Benoy Kumar Lahiri; Dibyendu Banerjee
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Mathematica Slovaca, Vol. 57 (2007), No. 3, [297]--299

Persistent URL: http://dml.cz/dmlcz/136955

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0023-6
Math. Slovaca **57** (2007), No. 3, 297–299

A NOTE ON EXPONENT OF CONVERGENCE OF ZEROS OF ENTIRE FUNCTIONS

B. K. Lahiri* — Dibyendu Banerjee**

(Communicated by Pavel Kostyrko)

ABSTRACT. If $f(z)$ is an entire function with $\rho_1 > 0$ as its exponent of convergence of zeros and if $0 \leq \alpha < \rho_1$, then we prove the existence of entire functions each having α as its exponent of convergence of zeros.

If $f(z)$ is an entire function having infinite number of zeros $z_1, z_2, \dots$ with $0 < |z_1| \leq |z_2| \leq \dots$ and $r_n = |z_n| \to \infty$ as $n \to \infty$, then it is well-known that the exponent of convergence of zeros of $f(z)$, ρ_1 is given by the formula

$$\rho_1 = \limsup_{n\to\infty} \frac{\log n}{\log r_n}\,.$$

If $\rho_1 > 0$ and $0 \leq \alpha < \rho_1$, we enquire in this note if there exists an entire function whose exponent of convergence of zeros is precisely α. We prove in the following theorem that the answer to our enquiry is affirmative. To prove the theorem we need a lemma from [1] which is vital in the sense that as soon as the lemma is known, the proof of the theorem is not difficult. However, to author's information, this problem has not been considered so far and so this note.

Theorem. *Let $f(z)$ be an entire function, with zeros $z_1, z_2, \dots$ where $0 < |z_1| \leq |z_2| \leq \dots$ and $|z_n| = r_n \to \infty$ as $n \to \infty$. Let $\rho_1 > 0$ be the exponent of convergence of the zeros of $f(z)$. If $0 \leq \alpha < \rho_1$, then there exists a continuum number of entire functions each having α as the exponent of convergence of its zeros.*

2000 Mathematics Subject Classification: Primary 30D20.
Keywords: entire functions, exponent of convergence of zeros.

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To prove the theorem we shal require the following lemma.

Lemma. ([1]) *Let $\{g_n\}$ be a non-decreasing sequence of real numbers with $g_n \to +\infty$ as $n \to \infty$. Let*

$$b - \limsup_{n \to} \frac{\log n}{g_n} > 0 \qquad (b \textit{ may be } +\infty)$$

and let $0 < t < b$. Then there exists a subsequence $\{g_{j_n}\}$ of $\{g_n\}$ such that

$$\limsup_{n \to \infty} \frac{\log n}{g_{j_n}} \quad t.$$

P r o o f o f T h e o r e m. We first prove that there exists an entire function having the desired property. Suppose that $\alpha \quad 0$. Since $r_n \to \infty$ as $n \to \infty$, there exists a sequence $k_1 < k_2 < \ldots$ of positive integers such that

$$r_{k_n} > n^n, \qquad n - 1, 2, \ldots$$

and then $\limsup\limits_{n \to \infty} \frac{\log n}{\log r_{k_n}} \quad 0$.

Let w_{k_n} be a point in the complex plane (w_{k_n} may be z_{k_n} also) with $|w_{k_n}| \quad r_{k_n}$, $n \quad 1, 2, \ldots$. By the Weierstrass theorem there exists an entire function $\phi(z)$, say having zeros only at the points w_{k_n}, $n - 1, 2, \ldots$ and so the function $\phi(z)$ serves the purpose.

Let now $0 < \alpha < \rho_1$. Let $g_n = \log r_n$. Then the sequence $\{g_n\}$ satisfies the conditions of the lemma. So there exists a subsequence $\{g_{k_n}\}$ of $\{g_n\}$ such that

$$\alpha - \limsup_{n \to \infty} \frac{\log n}{g_{k_n}} - \limsup_{n \to} \frac{\log n}{\log r_{k_n}}.$$

As in the preceding case, there exists an entire function $g(z)$, say having the desired property.

It is clear that the choice of w_{k_n} with $|w_{k_n}| = r_{k_n}$ is infinite, having the cardinality c. Moreover, if an entire function $f(z)$ has zeros at the points z , $n \quad 1, 2, \ldots; r_n \quad |z_n| \to \infty$ as $n \to \infty$, then any function of the form $e^{\phi(z)} f(z)$, where $\phi(z)$ is an entire function, also has zeros at the same set of points. We therefore obtain a family of entire functions having cardinality c, where each member of the family has α as exponent of convergence of its eros. This proves the theorem.

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Received 5. 8. 2005
Revised 11. 8. 2005

* *B-1/146 Kalyani*
West Bengal-741235
INDIA
E-mail: bklahiri@hotpop.com

** *Department of Mathematics*
Visva Bharati University
Santiniketan
West Bengal-731235
INDIA
E-mail: dibyendu192@rediffmail.com