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STABILITY OF STATIONARY SOLUTIONS OF PARABOLIC VARIATIONAL INEQUALITIES

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The stability of solutions of parabolic variational inequalities was studied by many authors (see e.g. [1,4,5,6,7,15,16]). In the present paper we study the Lyapunov stability of stationary solutions of semilinear parabolic variational inequalities in Hilbert spaces. In comparison to (some of) the papers cited above our assumptions are rather special, nevertheless we are able to prove exponential stability of a given stationary solution and also the instability results seem to be new. In our abstract setting we use the existence and regularity results of Brézis [2,3]; similar results should be possible to obtain also under another assumptions guaranteeing the necessary existence and regularity properties of the solution of the parabolic inequality.

Throughout this paper we assume that V or H is a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$ and the norm $\|\cdot\|$ or $|\cdot|$, respectively. We assume also that $V \subset H$, where the imbedding is dense and compact, and we denote by $(\cdot,\cdot)$ also the duality between V and its dual space V' induced by the imbedding $V \subset H = H' \subset V'$. Finally, let K be a closed convex set in V and $F: V \to H$ a (locally) Lipschitz continuous map. We define the operator $\mathcal{A}: V \to V'$ by $(\mathcal{A}u, v) := \langle u, v\rangle$ for any $u, v \in V$ and denote by A the H-realization of $\mathcal{A}$, i.e. the domain of A is $\mathcal{D}(A) := \{u \in V \mid \mathcal{A}u \in H\}$ and $Au := \mathcal{A}u$ for $u \in \mathcal{D}(A)$. Then A is a positive self-adjoint operator in H with a compact resolvent and the operator $\tilde{F} := \mathcal{A}^{-1}F : V \to V$ is compact. By X_α we denote the domains of the fractional powers A^α (particularly, $X_{1/2} = V$). We shall suppose that $u_o \in K$ is a stationary solution of the inequality

$$u(t) \in K \qquad (\frac{du}{dt}, v-u) + \langle u - \tilde{F}(u), v-u\rangle \geq 0 \qquad \forall v \in K \tag{1}$$
$$u(0) = u_o$$

i.e.

$$\langle u_o - \tilde{F}(u_o), v - u_o\rangle \geq 0 \qquad \forall v \in K$$

The inequality (1) can be written also in the form

$$u(t) \in K \qquad (\frac{du}{dt} + \mathcal{A}u - F(u), v-u) \geq 0 \qquad \forall v \in K$$

Analogously as for equations one can try to "linearize" this inequality and to study the stability of the stationary solution 0 for the "linearized" inequality

$$u(t) \in K_o : \qquad (\frac{du}{dt} + Au - Lu + F_o, v - u) \geq 0 \qquad \forall v \in K_o, \tag{1}_L$$

where K_o is the closure of the set $\bigcup_{\alpha>0} \alpha(K - u_o)$, $L : V \to H$ is a continuous linear operator such that $\|F(u) - F(u_o) - L(u - u_o)\|_{V'} = o(\|u - u_o\|)$ for $u \to u_o$, $F_o := Au_o - F(u_o)$. This approach does not yield always satisfactory results , however the following Theorems 1, 1B and 2B are true.

Theorem 1 ([12]). *Let* $\lambda_1 := \inf\{(Au - Lu, u) \,|\, u \in K_o, |u| = 1, (F_o, u) = 0\} > 0$. *Then* u_o *is an exponentially stable solution of (1) in* V.

If $F_o = 0$ and L is symmetric, then the number λ_1 defined in Theorem 1 is the least eigenvalue of the inequality

$$u \in K_o \qquad (Au - Lu, v - u) \geq \lambda(u, v - u) \qquad \forall v \in K_o, \tag{2}$$

i.e. there exists a nontrivial solution u_1 of (2). The function $u(t) := e^{-\lambda_1 t} u_1$ is a solution of the inequality $(1)_L$, hence $\lambda_1 < 0$ implies an instability result for $(1)_L$. However, if L is not symmetric then the condition $\lambda_1 \geq 0$ is, in general, not necessary for the (exponential) stability of u_o. A possible generalization of Theorem 1 for nonsymmetric L is the following

Theorem 1B ([13]). *Let* $B : H \to H$ *be a strictly positive, self-adjoint, continuous linear operator such that* $B(V) \subset V$, $B^2(K - u_o) \subset K - u_o$, $(id - B^2)(\partial(K - u_o)) \subset K - u_o$, $\langle u, B^2 u\rangle \geq \alpha\|u\|^2 - C|u|^2$, $(F_o, B^2 u) \geq c(F_o, u)$ *for some positive constants* α, c, C *and any* $u \in K$. *Then the condition* $\lambda_1^B := \inf\{(Au - Lu, B^2 u) \,|\, u \in K_o, |Bu| = 1, (F_o, u) = 0\} > 0$ *is sufficient for the exponential stability of the stationary solution* u_o *of (1) in* V.

Similarly, the instability result [12, Theorem 2] can be generalized to the following

Theorem 2B. *Let, in addition to our general assumptions,* K *be a cone with its vertex at* u_o, $F_o = 0$, $|F(u) - F(u_o) - L(u - u_o)| \leq C\|u - u_o\|^2$ *for* u *in a neighbourhood of* u_o *and let* B *fulfill the assumptions of Theorem 1B. Let* $u_1 \in K_o$, $|u_1| = 1$, *be an eigenvector of the inequality (2) with an eigenvalue* $\lambda_1 < 0$ *and let* $(Ay - Ly, B^2 y) \geq \lambda_1 |By|^2$ *for any* $y \in K_o - u_1$. *Then the stationary solution* u_o *of (1) is unstable in both of the topologies of* V *and* H.

The assumptions of Theorem 2B are rather restrictive, however in general one can not expect that the instability result for $(1)_L$ would imply an instability result for (1) (see [12, Example 1]). Thus we are led to the study of the nonlinear stationary inequality

$$u \in K \qquad \langle u - \tilde{F}(u), v - u\rangle \geq 0 \qquad \forall v \in K \tag{3}$$

which is obviously equivalent to the equation $u - P_K\tilde{F}(u) = 0$, where P_K is the projection in V onto K. We need some additional regularity hypothesis on F and K; we suppose that there exists a neighbourhood $\mathcal{U}$ of u_o in V, $\alpha > 0$ and a bounded (nonlinear) operator $G : \mathcal{U} \to X_{(1+\alpha)/2}$ such that $u + \lambda A^\alpha(u + G(u)) \in K \cap \mathcal{U}$ implies $u \in K$ for small $\lambda > 0$ (cf. [14]) and $\tilde{F} : \mathcal{U} \subset X_\beta \to V$ is Lipschitz continuous for some $\beta < 1/2$. By $d(u_o)$ we denote the Leray-Schauder degree $\deg(\mathrm{id} - P_K\tilde{F}, 0, \mathcal{U})$. Supposing $u \neq P_K\tilde{F}(u)$ for any $u \in \mathcal{U} \setminus \{u_o\}$ we have

Theorem 3 ([14]). *Let $d(u_o) \neq 1$. Then u_o is not asymptotically stable (neither in the topology of V nor in the topology of H). If, moreover, $\tilde{F}$ has a potential, then u_o is not stable.*

The assumption $d(u_o) \neq 1$ is, of course, not necessary for the instability of u_o. However, if $\tilde{F}$ has a potential $\mathcal{F}$ and the functional $\Phi(u) := \frac{1}{2}\|u\|^2 - \mathcal{F}(u)$ attains its local minimum at u_o with respect to K, then $d(u_o) = 1$ (if this degree exists) and u_o is an asymptotically stable solution of (1). The degree $d(u_o)$ was studied by the author in [8,9,10,11] and an application of Theorem 3 is given in [14].

Applications of Theorems 1 and 1B can be found in [12,13]; here we apply Theorem 2B to the system of reaction-diffusion equations

$$\begin{aligned}
&\frac{\partial u^1}{\partial t} = d\Delta u^1 - f_1(u^1, u^2) && \\
& && \text{in } (0,T)\times\Omega \\
&\frac{\partial u^2}{\partial t} = \Delta u^2 - f_2(u^1, u^2) && \\
&\frac{\partial u^1}{\partial n} = \frac{\partial u^2}{\partial n} = 0 && \text{on } (0,T)\times\partial\Omega \\
&u^1(0,\cdot) = u^1_o, \quad u^2(0,\cdot) = u^2_o &&
\end{aligned} \tag{4}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^N$, $d > 0$ and $f_i : \mathbb{R}^2 \to \mathbb{R}$ are C^1 functions such that $\left|\frac{\partial f_i}{\partial u_j}(u)\right| \leq C(1 + |u|^\gamma)$, $\gamma \leq \frac{2}{N-2}$ if $N > 2$. Let $u_o = (u^1_o, u^2_o) \in V := W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ be a stationary, spatially homogenous solution of (4) and denote $b_{ij} := \frac{\partial f_i}{\partial u_j}(u_o)$. Assuming $b_{11} > 0$, $b_{21} > 0$, $b_{11} + b_{22} < 0$, $b_{11}b_{22} > b_{12}b_{21}$ one can easily check (cf. [6,10]) that the condition $d > d^o := \max_{\lambda_i > 0} \frac{1}{\lambda_i}\left(b_{11} + \frac{b_{12}b_{21}}{\lambda_i - b_{22}}\right)$, where λ_i is the i-th eigenvalue of $-\Delta$ with Neumann boundary conditions, is sufficient for the exponential stability of u_o and the condition $d < d^o$ is sufficient for the instability of u_o.

Now suppose that we have an unilateral constraint for the solution u of (4), which leads to an inequality on a convex cone $K = (K_1 + u^1_o) \times (K_2 + u^2_o)$ with its vertex at u_o and suppose $K_2 \subset K^+$ and $-\mathbb{1} \in K_1$, where K^+ is the positive cone in $W^{1,2}(\Omega)$ and $\mathbb{1}$ is the function $\mathbb{1}(x) := 1$. Then one can easily check that $u_1 := (-\mathbb{1}, 0)$ is the eigenvector of (2) with $\lambda_1 = -b_{11} < 0$ and that the assumption $(Ay - Ly, B^2y) \geq \lambda_1 |By|^2$ is fulfilled for any $y \in V$ if we choose $B := \begin{pmatrix} \sqrt{b}I & 0 \\ 0 & I \end{pmatrix}$, where $b = -b_{21}/b_{12}$ and I is the identity in $L^2(\Omega)$. Hence the solution u_o of the corresponding inequality (1) is unstable for any $d > 0$.

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