v\|_{q'}} = C \sup_{0\neq v\in C_o^\infty(\mathbb{R}^n)} \frac{|<\Delta p,\partial_i v>|}{\|\nabla v\|_{q'}} \le C\|\Delta p\|_q$$

immediately leading to (3.9). ∎

An immediate consequence is the following density property.

<u>Corollary 5.4.</u> Let $1 < q < \infty$. Then $E^\infty(\mathbb{R}^n) := \{\nabla v : v \in C_o^\infty(\mathbb{R}^n)\}$ is dense in $E^q(\mathbb{R}^n)$ with respect to $\|\nabla.\|_q$-norm.

<u>Proof.</u> Suppose $E^\infty(\mathbb{R}^n)$ is not dense in $E^q(\mathbb{R}^n)$. Then there exists $F^* \in (E^q(\mathbb{R}^n))^*$ with $F^*|_{E^\infty} = 0$, $\|F^*\|_{(E^q)^*} > 0$. By Theorem 5.2 we conclude that $\mathbb{R}^n$ has property $P^1_a(s)$ for $s = q$ and q'. Therefore by Lemma 5.1 there exists a unique $\nabla u \in E^{q'}(\mathbb{R}^n)$ with $\|\nabla u\|_{q'} > 0$ such that $F^*(\nabla p) = <\nabla u,\nabla p>$ for all $\nabla p \in E^q(\mathbb{R}^n)$. But since $F^*(\nabla v) = 0$ for $\nabla v \in E^\infty(\mathbb{R}^n)$ by Lemma 5.2 we would conclude $\nabla u = 0$ contradicting $\|\nabla u\|_{q'} > 0$. ∎

Next we consider the half-space

(5.8) $H := \{x = (x',x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n < 0\}$.

For $1 < q < \infty$ and for $i = 0$ let $E^q_o(H)$ be defined by (2.8) and let $E^q_1(H) := E^q(H)$.
Further we put

$$(5.10)\quad \begin{cases} E_o^\infty(H) := \{\nabla p : p \in C_0^\infty(H)\} \\ E_1^\infty(H) := \{\nabla p : p \in C_o^\infty(\bar H)\} \equiv \{\nabla p|_{\bar H} : p \in C_o^\infty(\mathbb{R}^n)\}. \end{cases}$$

Given $\nabla p \in E_i^q(H)$ ($i = 0$ or 1) we put

$$(5.11)\quad p^i(x) := \begin{cases} p(x) & \text{for } x \in H \\ (-1)^{1+i} p(x',-x_n) & \text{for } x_n \geq 0 \end{cases} , \; i = 0 \text{ or } 1$$

and for $\phi \in C_o^\infty(\mathbb{R}^n)$ we put for $x \in H$ and $i = 0$ or 1

$$(5.12)\quad (T_i\phi)(x) := \phi(x) + (-1)^{1+i}\,\phi(x',-x_n).$$

Lemma 5.5. Let $1 < q < \infty$.

i) If $\nabla p \in E_i^q(H)$ then $\nabla p^i \in E^q(\mathbb{R}^n)$ and

$$(5.13)\quad \begin{cases} \partial_j p^i(x) = \begin{cases} \partial_j p(x) & \text{for } x \in H \\ (-1)^{1+i}\partial_j p(x',-x_n) & \text{for } x_n \geq 0 \end{cases} & \text{for } i = 0 \text{ or } 1 \text{ and } j = 1,..,n-1 \\ \partial_n p^i(x) = \begin{cases} (\partial_n p)(x) & \text{for } x \in H \\ (-1)^i(\partial_n p)(x',-x_n) & \text{for } x_n \geq 0 \end{cases} & i = 0 \text{ or } 1 \end{cases}$$

ii) For $\phi \in C_o^\infty(\mathbb{R}^n)$ we have

a) $\nabla(T_o\phi) \in E_1^\infty(H)$, $(T_o\phi)(x',0) = 0$ and therefore $\nabla(T_o\phi) \in E_o^q(H)$

b) $\nabla(T_1\phi) \in E_1^\infty(H)$

iii) Let $\nabla p \in E_i^q(H)$ ($i = 0$ or 1). Then for $\phi \in C_o^\infty(\mathbb{R}^n)$

$$(5.14)\quad \langle \nabla p^{(i)}, \nabla\phi\rangle_{\mathbb{R}^n} = \langle \nabla p, \nabla(T_i\phi)\rangle_H \quad (i = 0 \text{ or } 1)$$

$$(5.15)\quad \|\nabla p\|_{q,H} \leq \|\nabla p^{(i)}\|_{q,\mathbb{R}^n} \leq 2^{1/q}\|\nabla p\|_{q,H} \quad \text{for } \nabla p \in E_i^q(H)\ (i = 1 \text{ or } 2)$$

$$(5.16)\quad \|\nabla(T_i\phi)\|_{q,H} \leq 2\|\nabla\phi\|_{q,\mathbb{R}^n},\ i = 0,1,\ \phi \in C_o^\infty(\mathbb{R}^n)$$

Proof i) (5.13) and the integrability properties follow by elementary calculations. For $\nabla p \in E_o^q(H)$ observe definition (2.8).
ii) Let $\psi := T_o\phi$ then clearly $\psi(x',0) = 0$. Let $\rho \in C^\infty(\mathbb{R})$, $\rho(t) = 0$ for $|t| \leq 1$, $\rho(t) = 1$ for $|t| \geq 2$, $0 \leq \rho \leq 1$. For $k \in \mathbb{N}$ put $\rho_k(t) := \rho(kt)$. Define $\psi_k(x) := \rho_k(x_n)\psi(x)$. Then $\psi_k \in C_0^\infty(H)$.
If for some $R > 0$ supp $\psi_k \subset Z_R := \{x \in \mathbb{R}^n : |x'| \leq R, -R \leq x_n \leq 0\}$, because of $|\psi(x',x_n) - \psi(x',0)| \leq C(\nabla\psi)|x_n|$ we see $|\psi(x)\,\partial_n\rho_k(x_n)| \leq \text{const}(\nabla\psi)$ and since

$\psi\ \partial_n\rho_k$ is vanishing outside $Z_R \cap \{x = (x',x_n) \in \mathbb{R}^n : -\frac{2}{k} < x_n < -\frac{1}{k}\}$ we get $\|\psi\cdot\partial_n\rho_k\|_q \to 0$ $(k \to \infty)$. Therefore we immediately see $\|\nabla\psi-\nabla\psi_k\|_q \to 0$, therefore $\nabla\psi \in E^q_o(H)$. The remaining statements follow immediately by elementary calculations. ■

We need a further lemma seeming not to be obvious.

<u>Lemma 5.6.</u> Let $1 < q,r < \infty$, let $\nabla p \in E^r_o(H)$ and suppose $\nabla p \in L^q(H)$. Then $\nabla p \in E^q_o(H)$.

<u>Proof.</u> i) Let $p \in C^1(\bar{H})$, $p(x',0) = 0$ for $x' \in \mathbb{R}^{n-1}$ and let $\nabla p \in L^q(H)$. For $R > 0$ define $Z_R := \{x = (x',x_n) \in \mathbb{R}^n : -R < x_n < 0\}$. Then for $x \in Z_R$ we get $p(x',x_n) = -\int_{x_n}^{0} (\partial_n p)(x',t)dt$. Applying Hölder's inequality and integrating with respect to $x \in Z_R$ we get

$$(5.17)\quad \|p\|_{q,Z_R} \le R\|\partial_n p\|_{q,Z_R}.$$

Since (5.17) holds especially for $p \in C^\infty_o(H)$ we derive from the definition of $E^q_o(H)$ that (5.17) is true for $\nabla p \in E^q_o(H)$ too.

ii) Let now $p \in C^1_o(\bar{H})$, $p(x',0) = 0$. Consider ρ_k like as in part ii) of the proof of Lemma 5.5 and put $p_k(x) = \rho_k(x_n)p(x)$. Then $p_k \in C^1_o(H)$, $\partial_i p_k = \rho_k\partial_i p$ for $i = 1,\dots,n-1$ and $\partial_n p_k = \rho_k\partial_n p + p\cdot\partial_n\rho_k$. Clearly $\|\rho_k\partial_i p-\partial_i p\|_{q,H} \to 0$. Since $|\partial_n\rho_k| \le c\cdot k$ and vanishes outside $Z_{2k^{-1}}$ we get from (5.17)

$$\|\partial_n\rho_k\cdot p\|_{q,H} = \|\partial_n\rho_k\cdot p\|_{q,Z_{2k^{-1}}} \le c\cdot k\cdot 2k^{-1}\|\partial_n p\|_{q,Z_{2k^{-1}}} \to 0 \text{ and therefore}$$
$$\|\partial_n p_k-\partial_n p\|_{q,H} \to 0.$$

For $0 < \epsilon < k^{-1}$ we have $p_{k\epsilon} \in C^\infty_o(H)$, $\|\nabla p_k-\nabla p_{k,\epsilon}\|_q \to 0$.
Therefore $\nabla p \in E^q_o(H)$.

iii) Let now $p \in C^1(\bar{H})$, $p(x',0) = 0$, $\nabla p \in L^q(H)$.
Let $\eta \in C^\infty_o(\mathbb{R}^n)$, $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| \le 1$, $\eta(x) = 0$ for $|x| \ge 2$ and put $\eta_k(x) := \eta(k^{-1}x)$ for $k \in \mathbb{N}$. By ii) $\nabla(\eta_k\cdot p) \in E^q_o(H)$. Clearly $\|\eta_k\nabla p-\nabla p\|_{q,H} \to 0$. Again by (5.17) because of $|\nabla\eta_k(x)| \le Ck^{-1}$

$$(5.18)\quad \|p\cdot\nabla\eta_k\|_{q,H} = \|p\nabla\eta_k\|_{q,Z_{2k}} \le Ck^{-1}\cdot 2k\|\nabla p\|_{q,Z_{2k}} \le C\|\nabla p\|_{q,H}.$$

Therefore $\|\nabla(\eta_k p)\|_q \le C$ for all k. Clearly $\nabla p \in E^q(H) \supset E^q_o(H)$. We show now $\nabla(\eta_k p) \to \nabla p$ weakly in $E^q(H)$. Since $E^q_o(H)$ is weakly closed too this implies then $\nabla p \in E^q_o(H)$. Let $F^* \in E^q(H)^*$. We may consider $E^q(H)$ as a closed subspace of $L^q(H)^n$ and we may F^* extend normpreserving to an $\tilde{F}^* \in (L^q(H)^n)^*$. Then there is $f = (f_1,\dots,f_n) \in L^{q'}(H)$ such that $\tilde{F}^*(g) = \langle f,g\rangle_H = \sum_i \int_H f_i g_i$ for $g \in L^q(H)^n$.

Then $F^*(\nabla p-\nabla(\eta_k p)) = \int\limits_H \sum\limits_i f_i(1-\eta_k)\partial_i p - \int\limits_H \sum\limits_i f_i\partial_i\eta_k p$

Clearly the first integral tends to zero. By the properties of η_k the second integral reduces to an integral taken over $R_k \cap H = \{x \in H : k < |x| < 2k\}$ and therefore by (5.18) $\sum\limits_i|\int\limits_H f_i\partial_i\eta_k \cdot p| \le \sum\limits_i \|f_i\|_{q',R_k\cap H}\cdot C\|\nabla p\|_{q,H} \to 0$.

iv) Let now $\nabla p \in E^r_o(H)$ with $\nabla p \in L^q(H)$. By Lemma 5.5 the extended function satisfies $\nabla p^o \in E^r_o(\mathbb{R}^n)$. Consider the mollified $(p^o)_\epsilon$ with radial depending mollifier kernel j_ϵ. Since $\nabla(p^o)_\epsilon = (\nabla p^o)_\epsilon$ we see moreover $\nabla(p^o)_\epsilon \in L^q(\mathbb{R}^n)$ and $\|\nabla p-\nabla(p^o)_\epsilon\|_{p,H} = \|\nabla p^o|_H-\nabla(p^o)_\epsilon|_H\|_{q,H} \le \|\nabla p^o-\nabla(p^o)_\epsilon\|_{q,\mathbb{R}^n} \to 0$.

Observe by the properties of the mollifier and (5.11) for $i = 0$ that

$\nabla(p^o)_\epsilon(x',0) = 0$ for $x'\in \mathbb{R}^n$. By iii) we conclude $\nabla(p^o)_\epsilon|_H \in E^q_o(H)$ and therefore $\nabla p \in E^q_o(H)$ too. ∎

Remark 5.7 The linear space

$$E^q_v(H) := \{\nabla p : p \in C^1(\bar{H}),\ p(x',0) = 0 \text{ for } x'\in \mathbb{R}^{n-1} \text{ and } \nabla p \in L^q(H)\}$$

satisfies $E^\infty_o(H) \subset E^q_v(H) \subset E^q_o(H)$ and is therefore dense in $E^q_o(H)$.

Lemma 5.8 Let $1 < q < \infty$, $1 < r < \infty$.

i) Let $\nabla p \in E^r_o(H)$ and

$$d_o := \sup_{0\neq v\in C^\infty_o(H)} \frac{|<\nabla p,\nabla v>|}{\|\nabla v\|_{q',H}} < \infty$$

Then $\nabla p \in E^q_o(H)$ and

$$(5.19)\quad \|\nabla p\|_{q,H} \le C_2 \sup_{0\neq v\in C^\infty_o(H)} \frac{|<\nabla p,\nabla v>|}{\|\nabla v\|_{q',H}}$$

Here $C_2 = 2C_1$ with C_1 by Lemma 5.2.

ii) Let $\nabla p \in E^r(H)$ and

$$d_1 := \sup_{0\neq v\in C^\infty_o(\bar{H})} \frac{|<\nabla p,\nabla v>|}{\|\nabla v\|_{q',H}} < \infty$$

Then $\nabla p \in E^q(H)$ and

$$(5.20)\quad \|\nabla p\|_{q,H} \le C_2 \sup_{0\neq v\in C^\infty_o(\bar{H})} \frac{|<\nabla p,\nabla v>|}{\|\nabla v\|_{q',H}} .$$

Proof. In both cases we have with $p^{(i)}$ ($i = 0$ or 1) by (5.11)and by Lemma 5.5 that $\nabla p^{(i)} \in E^q(\mathbb{R}^n)$. For $\phi \in C^\infty_o(\mathbb{R}^n)$ we have by (5.14) for $i = 0$ or 1

$$<\nabla p^{(i)},\nabla\phi>_{\mathbb{R}^n} = <\nabla p,\nabla(T_i\phi)>_H$$

For $i = 0$ by Lemma 5.5 part ii) we get $(T_o\phi)(x',0) = 0$. Then with $v := T_o\phi$ we have $v \in C^1_o(\bar{H})$, $v(x',0) = 0$. Like as in part ii) of the proof of Lemma 5.6 consider $v_k(x) := \rho_k(x_n)\cdot v(x)$. Then $v_k \in C^1_o(H)$ and clearly $v_{k\epsilon} \in C^\infty_o(H)$ for $1 < \epsilon < k^{-1}$. Since $\|\nabla v_k - \nabla v_{k\epsilon}\|_{r'} \to 0$ and $\|\nabla v_k - \nabla v_{k\epsilon}\|_{q'} \to 0$ we see $\langle\nabla p,\nabla v_{k\epsilon}\rangle\cdot\|\nabla v_{k\epsilon}\|^{-1}_{q'} \to \langle\nabla p,\nabla v_k\rangle\|\nabla v_k\|^{-1}_{q'}$ since $\nabla p \in L^r(G)$. Analogously

(compare part ii) of proof of Lemma 5.6) we see $\|\nabla v - \nabla v_k\|_{r'} \to 0$, $\|\nabla v - \nabla v_k\|_{q'} \to 0$ and therefore $\langle\nabla p,\nabla v_k\rangle\|\nabla v_k\|^{-1}_{q'} \to \langle\nabla p,\nabla v\rangle\|\nabla v\|^{-1}_{q'}$. From assumption i) we therefore conclude

$$|\langle\nabla p^{(o)},\nabla\phi\rangle_{\mathbb{R}^n}| \le |\langle\nabla p,\nabla(T_o\phi)\rangle| \le d_o\|\nabla T_o\phi\|_{q',H} \le 2d_o\|\nabla\phi\|_{q,\mathbb{R}^n}.$$

For $i = 1$ analogously we get

$$|\langle\nabla p^{(1)},\nabla\phi\rangle_{\mathbb{R}^n}| \le 2d_1\|\nabla\phi\|_{q',\mathbb{R}^n}$$

By Lemma 5.2 we conclude $\nabla p^i \in L^q(\mathbb{R}^n)$.

Since $\nabla p^{(i)}|_H = \nabla p$ we see for $i = 0$ by Lemma 5.6 that $\nabla p \in E^q_o(H)$. For $i = 1$, from $\nabla p \in E^r(H)$ and $\nabla p \in L^q(H)$ we conclude like at the end of the proof of Lemma 5.2 that $\nabla p \in E^q(H)$. Estimates (5.19) and (5.20) then are trivial. Observe that in (5.19) the sup may be taken for $0 \ne \nabla v \in E^{q'}_o(H)$ and in (5.20) for $0 \ne \nabla v \in E^{q'}(H)$. ■

In the next step we consider a "bended" half-space. Let $\omega \in C^1(\mathbb{R}^{n-1})$ and $x' \in \mathbb{R}^{n-1}$. We suppose that there is some $R = R(\omega) > 0$ such that $\omega(x') = 0$ for $|x'| \ge R$. Then we define

(5.21) $$H_\omega := \{x = (x',x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\ x_n < \omega(x')\}$$

We want now to extend the results of Lemma 5.8 to H_ω. This will be done by a perturbation argument. For technical reasons we need a density result similar to Corollary 5.4.

Lemma 5.9. Let $1 < q < \infty$ and let Ω denote either H_ω or a bounded domain or an exterior domain G with boundary $\partial G \in C^1$. Then $E^\infty(\Omega) := \{\nabla v : v \in C^\infty(\bar{\Omega})\}$ is dense in $E^q(\Omega)$.

Sketch of proof. By well known techniques (see [5], [6]) given $\nabla v \in E^q(\Omega)$ there exists $\nabla\tilde{v} \in E^q(\mathbb{R}^n)$ such that $\nabla\tilde{v}|_\Omega = \nabla v$ and $\|\nabla\tilde{v}\|_q \le \|\nabla v\|_q$. Apply now Corollary 5.4. ■

Lemma 5.10. Let $1 < q < \infty$, $1 < r < \infty$. Then there exists a constant $K = K(q,r,n) > 0$ with the following property. If $\|\nabla\omega\|_\infty := \sup_{x'\in\mathbb{R}^{n-1}} |\nabla\omega(x')| \le K$ then

i) a) there are constants $C(s) = C(s,K,n)$ such that

$$\|\nabla p\|_{s,H_\omega} \le C(s) \sup_{0\neq v\in C_o^\infty(H_\omega)} \frac{|<\nabla p,\nabla v>|}{\|\nabla v\|_{s',H_\omega}} \tag{5.22}$$

holds for $\nabla p \in E_o^s(H_\omega)$ and $s = q,q',r,r'$ (here $s' = \frac{s}{s-1}$).

b) If $\nabla p \in E_o^r(H_\omega)$ and $D := \sup_{0\neq v\in C_o^\infty(H_\omega)} \frac{|<\nabla p,\nabla v>|}{\|\nabla v\|_{q',H_\omega}} < \infty$

then $\nabla p \in E_o^q(H_\omega)$ and (5.22) holds for $s = q$.

ii) The assertions of i) hold true if $E_o^s(H_\omega)$ is replaced by $E^s(H_\omega)$ and $C_o^\infty(H_\omega)$ is replaced by $C_o^\infty(\overline{H_\omega})$ $(s = r,q)$.

Proof: i) We define $y : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{cases} y_i(x) := x_i & \text{for } i = 1,\dots,n-1 \\ y_n(x) := x_n - \omega(x') \end{cases} \tag{5.23}$$

then y maps $\mathbb{R}^n$ one-to-one on $\mathbb{R}^n$, $y \in C^1(\mathbb{R}^n)$.
Further $y\big|_{H_\omega} : H_\omega \to H$ is onto, $y(x',\omega(x')) = (x',0)$ that is $y(\partial H_\omega) = \partial H$.
Further $J[y(x)] = 1$. The inverse map is given by $x_i(y) := y_i$ $(i = 1,..,n-1)$ and $x_n(y) := y_n + \omega(y')$. For $p \in C^1(H_\omega)$ we put $\tilde{p}(y) := p(x(y))$ for $y \in H$.

Then $\tilde{p} \in C^1(H)$, $p(x) = \tilde{p}(y(x))$ and

$$\begin{cases} \partial_i p(x) = (\partial_i\tilde{p})(y(x)) - (\partial_n\tilde{p})(y(x))\partial_i\omega(x') & \text{for } i = 1,\dots,n-1 \\ \partial_n p(x) = (\partial_n\tilde{p})(y(x)) \end{cases} \tag{5.24}$$

and conversely

$$(\partial_i\tilde{p})(y) = (\partial_i p)(x(y)) + (\partial_n p)(x(y))\partial_i\omega(y') \quad \text{for } i = 1,\dots,n-1$$

$$(\partial_n\tilde{p})(y) = (\partial_n p)(x(y))$$

With the aid of Lemma 5.9 we immediately conclude for s with $1 < s < \infty$: $\nabla p \in E^s(H_\omega)$ if and only if $\nabla\tilde{p} \in E^s(H)$ and $\nabla p \in E_o^s(H_\omega)$ if and only if $\nabla\tilde{p} \in E_o^s(H)$. From (5.24) we derive with a constant $d_1(s) = d_1(s,n) > 0$ for $\nabla p \in E^s(H_\omega)$

(5.25) $\|\nabla p\|_{s,H_\omega} \le d_1(s)(1 + \|\nabla\omega\|_\infty)\ \|\nabla\tilde{p}\|_{s,H}$

Let $\phi \in C_0^\infty(\bar{H}_\omega)$ and define $\tilde{\phi}(y) := \phi(x(y))$ for $y \in H$. Then $\tilde{\phi} \in C_0^1(\bar{H}_\omega)$. If $\nabla p \in E^s(H_\omega)$ then define

$$B_\omega[\nabla\tilde{p},\nabla\tilde{\phi}] := -\sum_{i=1}^{n-1}\int_H(\partial_n\tilde{p}(y)\partial_i\tilde{\phi}(y) + \partial_i\tilde{p}(y)\partial_n\tilde{\phi}(y))\partial_i\omega(y)dy$$

$$+ \int_H \sum_{i=1}^{n-1}(\partial_i\omega)^2(y)\partial_n\tilde{p}(y)\partial_n\tilde{\phi}(y)dy$$

and therefore with a constant $d_2(s) = d_2(s,n)$

(5.26) $|B_\omega[\nabla\tilde{p},\nabla\tilde{\phi}]| \le d_2(s)\|\nabla\omega\|_\infty(1+\|\nabla\omega\|_\infty)\|\nabla\tilde{p}\|_{s,H}\|\nabla\tilde{\phi}\|_{s',H}$

From (5.24) via the change of variables formula we immediately derive

(5.27) $\langle\nabla p,\nabla\phi\rangle_{H_\omega} = \langle\nabla\tilde{p},\nabla\tilde{\phi}\rangle_H + B_\omega[\nabla\tilde{p},\nabla\tilde{\phi}]$

and therefore by (5.25) for s' and (5.26) for $\nabla\phi \ne 0$

$$(5.28)\qquad \frac{|\langle\nabla p,\nabla\phi\rangle_{H_\omega}|}{\|\nabla\phi\|_{s',H_\omega}} \ge [d_1(s')(1+\|\nabla\omega\|_\infty)]^{-1}\{\frac{|\langle\nabla\tilde{p},\nabla\tilde{\phi}\rangle|}{\|\nabla\tilde{\phi}\|_{s',H}} - d_2(x)\|\nabla\omega\|_\infty(1+\|\nabla\omega\|_\infty)\|\nabla\tilde{p}\|_{s,H}\}.$$

ii) Choose now $K \le 1$ such that $0 < K \le \min\{(4C_2(s)d_2(s))^{-1} : s = q,q',r,r'\}$ with $C_2(s) > 0$ by Lemma 5.8.
If $\nabla p \in E_0^s(H_\omega)$ we then get from (5.19) and (5.28) if $\|\nabla\omega\|_\infty \le K \le 1$

$$\sup_{0\ne\phi\in C_0^\infty(H_\omega)} \frac{|\langle\nabla p,\nabla\phi\rangle|}{\|\phi\|_{q',H_\omega}} \ge (2d_1(s'))^{-1}\{\sup_{0\ne\tilde{\phi}\in C_0^\infty(H)} \frac{|\langle\nabla\tilde{p},\nabla\tilde{\phi}\rangle|}{\|\nabla\tilde{\phi}\|_{s',H}} - 2d_2(s)K\|\nabla\tilde{p}\|_{s,H}\} \ge$$

$$\ge (2d_1(s')^{-1}\{C_2(s)^{-1}\|\nabla\tilde{p}\|_{s,H} - 2d_2(s)\cdot K\|\nabla\tilde{p}\|_{s,H}\}$$

$$\ge (4d_1(s')C_2(s))^{-1}\|\nabla\tilde{p}\|_{s,H} \ge C(s)^{-1}\|\nabla p\|_{s,H}$$

with $C(s) := (8d_1(s')d_1(s)C_2(s))$.

iii) If $\nabla p \in E^q(H)$ and K is chosen like as in ii) then the analogous calculation using now (5.20) leads to (5.22) in that case too.

iv) In order to prove b) let $\nabla p \in E^r(H_\omega)$. We consider first the case $r \ge q$.

We use a cut-off procedure in order to reduce this case to the half-space. Let $R = R(\omega) > 0$ denotes the constant with $\omega(x') = 0$ for $|x'| \geq R$ and choose $R_1 \geq R$ such that $\max_{|x'|\leq R}|\omega(x')| \leq R_1$. Choose $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 0$ for $|x| \leq R_1$, $\varphi(x) = 1$ for $|x| \geq 2R_1$ and $0 \leq \varphi \leq 1$.

Let $L := \{x \in \mathbb{R}^n : x \in H,\ R_1 < |x| < 2R_1\} \subset H_\omega$. Given $h \in C_0^\infty(\bar{H})$ we choose $c(h) \in \mathbb{R}$ such that with $\tilde{h} := h+c(h)$ we have $\int_L \tilde{h}dx = 0$. By the properties of φ we get

$$(5.29)\qquad \langle\nabla(\varphi p),\nabla h\rangle = \langle\nabla(\varphi p),\nabla\tilde{h}\rangle = \\ = \langle\nabla p,\nabla(\varphi\tilde{h})\rangle - \langle\nabla p,\nabla\varphi\tilde{h}\rangle + \langle p\nabla\varphi,\nabla\tilde{h}\rangle$$

By the Poincaré-inequality (2.2) we get with a constant $c_1 > 0$

$$(5.30)\qquad \|\tilde{h}\|_{q',L} \leq c_1\|\nabla\tilde{h}\|_{q',L} \leq c_1\|\nabla h\|_{q'}$$

Since $r \geq q$ we have $\|\nabla p\|_{q,L} \leq c_3\|\nabla p\|_{r,L}$ and therefore

$$(5.31)\qquad |\langle\nabla p,\nabla\varphi\tilde{h}\rangle| \leq c_4\|\nabla p\|_{r,L}\cdot\|\nabla h\|_{q'}$$

Since $p \in L^r(L)$ and $r \geq q$ we have $p \in L^q(L)$,

$$(5.32)\qquad \begin{aligned} &|\langle p\nabla\varphi,\nabla\tilde{h}\rangle| \leq c_5\|p\|_{r,L}\cdot\|\nabla h\|_{q'} \\ &\|\nabla(\varphi\tilde{h})\|_{q'} \leq \|\nabla\varphi\tilde{h}\|_{q',L} + \|\varphi\nabla\tilde{h}\|_{q'} \leq c_6\|\nabla h\|_{q'} \end{aligned}$$

By definition $h \in C_0^\infty(\bar{H})$ if h is the restriction of a $C_0^\infty(\mathbb{R}^n)$-function to $\bar{H}$. Therefore $h \in C_0^\infty(\bar{H}_\omega)$. Clearly $(1-\varphi) \in C_0^\infty(\bar{H}_\omega)$. Then $\varphi_0 := \varphi h+c(h)(\varphi-1) \in C_0^\infty(\bar{H}_\omega)$ and since $\varphi\tilde{h} = \varphi_0+c(h)$, $\nabla(\varphi\tilde{h}) = \nabla\varphi_0$ and so

$$(5.33)\qquad |\langle\nabla p,\nabla(\varphi\tilde{h})\rangle| \leq \sup_{0\neq v\in C_0^\infty(\bar{H}_\omega)} \frac{|\langle\nabla p,\nabla v\rangle|}{\|\nabla v\|_{q'}} \|\nabla(\varphi\tilde{h})\|_{q'} \leq D\cdot c_6\|\nabla h\|_{q'}$$

Therefore we derive from (5.29) - (5.32)

$$\sup_{0\neq h\in C_0^\infty(\bar{H})} \frac{|\langle\nabla(\varphi p),\nabla h\rangle|}{\|\nabla h\|_{q'}} \leq c_3\|\nabla p\|_{r,L} + c_4\|p\|_{r,L} + c_5 D < \infty$$

Since $\nabla(\varphi p) \in E^r(H)$ by Lemma 5.8, ii) we conclude $\nabla(\varphi p) \in E^q(H)$ and therefore $\nabla(\varphi p) \in E^q(H_\omega)$. Because of $r \geq q$ clearly $\nabla[(1-\varphi)p] \in E^q(H_\omega)$.

v) In order to prove b) in the case $\nabla p \in E_0^r(H_\omega)$ for $r \geq q$ we proceed similar as in iv). We take any $h \in C_0^\infty(H)$ and put $\tilde{h} \equiv h$ in (5.29). By (5.17) we get

$$(5.34)\qquad \|h\|_{q',L} \leq \|h\|_{q',Z_{2R_1}} \leq 2R_1\|\nabla h\|_{q'}$$

If we replace (5.30) by (5.34) we get again (5.31), (5.32). Clearly

$\varphi h \in C_o^\infty(H_\omega)$ and therefore again $|<\nabla p,\nabla(\varphi h)>| \le Dc_5\|\nabla h\|_{q'}$,
and

$$\sup_{0\neq h\in C_o^\infty(H)} \frac{|<\nabla(\varphi p),\nabla h>|}{\|\nabla h\|_{q'}} \le c_3\|\nabla p\|_{r,L} + c_4\|p\|_{r,L} + c_5 D < \infty$$

By Lemma 5.8 i) we see $\nabla(\varphi p) \in E_o^q(H)$.

If $(p_k) \subset C_o^\infty(H)$ is a sequence such that $\|\nabla p_k - \nabla(\varphi p)\|_{q,H} \to 0$. Then clearly (observe (5.17)) $\|\nabla(\varphi p_k)-\nabla(\varphi p)\|_{q,H} \to 0$. Since $\varphi p_k \in C_o^\infty(H_\omega)$ we see $\nabla(\varphi p) \in E_o^q(H_\omega)$. Clearly $\nabla((1-\varphi)p) \in E_o^q(H_\omega) \subset E_o^r(H_\omega)$.

vi) By part ii) and iii) of proof H_ω has property $P_a^i(s)$ for i = 0 and 1 and for $s = r,r',q,q'$. Then by Lemma 5.1 H_ω has property $P_b^i(s)$ too. Again we write $E_i^s(H_\omega)$(i = 0 or 1) and $E_o^\infty(H_\omega) :=\{\nabla\phi : \phi \in C_o^\infty(H_\omega)\}$, $E_1^\infty(H_\omega) :=\{\nabla\phi : \phi \in C_o^\infty(\overline{H_\omega})\}$. We consider now the case $r < q$ in b). This case can be reduced to the previous one. Let now $\nabla p \in E_i^q(H_\omega)$ and $r < q$. According to the assumption in b) by $F^*(\nabla v) := <\nabla p,\nabla v>$ for $\nabla v \in E_i^\infty(H_\omega)$ a linear functional continuous with respect to $\|\nabla .\|_{q'}$-norm is defined on the dense subspace $E_i^\infty(H_\omega)$ of $E_i^{q'}(H_\omega)$. By property $P_b^i(q)$ the unique extension $\hat{F}^*$ of F^* to the whole space $E_i^{q'}(H_\omega)$ may be represented with a uniquely determined $\nabla\hat{p} \in E_i^q(H_\omega)$ in the form $\hat{F}^*(\nabla v) =<\nabla\hat{p},\nabla v>$ for $\nabla v \in E_i^{q'}(H_\omega)$. For $\nabla v \in E_i^\infty(H_\omega)$ we have $<\nabla p,\nabla v> = F^*(\nabla v) = \hat{F}^*(\nabla v) = <\nabla\hat{p},\nabla v>$. Since $\nabla p \in E^r(H_\omega)$ we see

$$\sup_{0\neq\nabla v\in E_i^\infty(H_\omega)} \frac{|<\nabla\hat{p},\nabla v>|}{\|\nabla v\|_{r'}} \le \|\nabla p\|_r$$

Since now $r < q$ we may apply parts iv) and v) of proof (with interchanged meaning of r and q) and conclude $\nabla\hat{p} \in E_i^r(H_\omega)$. Then $<\nabla p-\nabla\hat{p},\nabla v> = 0$ for $\nabla v \in E_i^\infty(H_\omega)$ and by part a) we conclude $\nabla p = \nabla\hat{p}$. For i = 1 we see immediately because of $\nabla\hat{p} \in E^q(H_\omega)$ that $\nabla p \in E^q(H_\omega)$. For i = 0 consider the function $\tilde{p}$ transformed like as in part i). Then $\nabla\tilde{p} \in E_o^r(H)$ and $\nabla\tilde{p} \in L^q(H)$. By Lemma 5.6 we conclude $\nabla\tilde{p} \in E_o^q(H)$ and transforming back $\nabla p \in E_o^q(H_\omega)$. ∎

<u>Lemma 5.11.</u> Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be a domain with boundary $\partial G \subset C^1$. Then for every $x_o \in \partial G$ there exists $R = R(x_o,\partial G,q,n) > 0$ and a constant $C = C(q,n,R) > 0$ with the following properties (write $G_R := G \cap B_R(x_o)$):
a) If $\nabla p \in E^q(G)$ then

$$(5.35)\qquad \|\nabla(\varphi p)\| \le C \sup_{\substack{v\in C_0^\infty(B_R(x_o))\\ 0\ne\nabla v \text{ on } G_R}} \frac{|<\nabla(\varphi p),\nabla v>_{G_R}|}{\|\nabla v\|_{q',G_R}}$$

holds for any $\varphi \in C^\infty(B_{R/2}(x_o))$

b) If $\nabla p \in E_0^q(G)$ then (5.35) holds if the sup is taken over all $0 \ne v \in C_0^\infty(G_R)$

c) If $\nabla p \in E^q(G)$ and $<\nabla p,\nabla\phi> = 0$ for all $\phi \in E^{q'}(G)$, then given $1 < s < \infty$, there is $0 < R' < R$, $R' = R'(s)$, such that $\nabla p \in E^s(G_{R'})$.

d) If $\nabla p \in E_0^q(G)$ and $<\nabla p,\nabla\phi> = 0$ for all $\phi \in E_0^{q'}(G)$, then given $1 < s < \infty$, there is $0 < R' < R$, $R' = R'(s)$, such that $\nabla(\varphi p) \in E_0^s(G_{R'})$ for each $\varphi \in C^\infty(B_R\ )$.

<u>Proof.</u> i) After a translation we may assume $x_o = 0$. Since $\partial G \in C^1$ there exists a $\rho > 0$ and a function $\sigma \in C^1(\bar B_\rho)$ with $(\nabla\sigma)(0) \ne 0$ such that $G \cap B_\rho = \{x \in B_\rho : \sigma(x) < 0\}$ and $\partial G \cap B_\rho = \{x \in B_\rho : \sigma(x) = 0\}$. A local parametrisation of ∂G most adequate to the problem under consideration is found by projecting $\partial G \cap B_\rho$ on the tangential hyperplane of ∂G at $x_o = 0$. Essentially this is done in the following. Observe that $|\nabla\sigma(0)|^{-1}\,\nabla\sigma(0)$ equals the exterior unit normal of ∂G at $x_o = 0$. This procedure enables us to reduce the situation to that of Lemma 5.10. There exists an orthogonal matrix S such that $S[\nabla\sigma(0)] = |\nabla\sigma(0)|e_n$, where $e_n = (\delta_{1n},\dots,\delta_{nn})$. Define $y(x) := Sx$ and put $\hat\sigma(y) := \sigma(S^{-1}y)$ for $y \in B_\rho$. Let $G_\rho := G \cap B_\rho$, $\hat G = SG$ and $\hat G_\rho := \hat G \cap B_\rho$. For $\nabla v \in E^s(G_\rho)$ we put $\hat v(y) := v(S^{-1}y)$ for $y \in \hat G_\rho$ $(1 < s < \infty)$. Then $\nabla\hat v \in E^s(\hat G_\rho)$ and the norms $\|\nabla\hat v\|_{s,\hat G_s}$ and $\|\nabla v\|_{s,G_\rho}$ are equivalent. Clearly $\nabla v \in E^s(G_\rho)$ if and only if $\nabla\hat v \in E_o^s(\hat G_\rho)$. The most important property (reflecting the invariance of Δ under orthogonal transforms) is that if $\nabla p \in E^s(G_\rho)$, $\nabla v \in E^{s'}(G_\rho)$ then $<\nabla\hat p,\nabla\hat v>_{\hat G_\rho} = <\nabla p,\nabla v>_{G_\rho}$. This is seen by a trivial calulation. Because of these properties we may omit in the sequel the distinction between $\hat v$ and v, $\hat G$ and G, $\hat\sigma$ and σ etc. and assume that the above rotation is performed. Since now $\nabla\sigma(0) = |\nabla\sigma(0)|e_n \ne 0$ by the implicit function theorem we find $0 < \rho' \le \rho$, $h > 0$ and a function $\psi \in C^1(\overline{B'}_{\rho'})$, where $B'_{\rho'} := \{y' \in \mathbb{R}^{n-1} : |y'| < \rho'\}$ with the following properties:

If $Z \equiv Z_{\rho',h} := \{y \in \mathbb{R}^n : |y'| < \rho',\ |y_n| < h\}$ then $Z \subset B_\rho$. For $y' \in B'_{\rho'}$ we have $(y',\psi(y')) \in Z$ and $\sigma(y',\psi(y')) = 0$. Further $\psi(0) = 0$, $(\nabla'\psi)(0) = 0$ (where $\nabla' = (\partial_1,\dots,\partial_{n-1})$) and $\partial G \cap Z = \{y \in Z : y_n = \psi(y')\}$, $G \cap Z = \{y \in Z : y_n < \psi(y')\}$. Let $\eta \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\eta(y') = 1$ for $|y'| \le 1$ and

$\eta(y') = 0$ for $|y'| \geq 2$, $0 \leq \eta \leq 1$ else. For $0 < \lambda < \rho'/2$ put $\eta_\lambda(y') := \eta(\lambda^{-1}y')$ and $\omega_\lambda(y') := \eta_\lambda(y')\psi(y')$ for $|y'| \leq \rho$ and $\omega_\lambda(y') = 0$ otherwise. Since $\psi(0) = |\nabla'\psi(0)| = 0$ we get $\sup\{|\nabla'\omega_\lambda(y')| : y' \in \mathbb{R}^{n-1}\} \longrightarrow 0$ for $\lambda \longrightarrow 0$. Let now $1 < q < \infty$ be given. Denote by $K_o := K(q,n) > 0$ the constant according to Lemma 5.10. We choose now $0 < \lambda < \rho'/2$ so small that $\|\nabla\omega_\lambda\|_\infty \leq K$ and define H_{ω_λ} according to (5.21). We choose any $0 < R < \lambda$ such that $B_R \subset\subset Z$.

ii) If $\nabla p \in E^q(G)$ then for $\varphi \in C_o^\infty(B_{R/2})$ clearly $\nabla(\varphi p) \in E^q(G_R)$ where $G_R := G \cap B_R$. By the choice of λ and R we have $G_R \subset H_{\omega_\lambda}$ and we may extend φp by zero to H_{ω_λ}. Denoting the extended function again by φp we have $\nabla(\varphi p) \in E^q(H_{\omega_\lambda})$. By Lemma 5.10 a)

$$(5.36)\quad \|\nabla(\varphi p)\| = \|\nabla(\varphi p)\|_{q,H_{\omega_\lambda}} \leq C_o \sup_{0\neq v\in C_o^\infty(\bar{H}_{\omega_\lambda})} \frac{|<\nabla(\varphi p),\nabla v>|}{\|\nabla v\|_{q',H_{\omega_\lambda}}}$$

Abbreviate the sup at the right hand side of 5.35) by d. Observe that the Poincaré inequality applies to G_R. Choose now $\psi \in C_o^\infty(B_r)$, $0 \leq \psi \leq 1$ such that $\psi = 1$ on $B_{R/2}$. Let $v \in C_o^\infty(\bar{H}_{\omega_\lambda})$ and let $c := |G_R|^{-1} \int_{G_R} v dy$.

Then because of the Poincaré-inequality there is $C' = C'(R,\psi)$ such that $\|\nabla(\psi(v-c))\|_{q',H_{\omega_\lambda}} \leq C'\|\nabla v\|_{q',H_{\omega_\lambda}}$. By definition of $C_o^\infty(\bar{H}_{\omega_\lambda})$ there is $\tilde{v} \in C_o^\infty(\mathbb{R}^n)$ with $\tilde{v}|_{H_{\omega_\lambda}} = v$. Then $\psi(\tilde{v}-c) \in C_o^\infty(B_R)$. Since $\psi \equiv 1$ on $\mathrm{supp}(\varphi p)$ we get

$$(5.37)\quad |<\nabla(\varphi p),\nabla v>| = |<\nabla(\varphi p),\nabla(v-c)>| = |<\nabla(\varphi p),\nabla(\psi(v-c))>| \leq$$
$$\leq d\|\nabla(\psi(v-c))\|_{q',G_R} \leq d\cdot C'\|\nabla v\|_{q'}$$

and (5.35) follows immediately from (5.36) and (5.37) with $C = C_o \cdot C'$.

iii) Let $Z_R := \{y = (y',y_n) \in \mathbb{R}^n : -R < x_n < \omega_\lambda(x')\}$. By means of the transform (5.23) we immediately see that (5.17) remains true for Z_R and $\nabla v \in E_o^{q'}(H_{\omega_\lambda})$. The proof of part b) is analogous to part ii) with the following changes: We use now (5.22). Given $v \in C_o^\infty(H_\omega)$, then by means of (5.17) we get $\|\nabla(\psi v)\|_{q',G_R} \leq C'\|\nabla v\|_{q',H_{\omega_\lambda}}$, $\psi v \in C_o^\infty(G_R)$ and instead of (5.37)

$$|<\nabla(\varphi p),\nabla v>| = |<\nabla(\varphi p),\nabla(\psi v)>| \leq dC'\|\nabla v\|_{q'}.$$

iv) The proof of c) and d) respectively is performed by induction using the Sobolev embedding theorem. We may assume $1 < q < 2 \le n$ and $\nabla p \in E^q(G_R)$. If $q \ge 2$ then clearly $\nabla p \in E^r(G_R)$ for $1 < r \le q$. Denote by k the biggest integer smaller than $\frac{n}{q}$. Then $k < \frac{n}{q} \le k+1$ and let $q_j := \frac{nq}{n-jq}$ for $j = 0,1,..,k$. Since $k + 1 \ge \frac{n}{q}$ we get $q_k \ge n \ge 2$. Let $K(q_{j-1},q_j) > 0$ for $j = 1,..,k$ denote the constants according to Lemma 5.10 and $K_1 := \min\{K(q_{j-1},q_j) : j = 1,..,k\} > 0$. Choose now $\lambda > 0$ in addition so small that $\|\nabla\omega_\lambda\|_q \le K$ and consider again H_{ω_λ}.
Let $0 < R < \lambda$ (R as above). Let $R_j := R2^{-(j+1)}$ for $j = 0,1,..,k+1$.
Choose $\varphi_j \in C_0^\infty(B_{R_j})$ such that $0 \le \varphi_j \le 1$ and $\varphi_j = 1$ on $B_{R_{j+1}}$.
Let $G_j := G \cap B_{R_j}$ for $j = 0,1,..,k,k+1$. Given $v \in C_0^\infty(\bar{H}_{\omega_\lambda})$ let $v_j := v - c_j(v)$ where $c_j(v) := |G_j|^{-1} \int_{G_j} v dx$. Then $\varphi_j v_j \in C_0^\infty(\bar{H}_{\omega_\lambda})$ and $\nabla(\varphi_j v_j) \in E^{q'}(G)$. Therefore

$$0 = <\nabla p, \nabla(\varphi_j v_j)> = <\nabla(\varphi_j p), \nabla v_j> - <p\nabla\varphi_j, \nabla v_j> + <\nabla p, v_j \nabla\varphi_j>$$

that is

$$(5.38) \qquad <\nabla(\varphi_j p), \nabla v_j> = <p\nabla\varphi_j, \nabla v_j> - <\nabla p, v_j\nabla\varphi_j>$$

We prove now by induction that $\nabla(\varphi_j p) \in E^{q_j}(H_{\omega_\lambda})$ for $j = 0,1,..,k$. The case $j = 0$ is clear. Let now $0 < j \le k$ and suppose $\nabla(\varphi_{j-1} p) \in E^{q_{j-1}}(H_{\omega_\lambda})$. Since $\varphi_{j-1} = 1$ on G_j we conclude $\nabla p|_{G_j} \in L^{q_{j-1}}(G_j)$ and by the Sobolev embedding theorem $p \in L^{q_j}(G_j)$ and

$$(5.39) \qquad \|p\|_{q_j,G_j} \le d_{1j}\|p\|_{H^{1,q_{j-1}}(G_j)}.$$

Therefore with $M_j := \|\nabla\varphi_j\|_\infty$

$$(5.40) \qquad |<p\nabla\varphi_j, \nabla v_j>| \le M_j\|p\|_{q_j,G_j}\|\nabla v_j\|_{q_j',G_j} \le d_{1j}M_j\|p\|_{H^{1,q_{j-1}}(G_j)}\|\nabla v_j\|_{q_j',G_j}$$

$$(5.41) \qquad |<\nabla p, v_j\nabla\varphi_j>| \le M_j\|\nabla p\|_{q_{j-1},G_j}\|v_j\|_{q_{j-1},G_j}$$

Since $\frac{n}{q} - n + 1 < 1 \le j$ we conclude $q_j' = \frac{nq}{nq-n+jq} < n$

and by the Sobolev theorem $v_j \in L^{q_j'^*}(G_j)$ where

$$q_j'^* = \frac{nq_j'}{n-q_j'} = q'_{j-1} \text{ and}$$

$$(5.42) \qquad \|v_j\|_{q'_{j-1},G_j} \le d_{2j}\|v_j\|_{H^{1,q_j'}(G_j)}$$

Since $\int_{G_j} v_j dx = 0$ we get by the Poincaré-inequality

(5.43) $\|v_j\|_{H^{1,q_j'}(G_j)} \le d_{3j}\|\nabla v_j\|_{q_j',G_j} \le d_{3j}\|\nabla v\|_{q_j',H_{\omega_\lambda}}$

By means of (5.40), (5.41) and (5.43) we get from (5.38) for $v \in C_o^\infty(H_{\omega_\lambda})$

$$|<\nabla(\varphi_j p),\nabla v>| = |<\nabla(\varphi_j p),\nabla v_j>| \le d_{4j}\|p\|_{H^{1,q_{j-1}}(G_j)}\|\nabla v\|_{q_j',H_{\omega_\lambda}} .$$

By Lemma 5.10 we conclude $\nabla(\varphi_j p) \in E^{q_j}(H_{\omega_\lambda})$ and therefore $\nabla p|_{G_{k+1}} \in L^{q_j}(G_{j+1})$. At the end follows $\nabla p|_{G_{k+1}} \in L^{q_k}(G_{k+1})$ where $q_k \ge n > 2$. Let now an arbitrary $s > n$ be given. Choose $0 < \epsilon < 1$ such that $s = \frac{n}{\epsilon}$. From the choice of k above we conclude $\frac{n}{n+1} \le q < \frac{n}{k}$. Define $\tilde{q} := \frac{n}{k+\epsilon}$. Then $\frac{n}{k+1} \le \tilde{q} < \frac{n}{k}$ and $k < \frac{n}{\tilde{q}} \le k + 1$ Since we originally assumed $1 < q < 2$ we have $k \ge \frac{n}{q} - 1 > \frac{n}{2} - 1 \ge 0$ we see $\tilde{q} \le \frac{n}{1+\epsilon} \le n \le q_k$ so that by the proof above we have $\nabla p|_{G_{k+1}} \in L^{\tilde{q}}(G_{k+1})$.

We repeat now the induction proof starting with $\tilde{q}_o = \tilde{q}$ and ending with

$\tilde{q}_k = \dfrac{n\tilde{q}}{n-k\tilde{q}} = \dfrac{n}{\epsilon} = s$. But observe that the constant $K_1 = K_1(\tilde{q})$ and therefore λ and especially R have to be taken depending on s.

v) Part d) is proven similiarily: Let $v \in C_o^\infty(H_{\omega_\lambda})$ be given. We no longer need to apply (5.42), instead we apply (5.17) for Z_R and $\nabla v \in E_o^{q'}(H_{\omega_\lambda})$ (see the beginning of part iii) of proof). We put now $v_j \equiv v$ and derive again (5.38). Since $\nabla(\varphi_{j-1}p) \in E_o^{q_{j-1}}(H_{\omega_\lambda}) \subset E_o^{q_{j-1}}(\mathbb{R}^n)$ we immediately get from the Sobolev estimate with a constant $d_{1j} = d_{1j}(q_j,n) > 0$

(5.44) $\|\varphi_{j-1}p\|_{q_j} \le d_{1j}\|\nabla(\varphi_{j-1}p)\|_{q_{j-1}}$

replacing now (5.39). Analogously

(5.46) $\|v\|_{q'_{j-1}} \le d_{2j}\|\nabla v\|_{q_j'}$

(observe $q'_{j-1} = q_j'^{*}$) replacing (5.42). Then observing $p\nabla\varphi_j \equiv (\varphi_{j-1}p)\nabla\varphi_j$ we get

for $v \in C_o^\infty(H_{\omega_\lambda})$

(5.47) $|<p\nabla\varphi_j,\nabla v>| \le M_j d_{1j}\|\nabla(\varphi_{j-1}p)\|_{q_{j-1}}\|\nabla v\|_{q_j'}$

(5.48) $|<\nabla p,v\nabla\varphi_j>| \le M_j d_{2j}\|\nabla(\varphi_{j-1}p)\|_{q_{j-1}}\|\nabla v\|_{q_j'}$

and so for $v \in C_o^\infty(H_{\omega_\lambda})$

$$|\langle\nabla(\varphi_j p),\nabla v\rangle| \le d_{4j}\|\nabla(\varphi_{j-1}p)\|_{q_{j-1}}\|\nabla v\|_{q_j'}.$$

Again by Lemma 5.10 i) b) $\nabla(\varphi_j p) \subset E_o^{q_j}(H_{\omega_\lambda})$ and since $\mathrm{supp}(\varphi_j p) \subset G_{j+1}$ we have $\nabla(\varphi_j p) \in E_o^{q_j}(G_{j+1})$. By induction we end with $\nabla(\varphi_k p) \in E_o^{q_k}(G_{k+1})$. Choose $R' = R_{k+1}$. If $\varphi \in C_o^\infty(B_{R'})$ then because of $\varphi_k = 1$ on B_{k+1} we have $\varphi\, \varphi_k p = \varphi p$. The remaining considerations are like as in iv). ■

The most difficult hard work is now done. For an easier later application we consider two further lemmas.

Lemma 5.12. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be a domain. Let $x_o \in G$ and let $R > 0$ be such that $B_R(x_o) \subset\subset G$.

a) Let $\nabla p \in E^q(G)$. Then for $\varphi \in C_o^\infty(B_{R/2}(x_o)), \nabla(\varphi p) \subset E_o^q(B_{R/2}(x_o)) \subset E_o^q(G)$ and with a constant $C = C(R, C_1(q)) > 0$, where C_1 is by Lemma 5.2, we have

$$(5.49) \quad \|\nabla(\varphi p)\|_q \le C \sup_{0 \ne v \in C_o^\infty(B_R)} \frac{|\langle\nabla(\varphi p),\nabla v\rangle|}{\|\nabla v\|_{q'}},$$

b) Let $\nabla p \in E^q(G)$ and $\langle\nabla p,\nabla\phi\rangle = 0$ for all $\phi \in C_o^\infty(G)$. Given $1 < s < \infty$ then there is a $0 < R' \le R$ with $R' = R'(s)$ such that $\nabla(\varphi p) \in E_o^s(B_{R'}(x_o))$ for each $\varphi \in C_o^\infty(B_{R'}(x_o))$.

Proof. The proof is almost identical with that of Lemma 5.12. In the sequel we abbreviate $B_r := B_r(x_o)$ for $r > 0$. i) If $x \in B_{R/2}$ and $0 < \epsilon < R/2$ then for the mollified p_ϵ we have $\nabla p_\epsilon(x) = (\nabla p)_\epsilon(x)$ and therefore $\|\nabla p-\nabla p_\epsilon\|_{q,B_{R/2}} \to 0$. For $\varphi \in C_o^\infty(B_{R/2})$ we have $\varphi p_\epsilon \in C_o^\infty(B_{R/2})$ and clearly $\|\nabla(\varphi p)-\nabla(\varphi p_\epsilon)\|_{q,B_{R/2}} \to 0$ proving $\varphi p \in E_o^q(B_{R/2}) \subset E_o^q(\mathbb{R}^n)$. Now we proceed like as in part ii) of the preceeding lemma applying Lemma 5.2 instead of Lemma 5.10: Choose again $\psi \in C_o^\infty(B_R)$ with $0 \le \psi \le 1$, $\psi = 1$ on $B_{R/2}$. If $v \in C_o^\infty(\mathbb{R}^n)$ put $c := |B_R|^{-1} \int_{G_R} v\,dy$ and use now the Poincaré-inequality for B_R. Consider again $\psi(v-c) \in C_o^\infty(B_R)$ and proceed analogously.

ii) The proof of b) is literally the same as part iv) of proof of Lemma 5.11, beginning with the 9th line before formula (5.38). Observe that $G_j = B_{R_j}$. ■

Lemma 5.13. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be an exterior domain, $G = \mathbb{R}^n \setminus \mathring{K}$ with $\emptyset \neq K \subset\subset \mathbb{R}^n$. Let $R > 0$ be such that $K \subset\subset B_R$, let $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(x) = 0$ for $|x| \leq 2R$, $\varphi(x) = 1$ for $|x| \geq 3R$.

a) If $\nabla p \in E^q(G)$ then $\nabla(\varphi p) \in E^q(G)$.

b) There is a constant $C = C(q,n,R) > 0$ such that

(5.50) $$\|\nabla(\varphi p)\|_q \leq C \sup_{0 \neq v \in C_o^\infty(\bar{G})} \frac{|<\nabla(\varphi p),\nabla v>|}{\|\nabla v\|_{q',G}}$$

c) If $\nabla p \in E^q(G)$ has the property

(5.51) $<\nabla p,\nabla v> = 0$ for all $v \in C_o^\infty(\bar{G})$.

Then $\nabla(\varphi p) \in E^s(G)$ for any $1 < s < \infty$.

Proof. i) Because of the properties of φ clearly $\nabla(\varphi p) \in E^q(G)$.

ii) (φp) may be extended by zero to the whole $\mathbb{R}^n$. Then $\nabla(\varphi p) \in E^q(\mathbb{R}^n)$. Since $\|\nabla v\|_{q',\mathbb{R}^n} \geq \|\nabla v\|_{q',G}$ for $v \in C_o^\infty(\mathbb{R}^n)$ estimate (5.50) follows immediately from (5.5).

iii) We will first show $\nabla(\varphi p) \in E^s(G)$ for $1 < s \leq q$. This is a priori by no means trivial. By definition of $E^q(G)$ we clearly have $\nabla p \in L^s(G_{3R}), p \in L^s(G_{3R})$ for $1 < s \leq q$. If $v \in C_o^\infty(\mathbb{R}^n)$ put $\tilde{v} := v-c$ where $c := |B_{3R}|^{-1} \int_{B_{3R}} v dx$. Then $(\varphi\tilde{v} + c) \in C_o^\infty(\bar{G})$ and by (5.51) we again get

$$0 = <\nabla p,\nabla(\varphi(v-c) + c)> = <\nabla p,\nabla(\varphi(v-c))> =$$

$$= <\nabla(\varphi p)\nabla v> - <p\nabla\varphi,\nabla v> + <\nabla p,\nabla\varphi\tilde{v}>$$

and therefore

$$<\nabla(\varphi p),\nabla v> = <\nabla(\varphi p),\nabla\tilde{v}> = <p\nabla\varphi,\nabla\tilde{v}> - <\nabla p,\tilde{v}\nabla\varphi>$$

Since by (2.2) $\|\tilde{v}\|_{s',B_{3R}} \leq C(s')\|\nabla\tilde{v}\|_{s',B_{3R}} \leq C(s')\|\nabla v\|_{s'}$

we get immediately with $C = C(\varphi,R,s,q) > 0$

$$|<(\varphi p),\nabla v>| \leq C(\|p\|_{q,G_{3R}} + \|\nabla p\|_{q,G_{3R}})\|\nabla v\|_{s'}$$

for $v \in C_o^\infty(\mathbb{R}^n)$ and by Lemma 5.2 $\nabla(\varphi p) \in E^s(\mathbb{R}^n)$.

iv) Because of iii) we may assume that $1 < q < 2 \leq n$. Then we proceed like as in part iv) of the proof of Lemma 5.11: Choose again $k \in \mathbb{N}$, $k < \frac{n}{q} \leq k+1$. Let $R_j := R + j\frac{R}{5k}$ for $j = 0,1,..,k+1$. Let $\varphi_j \in C^\infty(\mathbb{R}^n)$, $0 \leq \varphi_j \leq 1$, $\varphi_j(x) = 0$ for

$|x| \le R_j$, $\varphi_j(x) = 1$ for $|x| \ge R_{j+1}$. Given $v \in C_0^\infty(\mathbb{R}^n)$ let $v_j = v - c_j(v)$ where $c_j(v) := |B_{2R}|^{-1} \int_{B_{2R}} v dx$. Since $(\varphi_j v_j + c_j(v)) \in C_0^\infty(\bar{G})$ we can now go on like as in Lemma 5.11. Once we have shown $\nabla(\varphi p) \in E^q(G)$ and $\nabla(\varphi p) \in E^{q_k}(G)$, we see $\nabla(\varphi p) \in E^s(G)$ for $q \le s \le q_k$: Write $G' = \{x \in G: |\nabla(\varphi p)(x)| \ge 1\}$ and $G'' = \{x \in G: |\nabla(\varphi p)(x)| \le 1\}$
then

$$\int_G |\nabla(\varphi p)|^s dx = \int_{G'} |\nabla(\varphi p)|^s dx + \int_{G''} |\nabla(\varphi p)|^s dx \le$$
$$\le \int_{G'} |\nabla(\varphi p)|^{q_k} dx + \int_{G''} |\nabla(\varphi p)|^q dx. \quad \blacksquare$$

<u>Lemma 5.14.</u> Assume the same hypothesis as in Lemma 5.13.
a) If $\nabla p \in E_0^q(G)$ then $\nabla(\varphi p) \in E_0^q(G)$ and there is a constant $C = C(q,n,\mathbb{R}) > 0$ such that

$$(5.52) \quad \|\nabla(\varphi p)\|_q \le C \sup_{0 \ne v \in C_0^\infty(G)} \frac{|\langle\nabla(\varphi p), \nabla v\rangle|}{\|\nabla v\|_{q'}}$$

b) Suppose that $\nabla p \in E_0^q(G)$ satisfies

$$\langle\nabla p, \nabla v\rangle = 0 \text{ for all } v \in C_0^\infty(\bar{G}).$$

Then: i) If $1 < q \le \frac{n}{n-1}$ then $\nabla(\varphi p) \in E^s(G)$ for $1 < s < \infty$.

ii) If $q > \frac{n}{n-1}$ then $\nabla(\varphi p) \in E^s(G)$ for $\frac{n}{n-1} < s < \infty$.

<u>Proof.</u> i) Since $\nabla p \in E_0^q(G)$ there is a sequence $(p_i) \subset C_0^\infty(G)$ such that $\|\nabla p - \nabla p_i\|_q \to 0$. By (2.9) $\|p - p_i\|_{q,G_{3R}} \le C(R) \|\nabla p - \nabla p_i\|_q$. Therefore $\nabla(\varphi p_i) = p_i \nabla\varphi + \varphi\nabla p_i \to \nabla(\varphi p)$ in $L^q(G)$, $\varphi p_i \in C_0^\infty(G)$ and therefore $\nabla(\varphi p) \in E_0^q(G)$.
ii) We first show (5.52) for $1 < q \le \frac{n}{n-1}$. Then $q' \ge n$. Let $\psi \in C^\infty(\mathbb{R}^n)$, $0 \le \psi \le 1$, $\psi(x) = 1$ for $|x| \ge 2R$, $\psi(x) = 0$ for $|x| \le \frac{3}{2} R$. For $v \in C_0^\infty(\mathbb{R}^n)$ let $c_v = |B_{2R}|^{-1} \int_{B_{2R}} v dx$. We will show now that $\tilde{v} := \psi(v - c_v) \in E_0^{q'}(G)$ for $q' \ge n$. For this purpose let $\rho \in C_0^\infty(\mathbb{R}^n)$, $0 \le \rho \le 1$, $\rho(x) = 1$ for $|x| \le 1$, $\rho(x) = 0$ for $|x| \ge 2$ and put $\rho_k(x) := \rho(k^{-1}x)$ for $k \in \mathbb{N}$. Then $\operatorname{supp}(\nabla\rho_k) \subset R_k := \{x \in \mathbb{R}^n : k \le |x| \le 2k\}$. Then $\tilde{v}_k := \rho_k \cdot \tilde{v} \in C_0^\infty(G)$. There is $k_o \in \mathbb{N}$, $k_o \ge 2R$, such that $\operatorname{supp} v \subset B_{k_o}$. Then for $k \ge k_o$

$$\nabla\tilde{v}_k = \nabla\rho_k \cdot \psi(v - c_v) + \rho_k(\nabla\psi)(v - c_v) + \rho_k\psi \cdot \nabla v$$
$$= -\nabla\rho_k c_v + (v - c_v)\nabla\psi + \psi \cdot \nabla v.$$

Since $|\nabla\rho_k| \le C \cdot k^{-1}$ and vanishes outside R_k we get $\|\nabla\rho_k\|_{q',R_k} \le c \cdot k^{\frac{n}{q'}-1} \le$ const for $q' \geq n$. If $F^* \in E_o^{q'}(G)^*$ then it may be extended to a functional $\tilde{F}^* \in (L^{q'}(G)^n)^*$ and therefore represented with $f \in L^q(G)^n$ as $F^*(\nabla\phi) = \langle f,\nabla\phi\rangle$ for all $\nabla\phi \in E_o^{q'}(G)$. Since $|\langle f,\nabla\rho_k\rangle| \leq \|f\|_{q,R_k} \leq$ const $\|f\|_{q,R_k} \to 0$ we conclude $f^*(\nabla\tilde{v}_k) \to F^*(\nabla\tilde{v})$. Since $E_o^{q'}(G)$ is closed it is weakly closed too and therefore $\nabla\tilde{v} \in E_o^{q'}(G)$. Since $\nabla\psi = 0$ on the support of φ we see $\nabla(\varphi p)\cdot((v-c_v)\nabla\psi + \psi\cdot\nabla v) = \nabla(\varphi p)\cdot\nabla v$ and therefore $\langle\nabla(\varphi p),\nabla\tilde{v}\rangle = \langle\nabla(\varphi p),\nabla v\rangle$. By (2.2) (applied to B_{2R}) we see $\|\nabla\tilde{v}\|_{q'} \leq K\|\nabla v\|_{q'}$ and therefore

$$\frac{|\langle\nabla(\varphi p),\nabla v\rangle|}{\|\nabla v\|_{q'}} \leq K\,\frac{|\langle\nabla(\varphi p),\nabla\tilde{v}\rangle|}{\|\nabla\tilde{v}\|_{q'}} \text{ if } \nabla\tilde{v} \neq 0.$$

We abbreviate the sup at the right hand side of (5.52) by D. Since $0\neq\nabla\tilde{v} \in E_o^{q'}(G)$ there is a sequence $(v_i) \subset C_o^\infty(G)$ such that $\|\nabla v-\nabla v_i\|_{q'} \to 0$. Then

$$\frac{|\langle\nabla(\varphi p),\nabla\tilde{v}\rangle|}{\|\nabla\tilde{v}\|_{q'}} = \lim_{i\to\infty} \frac{|\langle\nabla(\varphi p),\nabla v_i\rangle|}{\|\nabla v_i\|_{q'}} \leq D.$$

Therefore $\dfrac{|\langle\nabla(\varphi p),\nabla v\rangle|}{\|\nabla v\|_{q'}} \leq KD$ for those $v \in C_o^\infty(\mathbb{R}^n)$ such that $\nabla\tilde{v} \neq 0$. If $\nabla\tilde{v}$ vanishes then $\langle\nabla(\varphi p),\nabla\varphi\rangle$ too. Therefore we derive (5.52) from Lemma 5.2.

iii) Let now $q > \frac{n}{n-1}$ and therefore $1 < q' < n$. Let $\psi \in C^\infty(\mathbb{R}^n)$ be defined as in ii). If $v \in C_o^\infty(\mathbb{R}^n)$ then $\psi\cdot v \in C_o^\infty(G)$ and $\|\nabla(\psi v)\|_{q'} \leq \|v\nabla\psi\|_{q'} + \|\nabla v\|_{q'}$.

Let $q'^* := \frac{nq'}{n-q'}$. Then by the Sobolev theorem and Hölder's inequality

$$\|v\nabla\psi\|_{q'} \leq c(\psi)\|v\|_{q',B_{2R}} \leq c(\psi)\|v\|_{q'^*,B_{2R}}|B_{2R}|^{1/n}$$
$$\leq c(\psi)|B_{2R}|^{1/n}\|\nabla v\|_{q'}$$

and so $\|\nabla(\psi v)\|_{q'} \leq c(R)\|\nabla v\|_{q'}$. Therefore

$$|\langle\nabla(\varphi p),\nabla v\rangle| = |\langle\nabla(\varphi p),\nabla(\psi v)\rangle| \leq D\|\nabla(\psi v)\|_{q'} \leq c(R)\cdot D\,\|\nabla v\|_{q'}$$

where D denotes the sup in (5.52) and again by Lemma 5.2 follows (5.52).

iv) We prove now b). Let $1 < q \leq \frac{n}{n-1}$. We first show

(5.53) $\langle\nabla p,\nabla\varphi\rangle = 0$ for φ by Lemma 5.13.

Choose ρ_k like as in part ii). Then for $k \geq 3R$ we have $\rho_k\varphi \in C_o^\infty(G)$ and by assumption

$$0 = \langle\nabla p,\nabla(\rho_k\varphi)\rangle = \langle\nabla p,\rho_k\nabla\varphi\rangle + \langle\nabla p,\varphi\nabla\rho_k\rangle = \langle\nabla p,\nabla\varphi\rangle + \langle\nabla p,\nabla\rho_k\rangle$$

Remember $\|\nabla\rho_k\|_{q'} \leq$ const. for $q' \geq n$. Therefore

$$|\langle\nabla p,\nabla\rho_k\rangle| = |\langle\nabla p,\nabla\rho_k\rangle_{R_k}| \leq C\,\|\nabla p\|_{q,R_k} \to 0.$$

v) For $v \in C_o^\infty(\mathbb{R}^n)$, $\varphi v \in C_o^\infty(G)$ and therefore admissible in (5.52). We then get

(5.54) $<\nabla(\varphi p),\nabla v> = <p\nabla\varphi,\nabla v> - <\nabla p, v\nabla\varphi>$.

Let $c(v) := |B_{3R}|^{-1}\int_{B_{3R}} v dx$ and put $\tilde{v} := v - c(v)$. Then by (5.53) $<\nabla p,\tilde{v}\nabla\varphi> = <\nabla p, v\nabla\varphi>$.

Let now $1 < s \le q \le \frac{n}{n-1}$. Then $p \in L^s(G_{3R})$ and $\nabla p \in L^s(G_{3R})$. We therefore get

$$|<\nabla p,\tilde{v}\nabla\varphi>| \le \|\nabla\varphi\|_\infty \|\nabla p\|_{s,G_{3R}} \|\tilde{v}\|_{s',B_{3R}}$$

and by the Poincaré-inequality

$$\|\tilde{v}\|_{s',B_{3R}} \le C\|\nabla\tilde{v}\|_{s'} = C\|\nabla v\|_{s'}$$

and so we derive from (5.54) for $v \in C_o^\infty(\mathbb{R}^n)$

$$|<\nabla(\varphi p),\nabla v>| \le C'(\|p\|_{s,G_{3R}} + \|\nabla p\|_{s,G_{3R}}) \|\nabla v\|_{s'}$$

and so by Lemma 5.2 we get $\nabla(\varphi p) \in E^s(\mathbb{R}^n)$

vi) If $n = q = 2$ then we are ready. If $n > 2$, then $1 < q \le \frac{n}{n-1} < 2$. Now we proceed similiarily as in parts iv) and v) of the proof of Lemma 5.11: Choose again $k \in \mathbb{N}$, $k < \frac{n}{q} \le k+1$. Let $R_j = R + j\frac{R}{5k}$ for $j = 0,1,\dots,2(k+1)$. Let $\varphi_j \in C^\infty(\mathbb{R}^n)$, $0 \le \varphi_j \le 1$, $\varphi_j(x) = 0$ for $|x| \le R_j$, $\varphi_j(x) = 1$ for $|x| \ge R_{j+1}$. Given $v \in C_o^\infty(\mathbb{R}^n)$, then $\varphi_j v \in C_o^\infty(G)$ and from the assumption we again derive (5.38) now with $v_j := v$. Let now again $0 < j \le k$ and assume that $\nabla(\varphi_{j-1}p) \in E^{q_{j-1}}(G)$. Since $\varphi_{j-1} = 1$ for $|x| \ge R_j$, we conclude $\nabla p \in L^{q_{j-1}}(\{|x| \ge R_j\})$. Since supp $(\nabla\varphi_j) \subset G_j := \{x \in \mathbb{R}^n\colon R_j \le |x| \le R_{j+1}\}$ again by the Sobolev embedding theorem we get (5.39). Since $q_j' < n$ instead of (5.42) we use the Sobolev theorem for $C_o^\infty(\mathbb{R}^n)$ functions

(5.55) $\|v\|_{q'_{j-1}} \le C\|\nabla v\|_{q_j'}$.

Then we can estimate (5.41) and end with

$$|<\nabla(\varphi_j p),\nabla v>| \le C\|p\|_{H^{1,q_{j-1}}(G_j)} \|\nabla v\|_{q_j'}.$$

The remaining arguments are the same as in part iv) of the proof of Lemma 5.11

vii) Assume now $q > \frac{n}{n-1}$. Let $\varphi_o \in C^\infty(\mathbb{R}^n)$, $0 \le \varphi_o \le 1$, $\varphi_o = 0$ on B_R, $\varphi_o(x) = 1$ for $|x| \ge \frac{3}{2}R$ and consider (5.54) with this φ_o. If $n \ge 3$ choose any q_1 with $\frac{n}{n-1} < q_1 < 2$ and $q_1 \le q$. Put $r := \frac{q_1 n}{n+q_1}$. Then $1 < r < \frac{2n}{n+2} < n$ (observe $\frac{2n}{n+2} > 1$ for $n \ge 3$). Further $r \le q$ and $q_1 = \frac{nr}{n-r} = r^*$, $q_1' = \frac{nr}{nr-n+r}$ and $q_1'^* = \frac{nq_1'}{n-q_1'} = r'$. Since $r \le q$ we have $p, \nabla p \in L^r(M_R)$, that is $p \in H^{1,r}(M_R)$, where $M_R := \{x \in \mathbb{R}^n: R < |x| < \frac{3}{2} R\}$. The Sobolev inequality gives

$$\|p\|_{q_1,M_R} = \|p\|_{r^*,M_R} \le C\|p\|_{H^{1,r}(M_R)}$$

Consider (5.54) for $v \in C_0^\infty(\mathbb{R}^n)$ and φ_o. Then with $C' := \|\nabla\varphi_o\|_\infty$

(5.56) $|<p\nabla\varphi,\nabla v>| \le C'\|p\|_{q_1(M_R)} \|\nabla v\|_{q_1'} \le C'C \|p\|_{H^{1,r}(M_R)}\|\nabla v\|_{q_1'}$

By the Sobolev inequality for $v \in C_0^\infty(\mathbb{R}^n)$ (see (5.55))

(5.57) $|<\nabla p, v\nabla\varphi>| \le C'\|\nabla p\|_{r,M_R} \|v\|_{r'} \le C'C\|\nabla p\|_{r,M_R}\|\nabla v\|_{q_1'}$

By Lemma 5.2 we see from (5.54) that $\nabla(\varphi_o p) \in E^{q_1}(\mathbb{R}^n)$ holds. Now we may start the iteration procedure from part iii).
If $n = 2$ and given $s > 2$, we put $\tilde{r} = \frac{2s}{2+s}$. Then $1 < \tilde{r} < 2 = n$, $\tilde{r}^* = \frac{2r}{2-r} = s$ and with estimates analogously to (5.56), (5.57) we conclude again via Lemma 5.2 $\nabla(\varphi_o p) \in E^s(G)$. ■

<u>Remark 5.15.</u>
If $G \subset \mathbb{R}^n$ is a <u>bounded</u> domain with sufficiently smooth boundary, say e.g. $\partial G \in C^1$ and if $1 < q < s < \infty$, if $\nabla p \in E_0^q(G)$ and $\nabla p \in L^s(G)$, then $\nabla p \in E_0^s(G)$. As we have seen in Lemma 5.6 this conclusion still holds for the half-space. But it is no longer true for an exterior domain: Let $K := B_1$, $G := \mathbb{R}^n\backslash \bar{K} = \{x \in \mathbb{R}^n: |x| > 1\}$ and consider for $x \in G$

$$(5.58) \qquad h(x) := \begin{cases} 1 - |x|^{2-n} & \text{if } n \ge 3 \\ \ln|x| & \text{if } n = 2 \end{cases}$$

Then $h \in C^\infty(\bar{G})$, $h|_{\partial G} = 0$, $\Delta h = 0$, $\nabla h \in L^q(G)$ for $q > \frac{n}{n-1}$. Consider again $\rho \in C_0^\infty(\mathbb{R}^n)$, $0 \le \rho \le 1$, $\rho(x) = 1$ for $|x| \le 1$, $\rho(x) = 0$ for $|x| \ge 2$ and $\rho_k(x) := \rho(k^{-1}x)$ for $k \in \mathbb{N}$. Since $h|_{\partial G} = 0$ it is easily seen that $h_k := \rho_k h \in E_0^q(G)$ for $q > \frac{n}{n-1}$. Since $\nabla h_k = h\nabla\rho_k + \rho_k\nabla h$ because of (5.58) and the properties of h one immediately verifies $\|\rho_k h\|_q \to 0$ if $q > n$ for $n \ge 2$. Clearly $\rho_k\nabla h \to \nabla h$ in $L^q(G)$. If $n \ge 3$ and $q = 2$ like as in part iii) of the proof of Lemma 5.6 one verifies $\nabla h_k \to \nabla h$ weakly in $E_0^q(G)$ and therefore $\nabla h \in E_0^q(G)$ in this case too. This rests on the property $\|\nabla\rho_k\|_q \le$ const for $q \ge n$ and $k \in \mathbb{N}$. If $q = n = 2$, $\nabla \ln|x| \notin L^2(G)$. That these are the optimal q for $\nabla h \in E_0^q(G)$ may be seen as follows: If $p \in C_0^\infty(G) \subset C_0^\infty(\mathbb{R}^n)$ by means of the Sobolev-inequality $\|p\|_{q^*} \le C\|\nabla p\|_q$ for $\nabla p \in E_0^q(G)$ and $1 \le q < n$. Clearly $h \notin L^s(G)$ vor $1 \le s < \infty$. ■

The best possible result we can expect therefore is

Lemma 5.16. Let $1 < q < \infty$ and let $G \subset \mathbb{R}^n$ be an exterior domain, $G = \mathbb{R}^n \setminus \bar{K}$, with $\phi \neq K \subset\subset \mathbb{R}^n$. Let $R > 0$ be such that $K \subset\subset B_R$. Suppose there is $1 < q < n$ and $\nabla p \in E_0^q(G)$. Assume in addition $p = 0$ in $G_{2R} := G \cap B_{2R}$. If there is any other $1 < r < \infty$ with $\nabla p \in L^r(G)$. Then $\nabla p \in E_0^r(G)$.

Proof. Without loss of generality we may assume $p \in C^\infty(G)$. Otherwise we consider the mollification p_ϵ. Since p vanishes on G_{2R}, $\nabla(p_\epsilon) = (\nabla p)_\epsilon$ in G for $0 < \epsilon < R$. Clearly $\nabla p_\epsilon \in E_0^q(G)$ too. Consider again our standard cut-off function $\rho \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \rho \leq 1$, $\rho(x) = 1$ for $|x| \leq 1$, $\rho(x) = 0$ for $|x| \geq 2$, $\rho_k(x) := \rho(k^{-1}x)$ and $R_k := \{x \in \mathbb{R}^n: k < |x| \leq 2k\}$. Then $\text{supp}(\nabla\rho_k) \subset R_k$. Since by assumption $\nabla p \in E_0^q(G)$ and $1 < q < n$, by the Sobolev embedding theorem $p \in L^{q^*}(G)$ with $q^* = \frac{nq}{n-1}$ and $\|p\|_{q^*} \leq C\|\nabla p\|_q$. Define $C_k := |R_k|^{-1} \int_{R_k} p dx$

By Hölder's inequality

$$|C_k| \leq |R_k|^{-1} \|p\|_{q^*,R_k} |R_k|^{\frac{1}{q^{*\prime}}} \leq C\|\nabla p\|_q |R_k|^{\frac{1}{q^{*\prime}}-1}.$$

Since $q^{*\prime} = \frac{q^*}{q^*-1} = \frac{nq}{nq-n+q}$ we have $\frac{1}{q^{*\prime}} - 1 = \frac{q-n}{q} < 0$ for $q < n$ and we get $|R_k|^{\frac{1}{q^{*\prime}}-1} \to 0$ $(k \to \infty)$ and therefore $|C_k| \to 0$. Observe the Poincaré-inequality (2.6) giving

$$(5.59) \qquad \|p-C_k\|_{r,R_k} \leq k \cdot C_1 \|\nabla p\|_{r,R_k}$$

(since we assumed $p \in C^1(G)$ clearly $p \in L^r(R_k)$). Define $\varphi(x) = 1-\rho_R(x)$ (with ρ like as above) and $p_k := \varphi(\rho_k(p-C_k)) \in C_0^\infty(G)$.
Then $\nabla p_k = \rho_k(p-C_k)\nabla\varphi + \varphi(p-C_k)\nabla\rho_k + \varphi\rho_k\nabla p$. Since $\varphi \equiv 1$ on $\text{supp}(\nabla p)$ we see $\varphi\rho_k\nabla p \to \nabla p$ in $L^r(G)$. Since p vanishes on $\text{supp}(\nabla\varphi) \subset B_{2R}$, $\rho_k \equiv 1$ for $k > R$, $\|\rho_k(p-C_k)\nabla\varphi\|_r = \|\nabla\varphi\|_\infty |C_k| |B_{2R}| \to 0$. By (5.59) and $|\nabla\rho_k| \leq C \cdot k^{-1}$, $\text{supp}(\nabla\rho_k) \subset R_k$ we see $\|\varphi(p-C_k)\nabla\rho_k\|_r \leq Ck^{-1}\|p-C_k\|_{r,R_k} \leq C \cdot C' \|\nabla p\|_{r,R_k} \to 0$ $(k \to \infty)$.
Therefore $\|\nabla p - \nabla p_k\|_r \to 0$. ∎

Remark 5.17. Let the assumption of Lemma 5.14 be satisfied, expecially (5.52) with $\nabla p \in E_0^q(G)$. If $1 < q \leq \frac{n}{n-1}$ and $n \geq 3$ (then $\frac{n}{n-1} < n$) we conclude via Lemma 5.15 that $\nabla(\varphi p) \in E_0^s(G)$ for $1 < s < \infty$. If $n = 2$ and $1 < q < 2$ then $\nabla(\varphi p) \in E_0^s(G)$ for $1 < s < \infty$. If $n \geq 3$ and $\frac{n}{n-1} < q < n$ then $\nabla(\varphi p) \in E_0^s(G)$ for $\frac{n}{n-1} < s < \infty$. These properties perfectly fits togehter with the observations made in Remark 5.15. ∎

The proof of estimates (5.35), **(5.49)**, **(5.50) was completely** elementary (e.g. we needed only Sobolev's embedding theorem and Hölder's inequality) but demanding lengthy hard work. It was done to prove in addition that solutions $\nabla p \in E^q(G)$ resp. $\nabla p \in E_o^q(G)$ of the homogeneous functional equations have integrability properties with respect to "other" exponents $1 < s < \infty$ and (compare Remark 5.17) belong under certain circumstances to $E_o^s(G)$ too. All this work we need to conclude via the trivial $L^2(G)$-uniqueness of the Dirichlet and Neumann problem the L^q-uniqueness. By means of a partition of unity the desired main theorems are then an easy consequence of the following uniqueness result. Concerning the Dirichlet problem we read off from Remark 5.15 that the uniqueness result is best possible. I'm very much indepted to my colleque Professor Dr. Michael Wiegner, who gave me the example of the "exceptional functions" h in (5.58) and drew my attention in a very early stage of the consideration of exterior problems in the appropriate direction.

Theorem 5.18. (Uniqueness) Let $G \subset \mathbb{R}^n$ be either a bounded or an exterior domain with boundary $\partial G \in C^1$ and let $1 < q < \infty$. Then:

a) Uniqueness of the weak Neumann problem:
   If $\nabla p \in E^q(G)$ satisies $\langle\nabla p,\nabla\phi\rangle = 0$ for all $\nabla\phi \in E^{q'}(G)$, then $\nabla p = 0$.

b) Uniqueness of the weak Dirichlet problem:
   If $\nabla p \in E_o^q(G)$ satisfies $\langle\nabla p,\nabla\phi\rangle = 0$ for all $\nabla\phi \in E_o^{q'}(G)$. Then:
   i) If G is bounded, then $\nabla p = 0$ (and therefore $p = 0$)
   ii) If G is an exterior domain, and
      if $n \geq 3$ and $1 < q < n$, then $\nabla p = 0$
      if $n = 2$ and $1 < q \leq 2 = n$, then $\nabla p = 0$
      (and $p = 0$ too).

Proof. i) By Lemma 5.11 for each $x_o \in \partial G$ there is $R' = R'(x_o) > 0$ such that for $G \cap B_{R'}(x_o)$ the properties c) respectively d) hold. By compactness of ∂G we find finitely many $x_i \in \partial G$ and $R_i := R'(x_i) > 0$, $i = 1,..,M$, such that $\partial G \subset \bigcup_{i=1}^{M} B_i$, where $B_i := B_{R_i}(x_i)$. If G is bounded, $G_1 := G \cap \bigcap_{i=1}^{M} \mathbb{R}^n\backslash B_i \subset G$ and is compact. By means of Lemma 5.12 we see that G_1 can be covered by finitely many balls $B_i = B_{R_i}(x_i) \subset\subset G$, $i = M+1,\ldots,N$. Then the $B_i, i = 1,\ldots,N$ form an open covering of $\bar{G}$. If G is an exerior domain, $G = \mathbb{R}^n\backslash\bar{K}$, where $\phi \neq K \subset\subset \mathbb{R}^n$, then we choose $R > 0$ such that $K \subset\subset B_R$ and put now $G_2 := G_1 \cap \bar{B}_{3R} \subset G_2$. Again G_2 is compact and may be covered by $B_i \subset\subset G$, $i = M+1,\ldots,N$.

Define $B_o := \{x \in \mathbb{R}^n : |x| > 2R\} \subset G$. Then again the system $B_i, i = 0,1,..,N$ forms an open covering of $\bar{G}$. Construct a partion of unity $\{\varphi_i : i = =,1,..,N\}$ such that $0 \le \varphi_i$, $\varphi_i \in C_0^\infty(B_i)$ for $i = 1,..,N$ and $\varphi_o \in C^\infty(B_o)$, $\varphi_o = 1$ for $|x| \ge 3R$, φ_o vanishing in a neighborhood of $|x| = 2R$, $\sum_{i=1}^N \varphi_i(x) = 1$ for $x \in \bar{G}$.

ii) In case a) we immediately conclude from the hypothesis and Lemmas 5.11 - 5.13 that $\nabla(\varphi_i p) \subset E^2(G)$ (continue $\varphi_i p$ by zero for $x \in G$, $x \notin G \cap B_i$) and therefore $\nabla p = \sum_{i=0}^N \nabla(\varphi_i p) \in E^2(G)$. Since $E^\infty(G) = \{\nabla\phi : \phi \in C_0^\infty(\bar{G})\}$ is dense in $E^s(G)$ for $1 < s < \infty$ from $\nabla p \in E^2(G)$ and $0 = \langle\nabla p, \nabla\phi\rangle$ for $\nabla\phi \in E^s(G)$ we see now $0 = \langle\nabla p, \nabla p\rangle$ and therefore $\nabla p = 0$.

iii) In case b) and G bounded we similiarly conclude by Lemmas 5.11 and 5.12 d) $\nabla(\varphi_i p) \in E_o^q(G)$ and therefore $\nabla p \in E_o^2(G)$ and again $\nabla p = 0$. Clearly, we conclude $p = 0$ too.

iv) If G is an exterior domain we consider Lemma 5.14 and 5.16: If $n \ge 3$ and $1 \le q \le \frac{n}{n-1} < n$, then $\nabla(\varphi_o p) \in E_o^s(G)$ for $1 < s < \infty$. If $\frac{n}{n-1} < q < n$, then again $\nabla(\rho_o p) \in E_o^s(G)$ for $1 < s < \infty$. In any case $\nabla(\varphi_o p) \in E_o^s(G)$. If $n = 2$ and $1 < q < 2$, then by Lemma 5.15 and 5.16 $\nabla(\varphi_o p) \in E_o^s(G)$. The case $q = 2$ is trivial. Since $\nabla(\varphi_i p) \in E_o^s(G)$ for $i = 0,..,N$ at the end $\nabla p = \sum_{i=0}^N \nabla(\varphi_i p) \in E_o^2(G)$ and again $0 = \langle\nabla p, \nabla\phi\rangle$ for all $\phi \in C_0^\infty(G)$ and by density of $\{\nabla\phi : \phi \in C_0^\infty(G)\}$ in $E_o^2(G)$ we again have $\nabla p = 0$ that is $p = 0$. ■

6. Proof of the main theorems.

Proof of Theorem 4.1. First we prove part i). Like as in part i) of the proof of Theorem 5.18 we construct a covering $B_i, i = 0,1,..,N$ of G ($B_o := \phi$ if G is bounded) and a partition of unity (φ_i) with respect to this covering such that

(5.35) holds for $i = 1,..,M$, (5.49) for $i = M+1,..,N$ and (5.50) for $i = 0$. Suppose that the statement a) of Theorem 4.1 is not true. Then there exists a sequence $(\nabla p_k) \subset E^q(G)$ such that $\|\nabla p_k\|_{q,G} = 1$ and with

$$(6.1) \quad \epsilon_k := \sup_{0 \neq \nabla v \in E^{q'}(G)} \frac{|\langle\nabla p_k, \nabla v\rangle|}{\|\nabla v\|_{q'}} \to 0 \ (k \to \infty).$$

Without loss of generality we may assume $\int_G p_k dx = 0$ if G is bounded and $\int_{G_{3R}} p_k dx = 0$ if G is an exterior domain. Then by the Poincaré- inequality we get

$\|p_k\|_{H^{1,q}(G)} \le$ const. if G is bounded, $\|p_k\|_{H^{1,q}(G_{3R})} \le$ const if G is an exterior domain. Since $E^q(G)$ is reflexive there is $\nabla p \in E^q(G)$ and a subsequence (again denoted by p_k) such that ∇p_k converges weakly to ∇p. for $\nabla v \in E^{q'}(G)$ we derive from (6.1) $\langle \nabla p, \nabla v\rangle = \lim_{k\to\infty} \langle \nabla p_k, \nabla v\rangle = 0$. By Lemma 5.18 a) we get $\nabla p = 0$. By the $H^{1,q}$-boundedness we see by means of Rellich's theorem that $p_k \to p$ strongly in $L^q(G)$ resp. $L^q(G_{3R})$. Then $\int_G p dx = 0$ ($\int_{G_{3R}} p dx = 0$) too and therefore $p = 0$, that is $p \to 0$ strongly in L^q on G resp. G_{3R}. Fix now any $i \in \{0,1,..,N\}$. If $i = 0$ let $\Omega := \mathbb{R}^n$, if $i = 1,\dots,N$ let $\Omega := B_{2R_i}(x_i)$. With a constant $C_i' > 0$ we have by (5.35), (5.49) and (5.50)

$$(6.2) \qquad C_i \|\nabla(\varphi_i p_k)\|_q \le \sup_{0 \ne v \in C_o^\infty(\Omega)} \frac{\langle \nabla(\varphi_i p_k), \nabla v\rangle}{\|\nabla v\|_{q', G\cap\Omega}} := d_k$$

For each $k \in \mathbb{N}$ there is $v_k \in C_o^\infty(\Omega)$, $\|\nabla v_k\|_{q'} = 1$ and

$$(6.3) \qquad 0 \le d_k - \langle \nabla(\varphi_i p_k), \nabla v_k\rangle \le \frac{1}{k}.$$

Let $\tilde{v}_k := v_k - c_k$, where $c_k := |\Omega|^{-1} \int_\Omega v_k dx$ if $i = 1,..,N$ and $c_k := |G_{3R}|^{-1} \int_{G_{3R}} v_k dx$ for $i = 0$. Then by the Poincaré-inequality we conclude $\|\tilde{v}_k\|_{H^{1,q'}(\Omega)} \le$ const. for $i = 1,\dots,N$ and $\|\tilde{v}_k\|_{H^{1,q'}(G_{3R})} \le$ const for $i = 0$. Again we select a subsequence $(\tilde{v}_k)$ and find $\nabla\tilde{v}_k \to \nabla v$ weakly in $E^{q'}(G)$ and $\tilde{v}_k \to v$ strongly in $L^q(\Omega)$ resp. $L^q(G_{3R})$. By (6.3)

$$\begin{aligned} d_k &\le \frac{1}{k} + \langle \nabla(\varphi_i p_k), \nabla\tilde{v}_k\rangle \\ &= \frac{1}{k} + \langle \nabla p_k, \nabla(\varphi_i \tilde{v}_k)\rangle + \langle p_k \nabla\varphi_i, \nabla\tilde{v}_k\rangle - \langle \nabla p_k, \tilde{v}_k \nabla\varphi_i\rangle \\ &\le \frac{1}{k} + \epsilon_k \|\nabla(\varphi_i \tilde{v}_k)\|_{q'} + |\langle p_k \nabla\varphi_i, \nabla\tilde{v}_k\rangle| + |\langle \nabla p_k, \tilde{v}_k \nabla\varphi_i\rangle| \end{aligned}$$

Since at the support of $\nabla\varphi_i$ we have $p_k \to 0$ strongly in L^q and $\nabla\tilde{v}_k \to \nabla v$ weakly we see $\langle p_k \nabla\varphi_i, \nabla\tilde{v}_k\rangle \to 0$. Analogously $\langle \nabla p_k, \tilde{v}_k \nabla\varphi\rangle \to 0$. Further $\|\nabla(\varphi_i \tilde{v}_k)\|_{q'} \le$ const and $\epsilon_k \to 0$. So we conclude by (6.2) $\|\nabla(\varphi_i p_k)\|_q \to 0$ ($k \to \infty$) and for $i = 0,1,..,N$. Since $\nabla p_k = \sum_{i=0}^{n} \nabla(\varphi_i p_k)$ we get $\|\nabla p_k\|_q = 1$ forming a contradiction. Part b): By part a) G has property $P_a^1(s)$ for $s = q$ and q', therefore by Lemma 5.1 G has property $P_b^1(s)$ for $s = q$ and q', that is b). ∎

<u>Proof of Theorem 4.2.</u> Part a): Like as in the proof of Theorem 4.1 we construct a covering $B_i = B_{R_i}(x_i), i = 0,1,..,N$ of G and a partition of unity

such that for i = 0,..,N e.g. (5.35) holds for $\varphi \in C_0^\infty(B_1)$ and the sup is taken over $v \in C_0^\infty(G \cap B_{2R_1}(x_1))$, analogously for (5.49). For (5.52) the sup is taken over $v \in C_0^\infty(G)$. If G is bounded, let $1 < q < \infty$. If G is an exterior domain and $n \geq 3$, let $1 < q < n$ and if $n = 2$, let $1 < q \leq 2$. Suppose again that (4.4) is not true. Then there is a sequence $(\nabla p_k) \subset E_o^q(G)$ such that $\|\nabla p_k\|_q = 1$ and

$$(6.4) \qquad \epsilon_k := \sup_{0 \neq \nabla v \in E_o^{q'}(G)} \frac{<\nabla p_k, \nabla v>}{\|\nabla v\|_{q'}} \to 0.$$

For G bounded, by (2.1)

$$(6.5) \qquad \|p_k\|_q \leq C(G)\ \|\nabla p_k\|_q$$ and for G an exterior domain by (2.9)

$$(6.6) \qquad \|p_k\|_{q,G_{3R}} \leq C(R)\ \|\nabla p_k\|_{q,G}.$$

Again by reflexivity we find a subsequence (again denoted p_k) such that $\nabla p_k \to \nabla p \in E_o^q(G)$ weakly. Since by (6.2) $<\nabla p, \nabla v> = 0$ for all $\nabla v \in E_o^{q'}(G)$, by Theorem 5.18 $\nabla p = 0$. By (6.5), (6.6) we conclude $p = 0$. Then by Rellich's theorem $p_k \to 0$ strongly in $L^q(G)$ resp. $L^q(G_{3R})$. Now we proceed like as in the proof of Theorem 4.1: Consider (6.2). Observe that if $x_1 \in \partial G$ then the sup has to be taken over $v \in C_0^\infty(G \cap B_{2R_1}(x_1))$ in (6.2). Again we find v_k with $\|\nabla v_k\|_{q'} = 1$ and (6.3). The use of the Poincaré inequality is replaced by (6.5) and (6.6) for v_k instead of p_k and q' instead of q. The remaining arguments are the same. Part b): If G is bounded, by part a) G has property $P_a^o(s)$ for s = q and q' if $1 < q < \infty$. If G is an exterior domain and $n \geq 3$ G has property $P_a^o(s)$ for s = q and q' if $\frac{n}{n-1} < q < n$. If n = 2 the exterior domain G has property $P_a^o(s)$ for s = q and q' if and only if q = n = 2. Via Lemma 5.1 part b) follows. ■

7. The exceptional spaces for the Dirichlet problem in exterior domains.

By Theorem 5.18 b) from $\nabla p \in E_o^q(G)$, $G \subset \mathbb{R}^n (n \geq 3)$ exterior domain, and $<\nabla p, \nabla \phi> = 0$ for $\nabla \phi \in E_o^{q'}(G)$ we can conclude $\nabla p = 0$ only if $1 < q < n$ and the functional representation (Theorem 4.2) holds only for $\frac{n}{n-1} < q < n$. In the case $G := \{x \in \mathbb{R}^n : |x| > 1\}$ the reason is clear by Remark 5.15. For an arbitrary exterior domain G with $\partial G \in C^1$ we see that the situation in Remark 5.15 is typical:

Theorem 7.1. Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain with boundary $\partial G \in C^1$, $G = \mathbb{R}^n \setminus \bar{K}$, $\phi \neq K \subset\subset \mathbb{R}^n$. Without loss of generality assume $0 \in K$. Then there exists $h \in C^{\infty}(G) \cap C^{0}(\bar{G})$ such that $\Delta h = 0$ in G, $h|_{\partial G} = 0$ and $\nabla h \in E_o^q(G)$ for all q with $n \le q < \infty$ if $n \ge 3$ and $q > 2$ if $n = 2$. Let $0 < r < 1$ such that $B_r \subset\subset G$. Then further there is a harmonic function u in B_r with $u(0) = 0$ and constants $a,b \in \mathbb{R}$, $a \ne 0$, $b \ne 0$ such that for $|x| \ge \frac{1}{r}$

$$(7.1) \qquad h(x) = \begin{cases} a + \frac{b}{|x|^{n-2}} + \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}) & \text{for } n \ge 3 \\ a + b \ln|x| + u(\frac{x}{|x|^2}) & \text{for } n = 2 \end{cases}$$

Conversely if $\nabla\tilde{h} \in E_o^q(G)$, where q is subordinate to the restrictions above, and $\langle\nabla\tilde{h},\nabla\phi\rangle = 0$ for all $\phi \in C_0^{\infty}(G)$, then there exists $\alpha \in \mathbb{R}$ sucht that $\tilde{h} = \alpha h$.■

The proof (see [9]) is not difficult but somehow lengthy. The main tools are the mean value properties (3.1), (3.2), Weyl's Lemma (= Theorem 3.1), the Kelvin-transform etc.

This theorem has a lot of consequences. To avoid the cumbersome distinction $n \ge 3$ and $n = 2$ we restrict ourselfes in the following considerations to the case $n \ge 3$.

Let $H(G) := \{\alpha\nabla h : \alpha \in \mathbb{R}\}$. For $q \ge n$ let $\nabla h_q := \dfrac{\nabla h}{\|\nabla h\|_q} \in E_o^q(G)$.

Then $H(G) := \{\alpha\nabla h_q : \alpha \in \mathbb{R}\}$. Define $F_q^* : H(G) \to \mathbb{R}$ by

(7.2) $\quad F_q^*(\nabla g) = \alpha \quad$ for $g = \alpha h_q$.

Then $\|F_q^*\| = \sup\limits_{0 \ne g \in H} \dfrac{F_o^*(\nabla g)}{\|\nabla g\|_q} = 1$. Extend F_q^* norm-preserving by the Hahn-Banach theorem to a continuous linear functional defined on the whole space $E_o^q(G)$ and denote if again by F_q^*. Define

(7.3) $\quad P_q(\nabla p) := F_q^*(\nabla p)\nabla h_q \quad$ for $\nabla p \in E_o^q(G)$.

Then $P_q : E_o^q(G) \to H(G)$ is a projection, that is $P_q^2 = P_q$, with the additional property $\|P_q\| = 1$.

Define

(7.4) $\quad P_{c,q} := I - P_q$

and

(7.5) $\quad E_{o,c}^q(G) := P_{c,q}(E_o^q(G))$.

Then we have in the sense of a direct sum

(7.6) $\qquad E_o^q(G) = E_{o,c}^q(G) \oplus H(G)$

and there is a constant $K > 0$ such that

(7.7) $$\|\nabla p\|_q \geq K(\|P_{c,q}(\nabla p)\|_q + \|P_q(\nabla p)\|_q)$$

for all $\nabla p \in E_o^q(G)$. Clearly $E_{o,c}^q(G)$ is topologically equivalent to the quotient space $E_o^q(G)/H(G)$.

Concerning the estimates from Theorem 4.2, we have the following extension of Theorem 4.2:

Theorem 7.2. For G assume the same as in Theorem 7.1 and let $n \geq 3$. Then for $1 < q < \infty$ there exists a constant $C = C(G,q) > 0$ such that

a) for $1 < q \leq \frac{n}{n-1}$ (then $q' \geq n$) and all $\nabla p \in E_o^q(G)$

(7.8) $$\|\nabla p\|_q \leq C \sup_{0 \neq \nabla\phi \in E_{o,c}^{q'}(G)} \frac{|\langle\nabla p, \nabla\phi\rangle|}{\|\nabla\phi\|_{q'}} .$$

b) for $\frac{n}{n-1} < q < n$ estimate (4.4) holds

c) for $q \geq n$ (then $q' \leq \frac{n}{n-1}$) and all $\nabla p \in E_{o,c}^q(G)$

(7.9) $$\|\nabla p\|_q \leq C \sup_{0 \neq \nabla\phi \in E^{q'}(G)} \frac{|\langle\nabla p, \nabla\phi\rangle|}{\|\nabla\phi\|_{q'}} . \quad \blacksquare$$

Observe that (7.8) is sharper then (4.4), because the variational class is smaller. (7.9) is an extension of (4.4) to the case $q > n$ but for the narrower class $E_{o,c}^q(G) \subsetneq E_o^q(G)$.

Concerning the functional representation we get:

Theorem 7.3. For $G \subset \mathbb{R}^n$ assume the same as in Theorem 7.1 and let $n \geq 3$. Then for $1 < q < \infty$ with the constants $C = C(G,q) > 0$ by Theorem 7.2 holds:

a) If $1 < q \leq \frac{n}{n-1}$ and given $F^* \in (E_{o,c}^{q'}(G))^*$,

$$\|F^*\|_{(E_{o,c}^{q'})^*} := \sup_{0 \neq \nabla\phi \in E_{o,c}^{q'}} \frac{|F^*(\nabla\phi)|}{\|\nabla\phi\|_q}, \text{ then}$$

there is a unique $\nabla p \in E_o^q(G)$ such that

(7.10) $$F^*(\nabla\phi) = \langle\nabla p, \nabla\phi\rangle \text{ for all } \nabla\phi \in E_{o,c}^{q'}(G)$$

Further

(7.11) $$\|\nabla p\|_q \leq C\|F^*\|_{(E_{o,c}^{q'})^*} \leq C\|\nabla p\|_q$$

b) If $\frac{n}{n-1} < q < n$ then Theorem 4.2 b) holds.

c) If $q \ge n$ and given $F^* \in (E_o^{q'}(G))^*$, then there is a unique $\nabla p \in E_{o,c}^q(G)$ such that

(7.12) $\quad F^*(\nabla\phi) = \langle\nabla p,\nabla\phi\rangle$ for all $\nabla\phi \in E_o^{q'}(G)$

Further $\|\nabla p\|_q \le C\|F^*\|_{(E_o^{q'})^*} \le C\|\nabla p\|_q$.

For case a) we have in addition

Theorem 7.4. Let the same assumptions as in Theorem 7.1 hold and let $1 < q \le \frac{n}{n-1}$ $(n \ge 3)$. Given $F^* \in (E_o^{q'}(G))^*$, then there exists $\nabla p \in E_o^q(G)$ with $F^*(\nabla\phi) = \langle\nabla p,\nabla\phi\rangle$ for all $\nabla\phi \in E_o^{q'}(G)$ if and only if $F^*(\nabla h) = 0$ (h by Theorem 7.1). Then in addition $\|\nabla p\|_q \le C\|F^*\|_{(E_o^q(G))^*} \le C\|\nabla p\|_q$.

8. Applications, concluding remarks. We are now able to prove existence of weak solutions for the Neumann- and Dirichlet problem in bounded as well as in exterior domains. E.g. let G be an exterior domain and let $f \in C_o^o(G)$ be given. Let $F(\phi) := \langle f,\phi\rangle$ for $\phi \in C_o^\infty(G)$. Suppose $\mathrm{supp}(f) \subset B_R$ for some $R > 0$. By (2.9) we get $|F(\phi)| \le \|f\|_{q,G_R}\|\phi\|_{q',G_R} \le C(R)\|f\|_q \|\nabla\phi\|_{q'}$.
Then for $\frac{n}{n-1} < q < \infty$ there exists $\nabla p \in E_o^q(G)$ with

(7.13) $\quad \langle\nabla p,\nabla\phi\rangle = F(\phi) = \langle f,\phi\rangle$ for $\phi \in E_o^{q'}(G)$.

If in addition $\int_G fh\,dx = 0$, then for $1 < q \le \frac{n}{n-1}$ there is again $\nabla p \in E_o^q(G)$ with (7.13). Clearly p is a weak solution of the Dirichlet problem "$-\Delta p = f$ in G and $p|_{\partial G} = 0$". It is not difficult to see that for $|x| \ge R_o$, $R_o > R$ sufficiently big, a representation like as in (7.1) holds (since $\Delta p = 0$ for $|x| > R$).
In case $1 < q \le \frac{n}{n-1}$ follows $a = b = 0$. That means $|p(x)| \le \frac{c}{|x|^{n-1}}$.
In case $\frac{n}{n-1} < q < n$ follows $a = 0$ and $|p(x)| \le \frac{c}{|x|^{n-2}}$. Analogous results hold for the Neumann problem.
Most important applications are in connection with the Stokes problem in bounded as well as in exterior domains. With ideas similar to that one used here, Galdi and Simader [3] proved existence, uniqueness and L^q-estimates for the Stokes problem in exterior domains $G \subset \mathbb{R}^3$.
A most convincing application of Theorem 4.1 is given in Simader and Sohr [8] in their proof of the Helmholtz decomposition. Moreover, in turn the Helmholtz

decomposition is equivalent (see [8]) to Theorem 4.1. The results in [8] extend those given by Fujiwara and Morimoto [2] to unbounded domains too.
It is well known (see e.g. [5], p. 337 and p. 341 or [6], p. 99 and p. 103) that for bounded domains G with $\partial G \in C^1$ and $1 < s < \infty$ there is a well defined continuous linear trace operator $V_s : H^{1,s}(G) \to W^{1-1/s,s}(\partial G)$ such that for $p \in C^1(\bar{G})$ we have $V_s p = p|_{\partial G}$. If $W^{-1/s',s'}(\partial G) := (W^{1-1/s,s}(\partial G))^*$ equipped with the "dual space norm", then in [8] is shown that for the subspace $F^{s'}(G) := \{\nabla p \in E^{s'}(G) : \Delta p \in L^{s'}(G)\}$ of $E^{s'}(G)$ equipped with norm $\|\nabla p\|_{F^{s'}} := (\|\nabla p\|_{s'}^{s'} + \|\Delta p\|_{s'}^{s'})^{1/s'}$ there is a continuous linear trace operator $S_{s'} : F^{s'}(G) \to W^{-1/s',s'}(\partial G)$ such that for $p \in C^\infty(\bar{G})$ we have $S_{s'}(\nabla p) = \partial_N p|_{\partial G}$, where $\partial_N p|_{\partial G}(x) = \sum_{i=1}^{n} N_i(x)\partial_i p(x)|_{\partial G}$ and N(x) denotes the outward unit normal vector in x_0 at ∂G.

Via difference quotient methods, using (4.1) and (4.4) respectively, like as in the case q = 2 higher differentiability properties of weak solutions of e.g. equation (7.13) can be proved (compare e.g. [7].
The case of arbitrary elliptic operators of second order for G bounded, and under additional asymptotic assumptions for the coefficients for exterior domains, is reduced to Δ by elementary coordinate transform and standard localization procedures.

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