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SOLUTIONS TO A DIFFERENTIAL INCLUSION OF ORDER n

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We will consider the differential inclusion

(E) $$L_n x(t) \in F(t,x(\varphi(t))),\ n>1$$

where $L_n x(t)$ is the n-th quasiderivative of $x(t)$ with respect to the continuous functions $a_i(t):J=[t_o,\infty)\to(0,\infty), i=0,1,\dots,n$, $\int_{t_o}^{\infty} a_i^{-1}(t)dt=\infty$, $i=0,1,\dots,n-1$, $L_o x(t)=a_o(t)x(t)$, $L_i x(t)=a_i(t)(L_{i-1}x(t))'$, $i=1,2,\dots,n$; $F(t,x):J\times R\to$ {nonempty convex compact subsets of R}, $R=(-\infty,\infty)$; $\varphi:J\to R$ a continuous function, $\lim \varphi(t) = \infty$ as $t\to\infty$.

Under a solution $x(t)\in(E)$ we will understand a proper solution existing on some ray $[T_x,\infty)$.

__Notations.__ $F(t,x)x>0$ (<0) means : $yx>0$ (<0) for each $y \in F(t,x)$; if $h:J\times R\to R$, then $F(t,x)\geq (\leq)\ h(t,x)$ means : $y\geq (\leq)\ h(t,x)$ for each $y \in F(t,x)$; if $B \subset R$, then $|B| = \sup\{|x|:x \in B\}$, $\|B\| = \inf\{|x|:x \in B\}$.

For $t_o\leq c \leq t$

$$P_o(t,c) = 1,\quad P_i(t,c) = \int_c^t a_1^{-1}(s_1) \int_c^{s_1} a_2^{-1}(s_2)\dots\int_c^{s_{i-1}} a_i^{-1}(s_i)ds_i\dots ds_1,$$

$$Q_n(t,s)=1,\quad Q_i(t,c) =\int_c^t a_{n-1}^{-1}(s_{n-1}) \int_c^{s_{n-1}} a_{n-2}^{-1}(s_{n-2})\dots\int_c^{s_{i+1}} a_i^{-1}(s_i)ds_i\dots ds_{n-1},$$

$i = 1,2,\dots,n-1$.

__The basic assumptions.__ 1. $F(t,x)$ is upper semicontinuous on $J\times R$; 2. $F(t,0)=\{0\}$; 3. $F(t,x)<0$ for each $(t,x) \in J\times R$, $x \neq 0$ or 4. $F(t,x)>0$ for each $(t,x) \in J\times R$, $x \neq 0$.

The notions of oscillatory and nonoscillatory solutions will be used in the usual sense.

Let $x(t)$ be a nonoscillatory solution of (E) existing on $[T_x,\infty)$. Then from the assumption $\lim_{t\to\infty} \varphi(t) = \infty$ and from the assumptions 1.- 4. it follows the existence of such $t_1\geq T_x$ that $L_i x(t) \neq 0$, $i=0,1,\dots,n$, on $[t_1,\infty)$, $x(t)L_n x(t) < 0$ (>0) if 3. (if 4.) is satisfied. Therefore, all $L_i x(t)$, $i=0,1,\dots,n-1$, are monotone and $\lim_{t\to\infty} L_i x(t)$ exist in the extended sense. Only two cases are possible: a) $\lim_{t\to\infty} |L_i x(t)| = \infty$; b) there exists $k \in \{0,1,\dots,n-1\}$ such that $\lim_{t\to\infty} L_k x(t)$ is finite, $\lim_{t\to\infty} L_i x(t) = \infty \operatorname{sgn} x(t)$, $i=0,1,\dots,k-1$, $\lim_{t\to\infty} L_i x(t) =0$, $i=k+1,\dots,n-1$.

Thus, the set of all nonoscillatory solutions of (E) can be divided into disjoint classes defined in the following way: A nonoscillatory solution $x(t)$ of (E) belongs to class V_n if the case a) occurs, and it belongs to the class V_k, $k \in \{0,1,\dots,n-1\}$, if the case b) occurs.

Lemma 1. ([1], Lemma 4 and Lemma 6, [2], Lemma 3). Let $x(t) \in V_k$, $k \in \{0,1,\dots,n-1\}$. Then there exists $T_1 > t_0$ such that $\operatorname{sgn} x(t) = \operatorname{sgn} L_k x(t)$ for $t \ge T_1$. If $x(t)L_n x(t) < 0$ on $[T_1,\infty)$, then for $n+k$ even (odd) $|L_k x(t)|$ increases (decreases) on $[T_1,\infty)$. If $x(t)L_n x(t) > 0$, then for $n+k$ even (odd) $|L_k x(t)|$ decreases (increases) on $[T_1,\infty)$. If $\lim_{t\to\infty} L_k x(t) = c_k \ne 0$, then there exist two constants $0 < \alpha_k \le |c_k| \le \beta_k$ and $T_k \ge T_1$ such that $\alpha_k P_k(t,c) \le a_0(t)|x(t)| \le \beta_k P_k(t,c)$, $t \ge T_k$.

Our aims are : to state the conditions which guarantee that $\lim_{t\to\infty} L_k x(t) = 0$ for each $x(t) \in V_k$, $k \in \{0,1,\dots,n-1\}$ and also to state the conditions which guarantee that the class V_k, $k \in \{0,1,\dots,n-1\}$, is empty.

These problems for the case that instead of the inclusion (E) we have an equation were discussed in [1],[2],[3] and for (E) in [4],[5].

Theorem 1. Let the conditions 1.- 4. be satisfied. Let $G(t,u) : J\times[0,\infty) \to [0,\infty)$ be continuous and for each fixed $t\in J$ nondecreasing in u such that

(1) $$G(t,|x|) \le \|F(t,x)\| , \quad x \in R .$$

Let $k \in \{0,1,\dots,n-1\}$ and let

(2) $$\int_t^\infty a_n^{-1}(s)\, Q_{k+1}(s,t)\, G(s,\alpha a_0^{-1}(\varphi(s))P_k(\varphi(s),c))\, ds = \infty$$

for all $t \ge T_k$ such that $\varphi(s) > c$ for $s \ge T_k$, $c \ge t_0$ and each $\alpha > 0$ or

(3) $$\limsup_{t\to\infty} \int_t^\infty a_n^{-1}(s)\, Q_{k+1}(s,t)\, G(s,\alpha a_0^{-1}(\varphi(s))P_k(\varphi(s),c))\, ds > 0$$

for each $\alpha > 0$. Then for each $x(t) \in V_k$ we have $\lim_{t\to\infty} L_k x(t) = 0$.

Sketch of the proof. Using the properties of $x(t) \in V_k$, Lemma 1 and (1) we get

$$0 \le \int_t^\infty a_n^{-1}\, Q_{k+1}(s,t)\, G(s,\alpha_k a_0^{-1}(\varphi(s))P_k(\varphi(s),c))\, ds \le |L_k x(t) - c|$$

which leads to a contradiction.

Theorem 2. Let all assumptions of Theorem 1 be satisfied. Then, if 3. is satisfied, the sets V_k for $n+k$ even are empty. If 4. is satisfied, then the sets V_k for $n+k$ odd are empty.

Denote $\gamma(t) = \sup\{s \ge t_0 : \varphi(s) \le t\}$, $m(t) = \max\{\gamma(t),t\}$, $t \ge t_0$.

Theorem 3. Let the assumptions 1.- 4. be satisfied and suppose that :

(H_1) To each measurable function $z(t) : J \to R$ there exists a measurable selector $v(t) : J \to R$ such that $v(t) \in F(t,z(t))$ a.e. on J.

(H_2) There exists a continuous function $G_1(t,u) : J\times[0,\infty) \to [0,\infty)$ such that : a) $G_1(t,u)$ is nondecreasing in u for each fixed $t\in J$;

b) $|F(t,z)| \le G_1(t,z)$ for each $(t,z) \in J\times R$;

c) $\int_t^\infty a_n^{-1}(s)\, Q_{k+1}(s,t_o)\, G_1(s,\alpha a_o^{-1}(\varphi(s))P_k(\varphi(s),t_o))\, ds < \infty$ for some $\alpha > 0$ and each $t \in J$.

Then (E) has a solution $x(t) \in V_k$ defined on some interval $[T_o,\infty)$, $T_o \ge t_o$ such that $\lim_{t\to\infty} L_k x(t) = c_k \ne 0$.

Sketch of the proof. Let n-k be even, let 3. be satisfied and let $c_k > 0$. To t_o we can find $T_o \ge \gamma(t_o)$ such that $\varphi(t) > t_o$ for each $t>T_o$. We seek the desired solution in the set

$$Y = \{ u(t) \in C[t_o,\infty) : \alpha_k P_k(t,t_o) \le a_o(t)u(t) \le \beta_k P_k(t,t_o) \ , \ \alpha_k<c_k<\beta_k\}$$

as a fixed point of the operator A : $u(t) \in Y$

$$Au(t) = a_o^{-1}(t) \{ c_k P_k(t,t_o) + \int_{T_o}^{t} a_1^{-1}(s_1) \int_{T_o}^{s_1} a_2^{-1}(s_2) \dots \int_{T_o}^{s_{k-1}} a_k^{-1}(s_k)$$

$$\int_{s_k}^{\infty} a_n^{-1}(s)\, Q_{k+1}(s,s_k)\, v(\varphi(s))\, ds\, ds_k \dots ds_1,\ v(\varphi(t)) \in M(u(\varphi(t))) \},\ t\ge T_o$$

$$Au(t) = a_o^{-1}(t)\, c_k P_k(t,t_o),\ t_o \le t \le T_o,$$

where $M(u(\varphi(t)))$ is the set of all measurable selectors from $F(t,u(\varphi(t)))$.

Assume now that all assumptions of Theorem 1 are satisfied. Let $x(t) \in V_k$, $k \in \{1,2,\dots,n-1\}$. Then we have

$$(4) \qquad 0 \le \int_t^\infty a_n^{-1}(s)\, G(s,|x(\varphi(s))|)\, ds \le |L_{n-1}x(t)| < \infty .$$

Our following considerations are based on this fact. Succesive integrations of (4), by respecting the fact that $\lim_{t\to\infty} L_i x(t) = 0$, $i=k+1,\dots,n-1$, and monotonicity of G and $L_o x(t)$, the properties of $\gamma(t)$ and $m(t)$ lead to the inequality

$$(5) \qquad 0 \le R_k(v,u) \int_{m(v)}^\infty a_n^{-1}(s)\, G(s,a_o^{-1}(\varphi(s))|L_o x(v)|)\, ds \le |L_o x(v)|$$

for $(t_o \le)\ u < v$, where

$$R_k(v,u) = \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \dots \int_u^{t_{k-1}} a_k^{-1}(t)\, Q_{k+1}(t_{k-1},t)\, dt\, dt_{k-1} \dots dt_1 .$$

Let

$$p(v) = \int_{m(v)}^\infty a_n^{-1}(s)\, G(s,a_o^{-1}(\varphi(s))|L_o x(v)|)\, ds .$$

Then respecting once more the monotonicity of G we get

$$(6) \quad 0 \le \int_{m(v)}^\infty a_n^{-1}(s)\, G(s,a_o^{-1}(\varphi(s))R_k(v,u)p(v))\, ds \le p(v) .$$

On the basis of (5) and (6) we are able to prove the following theorems ([4]).

Theorem 4. Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for each fixed $t \ge t_o$

(7) $\qquad z^{-1}G(t,z)$ is nondecreasing in z , $z > 0$

and for $k \in \{1,2,\dots,n-1\}$

$$(8) \qquad \limsup_{v\to\infty} R_k(v,u) \int_{m(v)}^{\infty} a_n^{-1}(s)\, c^{-1} G(s, a_o^{-1}(\varphi(s))c)\, ds > 1$$

for some $c > 0$. Then the set V_k is empty.

Theorem 5. Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for each fixed $t \geq t_o$

$$(9) \qquad z^{-1}G(t,z) \text{ is nonincreasing in } z \text{ , } z > 0$$

and for $k \in \{1,2,\dots,n-1\}$

$$(10) \qquad \limsup_{v\to\infty} \int_{m(v)}^{\infty} a_n^{-1}(s)\, c^{-1} G(s, R_k(v,u) a_o^{-1}(\varphi(s))c)\, ds > 1$$

for some $c > 0$. Then the set V_k is empty.

From Theorems 1.,2.,4.,5. we get the final theorem.

Theorem 6. Let all assumptions of Theorem 1 be satisfied.

a) If the assumptions 1.,2.,3. hold and if (7) and (8) or (9) and (10) hold for $k = 1,2,\dots,n-1$, then for n even all solutions of (E) are oscillatory and for n odd each solution x(t) of (E) is either oscillatory or $\lim_{t\to\infty} L_i x(t) = 0$, $i = 0,1,\dots,n-1$.

b) If the assumptions 1.,2.,4. hold and if (7) and (8) or (9) and (10) hold for $k = 1,2,\dots,n-1$, then for n even each solution x(t) of (E) is either oscillatory or $\lim_{t\to\infty} L_i x(t) = 0$, $i = 0,1,\dots,n-1$ or it belongs to the class V_n and for n odd each solution x(t) of (E) is oscillatory or belongs to the class V_n.

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