Mathematica Slovaca

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Mathematica Slovaca, Vol. 57 (2007), No. 2, [157]--170

Persistent URL: http://dml.cz/dmlcz/136944

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0006-7
Math. Slovaca **57** (2007), No 2, 157–170

OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF A NONLINEAR SECOND ORDER NEUTRAL DIFFERENTIAL EQUATION

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(*Communicated by Michal Fečkan*)

ABSTRACT. In this paper, necessary and sufficient conditions for the oscillation and asymptotic behaviour of solutions of the second order neutral delay differential equation (NDDE)

$$\left[r(t)\big(y(t)-p(t)y(t-\tau)\big)'\right]'+q(t)G\big(y(h(t))\big)=0$$

are obtained, where $q,h\in C\big([0,\infty),\mathbb{R}\big)$ such that $q(t)\geq 0$, $r\in C^{(1)}\big([0,\infty),(0,\infty)\big)$, $p\in C\big([0,\infty),\mathbb{R}\big)$, $G\in C(\mathbb{R},\mathbb{R})$ and $\tau\in\mathbb{R}^+$. Since the results of this paper hold when $r(t)\equiv 1$ and $G(u)\equiv u$, therefore it extends, generalizes and improves some known results.

1. Introduction

In this paper we find sufficient conditions for every solution of

$$\left[r(t)\big(y(t)-p(t)y(t-\tau)\big)'\right]'+q(t)G\big(y(h(t))\big)=0 \tag{E}$$

and necessary conditions for every solution of

$$\left[r(t)\big(y(t)-p(t)y(t-\tau)\big)'\right]'+q(t)G\big(y(h(t))\big)=f(t) \tag{F}$$

to oscillate or tend to zero as $t\to\infty$, where $q\in C\big([0,\infty),\mathbb{R}^+\big)$, $f,h\in C\big([0,\infty),\mathbb{R}\big)$, $r\in C^1\big([0,\infty),(0,\infty)\big)$, $p\in C\big([0,\infty),\mathbb{R}\big)$, $G\in C(\mathbb{R},\mathbb{R})$, $h(t)\leq t$, $\lim\limits_{t\to\infty}h(t)=\infty$, $\tau\in\mathbb{R}^+$. We need the following assumptions for our use in the sequel:

2000 Mathematics Subject Classification: Primary 34C10, 34C15, 34K40.
Keywords: neutral differential equation, oscillation, non-oscillation.

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(H_1) There exists $F \in C^{(1)}\big([0,\infty),\mathbb{R}\big)$ such that $F'(t) = f(t)$ and $\lim\limits_{t\to\infty} F(t) = 0$.

(H_2) G is non-decreasing, $uG(u) > 0$ for $u \neq 0$.

(H_3) $\int\limits_0^\infty q(t)\,\mathrm{d}t = \infty$.

(H_4) $\int\limits_0^\infty \frac{\mathrm{d}t}{r(t)} = \infty$.

(H_5) $\int\limits_0^\infty \frac{\mathrm{d}t}{r(t)} < \infty$.

(H_6) For every sequence $\langle\sigma_i\rangle \subset (0,\infty)$, $\sigma_i \to \infty$ as $i \to \infty$ and for every $\beta > 0$ such that the intervals $(\sigma_i - \beta, \sigma_i + \beta)$, $i = 1, 2, \dots$, are non overlapping,

$$\sum_{i=1}^{\infty} \int\limits_{\sigma_i-\beta}^{\sigma_i+\beta} q(t)\,\mathrm{d}t = \infty\,.$$

(H_7) $\int\limits_0^\infty \frac{1}{r(t)}\Big(\int\limits_0^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t = \infty$.

(H_8) $\Big|\int\limits_0^\infty f(t)\,\mathrm{d}t\Big| < \infty$.

(H_9) Suppose that G is Lipschitzian in every interval $[a,b]$, $0 < a < b$.

Remark 1.

(i) Since $r(t) > 0$, therefore one and only one of (H_4) and (H_5) holds.

(ii) (H_6) implies (H_3).

(iii) If (H_5) holds, then (H_7) $\implies$ (H_3).

(iv) (H_1) $\iff$ (H_8).

(v) If (H_4) holds, then (H_3) $\implies$ (H_7).

We assume that $p(t)$ lies in one of the following ranges in this work.

(A_1) $0 \le p(t) \le p_1 < 1$,
(A_2) $-1 < -p_2 \le p(t) \le 0$,
(A_3) $-p_4 \le p(t) \le -p_3 < -1$,
(A_4) $0 \le p(t) \le p_4$,
(A_5) $1 \le p(t) \le p_4$,
(A_6) $p_4 > p(t) \ge p_3 > 1$,
(A_7) $-p_4 \le p(t) \le 0$,
where p_i $(i = 1, \dots, 4)$ is a positive real number.

In recent years, oscillatory and asymptotic behaviour of solutions of NDDEs

$$\big(y(t) - p(t)y(t-\tau)\big)' + q(t)G\big(y(t-\sigma)\big) = f(t) \tag{1}$$

and

$$\big(y(t) - p(t)y(t-\tau)\big)^{(n)} + q(t)G\big(y(t-\sigma)\big) = f(t) \tag{2}$$

are studied by many authors (see [3], [8]–[14]) for both n odd and even. But most of the results are concerned with either (A_1) or (A_2) as ranges of the coefficient function $p(t)$. Second order NDDEs have applications in problems dealing with vibrating masses attached to an elastic and also appear as the Euler equation, in some vibrational problems (see Driver [4] and Hale [7]). The second order and in general even order neutral equations are not as often studied in detail as the odd order NDDEs (1) and (2). It is well known that behaviour of solutions of odd order and even order NDDEs are quite different at times. In [1], [2], [5], [8], [9] the authors have studied the behaviour of solutions of NDDEs of second order. It seems that [8] is the only result about the oscillatory behaviour of solutions of second order neutral equation (E), available in the literature. In [8] the authors consider

$$\Big(r(t)\big(y(t) - p(t)y(t-\tau)\big)'\Big)' + q(t)G\big(y\big(h(t)\big)\big) = 0 \tag{3}$$

and prove the following theorem.

THEOREM 1.1. *Assume that $-1 \le p(t) \le 0$ and $r(t) > 0$. Further, suppose that (H_2) and (H_4) hold, and $q(t) \ge 0$ for $t \ge t_0$, and*

$$\frac{G(x)}{x} \ge \gamma > 0 \qquad \text{for} \quad x \ne 0, \tag{4}$$

$$t - \tau \le h(t) \le t, \quad h'(t) \ge 0, \tag{5}$$

$$\int_0^\infty q(s)\big(1 + p\big(h(s)\big)\big)\,\mathrm{d}s = \infty. \tag{6}$$

Then every solution of (3) *oscillates.*

Again, if we put $f(t) \equiv 0$ in [2, Theorem 1], we get the following result.

THEOREM 1.2. *If (H_2), (H_5), and (H_7) hold, then every solution of*

$$\big(r(t)x'(t)\big)' + q(t)G\big(x\big(h(t)\big)\big) = 0 \tag{7}$$

is either oscillatory or tends to zero as $t \to \infty$.

Finally we note another result ([9, Theorem 2.8] for $r(t) = t - \tau$ and $f(t) \equiv 0$) which is:

THEOREM 1.3. *Suppose that* (H_3) *holds. Further if*

$$p(t)p(t-\tau) \geq 0 \qquad \text{and} \qquad -1 < -p_2 \leq p(t) \leq p_1 < 1 \tag{8}$$

where p_1 *and* p_2 *are positive constants,*

$$|u| > \delta \implies |G(u)| > \eta \qquad \text{where} \quad \delta > 0 \ \text{ and } \ \eta > 0\,. \tag{9}$$

Then every solution of

$$\big(y(t) - p(t)y(t-\tau)\big)'' + q(t)G\big(y(t-\sigma)\big) = 0 \tag{10}$$

is oscillatory or tends to zero as $t \to \infty$.

In [8] and [9] only one range of $p(t)$ is considered and the results there hold for G satisfying either (4) or (9). In this paper an attempt is made to extend $p(t)$ to all possible ranges. Further we do not have any restriction on G. Also this paper generalises the results of [2] that is from delay differential equation to neutral delay differential equation. In the literature the conditions assumed differ from author to author due to the different technique they use and different equation they consider. Even the conditions assumed by different authors for similar type of equations are often not comparable. Because of the simplicity of the hypothesis assumed in this paper we ask for a comparison of our result with some of the work of [2] and [9]. While considering $p(t)$ in a particular range we tried to give two results one with (H_4) and another with (H_5). The results with (H_4) allow us to take $r(t) \equiv 1$ and thus it has the scope to generalize some of the existing results available in the literature. Last but not least, our Theorem 2.11 answers the problem (10.10.2) of [6, p. 287].

Let $T_y \geq 0$ and $T_0 = \min\{h(T_y), T_y - \tau\}$. Suppose $\phi \in C\big([T_0, T_y], \mathbb{R}\big)$. By a solution of (E) we mean a real valued continuous function $y \in C^{(2)}\big([T_y, \infty), \mathbb{R}\big)$ such that $y(t) = \phi(t)$ for $T_0 \leq t \leq T_y$ and for $t \geq T_y$, $\big(y(t) - p(t)y(t-\tau)\big)$ is differentiable, $r(t)\big(y(t) - p(t)y(t-\tau)\big)'$ is again differentiable and then (E) is satisfied. Such a solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory.

In the sequel, for convenience, when we write a functional equation or inequation with out specifying its domain of validity, we assume that it holds for all sufficiently large t.

2. Sufficient conditions

In this section we give the sufficient conditions for every solution of (E) to be oscillatory or tending to zero.

We need the following Lemma ([6, Lemma 1.5.2]) for our work:

Lemma 2.1. *Let F^*, G^*, $p\colon [0,\infty) \to \mathbb{R}$ be such that*

$$F^*(t) = G^*(t) - p(t)G^*(t-c), \qquad t \ge c,$$

where $c \ge 0$. Suppose that $p(t)$ is in one of the ranges given by $(\mathrm{A}_1) - (\mathrm{A}_6)$. If $G^*(t) > 0$ for $t \ge 0$, $\liminf\limits_{t\to\infty} G^*(t) = 0$ and $\lim\limits_{t\to\infty} F^*(t) = L \in \mathbb{R}$ exists, then $L = 0$.

Note. If $G^*(t) < 0$, then lim inf is replaced by lim sup in the above result.

Note. We assume that (H_2) holds in all the results to follow in this work though explicitly we do not mention it.

Lemma 2.2. *Suppose that $p(t)$ satisfies (A_1) or (A_2). If (H_3) and (H_4) hold and $y(t)$ is a non-oscillatory solution of (E) for $t \ge T_y$, then setting*

$$z(t) = y(t) - p(t)y(t-\tau) \tag{11}$$

for large $t > t_0$, we conclude that $\lim\limits_{t\to\infty} z(t) = 0$.

Proof. Let $y(t)$ be a positive solution of (E) for $t \ge T_y$. Then setting

$$r(t)z'(t) = w(t) \tag{12}$$

for $t > t_1 \ge t_0 + \tau$ we obtain

$$w'(t) = -q(t)G\big(y(h(t))\big) \le 0. \tag{13}$$

Then $\lim_{t\to\infty} w(t) = l$ where $-\infty \le l < \infty$. Consider the first case when l is finite. We claim that $\liminf_{t\to\infty} y(t) = 0$. Otherwise $y(t) > m > 0$ for $t \ge t_3 > t_2$, which implies $\int_{t_3}^{\infty} q(s)G\big(y\big(h(s)\big)\big)\,\mathrm{d}s = \infty$, by ($H_3$). But integrating (13) from t_3 to t and then taking limit as $t \to \infty$ we obtain $\int_{t_3}^{\infty} q(s)G\big(y\big(h(s)\big)\big)\,\mathrm{d}s < \infty$, a contradiction. Thus our claim holds. From (13) we have $w(t) > 0$ or $w(t) < 0$. Again from (12) and the fact that $r(t) > 0$ it follows that $z'(t) > 0$ or $z'(t) < 0$. Consequently $z(t) > 0$ or $z(t) < 0$. Hence $-\infty \le \lim_{t\to\infty} z(t) \le \infty$. If $-\infty \le \lim_{t\to\infty} z(t) < 0$, then $y(t)$ is bounded, which implies $\lim_{t\to\infty} z(t) \ne -\infty$. Hence $\lim_{t\to\infty} z(t) = 0$ by Lemma 2.1. Again if $\lim_{t\to\infty} z(t) = \infty$, then in this case $z'(t) > 0$ and $\liminf_{t\to\infty}\big(y(t)/z(t)\big) = 0$. Now $\lim_{t\to\infty}\left(\frac{y(t)}{z(t)} - \frac{p*(t)y(t-\tau)}{z(t-\tau)}\right) = 1$ where $p^*(t) = p(t)z(t-\tau)/z(t)$ and $p^*(t)$ lies in the same range as $p(t)$. Hence we get a contradiction due to Lemma 2.1. Thus $\lim_{t\to\infty} z(t)$ is finite and equal to 0 by Lemma 2.1. Next consider the second case $l = -\infty$. This implies $w < 0$ and $z' < 0$ and consequently $-\infty \le \lim_{t\to\infty} z(t) < \infty$. If $\lim_{t\to\infty} z(t) = -\infty$, then $p(t)$ can only be in (A_1), but not in (A_2). In that case $z(t) < 0$ and consequently $y(t)$ is bounded, a contradiction. Hence $\lim_{t\to\infty} z(t)$ is finite. But next we prove that this also is not possible. We observe that $w(t) < 0$ and is decreasing. Hence for $t \ge t_2$ it follows that $w(t) \le w(t_2)$. From this, we find t_3 such that $t \ge t_3 > t_2$ implies $z'(t) \le w(t_2)/r(t)$. Integrating from t_3 to t then taking limit $t \to \infty$, we obtain $z(t) \to -\infty$ by (H_4), a contradiction. The case for $y(t) < 0$ is similar. Thus the lemma is proved. □

Remark 2. We don't need (H_4) in the proof of the above lemma for the first case that is when l is finite.

Lemma 2.3. *Suppose that $p(t)$ satisfies* (A_1) *or* (A_2). *If* (H_5) *and* (H_7) *hold and $y(t)$ be a non-oscillatory solution of* (E) *for $t \ge T_y$, then setting $z(t)$ as in* (11) *we conclude that $\lim_{t\to\infty} z(t) = 0$.*

Proof. Using Remarks 1(iii) we observe that (H_3) holds. Next we proceed as in Lemma 2.2 and see that the proof for the first case when l is finite is similar. In the second case also that is when $l = -\infty$, we proceed on similar lines and prove that $\lim_{t\to\infty} z(t) = a$ is finite. Next we claim $\liminf_{t\to\infty} y(t) = 0$. Otherwise,

$y(t) > m > 0$ for $t \geq t_4 > t_3$. Hence from (12) we obtain

$$w(t) - w(t_4) = -\int_{t_4}^{t} q(s)G\big(y\big(h(s)\big)\big)\,\mathrm{d}s < -G(m)\int_{t_4}^{t} q(s)\,\mathrm{d}s.$$

This further implies

$$r(t)z'(t) - w(t_4) \leq -G(m)\int_{t_4}^{t} q(s)\,\mathrm{d}s\,.$$

Then we get $z'(t) < -\frac{G(m)}{r(t)}\int_{t_4}^{t} q(s)\,\mathrm{d}s$ for $t \geq t_5 > t_4$.

Integrating this inequality between t_4 and t we obtain

$$z(t) < z(t_4) - G(m)\int_{t_4}^{t} \frac{1}{r(u)}\bigg(\int_{t_4}^{u} q(s)\,\mathrm{d}s\bigg)\,\mathrm{d}u\,.$$

Then taking limit as $t \to \infty$, and using (H_7) we see that $z(t)$ tends to $-\infty$, a contradiction. Hence our claim holds. Then Lemma 2.1 yields $\lim\limits_{t\to\infty} z(t) = a = 0$. Thus the lemma is proved. □

LEMMA 2.4. *Suppose that $p(t)$ satisfies (A_5). If (H_3) and (H_4) hold and $y(t)$ be a non-oscillatory solution of (E) for $t \geq T_y$, then setting $z(t)$ as in (11), we conclude that $\lim\limits_{t\to\infty} z(t) = 0$ or $\lim\limits_{t\to\infty} z(t) = -\infty$.*

Proof. We proceed as in Lemma 2.2 for the first case, that is when $\lim\limits_{t\to\infty} w(t)$ is finite, to prove $\liminf\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} z(t) = a$ where $-\infty \leq a \leq \infty$. If $a > 0$, then by (A_5) it follows that $\liminf\limits_{t\to\infty} y(t) > 0$, a contradiction. Again if $a < 0$ but finite, then by Lemma 2.1 we get $a = 0$, a contradiction. Hence $\lim\limits_{t\to\infty} z(t) = -\infty$ or 0.

Next consider the second case, i.e $\lim\limits_{t\to\infty} w(t) = -\infty$. In this case $z'(t) < 0$ and $\lim\limits_{t\to\infty} z(t) = a$ where a is finite or $-\infty$. Suppose that a is not $-\infty$. Then we see that (H_7) holds by Remark 1(v). Using (H_7) we proceed as in Lemma 2.3 to prove $z(t) \to -\infty$ if $\liminf\limits_{t\to\infty} y(t) \neq 0$, which is a contradiction. Hence $\liminf\limits_{t\to\infty} y(t) = 0$. Then by Lemma 2.1 we get $\lim\limits_{t\to\infty} z(t) = 0$. Since $z(t)$ is monotonic decreasing therefore $z(t) > 0$. This implies $\liminf\limits_{t\to\infty} y(t) > 0$, a contradiction. Hence the only possibility left out is $\lim\limits_{t\to\infty} z(t) = -\infty$. Thus the lemma is proved. □

Lemma 2.5. *Suppose that $p(t)$ satisfies* (A_5). *If* (H_5) *and* (H_7) *hold and $y(t)$ be a non-oscillatory solution of* (E) *for $t \ge T_y$, then setting $z(t)$ as in* (11), *we conclude that $\lim_{t\to\infty} z(t) = 0$ or $\lim_{t\to\infty} z(t) = -\infty$.*

Proof. The proof is similar to that of Lemma 2.4 if we use Remarks 1(iii) and 2. □

Theorem 2.6. *Suppse that* (H_3) *and* (H_4) *hold. If $p(t)$ is in the range* (A_1) *or* (A_2), *then every solution of* (E) *oscillates or tends to zero as $t \to \infty$.*

Proof. If $y(t)$ is eventually a positive solution for large t, then setting $z(t)$ and $w(t)$ as in (11) and (12), respectively, we obtain $\lim_{t\to\infty} z(t) = 0$ by Lemma 2.2. Hence, if $p(t)$ is in (A_1), then we have

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} \bigl[y(t) - p(t)y(t-\tau)\bigr] \\ &\ge \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} \bigl(-p_1 y(t-\tau)\bigr) \\ &\ge (1-p_1) \limsup_{t\to\infty} y(t)\,. \end{aligned}$$

Thus $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$. Again, if $p(t)$ is in (A_2), then $\lim_{t\to\infty} y(t) = 0$ follows from the fact that $y(t) < z(t)$. If $y(t) < 0$ for $t > t_0$, then one may proceed as above and prove $\lim_{t\to\infty} y(t) = 0$. Hence the theorem is proved. □

Remark 3. If in the above theorem we take $r(t) \equiv 1$, then we get a result which improves Theorem 1.3.

Example 1. The NDDE

$$\left(\mathrm{e}^{-2t}\left(y(t) - \frac{1}{2}y(t-\ln 3)\right)'\right)' + \frac{3}{16}y^3(t-\ln 2) = 0 \tag{14}$$

satisfies all the conditions of Theorem 2.6. Hence all non-oscillatory solutions of (14) tend to zero as $t \to \infty$. In particular $y(t) = \mathrm{e}^{-t}$ is such a solution. Here $G(u) = u^3$ is superlinear which satisfies the general superlinear condition $\int_k^\infty \mathrm{d}u/G(u) < \infty$.

Remark 4. Theorem 2.6 is an extension and generalisation of Theorem 1.1 under (A_2) in view of the fact that $(H_3) \iff (6)$. We do not require (4) or (5) for Theorem 2.6 though Theorem 1.1 requires all these conditions. Further Theorem 1.1 does not hold when G is sublinear whereas Theorem 2.6 holds for all types of G.

COROLLARY 2.7. *Suppse that* (H_5) *and* (H_7) *hold. If* $p(t)$ *is in the range* (A_1) *or* (A_2), *then every solution of* (E) *oscillates or tends to zero as* $t \to \infty$.

Proof. The proof is similar to that of Theorem 2.6, and the difference here is that Lemma 2.3 is to be applied in place of Lemma 2.2. □

Remark 5. If we put $p(t) = 0$ in the above result, then Theorem 1.2 follows from Corollary 2.7.

THEOREM 2.8. *Suppose that* $p(t)$ *satisfies* (A_6). *If* (H_3) *and* (H_4) *hold, then every bounded solution of* (E) *oscillates or tends to zero as* $t \to \infty$.

Proof. From Lemma 2.4, it follows that if $y(t)$ is an eventually positive bounded solution of (E), then $z(t)$ is bounded. Hence by Lemma 2.4 we observe that $\lim_{t\to\infty} z(t) = -\infty$ is not possible. Hence

$$0 = \lim_{t\to\infty} z(t) = \liminf_{t\to\infty}\big[y(t) - p(t)y(t-\tau)\big] \le (1-p_3)\limsup_{t\to\infty} y(t).$$

Hence $\lim_{t\to\infty} y(t) = 0$ and the proof for the case $y(t) < 0$ is similar. □

COROLLARY 2.9. *Suppose that* $p(t)$ *satisfies* (A_6). *If* (H_5) *and* (H_7) *hold, then every bounded solution of* (E) *oscillates or tends to zero as* $t \to \infty$.

Proof. The proof is similar to that of Theorem 2.8 and Lemma 2.5 is to be applied here in place of Lemma 2.4. □

THEOREM 2.10. *Suppose that* (H_4), (H_6) *hold and* $p(t)$ *satisfies* (A_5). *Then every unbounded solution of* (E) *oscillates or tends to* $\pm\infty$ *as* $t \to \infty$ *and every bounded solution of* (E) *oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a positive bounded solution of (E) for $t > t_0$. Then setting $z(t)$ and $w(t)$ as in (11) and (12) respectively and applying Lemma 2.4 we get $\lim_{t\to\infty} z(t) = 0$. If $\limsup_{t\to\infty} y(t) = \alpha$, with $\alpha > 0$, then there exists a sequence $\langle t_n\rangle$ such that $y(t_n) > M > 0$ for $n \ge N_1 > 0$. From the continuity of y it follows that there exists $\delta_n > 0$ with $\liminf_{n\to\infty} \delta_n > 0$ such that $y(t) > M$ for

$t \in (t_n - \delta_n, t_n + \delta_n)$. Then choosing n large enough such that $\delta_n > \delta > 0$ for $n \ge N > N_1$, we obtain

$$\begin{aligned}
\int_{T_2}^{\infty} q(t)G\big(y\big(h(t)\big)\big)\,\mathrm{d}t &\ge \sum_{n=N}^{\infty} \int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} q(t)G\big(y\big(h(t)\big)\big)\,\mathrm{d}t \\
&> G(M) \sum_{n-N}^{\infty} \int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} q(t)\,\mathrm{d}t \\
&> G(M) \sum_{n=N}^{\infty} \int_{t_n-\delta+\sigma}^{t_n+\delta+\sigma} q(t)\,\mathrm{d}t\,.
\end{aligned}$$

From the given hypothesis (H_6), it follows that

$$\int_{T_2}^{\infty} q(t)G\big(y\big(h(t)\big)\big)\,\mathrm{d}t = \infty\,.$$

Since $\lim\limits_{t\to\infty} z(t) = 0$ therefore from the proof of Lemma 2.4 it is clear that $\lim\limits_{t\to\infty} w(t)$ exists. Hence integrating (13) we get

$$\int_{T_2}^{\infty} q(t)G\big(y\big(h(t)\big)\big)\,\mathrm{d}t < \infty\,,$$

a contradiction. Hence $\limsup\limits_{t\to\infty} y(t) = 0$, which implies $\lim\limits_{t\to\infty} y(t) - 0$. If $y(t) < 0$ for large t and bounded, then we proceed as above to show that $\lim\limits_{t\to\infty} y(t) = 0$.

Next let $y(t)$ be an unbounded positive solution of (E) for large t. Then we apply Lemma 2.4 and obtain $\lim\limits_{t\to\infty} z(t) = 0$ or $\lim\limits_{t\to\infty} z(t) = -\infty$. If $\lim\limits_{t\to\infty} z(t) \quad 0$, then as in the above we prove $\lim\limits_{t\to\infty} y(t) = 0$. If $\lim\limits_{t\to\infty} z(t) = -\infty$, then from ($A_5$) we get $y(t-\tau) \ge -z(t)/p_4$ and this implies $\lim\limits_{t\to\infty} y(t) = \infty$. The proof for the case when $y(t) < 0$ for $t > t_0$ is similar. □

Theorem 2.11. *Suppose that* (H_4) *holds and* $p(t)$ *satisfies* (A_7). *Again assume that* $h(t) = t - \sigma$, σ *is a nonnegative constant. If* $\rho = \max\{\tau, \sigma\}$ *and*

(H_{10}) $\int_{\rho}^{\infty} q^*(t)\,\mathrm{d}t = \infty$, *where* $q^*(t) = \min\{q(t), q(t-\tau)\}$,

(H_{11}) *for* $u > 0$, $\nu > 0$, *there exists* $\delta > 0$ *such that* $G(u) + G(\nu) \geq \delta G(u + \nu)$,

(H_{12}) $G(-u) = -G(u)$

(H_{13}) *for* $u > 0$, $\nu > 0$, $G(u)G(\nu) \geq G(u\nu)$,

then every solution of (E) *oscillates or tends to zero as* $t \to \infty$.

Remark 6. (H_{10}) implies (H_3).

Remark 7. The prototype of G satisfying (H_2), (H_{11})–(H_{13}) is $G(u) = \big(\beta + |u|^{\mu}\big)|u|^{\lambda}\operatorname{sgn} u$, where $\lambda > 0$, $\mu > 0$, $\beta \geq 1$.

Proof of Theorem 2.11. Let $y(t)$ be a positive solution of (E) for $t > t_0$. Then setting $z(t)$ and $w(t)$ as in (11) and (12) respectively, we arrive at (13). Then $\lim_{t\to\infty} w(t) = l$ where $-\infty \leq l < \infty$. If we consider the first case $l \neq -\infty$, then consequently since $r(t) > 0$, we obtain $z'(t) > 0$ or $z'(t) < 0$. This implies $\lim_{t\to\infty} z(t) = a$ where $-\infty \leq a \leq \infty$. We see that $z(t) > 0$ because of (A_7). Hence $a < 0$ is not possible. If $a = 0$, then we are happy to have our necessary conclusion that $\lim_{t\to\infty} y(t) = 0$ because $y(t) \leq z(t)$. If $a > 0$, then $z(t) > \alpha > 0$ for $t \geq t_2 > t_1$. Then using (H_{11}) and (H_{13}), we obtain for $t \geq t_3 > t_2$

$$\begin{aligned}
0 &= w'(t)+q(t)G\big(y(t-\sigma)\big)+G\big(-p(t-\sigma)\big)\big[w'(t-\tau) + q(t-\tau)G\big(y(t-\tau-\sigma)\big)\big] \\
&\geq w'(t) + G(p_4)w'(t-\tau) + \delta q^*(t)\big[G\big(z(t-\sigma)\big)\big] \\
&\geq w'(t) + G(p_4)w'(t-\tau) + G(\alpha)\delta q*(t).
\end{aligned}$$

Integrating from t_3 to t and then taking limit as $t \to \infty$ we arrive at a contradiction due to (H_{10}). Next consider the second case $l = -\infty$. Then $w(t) < 0$ and consequently $z' < 0$. Because of (A_7) we have $\lim_{t\to\infty} z(t) = a$ where $a \geq 0$ and is finite. Then as in Lemma 2.2 we use (H_4) and obtain $z(t) \to -\infty$, a contradiction. The proof for the case when $y(t) < 0$ for $t > t_0$ is similar. It may be noted that (H_{12}) is needed for the case $y(t) < 0$. Thus the theorem is proved. □

Remark 8. In [6, p. 287], the authors have proposed the following open problem.

(10.10.2): *Extend the results of* Section 10.4 *to equations where the coefficient function* $p(t)$ *lies in different ranges.*

The results of that section deal with (E) where $G(u) = u$ and $r(t) = 1$. The ranges of $p(t)$ they have considered in that section is (A_1), (A_2), (A_5). Hence our Theorem 2.11 answers that problem.

3. Necessary conditions

In this section we prove that either (H_3) or (H_4) are necessary for every solution of (F) to be oscillatory or tending to zero.

THEOREM 3.1. *Let* (H_1), (H_2), *and* (H_9) *hold. Suppose that* $p(t)$ *is in* (A_2) *and* $h(t) = t - \sigma$ *where* σ *is a positive constant. Then every solution of* (F) *oscillates or tends to zero as* $t \to \infty$ *implies* (H_3) *or* (H_4) *holds.*

Proof. Suppose it is not true that (H_3) or (H_4) holds. Then by Remark 1 we have (H_5) holds and

$$\int_0^\infty q(t)\,\mathrm{d}t < \infty \tag{16}$$

holds. From (H_1) and (H_5) and (16) we find $T > 0$ such that for $t \ge T$

$$k\int_t^\infty q(s)\,\mathrm{d}s < \frac{1-p_2}{10}, \qquad |F(t)| < \frac{1-p_2}{10} \qquad \text{and} \qquad \int_t^\infty \frac{\mathrm{d}s}{r(s)} < \frac{1}{2},$$

where $k = \max\{G(1), k_1\}$, k_1 is the Lipschitz constant. We take

$$X = \Big\{u \in BC\big([T,\infty),\mathbb{R}\big) : \tfrac{1-p_2}{10} \le u(t) \le 1\Big\}.$$

Define S on X as

$$Sy(t) = \begin{cases} p(t)y(t-\tau) - \int_t^\infty \frac{1}{r(s)}\Big(\int_s^\infty q(u)G\big(y(h(u))\big)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ \quad + \frac{4p_2+1}{5} - \int_t^\infty \frac{F(s)}{r(s)}\,\mathrm{d}s, & t > T+\rho, \\ Sy(T+\rho), & T \le t < T+\rho, \end{cases}$$

where $\rho = \max\{\tau, \sigma\}$. Then we apply Banach contraction principle ([6, p. 30] and show $S(X) \subset X$ and $\|Sy_1 - Sy_2\| \le \mu\|y_1 - y_2\|$ where $\mu = (19p_2 + 1)/20 < 1$. Thus S is a contraction, admitting a fixed point y_0 which is the required positive solution with $\liminf\limits_{t\to\infty} y_0(t) > (1-p_2)/10$ □

Remark 9. One may easily develop a similar theorem when $p(t)$ is in (A_1).

Remark 10. For the ranges $1 < p_3 < p(t) \leq p_4$ and $-p_4 \leq p(t) \leq -p_3 < -1$ also we can find a positive solution of (F) which does not tend to zero, as we have done in Theorem 3.1. Here we have to define the contraction mapping S as

$$Sy(t) = \begin{cases} \frac{y(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} \frac{1}{r(s)} \left(\int\limits_{s}^{\infty} q(u)G\bigl(y(h(u))\bigr)\,\mathrm{d}u \right) \mathrm{d}s & \\ + \frac{1}{p(t+\tau)} \int\limits_{t+\tau}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s + \frac{K(p_3,p_4)}{p(t+\tau)}\,, & t \geq T+\tau\,, \\ Sy(T+\tau)\,, & T \leq t \leq T+\tau\,, \end{cases}$$

where $K(p_3, p_4)$ is a constant depending on p_3 and p_4.

Remark 11. In Theorem 3.1, if we take $f(t) \equiv 0$ (which is admissible), then we conclude that the conditions (H_3) or (H_4) is necessary for all solutions of (E) to oscillate or tend to zero under the assumptions (H_2) and (H_9).

Acknowledgement. The authors are very much thankful and obliged to the referee for his comments which helped to improve the presentation of the paper.

REFERENCES

[1] DAS, P: *Oscillations and asymptotic behaviour of solutions for second order neutral delay differential equations*, J. Indian Math. Soc. (N.S.) **60** (1994), 159–170.

[2] DAS, P: *On the behaviour of the solutions of* $\bigl(r(t)x'\bigr)' + p(t)G\bigl(x(g(t))\bigr) = f(t)$, J. Math. Anal. Appl. **187** (1994), 752 768.

[3] DAS, P.—MISRA, N: *A necessary and sufficient condition for the solution of a functional differential equation to be oscillatory or tend to zero*, J. Math. Anal. Appl. **204** (1997), 78 87.

[4] DRIVER, R. D: *Ordinary and Delay Differential Equations*, Springer Verlag, New York, 1976.

[5] GRAMMATIKOPOULOS, M. K.—LADAS, G.—MEIMARIDOU, A: *Oscillation and asymptotic behaviour of second order neutral differential equations*, Ann. Mat. Pura Appl. (4) **148** (1987), 29–40.

[6] GYORI, I.—LADAS, G: *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[7] HALE, J. K: *Theory of Functional Differential Equations*, Springer Verlag, New York, 1977.

[8] LALLI, B. S: *Oscillation of nonlinear second order neutral delay differential equations*, Bull. Inst. Math. Acad. Sinica **18** (1990), 233 238.

[9] PARHI, N. MOHANTY, P. K: *On oscillatory forced neutral differential equations of second order*, J. Indian. Math. Soc. (N.S.) **61** (1995), 241 254.

[10] PARHI, N.—MOHANTY, P. K: *Oscillation of solutions of forced nutral differential equations of n-th order*, Czechoslovak Math. J. **45** (1995), 413 433.

[11] PARHI, N.—RATH, R. N: *Oscillation criteria for forced first order neutral differential equations with variable coefficients*, J. Math. Anal. Appl. **256** (2001), 525–541.
[12] PARHI, N.—RATH, R. N: *On oscillation of solutions of forced nonlinear neutral differential equations of higher order II*, Ann. Polon. Math. **81** (2003), 101–110.
[13] PARHI, N.—RATH, R. N: *On oscillation and asymptotic behaviour of solutions of forced first order neutral differential equations*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), 337–350.
[14] RATH, R. N: *Necessary and sufficient conditions for the oscillatory behaviour of solutions of a first order neutral equation with several delays*, Math. Slovaca **53** (2003), 78–89.
[15] ROYDEN, H. L: *Real Analysis* (3rd ed.) MacMillan Publ. Co., New York, 1989.

Received 22. 12. 2003
Revised 15. 7. 2005

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